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Research Article

Hyers–Ulam–Mittag-Leffler Stability for a System of Fractional Neutral Differential Equations

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This article concerns with the existence and uniqueness for a new model of implicit coupled system of neutral fractional differential equations involving Caputo fractional derivatives with respect to the Chebyshev norm. In addition, we prove the Hyers–Ulam–Mittag-Leffler stability for the considered system through the Picard operator. For application of the theory, we add an example at the end. The obtained results can be extended for the Bielecki norm.

1. Introduction

Models of fractional differential equations (FDEs) have many applications in various fields of engineering and science such as mechanics, electricity, biology, chemistry, physics, and control and signal processing[1, 2]. For the different materials and phenomena, the heredity characteristics are well explained by FDEs, and as a result, many research papers and books have been published in this field [3–7]. FDEs in which the highest fractional derivatives (FDs) of unknown term appear both with and without delays are known as neutral FDEs. In the last few years, the study of neutral FDEs have developed dramatically. This is due the fact that the qualitative behavior of the aforesaid equations are quite different from those of nonneutral FDEs. Neutral FDEs also play a key role and has many advantages. For instance, they give more better description over population fluctuations. Also, neutral FDEs with delay appear in models of electrical networks containing lossless transmission lines, for more details see [8]. Different attempts have been made for the investigation of solutions of fractional and neutral FDEs [9–17].

Another imperative and more remarkable area of research is committed to the stability analysis of the solutions for the FDEs and ordinary orders. Many targets are achieved in this regard, for some recent work we refer the reader to see [18–35]. Niazi et al. [36] investigated the existence, uniqueness (EU), and Hyers–Ulam–Mittag-Leffler (HUML) stability for the following neutral FDEs:

$$\begin{cases} {}^{c}\mathbf{D}_0^{\kappa} u(\omega) = g(\omega, u_\omega, {}^{c}\mathbf{D}_0^{\sigma} u_\omega), & \omega \in [0,1], \\ u(\omega) = \varphi_1(\omega), & \omega \in [-\xi, 0], \end{cases} \quad (1)$$

where $1 < \kappa \leq 2$, $0 < \sigma < 1$, ${}^{c}\mathbf{D}_0^{\kappa}$ and ${}^{c}\mathbf{D}_0^{\sigma}$ are the Caputo derivatives, and $g\colon [0,1] \times C_\xi \times C_\xi \longrightarrow \mathbb{R}$ is continuous. Here, C_ξ denotes the Banach space of all continuous functions $\varphi_1\colon [-\xi, 1] \longrightarrow \mathbb{R}$ with norm

$$\|\varphi_1\| = \max\left\{|\varphi_1(\omega)|\colon -\xi \leq \omega \leq 1\right\}. \quad (2)$$

If $u_\omega \in C_\xi$, then is defined by

$$u_\omega(\omega) = u(\omega + \tau), \quad \tau \in [-\xi, 1], \omega \in [0,1]. \quad (3)$$

As far as we know, results on EU and HUML stability for a coupled system of neutral FDEs have not been investigated by the researchers. Many real-world problems required to be modeled into a coupled system of FDEs as they cannot be described into a single fractional differential equation, see [37] and references cited therein.

Motivated by the abovementioned work, in this article, we study the EU and HUML stability of the following implicit coupled neutral FDEs system involving Caputo FDs. The proposed coupled system (CS) is given by

$$\begin{cases} {}^c\mathbf{D}_0^{\kappa}u(\omega) = g(\omega, u_\omega, {}^c\mathbf{D}_0^{\sigma}v_\omega), & \omega \in \mathbf{J}, \\ {}^c\mathbf{D}_0^{\sigma}v(\omega) = h(\omega, {}^c\mathbf{D}_0^{\kappa}u_\omega, v_\omega), & \omega \in \mathbf{J}, \\ u(\omega) = \varphi_1(\omega), & \omega \in [-\xi, 0], \\ v(\omega) = \varphi_2(\omega), & \omega \in [-\xi, 0], \end{cases} \quad (4)$$

where $\mathbf{J} = [0,1]$, $1 < \kappa, \sigma \leq 2$ and ${}^c\mathbf{D}_0^{\kappa}$ and ${}^c\mathbf{D}_0^{\sigma}$ represent the Caputo FDs of orders κ and σ, respectively. The set of all continuous functions φ_1, φ_2: $[-\xi, 1] \longrightarrow \mathbb{R}$ with norms $\|\varphi_1\| = \max\{|\varphi_1(\omega)|: -\xi \leq \omega \leq 1\}$ and $\|\varphi_2\| = \max\{|\varphi_2(\omega)|: -\xi \leq \omega \leq 1\}$ is a Banach space denoted by C_ξ, and hence their product is also a Banach space. The functions g, h: $\mathbf{J} \times C_\xi \times C_\xi \longrightarrow \mathbb{R}$ are continuous. If u, v: $[-\xi, 1] \longrightarrow \mathbb{R}$ are continuous, then $\forall \omega \in \mathbf{J}$, and the functions $u_\omega, v_\omega \in C_\xi$ are presented by $u_\omega(\tau) = u(\omega + \tau), v_\omega(\tau) = v(\omega + \tau), \tau \in [-\xi, 1], -\xi < 0$.

2. Preliminaries

This section is concerned with some notions, definitions, and preliminary results used throughout the article.

Definition 1 (see [38]). The $\kappa > 0$ order Riemann–Liouville integral for θ^* is

$$\mathbf{I}_a^{\kappa}\theta^*(\omega) = \frac{1}{\Gamma(\kappa)}\int_a^{\omega}(\omega - s)^{\kappa-1}\theta^*(s)\mathrm{d}s, \quad (5)$$

where the integral is point-wise convergent.

Definition 2 (see [38]). Suppose θ^* be a given function on $[a, b]$, the $\kappa \in (n-1, n]$ order Caputo derivative of θ^* is stated by

$${}^c\mathbf{D}_a^{\kappa}\theta^*(\omega) = \int_a^{\omega}\frac{(\omega - s)^{n-\kappa-1}}{\Gamma(n-\kappa)}\left(\frac{\mathrm{d}^n\theta^*(s)}{\mathrm{d}s^n}\right)\mathrm{d}s, \quad (6)$$

with $[\kappa] = n - 1$. Furthermore, if domain of θ^* is $[a, b]$ and $\kappa \in (0, 1]$, then

$${}^c\mathbf{D}_a^{\kappa}\theta^*(\omega) = \frac{1}{\Gamma(1-\kappa)}\int_a^{\omega}\frac{\theta^{*\prime}(s)}{(\omega - s)^{\kappa}}\mathrm{d}s, \quad (7)$$

where $\theta^{*\prime}(s) = ((\mathrm{d}\theta^*(s))/\mathrm{d}s)$.

Theorem 1 (see [39]). *Let $\kappa \in (n-1, n]$, then $\theta^* \in C([a, b], \mathbb{R})$, which implies $(d^{\kappa}/dt^{\kappa})\theta^*(\omega) = 0$ has the formula*

$$\theta^*(\omega) = \sum_{k=0}^{[\kappa]} c_k\omega^k, \quad (8)$$

where $c_k \in \mathbb{R}$, $k = 1, 2, \ldots, [\kappa]$, $[\kappa] + 1 = n$.

Definition 3 (see [40]). For a metric space $(\mathbf{Y}_{\kappa,\sigma}, d)$, an operator $\mathbf{A}_{\kappa,\sigma}$: $\mathbf{Y}_{\kappa,\sigma} \longrightarrow \mathbf{Y}_{\kappa,\sigma}$ is said to be Picard operator (PO) if there is $x^* \in \mathbf{Y}_{\kappa,\sigma}$ so that

(a) $\mathbf{F}_{A_{\kappa,\sigma}} = \{x^*\}$ where $\mathbf{F}_{A_{\kappa,\sigma}} = \{x \in \mathbf{Y}_{\kappa,\sigma}: \mathbf{A}_{\kappa,\sigma}(x) = x\}$.

(b) The sequence $((\mathbf{A}_{\kappa,\sigma})^n(x_0))_{n\in N}$ has the limit $x^* \forall x_0 \in \mathbf{Y}_{\kappa,\sigma}$.

Definition 4 (see [40]). For an ordered metric space $(\mathbf{Y}_{\kappa,\sigma}, d, \preccurlyeq)$, if $\mathbf{A}_{\kappa,\sigma}$: $\mathbf{Y}_{\kappa,\sigma} \longrightarrow \mathbf{Y}_{\kappa,\sigma}$ is increasing PO with $\mathbf{A}_{\kappa,\sigma}(x^*) = x^*$, then $\forall x \in \mathbf{Y}_{\kappa,\sigma}$, $x \leq \mathbf{A}_{\kappa,\sigma}(x)$ implies $x \leq x^*$ and $x \geq \mathbf{A}_{\kappa,\sigma}(x)$ implies $x \geq x^*$.

Lemma 1 (see [41]). *Let $\theta_1, \theta_2 \in C(\mathbf{J}, \mathbb{R}_+)$. If $\kappa > 0$, θ_2 is increasing and there is $m \geq 0$ so that*

$$\theta_1(\omega) \leq \theta_2(\omega) + m\int_0^{\omega}(\omega - s)^{\kappa-1}\theta_1(s)\mathrm{d}s, \quad \omega \in \mathbf{J}. \quad (9)$$

Then,

$$\theta_1(\omega) \leq \theta_2(\omega) + m\int_0^{\omega}\left(\sum_{k=1}^{\infty}\frac{(m\Gamma(\kappa))^n}{\Gamma(n\kappa)}\right)(\omega - s)^{n\kappa-1}\theta_1(s)\mathrm{d}s, \quad \omega \in \mathbf{J}. \quad (10)$$

Lemma 2 (see [42]). *For $\lambda > 0$ and $\kappa > -1$, we have*

$$\int_0^{\omega}(\omega - s)^{\kappa-1}s^{\lambda}\mathrm{d}s = \frac{\Gamma(\lambda)\Gamma(\kappa+1)}{\Gamma(\lambda+\kappa+1)}\omega^{\kappa+\lambda}. \quad (11)$$

3. Main Results

In this section, we provide results regarding the EU and HUML stability for the solution of the considered system on the compact interval $\mathbf{J} = [0, 1]$, using the PO and Henry–Gronwall lemma [41]. Suppose g, h: $\mathbf{J} \times C_\xi \times C_\xi \longrightarrow \mathbb{R}$ be continuous and C_ξ be a Banach space of all the functions φ_1, φ_2: $[-\xi, 1] \longrightarrow \mathbb{R}$ which are continuous with norms $\|\varphi_1\| = \max\{|\varphi_1(\omega)|: -\xi \leq \omega \leq 1\}$ and $\|\varphi_2\| = \max\{|\varphi_2(\omega)|: -\xi \leq \omega \leq 1\}$. Consider the system

$$\begin{cases} {}^c\mathbf{D}_0^{\kappa}u(\omega) = g(\omega, u_\omega, {}^c\mathbf{D}_0^{\sigma}v_\omega), & \omega \in \mathbf{J}, \\ {}^c\mathbf{D}_0^{\sigma}v(\omega) = h(\omega, {}^c\mathbf{D}_0^{\kappa}u_\omega, v_\omega), & \omega \in \mathbf{J}, \\ u(\omega) = \varphi_1(\omega), & \omega \in [-\xi, 0], \\ v(\omega) = \varphi_2(\omega), & \omega \in [-\xi, 0], \end{cases} \quad (12)$$

where $1 < \kappa, \sigma \leq 2$ and $\varphi_1, \varphi_2 \in C([-\xi, 1], \mathbb{R}_+)$. For ϵ_1, ϵ_2, we have the following inequalities:

$$\begin{cases} \left|{}^c\mathbf{D}_0^{\kappa}u(\omega) - g(\omega, u_\omega, {}^c\mathbf{D}_0^{\sigma}v_\omega)\right| \leq \epsilon_1\mathbf{E}_{\kappa}(\omega^{\kappa}), & \omega \in \mathbf{J}, \\ \left|{}^c\mathbf{D}_0^{\sigma}v(\omega) - h(\omega, {}^c\mathbf{D}_0^{\kappa}u_\omega, v_\omega)\right| \leq \epsilon_2\mathbf{E}_{\sigma}(\omega^{\sigma}), & \omega \in \mathbf{J}, \end{cases} \quad (13)$$

where $\mathbf{E}_{\kappa}(\cdot)$ and $\mathbf{E}_{\sigma}(\cdot)$ represent the Mittag-Leffler (ML) function defined by

$$\mathbf{E}_\kappa(\cdot) = \sum_{k=0}^{\infty} \frac{(\cdot)^k}{\Gamma(k\kappa+1)},$$
$$\mathbf{E}_\sigma(\cdot) = \sum_{k=0}^{\infty} \frac{(\cdot)^k}{\Gamma(k\sigma+1)}, \tag{14}$$
$$(\cdot) \in \mathbb{C},\ \mathrm{Re}(\kappa),\ \mathrm{Re}(\sigma) > 0.$$

Definition 5. Extending the definition of Hyers–Ulam (HU) stability for a CS [43], we say that system (12) is HUML stable with respect to $\mathbf{E}_\lambda(\omega^\lambda) = \max\{\mathbf{E}_\kappa(\omega^\kappa), \mathbf{E}_\sigma(\omega^\sigma)\}$ if there is $c_{\mathbf{E}_\lambda} = \max\{c_{\mathbf{E}_\kappa}, c_{\mathbf{E}_\sigma}\} > 0$ so that for every $\varepsilon = \max\{\epsilon_1, \epsilon_2\} > 0$ and each solution $(\overline{u}, \overline{v}) \in C_\xi \times C_\xi$ of the inequalities (13) there is a unique solution $(u, v) \in C_\xi \times C_\xi$ so that

$$|(u,v) - (\overline{u}, \overline{v})| \le c_{E_\lambda} \epsilon \mathbf{E}_\lambda(\omega^\lambda), \quad \lambda = \max\{\kappa, \sigma\}. \tag{15}$$

Remark 1. Functions $\overline{u}, \overline{v} \in C^1(\mathbf{J}, \mathbb{R})$ are, respectively, the solutions of the above inequalities $\Longleftrightarrow$ there is $\theta_1, \theta_2 \in C^1(\mathbf{J}, \mathbb{R})$ so that

(1) $|\theta_1(\omega)| \le \epsilon_1 \mathbf{E}_\kappa(\omega^\kappa)$, $|\theta_2(\omega)| \le \epsilon_2 \mathbf{E}_\sigma(\omega^\sigma)$, $\omega \in \mathbf{J}$;

(2) $\begin{cases} {}^c\mathbf{D}_0^\kappa u(\omega) = g(\omega, u_\omega, {}^c\mathbf{D}^\sigma v_\omega) + \theta_1(\omega), & \omega \in \mathbf{J}, \\ {}^c\mathbf{D}_0^\sigma v(\omega) = h(\omega, {}^c\mathbf{D}^\kappa u_\omega, v_\omega) + \theta_2(\omega), & \omega \in \mathbf{J}. \end{cases}$

Theorem 2. *For $1 < \kappa, \sigma \le 2$, if $\overline{u}, \overline{v} \in C^1(\mathbf{J}, \mathbb{R}_+)$ are, respectively, the solutions of*

$$\begin{cases} {}^c\mathbf{D}_0^\kappa u(\omega) = g(\omega, u_\omega, {}^c\mathbf{D}_0^\sigma v_\omega), & \omega \in \mathbf{J}, \\ u(\omega) = \varphi_1(\omega), & \omega \in [-\xi, 0], \end{cases}$$
$$\begin{cases} {}^c\mathbf{D}_0^\sigma v(\omega) = h(\omega, {}^c\mathbf{D}_0^\kappa u_\omega, v_\omega), & \omega \in \mathbf{J}, \\ v(\omega) = \varphi_2(\omega), & \omega \in [-\xi, 0]. \end{cases} \tag{17}$$

Then, $(\overline{u}, \overline{v})$ satisfies the following integral inequalities:

$$\begin{cases} \left|\overline{u}(\omega) - \overline{u}(0) - \overline{u}'(0)\omega - \frac{1}{2}\overline{u}''(0)\omega^2 - \frac{1}{\Gamma(\kappa)}\int_0^\omega (\omega - s)^{\kappa-1} g(s, \overline{u}_s, {}^c\mathbf{D}_0^\sigma \overline{v}_s) \mathrm{d}s\right| \le \epsilon_1 \mathbf{E}_\kappa(\omega^\kappa), \\ \left|\overline{v}(\omega) - \overline{v}(0) - \overline{v}'(\omega)\omega - \frac{1}{2}\overline{v}''(0)\omega^2 - \frac{1}{\Gamma(\sigma)}\int_0^\omega (\omega - s)^{\sigma-1} h(s, {}^c\mathbf{D}_0^\kappa \overline{u}_s, \overline{v}_s) \mathrm{d}s\right| \le \epsilon_2 \mathbf{E}_\sigma(\omega^\sigma). \end{cases} \tag{18}$$

Proof. From Remark 1, we have

$${}^c\mathbf{D}_0^\kappa \overline{u}(\omega) = g(\omega, \overline{u}_\omega, {}^c\mathbf{D}_0^\sigma \overline{v}_\omega) + \theta_1(\omega), \quad 1 < \kappa,\ \sigma \le 2,\ \omega \in \mathbf{J}. \tag{19}$$

Then,

$$\overline{u}(\omega) - \overline{u}(0) - \overline{u}'(0)\omega - \frac{1}{2}\overline{u}''(0)\omega^2 = \frac{1}{\Gamma(\kappa)}\int_0^\omega (\omega - s)^{\kappa-1} g(s, \overline{u}_s, {}^c\mathbf{D}_0^\sigma \overline{v}_s)\mathrm{d}s + \frac{1}{\Gamma(\kappa)}\int_0^\omega (\omega - s)^{\kappa-1}\theta_1(s)\mathrm{d}s. \tag{20}$$

So, we have

$$\begin{aligned} &\left|\overline{u}(\omega) - \overline{u}\varepsilon - \overline{u}'(0)\varepsilon\omega - \frac{1}{2}\overline{u}''(0)\omega^2 - \frac{1}{\Gamma(\kappa)}\int_0^\omega (\omega - s)^{\kappa-1} g(s, \overline{u}_s, {}^c\mathbf{D}_0^\sigma \overline{v}_s)\mathrm{d}s\right| \\ &\le \frac{1}{\Gamma(\kappa)}\int_0^\omega (\omega - s)^{\kappa-1}|\theta_1(s)| \\ &\le \frac{\epsilon_1}{\Gamma(\kappa)}\int_0^\omega (\omega - s)^{\kappa-1}\mathbf{E}_\kappa(s^\kappa) \\ &= \frac{\epsilon_1}{\Gamma(\kappa)}\sum_{k=0}^{\infty}\frac{1}{\Gamma(k\kappa+1)}\int_0^\omega (\omega - s)^{\kappa-1}s^{k\kappa}\mathrm{d}s \\ &= \frac{\epsilon_1}{\Gamma(\kappa)}\sum_{k=0}^{\infty}\frac{\omega^{(k+1)\kappa}\Gamma(\kappa)\Gamma(k\kappa+1)}{\Gamma(k\kappa+1)\Gamma((k+1)\kappa+1)} \\ &\le \frac{\epsilon_1}{\Gamma(\kappa)}\sum_{m=0}^{\infty}\frac{\omega^{m\kappa}}{\Gamma(m\kappa+1)} \le \epsilon_1\mathbf{E}_\kappa(\omega^\kappa). \end{aligned} \tag{21}$$

Using the same technique, we can get

$$\left|\overline{v}(\omega) - \overline{v}(0) - \overline{v}'(0)\omega - \frac{1}{2}\omega^2\overline{v}''(0) - \frac{1}{\Gamma(\sigma)}\int_0^\omega (\omega - s)^{\sigma-1} h(s, {}^c\mathbf{D}_0^\kappa \overline{u}_s, \overline{v}_s)\mathrm{d}s\right| \le \epsilon_2\mathbf{E}_\sigma(\omega^\sigma). \tag{22}$$

□

Theorem 3. *Let the following assumptions hold:*

$(\mathbf{M}_1)$ $g, h \in C(\mathbf{J} \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$, $\theta_1, \theta_2 \in C^1(\mathbf{J}, \mathbb{R})$, $|{}^c\mathbf{D}_0^\kappa u(\omega)| \le (1/(\Gamma(2-\kappa)))|u(\omega)|$, $|{}^c\mathbf{D}_0^\sigma v(\omega)| \le (1/(\Gamma(2-\sigma)))|v(\omega)|$;

$(\mathbf{M}_2)$ *There are $\mathcal{Q}_1, \mathcal{Q}_2 > 0$, so that $\forall \omega \in \mathbf{J}$ and $u_j, v_j \in \mathbb{R}$, $j = 1, 2$*

$$\left|g(\omega, u_1, u_2) - g(\omega, v_1, v_2)\right| \le \mathcal{Q}_1\sum_{j=1}^{2}\left|u_j - v_j\right|,$$
$$\left|h(\omega, u_1, u_2) - h(\omega, v_1, v_2)\right| \le \mathcal{Q}_2\sum_{j=1}^{2}\left|u_j - v_j\right|; \tag{23}$$

$(\mathbf{M}_3)$

$$\max_{\omega\in\mathbf{J}}\left\{\frac{\mathcal{Q}_1}{\Gamma(\kappa-1)}\left[\frac{1}{\kappa(\kappa-1)}+\frac{1}{\Gamma(3-\sigma)}\right],\right.$$
$$\left.\frac{\mathcal{Q}_2}{\Gamma(\sigma-1)}\left[\frac{1}{\sigma(\sigma-1)}+\frac{1}{\Gamma(3-\kappa)}\right]\right\}=\mathbf{K}<1. \tag{24}$$

Then,

(i) System (12) has a unique solution in $C^2([-\xi,1],\mathbb{R})\cap C^2(\mathbf{J},\mathbb{R})$;

(ii) roblem (4) is HUML stable.

Proof.

(i) Consider

$$ {}^c\mathbf{D}_0^\kappa u(\omega)=g(\omega,u_\omega,{}^c\mathbf{D}_0^\sigma v_\omega),\quad 1<\kappa,\ \sigma\le 2,\ \omega\in\mathbf{J}. \tag{25}$$

The solution is equivalent to

$$u(\omega)-u(0)-u'(0)\omega-\frac{1}{2}u''(0)\omega^2=\frac{1}{\Gamma(\kappa)}\int_0^\omega(\omega-s)^{\kappa-1}g(s,u_s,{}^c\mathbf{D}_0^\sigma v_s)\mathrm{d}s,$$
$$u(\omega)=u(0)+u'(0)\omega+\frac{1}{2}u''(0)\omega^2+\frac{1}{\Gamma(\kappa)}\int_0^\omega(\omega-s)^{\kappa-1}g(s,u_s,{}^c\mathbf{D}_0^\sigma v_s)\mathrm{d}s. \tag{26}$$

Thus,

$$u(\omega)=\begin{cases}u(0)+u'(0)\omega+\frac{1}{2}u''(0)\omega^2\\ +\frac{1}{\Gamma(\kappa)}\int_0^\omega(\omega-s)^{\kappa-1}g(s,u_s,{}^c\mathbf{D}_0^\sigma v_s)\mathrm{d}s, & \omega\in\mathbf{J},\\ \varphi_1(\omega), & \omega\in[-\xi,1].\end{cases} \tag{27}$$

Similarly, on considering

$$ {}^c\mathbf{D}_0^\sigma v(\omega)=h(\omega,{}^c\mathbf{D}_0^\kappa u_\omega,v(\omega)),\quad 1<\kappa,\ \sigma\le 2,\ \omega\in\mathbf{J}, \tag{28}$$

we can obtain

$$v(\omega)=\begin{cases}v(0)+v'(0)\omega+\frac{1}{2}v''(0)\omega^2\\ +\frac{1}{\Gamma(\sigma)}\int_0^\omega(\omega-s)^{\sigma-1}h(s,{}^c\mathbf{D}_0^\kappa u_s,v_s)\mathrm{d}s, & \omega\in\mathbf{J},\\ \varphi_2(\omega), & \omega\in[-\xi,1].\end{cases} \tag{29}$$

Let $\mathbf{Y}_\xi=\{u\in C([-\xi,1],\mathbb{R});{}^c\mathbf{D}_0^\kappa u\in C^2([-\xi,1],\mathbb{R})\}$ and define norm on $\mathbf{Y}_\xi$ by $\|u\|=\max_{\omega\in J}|u(\omega)|+\max_{\omega\in J}|{}^c\mathbf{D}_0^\kappa u(\omega)|$ and $\mathbf{Y}_\xi=\{v\in C([-\xi,1],\mathbb{R});\mathbf{D}_0^\sigma v\in C^2([-\xi,1],\mathbb{R})\}$ define norm on $\mathbf{Y}_\xi$ by $\|v\|=\max_{\omega\in J}|v(\omega)|+\max_{\omega\in J}|{}^c\mathbf{D}_0^\sigma v(\omega)|$, where $C([-\xi,1],\mathbb{R})$ and $C^2([-\xi,1],\mathbb{R})$, respectively, represent the class of continuous and continuously differentiable functions from $[-\xi,1]$ to $\mathbb{R}$. The norm is defined in such a way that the norm of each term depends on the derivatives of the fractional order of the other terms of $\mathbf{Y}_\xi$. The product $\mathbf{Y}_\xi\times\mathbf{Y}_\xi$ is also a Banach space with norm $\|(u,v)\|=\|u\|+\|v\|$.

Define operators $\mathbb{K}=(\mathbf{H},\mathbf{G})$: $\mathbf{Y}_\xi\times\mathbf{Y}_\xi\longrightarrow\mathbf{Y}_\xi\times\mathbf{Y}_\xi$ by

$$\mathbf{H}(u,v)(\omega)=\begin{cases}u(0)u'(0)\omega+\frac{1}{2}u''(0)\omega^2\\ +\frac{1}{\Gamma(\kappa)}\int_0^\omega(\omega-s)^{\kappa-1}g(s,u_s,{}^c\mathbf{D}_0^\sigma v_s)\mathrm{d}s, & \omega\in\mathbf{J},\\ \varphi_1(\omega), & \omega\in[-\xi,0],\end{cases}$$
$$\mathbf{G}(u,v)(\omega)=\begin{cases}v(0)+v'(0)\omega+\frac{1}{2}v''(0)\omega^2\\ +\frac{1}{\Gamma(\sigma)}\int_0^\omega(\omega-s)^{\sigma-1}h(s,{}^c\mathbf{D}_0^\kappa u_s,v_s)\mathrm{d}s, & \omega\in\mathbf{J},\\ \varphi_2(\omega), & \omega\in[-\xi,0].\end{cases} \tag{30}$$

First, we show that $\mathbb{K}$ is a contraction mapping. It is clear that

$$\begin{aligned}&|\mathbb{K}(u,v)(\omega)-\mathbb{K}(\overline{u},\overline{v})(\omega)|\\ &=|(\mathbf{H}(u,v)(\omega),\mathbf{G}(u,v)(\omega))-(\mathbf{H}(\overline{u},\overline{v})(\omega),\mathbf{G}(\overline{u},\overline{v})(\omega))|\\ &\le|\mathbf{H}(u,v)(\omega)-\mathbf{H}(\overline{u},\overline{v})(\omega)|+|\mathbf{G}(u,v)(\omega)-\mathbf{G}(\overline{u},\overline{v})(\omega)|\\ &=|\varphi_1(\omega)-\varphi_1(\omega)|+|\varphi_2(\omega)-\varphi_2(\omega)|=0,\\ &\qquad\forall\omega\in[-\xi,0].\end{aligned} \tag{31}$$

For $\omega\in\mathbf{J}$, we have

$$\begin{aligned}&\max_{0\le s\le\omega}|(u_s,v_s)-(\overline{u}_s,\overline{v}_s)|\\ &\le\max_{0\le s\le\omega}|u_s-\overline{u}_s|+\max_{0\le s\le\omega}|v_s-\overline{v}_s|\\ &\le\max_{0\le s\le\omega}|u(s+\tau)-\overline{u}(s+\tau)|+\max_{0\le s\le\omega}|v(s+\tau)-\overline{v}(s+\tau)|\\ &=\max_{\tau\le s+\tau\le\omega+\tau}|u(s+\tau)-\overline{u}(s+\tau)|\\ &\quad+\max_{\tau\le s+\tau\le\omega+\tau}|v(s+\tau)-\overline{v}(s+\tau)|\\ &\le\max_{-\xi\le s^*\le\omega}|u(s^*)-\overline{u}(s^*)|+\max_{-\xi\le s^*\le\omega}|v(s^*)-\overline{v}(s^*)|\\ &\le\max_{-\xi\le s^*\le 1}|u(s^*)-\overline{u}(s^*)|+\max_{-\xi\le s^*\le 1}|v(s^*)-\overline{v}(s^*)|\\ &=|u(s^*)-\overline{u}(s^*)|+|v(s^*)-\overline{v}(s^*)|=\|(u,v)-(\overline{u},\overline{v})\|,\end{aligned} \tag{32}$$

where $s^* = s + \tau$ and $-\xi \le \tau < 0$. Thus,

$$
\begin{aligned}
|\mathbb{K}(u,v) - \mathbb{K}(\overline{u},\overline{v})| &= |(\mathbf{H},\mathbf{G})(u,v) - (\mathbf{H},\mathbf{G})(\overline{u},\overline{v})| \\
&\le |\mathbf{H}(u,v) - \mathbf{H}(\overline{u},\overline{v})| + |\mathbf{G}(u,v) - \mathbf{G}(\overline{u},\overline{v})| \\
&\le \frac{1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa-1} \left| g(s, u_s, {}^c\mathbf{D}_0^\sigma v_s) - g(s, \overline{u}_s, {}^c\mathbf{D}_0^\sigma \overline{v}_s) \right| \mathrm{d}s \\
&\quad + \frac{1}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma-1} \left| h(s, {}^c\mathbf{D}_0^\kappa u_s, v_s) - h(s, {}^c\mathbf{D}_0^\kappa \overline{u}_s, \overline{v}_s) \right| \mathrm{d}s \\
&\le \frac{\mathcal{Q}_1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa-1} \max_{0\le s\le\omega} |u(s+\tau) - \overline{u}(s+\tau)| + \max_{0\le s\le\omega} \left| {}^c\mathbf{D}_0^\sigma v(s+\tau) - {}^c\mathbf{D}_0^\sigma \overline{v}(s+\tau) \right| \mathrm{d}s \\
&\quad + \frac{\mathcal{Q}_2}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma-1} \max_{0\le s\le\omega} \left| {}^c\mathbf{D}_0^\kappa u(s+\tau) - {}^c\mathbf{D}_0^\kappa \overline{u}(s+\tau) \right| + \max_{0\le s\le\omega} |v(s+\tau) - \overline{v}(s+\tau)| \mathrm{d}s \\
&\le \frac{\mathcal{Q}_1}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa-1} \max_{-\xi\le s^*\le 1} \left| u(s^*) - \overline{u}(s^*) \right| + \max_{-\xi\le s^*\le 1} \left| {}^c\mathbf{D}_0^\sigma v(s^*) - {}^c\mathbf{D}_0^\sigma \overline{v}(s^*) \right| \mathrm{d}s \\
&\quad + \frac{\mathcal{Q}_2}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma-1} \max_{-\xi\le s^*\le 1} \left| {}^c\mathbf{D}_0^\kappa u(s^*) - {}^c\mathbf{D}_0^\kappa \overline{u}(s^*) \right| + \max_{-\xi\le s^*\le 1} \left| v(s^*) - \overline{v}(s^*) \right| \mathrm{d}s \\
&\le \frac{\mathcal{Q}_1 \left(\max_{-\xi\le s^*\le 1} |u(s^*) - \overline{u}(s^*)| + \max_{-\xi\le s^*\le 1} {}^c\mathbf{D}_0^\sigma |v(s^*) - \overline{v}(s^*)| \right)}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa-1} \mathrm{d}s \\
&\quad + \frac{\mathcal{Q}_2 \left(\max_{-\xi\le s^*\le 1} |v(s^*) - \overline{v}(s^*)| + \max_{-\xi\le s^*\le 1} {}^c\mathbf{D}_0^\kappa |(u(s^*) - \overline{u}(s^*)| \right.}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma-1} \mathrm{d}s \\
&\le \frac{\mathcal{Q}_1 \|u - \overline{u}\|}{\Gamma(\kappa+1)} + \frac{\mathcal{Q}_2 \|v - \overline{v}\|}{\Gamma(\sigma+1)}.
\end{aligned} \tag{33}
$$

Also,

$$
\begin{aligned}
&\left\| \left({}^c\mathbf{D}_0^\sigma, {}^c\mathbf{D}_0^\kappa\right)\mathbb{K}(u,v) - \left({}^c\mathbf{D}_0^\sigma, {}^c\mathbf{D}_0^\kappa\right)\mathbb{K}(\overline{u},\overline{v}) \right\| \\
&\quad = \max_{\omega\in\mathbf{J}} \left| \left({}^c\mathbf{D}_0^\sigma \mathbf{H}(u,v), {}^c\mathbf{D}_0^\kappa \mathbf{G}(u,v)\right) - \left({}^c\mathbf{D}_0^\sigma \mathbf{H}(\overline{u},\overline{v}), {}^c\mathbf{D}_0^\kappa \mathbf{G}(\overline{u},\overline{v}\right) \right| \\
&\quad \le \max_{\omega\in\mathbf{J}} \left| {}^c\mathbf{D}_0^\sigma \mathbf{H}(u,v) - {}^c\mathbf{D}_0^\sigma \mathbf{H}(u,v) \right| + \max_{\omega\in\mathbf{J}} \left| {}^c\mathbf{D}_0^\kappa \mathbf{G}(u,v) - {}^c\mathbf{D}_0^\kappa \mathbf{G}(\overline{u},\overline{v}) \right| \\
&\qquad \cdot \frac{1}{\Gamma(2-\sigma)} \max_{\omega\in\mathbf{J}} \left| \int_0^\omega (\omega - s)^{1-\sigma} \left(\frac{(\kappa-1)(\kappa-2)}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa-3} g(s, u_s, {}^c\mathbf{D}_0^\sigma v_s) \mathrm{d}s \right.\right. \\
&\qquad \left.\left. - \frac{(\kappa-1)(\kappa-2)}{\Gamma(\kappa)} \int_0^\omega (\omega - s)^{\kappa-3} g(s, \overline{u}_s, {}^c\mathbf{D}_0^\sigma \overline{v}_s) \mathrm{d}s \right) \right| \\
&\qquad + \frac{1}{\Gamma(2-\kappa)} \max_{\omega\in\mathbf{J}} \left| \int_0^\omega (\omega - s)^{1-\kappa} \left(\frac{(\sigma-1)(\sigma-2)}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma-3} h(s, {}^c\mathbf{D}_0^\kappa u_s, v_s) \mathrm{d}s \right.\right. \\
&\qquad \left.\left. - \frac{(\sigma-1)(\sigma-2)}{\Gamma(\sigma)} \int_0^\omega (\omega - s)^{\sigma-3} h(s, {}^c\mathbf{D}_0^\kappa \overline{u}_s, \overline{v}_s) \mathrm{d}s \right) \right| \\
&\quad \le \frac{\mathcal{Q}_1}{\Gamma(2-\sigma)\Gamma(\kappa-1)} \|u - \overline{u}\| \int_0^\omega (\omega - s)^{1-\sigma} \mathrm{d}s + \frac{\mathcal{Q}_2}{\Gamma(2-\kappa)\Gamma(\sigma-1)} \|v - \overline{v}\| \int_0^\omega (\omega - s)^{1-\kappa} \mathrm{d}s \\
&\quad \le \frac{\mathcal{Q}_1}{\Gamma(3-\sigma)\Gamma(\kappa-1)} \|u - \overline{u}\| + \frac{\mathcal{Q}_2}{\Gamma(3-\kappa)\Gamma(\sigma-1)} \|v - \overline{v}\|.
\end{aligned} \tag{34}
$$

So, we obtain

$$
\begin{aligned}
&\|\mathbb{K}(u,v)-\mathbb{K}(\overline{u},\overline{v})\|\\
&\quad\leq \frac{\mathcal{Q}_1\|u-\overline{u}\|}{\Gamma(\kappa+1)}+\frac{\mathcal{Q}_2\|v-\overline{v}\|}{\Gamma(\sigma+1)}+\frac{\mathcal{Q}_1}{\Gamma(3-\sigma)\Gamma(\kappa-1)}\|u-\overline{u}\|\\
&\qquad+\frac{\mathcal{Q}_2}{\Gamma(3-\kappa)\Gamma(\sigma-1)}\|v-\overline{v}\|\\
&\quad\leq \frac{\mathcal{Q}_1}{\Gamma(\kappa-1)}\left[\frac{1}{\kappa(\kappa-1)}+\frac{1}{\Gamma(3-\sigma)}\right]\|u-\overline{u}\|\\
&\qquad+\frac{\mathcal{Q}_2}{\Gamma(\sigma-1)}\left[\frac{1}{\sigma(\sigma-1)}+\frac{1}{\Gamma(3-\kappa)}\right]\|v-\overline{v}\|\\
&\quad\leq \mathbf{K}\|(u,v)-(\overline{u},\overline{v})\|.
\end{aligned}
\tag{35}
$$

This shows that $\mathbb{K}$ is a contraction operator.

(ii) Next, let $(\overline{u},\overline{v})\in C^2(\mathbf{J},\mathbb{R})$ be the approximate solution of (13) and $(u,v)\in C^2(\mathbf{J},\mathbb{R})$ be the unique solutions of system (12), that is,

$$
\begin{cases}
{}^c\mathbf{D}_0^{\kappa}u(\omega)=g\left(\omega,u_\omega,{}^c\mathbf{D}^{\sigma}v_\omega\right), & \omega\in\mathbf{J},\ 1<\kappa,\ \sigma\leq 2,\\
{}^c\mathbf{D}_0^{\sigma}v(\omega)=h\left(\omega,{}^c\mathbf{D}_{0^+}^{\kappa}u_\omega,v_\omega\right), & \omega\in\mathbf{J},\ 1<\kappa,\ \sigma\leq 2,\\
u(\omega)=\overline{u}(\omega),v(\omega)=\overline{v}(\omega), & \omega\in[-\xi,0],
\end{cases}
\tag{36}
$$

then

$$
\begin{aligned}
u(\omega)&=u(0)+u'(0)\omega+\frac{1}{2}u''(0)\omega^2\\
&\quad+\frac{1}{\Gamma(\kappa)}\int_0^\omega(\omega-s)^{\kappa-1}g\left(s,u_s,{}^c\mathbf{D}_0^{\sigma}v_s\right)\mathrm{d}s\\
v(\omega)&=v(0)+v'(0)\omega+\frac{1}{2}v''(0)\omega^2\\
&\quad+\frac{1}{\Gamma(\sigma)}\int_0^\omega(\omega-s)^{\sigma-1}h\left(s,{}^c\mathbf{D}_0^{\kappa}u_s,v_s\right)\mathrm{d}s.
\end{aligned}
\tag{37}
$$

For $\omega\in[-\xi,0]$, we have

$$
|(\overline{u},\overline{v})-(u,v)|\leq|\overline{u}-u|+|\overline{v}-v|=\left|\varphi_1-\varphi_1\right|+\left|\varphi_2-\varphi_2\right|=0.
\tag{38}
$$

Also, for $\omega\in\mathbf{J}$, we have

$$
|(u,v)(\omega)-(\overline{u},\overline{v})(\omega)|\leq|u(\omega)-\overline{u}(\omega)|+|v(\omega)-\overline{v}(\omega)|.
\tag{39}
$$

Now,

$$
\begin{aligned}
&|u(\omega)-\overline{u}(\omega)|\\
&\quad=\left|u(\omega)-u(0)-u\prime(0)\omega-\frac{1}{2}u''\omega^2-\frac{1}{\Gamma(\kappa)}\int_0^\omega(\omega-s)^{\kappa-1}g\left(s,\overline{u}_s,{}^c\mathbf{D}_0^{\sigma}\overline{v}_s\right)\mathrm{d}s\right|\\
&\quad\leq\left|u(\omega)-u(0)-u\prime(0)\omega-\frac{1}{2}u''\omega^2-\frac{1}{\Gamma(\kappa)}\int_0^\omega(\omega-s)^{\kappa-1}g\left(s,u_s,{}^c\mathbf{D}_0^{\sigma}v_s\right)\mathrm{d}s\right|\\
&\qquad+\left|\frac{1}{\Gamma(\kappa)}\int_0^\omega(\omega-s)^{\kappa-1}g\left(s,u_s,{}^c\mathbf{D}_0^{\sigma}v_s\right)\mathrm{d}s-\frac{1}{\Gamma(\kappa)}\int_0^\omega(\omega-s)^{\kappa-1}g\left(s,\overline{u}_s,{}^c\mathbf{D}_0^{\sigma}\overline{v}_s\right)\mathrm{d}s\right|\\
&\quad\leq\epsilon_1\mathbf{E}_\kappa\left(\omega^\kappa\right)+\frac{1}{\Gamma(\kappa)}\int_0^\omega(\omega-s)^{\kappa-1}\left|g\left(s,u_s,{}^c\mathbf{D}_0^{\kappa}v_s\right)-g\left(s,\overline{u}_s,{}^c\mathbf{D}_0^{\sigma}\overline{v}_s\right)\right|\mathrm{d}s\\
&\quad\leq\epsilon_1\mathbf{E}_\kappa\left(\omega^\kappa\right)+\frac{\mathcal{Q}_1}{\Gamma(\kappa)}\int_0^\omega(\omega-s)^{\kappa-1}\left|u_s-\overline{u}_s\right|+\left|{}^c\mathbf{D}_0^{\sigma}v_s-{}^c\mathbf{D}_0^{\sigma}\overline{v}_s\right|\mathrm{d}s\\
&\quad\leq\epsilon_1\mathbf{E}_\kappa\left(\omega^\kappa\right)+\frac{\mathcal{Q}_1}{\Gamma(\kappa)}\left(\int_0^\omega(\omega-s)^{\kappa-1}\left|u_s-\overline{u}_s\right|\mathrm{d}s+\int_0^\omega(\omega-s)^{\kappa-1}\left|{}^c\mathbf{D}_0^{\sigma}v_s-{}^c\mathbf{D}_0^{\sigma}\overline{v}_s\right|\right)\mathrm{d}s.
\end{aligned}
\tag{40}
$$

Similarly, we can get

$$
|v(\omega)-\overline{v}(\omega)|\leq\epsilon_2\mathbf{E}_\sigma\left(\omega^\sigma\right)+\frac{\mathcal{Q}_2}{\Gamma(\sigma)}\left(\int_0^\omega(\omega-s)^{\sigma-1}\left|{}^c\mathbf{D}_0^{\kappa}u_s-{}^c\mathbf{D}_0^{\kappa}\overline{u}_s\right|\mathrm{d}s+\int_0^\omega(\omega-s)^{\sigma-1}\left|v_s-\overline{v}_s\right|\mathrm{d}s\right).
\tag{41}
$$

Let $\chi, \vartheta \in C([-\xi, 1], \mathbb{R}_+)$ and in view of inequalities (40) and (41) take $\mathbf{A}_{\kappa,\sigma} = (\mathbf{A}_\kappa, \mathbf{A}_\sigma)$, where $\mathbf{A}_\kappa, \mathbf{A}_\sigma: C([-\xi, 1] \times [-\xi, 1], \mathbb{R}_+) \longrightarrow C([-\xi, 1], \mathbb{R}_+)$ are defined by

$$\mathbf{A}_\kappa(\chi,\vartheta)(\omega) = \begin{cases} \epsilon_1 \mathbf{E}_\kappa(\omega^\kappa) + \dfrac{\mathcal{Q}_1}{\Gamma(\kappa)}\left(\displaystyle\int_0^\omega (\omega - s)^{\kappa-1}\chi_s \mathrm{d}s + \int_0^\omega (\omega - s)^{\kappa-1c}\mathbf{D}_0^\sigma \vartheta_s\right)\mathrm{d}s, & \omega \in \mathbf{J}, \\ 0, \quad \omega \in [-\xi, 0], \end{cases}$$
$$\mathbf{A}_\sigma(\chi,\vartheta)(\omega) = \begin{cases} \epsilon_2 \mathbf{E}_\sigma(\omega^\sigma) + \dfrac{\mathcal{Q}_2}{\Gamma(\sigma)}\left(\displaystyle\int_0^\omega (\omega -)^{\sigma-1}\vartheta_s \mathrm{d}s + \int_0^\omega (\omega - s)^{\sigma-1c}\mathbf{D}_0^\kappa \chi_s)\right)\mathrm{d}s, & \omega \in \mathbf{J}, \\ 0, \quad \omega \in [-\xi, 0]. \end{cases} \tag{42}$$

We will show that $\mathbf{A}_{\kappa,\sigma}$ is PO and consequently need to show that $\mathbf{A}_{\kappa,\sigma}$ is a contraction. Consider

$$\begin{aligned} &\left|\mathbf{A}_{\kappa,\sigma}(\chi,\vartheta)(\omega) - \mathbf{A}_{\kappa,\sigma}(\chi^*,\vartheta^*)(\omega)\right| \\ &\quad\le \left|\mathbf{A}_\kappa(\chi,\vartheta)(\omega) - \mathbf{A}_\kappa(\chi^*,\vartheta^*)(\omega)\right| + \left|\mathbf{A}_\sigma(\chi,\vartheta)(\omega) - \mathbf{A}_\sigma(\chi^*,\vartheta^*)(\omega)\right| \\ &\quad\le \frac{\mathcal{Q}_1}{\Gamma(\kappa)}\left(\int_0^\omega (\omega - s)^{\kappa-1}\left|\chi_s - \chi_s^*\right|\mathrm{d}s + \int_0^\omega (\omega - s)^{\kappa-1}\left|{}^c\mathbf{D}_0^\sigma\vartheta_s - {}^c\mathbf{D}_0^\sigma\vartheta_s^*\right|\right)\mathrm{d}s \\ &\qquad + \frac{\mathcal{Q}_2}{\Gamma(\sigma)}\left(\int_0^\omega (\omega - s)^{\sigma-1}\left|\vartheta_s - \vartheta_s^*\right| ds + \int_0^\omega (\omega - s)^{\sigma-1}\left|{}^c\mathbf{D}_0^\kappa\chi_s - {}^c\mathbf{D}_0^\kappa\chi_s^*\right|\right)\mathrm{d}s \\ &\quad\le \frac{\mathcal{Q}_1}{\Gamma(\kappa)}\left(\max_{0\le s\le\omega}\left|\chi_s - \chi_s^*\right| + \max_{0\le s\le\omega}\left|{}^c\mathbf{D}_0^\sigma\vartheta_s - {}^c\mathbf{D}_0^\sigma\vartheta_s^*\right|\right)\int_0^\omega (\omega - s)^{\kappa-1}\mathrm{d}s \\ &\qquad + \frac{\mathcal{Q}_2}{\Gamma(\sigma)}\left(\max_{0\le s\le\omega}\left|\vartheta_s - \vartheta_s^*\right| + \max_{0\le s\le\omega}\left|{}^c\mathbf{D}_0^\kappa\chi_s - {}^c\mathbf{D}_0^\sigma\chi_s^*\right|\right)\int_0^\omega (\omega - s)^{\sigma-1}\mathrm{d}s \\ &\quad\le \frac{\mathcal{Q}_1}{\Gamma(\kappa+1)}\left\|\chi - \chi^*\right\| + \frac{\mathcal{Q}_2}{\Gamma(\sigma+1)}\left\|\vartheta - \vartheta^*\right\|. \end{aligned} \tag{43}$$

Moreover,

$$\begin{aligned} &\left|\left({}^c\mathbf{D}_0^\sigma, {}^c\mathbf{D}_0^\kappa\right)\mathbf{A}_{\kappa,\sigma}(\chi,\vartheta) - \left({}^c\mathbf{D}_0^\sigma, {}^c\mathbf{D}_0^\kappa\right)\mathbf{A}_{\kappa,\sigma}(\chi^*,\vartheta^*)\right| \\ &\quad = \left|\left({}^c\mathbf{D}_0^\sigma\mathbf{A}_\kappa(\chi,\vartheta), {}^c\mathbf{D}_0^\kappa\mathbf{A}_\sigma(\chi,\vartheta)\right) - \left({}^c\mathbf{D}_0^\sigma\mathbf{A}_\kappa(\chi^*,\vartheta^*), {}^c\mathbf{D}_0^\kappa\mathbf{A}_\sigma(\chi^*,\vartheta^*)\right)\right| \\ &\quad \le \left|{}^c\mathbf{D}_0^\sigma\mathbf{A}_\kappa(\chi,\vartheta) - {}^c\mathbf{D}_0^\sigma\mathbf{A}_\kappa(\chi^*,\vartheta^*)\right| + \left|{}^c\mathbf{D}_0^\kappa\mathbf{A}_\sigma(\chi,\vartheta) - {}^c\mathbf{D}_0^\kappa\mathbf{A}_\sigma(\chi^*,\vartheta^*)\right| \\ &\quad \le \frac{1}{\Gamma(2-\sigma)}\int_0^\omega (\omega - s)^{1-\sigma}\left(\frac{(\kappa-1)(\kappa-2)}{\Gamma(\kappa)}\int_0^\omega (\omega - s)^{\kappa-3}\max_{\omega\in\mathbf{J}}\left|\chi_s - \chi_s^*\right|\mathrm{d}s\right. \\ &\qquad \left. + \frac{(\kappa-1)(\kappa-2)}{\Gamma(\kappa)|}\int_0^\omega (\omega - s)^{\kappa-3}\max_{\omega\in\mathbf{J}}\left|{}^c\mathbf{D}_0^\sigma\vartheta_s - {}^c\mathbf{D}_0^\sigma\vartheta_s^*\right|\mathrm{d}s\right) \\ &\qquad + \frac{1}{\Gamma(2-\kappa)}\int_0^\omega (\omega - s)^{1-\kappa}\left(\frac{(\sigma-1)(\sigma-2)}{\Gamma(\sigma)}\int_0^\omega (\omega - s)^{\sigma-3}\max_{\omega\in\mathbf{J}}\left|\vartheta_s - \vartheta_s^*\right|\mathrm{d}s\right. \\ &\qquad \left. + \frac{(\sigma-1)(\sigma-2)}{\Gamma(\sigma)}\int_0^\omega (\omega - s)^{\sigma-3}\max_{\omega\in\mathbf{J}}\left|{}^c\mathbf{D}_0^\kappa\chi_s - {}^c\mathbf{D}_0^\kappa\chi_s^*\right|\mathrm{d}s\right) \\ &\quad \le \frac{\mathcal{Q}_1}{\Gamma(3-\sigma)\Gamma(\kappa-1)}\left\|\chi - \chi^*\right\| + \frac{\mathcal{Q}_2}{\Gamma(3-\kappa)\Gamma(\sigma-1)}\left\|\vartheta - \vartheta^*\right\|. \end{aligned} \tag{44}$$

Thus, we get

$$\begin{aligned}&\left\|\mathbf{A}_{\kappa,\sigma}(\chi,\vartheta)(\omega)-\mathbf{A}_{\kappa,\sigma}\left(\chi^{*},\vartheta^{*}\right)(\omega)\right\|\\&\leq\frac{\mathcal{Q}_1}{\Gamma(\kappa-1)}\left[\frac{1}{\kappa(\kappa-1)}+\frac{1}{\Gamma(3-\sigma)}\right]\left\|\chi-\chi^{*}\right\|\\&\quad+\frac{\mathcal{Q}_2}{\Gamma(\sigma-1)}\left[\frac{1}{\Gamma(3-\kappa)}+\frac{1}{\sigma(\sigma-1)}\right]\left\|\vartheta-\vartheta^{*}\right\|.\end{aligned}\tag{45}$$

Therefore, $\|\mathbf{A}_{\kappa,\sigma}(\chi,\vartheta)(\omega)-\mathbf{A}_{\kappa,\sigma}(\chi^{*},\vartheta^{*})(\omega)\|\leq\mathbf{K}\|(\chi,\vartheta)-(\chi^{*},\vartheta^{*})\|\ \forall\chi,\vartheta,\chi^{*},\vartheta^{*}\in C([-\xi,1],\mathbb{R}_{+})$, showing that $\mathbf{A}_{\kappa,\sigma}$ is a contraction operator. By Definition 3, $\mathbf{A}_{\kappa,\sigma}$ is a PO and $\mathbf{F}_{\mathbf{A}_{\kappa,\sigma}}=\{(x^{*},y^{*})\}$, then

$$\begin{aligned}x^{*}(\omega)&\leq\epsilon_1\mathbf{E}_{\kappa}\left(\omega^{\kappa}\right)+\frac{\mathcal{Q}_1}{\Gamma(\kappa)}\int_0^{\omega}(\omega-s)^{\kappa-1}x_s^{*}ds\\&\quad+\frac{\mathcal{Q}_1}{\Gamma(\kappa)}\int_0^{\omega}(\omega-s)^{\kappa-1}{}^{c}\mathbf{D}_0^{\sigma}y_s^{*}ds,\\y^{*}(\omega)&\leq\epsilon_2\mathbf{E}_{\sigma}\left(\omega^{\sigma}\right)+\frac{\mathcal{Q}_2}{\Gamma(\sigma)}\int_0^{\omega}(\omega-s)^{\sigma-1}y_s^{*}ds\\&\quad+\frac{\mathcal{Q}_2}{\Gamma(\sigma)}\int_0^{\omega}(\omega-s)^{\sigma-1}{}^{c}\mathbf{D}_0^{\sigma}x_s^{*}ds.\end{aligned}\tag{46}$$

We try to show that $x^{*}(\omega)$ and $y^{*}(\omega)$ are increasing, and for this, take $\hbar_1,\hbar_2\in(0,1]$, $\hbar_1<\hbar_2$ and $q_1^{*}=\min_{s\in\mathbf{J}}[x_s^{*}+{}^{c}\mathbf{D}_0^{\sigma}y_s^{*}]$, $q_2^{*}=\min_{s\in\mathbf{J}}[y_s^{*}+{}^{c}\mathbf{D}_0^{\kappa}x_s^{*}]$. Thus,

$$\begin{aligned}x^{*}(\hbar_2)-x^{*}(\hbar_2)&\leq\epsilon_1\left(\mathbf{E}_{\kappa}\left(\hbar_2^{\kappa}\right)-\mathbf{E}_{\kappa}\left(\hbar_2^{\kappa}\right)\right)+\frac{\mathcal{Q}_1}{\Gamma(\kappa)}\int_0^{\hbar_1}\left[(\hbar_2-s)^{\kappa-1}-(\hbar_1-s)^{\kappa-1}\right]\left[x_s^{*}+{}^{c}\mathbf{D}_0^{\sigma}y_s^{*}\right]ds\\&\quad+\frac{\mathcal{Q}_1}{\Gamma(\kappa)}\int_0^{\hbar_2}(\hbar_2-s)^{\kappa-1}\left[x_s^{*}+{}^{c}\mathbf{D}_0^{\sigma}y_s^{*}\right]ds\\&\geq\epsilon_1\left(\mathbf{E}_{\kappa}\left(\hbar_2^{\kappa}\right)-\mathbf{E}_{\kappa}\left(\hbar_2^{\kappa}\right)\right)+\frac{q_1^{*}\mathcal{Q}_1}{\Gamma(\kappa)}\int_0^{\hbar_1}\left[(\hbar_2-s)^{\kappa-1}-(\hbar_1-s)^{\kappa-1}\right]ds\\&\quad+\frac{q_1^{*}\mathcal{Q}_1}{\Gamma(\kappa)}\int_0^{\hbar_2}(\hbar_2-s)^{\kappa-1}ds\\&\geq\epsilon_1\left(\mathbf{E}_{\kappa}\left(\hbar_2^{\kappa}\right)-\mathbf{E}_{\kappa}\left(\hbar_2^{\kappa}\right)\right)+\frac{q_1^{*}\mathcal{Q}_1}{\Gamma(\kappa+1)}\left(\hbar_2^{\kappa}-\hbar_1^{\kappa}\right)>0.\end{aligned}\tag{47}$$

Similarly, we can get

$$y^{*}(\omega)\geq\epsilon_2\left(\mathbf{E}_{\sigma}\left(\hbar_2^{\sigma}\right)-\mathbf{E}_{\sigma}\left(\hbar_2^{\sigma}\right)\right)+\frac{q_2^{*}\mathcal{Q}_2}{\Gamma(\sigma+1)}\left(\hbar_2^{\sigma}-\hbar_1^{\sigma}\right)>0.\tag{48}$$

So that the solution $(x^{*}(\omega),y^{*}(\omega))$ is increasing and ${}^{c}\mathbf{D}_0^{\kappa}x^{*},{}^{c}\mathbf{D}_0^{\sigma}y^{*}>0$. Also

$$\begin{aligned}x^{*}(\omega)&\leq\epsilon_1\mathbf{E}_{\kappa}\left(\omega^{\kappa}\right)+\frac{\mathcal{Q}_1}{\Gamma(\kappa)}\int_0^{\omega}(\omega-s)^{\kappa-1}x^{*}(s)ds\\&\quad+\frac{\mathcal{Q}_1}{\Gamma(\kappa)\Gamma(2-\sigma)}\int_0^{\omega}(\omega-s)^{\kappa-1}x^{*}(s)ds\\&\leq\epsilon_1\mathbf{E}_{\kappa}\left(\omega^{\kappa}\right)+\frac{\mathcal{Q}_1}{\Gamma(\kappa)}\left(1+\frac{1}{\Gamma(2-\sigma)}\right)\\&\quad\cdot\int_0^{\omega}(\omega-s)^{\kappa-1}x^{*}(s)ds.\end{aligned}\tag{49}$$

Thus, by using Gronwall's lemma we have

$$x^{*}(\omega)\leq c_{E_{\kappa}}\epsilon_1\mathbf{E}_{\kappa}\left(\omega^{\kappa}\right),\tag{50}$$

where $c_{E_{\kappa}}=\mathbf{E}_{\kappa}((\mathcal{Q}_1/\Gamma(\kappa))(1+(1/\Gamma(2-\sigma))))$. In the same way

$$y^{*}(\omega)\leq c_{E_{\sigma}}\epsilon_2\mathbf{E}_{\sigma}\left(\omega^{\sigma}\right),\tag{51}$$

where $c_{E_{\sigma}}=\mathbf{E}_{\sigma}((\mathcal{Q}_2/\Gamma(\sigma))(1+(1/\Gamma(2-\kappa))))$, $\omega\in\mathbf{J}$. Particularly, if $x=|\overline{u}-u|$ and $y=|\overline{v}-v|$, then by (40) and (41), we have $x(\omega)\leq\mathbf{A}_{\kappa}$ and $y(\omega)\leq\mathbf{A}_{\sigma}$; therefore, by Gronwall's lemma we obtain

$$\begin{aligned}|u(\omega)-\overline{u}(\omega)|&\leq c_{E_{\kappa}}\epsilon_1\mathbf{E}_{\kappa}\left(\omega^{\kappa}\right),\\|v(\omega)-\overline{v}(\omega)|&\leq c_{E_{\sigma}}\epsilon_2\mathbf{E}_{\sigma}\left(\omega^{\sigma}\right).\end{aligned}\tag{52}$$

Thus, from the above we have

$$\begin{aligned}&\|(u,v)-(\overline{u},\overline{v})\|\leq\|u-\overline{u}\|+\|v-\overline{v}\|\\&\quad\leq c_{E_{\kappa}}\epsilon_1\mathbf{E}_{\kappa}\left(\omega^{\kappa}\right)+c_{E_{\sigma}}\epsilon_2\mathbf{E}_{\sigma}\left(\omega^{\sigma}\right)\leq 2c_{E_{\lambda}}\epsilon\mathbf{E}_{\lambda}\left(\omega^{\lambda}\right).\end{aligned}\tag{53}$$

Hence, it follows that (4) is ML stable. □

4. Illustrative Example

As an application of our results, we consider the following example.

Example 1. Consider the problem

$$\begin{cases} {}^c\mathbf{D}_0^{3/2}u(\omega) = \dfrac{1}{4e^{\omega}|u(\omega-1)| + \left|{}^c\mathbf{D}_0^{5/3}v(\omega-1)\right|}, \\ \omega \in \mathbf{J} = [0,1], \\ {}^c\mathbf{D}_0^{5/3}v(\omega) = \dfrac{1}{10e^{2\omega} + \left(1 + \left|{}^c\mathbf{D}_0^{3/2}u(\omega-1)\right| + |v(\omega-1)|\right)}, \\ \omega \in \mathbf{J} = [0,1], \\ u(\omega) = 0,\ v(\omega) = 0,\ \omega \in [-1,0]. \end{cases} \tag{54}$$

Set the functions as

$$\begin{aligned} g_{3/2}(\omega, u(\omega), v(\omega)) &= \frac{1}{4e^{\omega}(1 + |u| + |v|)}, \\ h_{5/3}(\omega, u(\omega), v(\omega))) &= \frac{1}{10e^{2t} + (1 + |u| + |v|)}, \end{aligned} \tag{55}$$

and then for every $u, v, \overline{u}, \overline{v}$ from $\mathbb{R}$, $\omega \in [0,1]$ we have

$$\begin{cases} \left|g_{3/2}(\omega, u(\omega), v(\omega)) - g_{3/2}(\omega, \overline{u}(\omega), \overline{v}(\omega))\right| \\ \leq \dfrac{1}{4}|u - \overline{u}| + \dfrac{1}{4}|v - \overline{v}|, \\ \left|h_{5/3}(\omega, u(\omega), v(\omega)) - h_{5/3}(\omega, \overline{u}(\omega), \overline{v}(\omega))\right| \\ \leq \dfrac{1}{10}|u - \overline{u}| + \dfrac{1}{10}|v - \overline{v}|. \end{cases} \tag{56}$$

Therefore, $(\mathbf{M}_2)$ is satisfied with $\kappa = 3/2$ and $\sigma = 5/3$, and from the inequalities in (56), $\mathcal{Q}_1 = 1/4$ and $\mathcal{Q}_2 = 1/10$. Thus, we have $\mathbf{K} = 0.34601 < 1$. Therefore, by Theorem 2, (54) has a unique solution. After calculations, we obtain

$$\|(u, v) - (\overline{u}, \overline{v})\| \leq c_{3/2}\epsilon \mathbf{E}_{3/2}\left(\omega^{3/2}\right), \quad \omega \in [0,1]. \tag{57}$$

Using Theorem 3, the solution of (54) is Hyers–Ulam–ML stable.

Remark 2. If we consider another space $B_{u,v} = \{u, v\colon u, v \in C[-\xi, 1], {}^c\mathbf{D}_0^{\kappa}u, {}^c\mathbf{D}_0^{\sigma}v \in C^1[-\xi, 1]\}$ with modified Bielecki's norms defined by

$$\begin{aligned} \|u\| &= \max_{\omega \in \mathbf{J}}\left(|u(\omega)|e^{-\lambda\omega} + \left|{}^c\mathbf{D}_0^{\sigma}v(\omega)\right|e^{-\rho\omega}\right), \\ \|v\| &= \max_{\omega \in \mathbf{J}}\left(|v(\omega)|e^{-\lambda\omega} + \left|{}^c\mathbf{D}_0^{\kappa}u(\omega)\right|e^{-\rho\omega}\right), \end{aligned} \tag{58}$$

where $0 < \rho < \lambda$, $\mathbf{J} = [0,1]$, then the results similar to Theorem 3 can be obtained for the solution of (4).

5. Conclusion

We gave sufficient conditions for the EU of the solutions to the nonlinear implicit CS of neutral FDEs. Our main tool was the Banach contraction principle. Likewise under specific conditions, we have found the HUML stability results for the solution of the CS given in (4).

Data Availability

Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this article.

Authors' Contributions

The main idea of the paper was given by Zada. Results were proved by Ahmad and Zada. Ali helped in the example. Fu and Xu drafted the paper, and Jiang helped in the revision and gave financial support for publishing.

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References

[1] D. Baleanu, J. Machado, and A. Luo, *Fractional Dynamics and Control*, Springer, Berlin, Germany, 2012.

[2] I. Podlubny, *Fractional Differential Equations*, Academic Press, Cambridge, MA, USA, 1999.

[3] K. Balachandran and J. Y. Park, "Controllability of fractional integrodifferential systems in Banach spaces," *Nonlinear Analysis: Hybrid Systems*, vol. 3, no. 4, pp. 363–367, 2009.

[4] K. Diethelm, "The analysis of fractional differential equations," *Lecture Notes in Mathematics*, Springer, Berlin, Germany, 2010.

[5] V. Lakshmikantham, S. Leela, and J. V. Devi, *Theory of Fractional Dynamic Systems*, Cambridge Scientific Publishers, Cambridge, UK, 2009.

[6] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Differential Equations*, John Wiley, Hoboken, NJ, USA, 1993.

[7] Y. Zhou, *Basic Theory of Fractional Differential Equations*, World Scientific, Singapore, 2014.

[8] W.-H. Chen and W. X. Zheng, "Delay-dependent robust stabilization for uncertain neutral systems with distributed delays," *Automatica*, vol. 43, no. 1, pp. 95–104, 2007.

[9] X. Zhang, J. Wu, L. Liu, Y. Wu, and Y. Cui, "Convergence analysis of iterative scheme and error estimation of positive solution for a fractional differential equation," *Mathematical Modelling and Analysis*, vol. 23, no. 4, pp. 611–626, 2018.

[10] J. Wu, X. Zhang, L. Liu, Y. Wu, and Y. Cui, "The convergence analysis and error estimation for unique solution of a *p*-Laplacian fractional differential equation with singular decreasing nonlinearity," *Boundary Value Problems*, vol. 2018, no. 1, p. 82, 2018.

[11] J. He, X. Zhang, L. Liu, Y. Wu, and Y. Cui, "Existence and asymptotic analysis of positive solutions for a singular fractional differential equation with nonlocal boundary conditions," *Boundary Value Problems*, vol. 2018, no. 1, p. 189, 2018.

[12] W. Cheng, J. Xu, D. O'Regan, and Y. Cui, "Positive solutions for a nonlinear discrete fractional boundary value problem

with a p-Laplacian operator," *Journal of Applied Analysis and Computation*, vol. 9, no. 5, pp. 1959–1972, 2019.

[13] W. Cheng, J. Xu, Y. Cui, and Q. Ge, "Positive solutions for a class of fractional difference systems with coupled boundary conditions," *Advances in Difference Equations*, vol. 2019, no. 1, p. 249, 2019.

[14] J. Xu, C. S. Goodrich, and Y. Cui, "Positive solutions for a system of first-order discrete fractional boundary value problems with semipositone nonlinearities," *Revista de la Real Academia de Ciencias Exactas, Físicas y Naturales: Serie A. Matemáticas*, vol. 113, no. 2, pp. 1343–1358, 2019.

[15] H. Zhang, Y. Li, and J. Xu, "Positive solutions for a system of fractional integral boundary value problems involving Hadamard-type fractional derivatives," *Complexity*, vol. 2019, Article ID 2671539, 11 pages, 2019.

[16] Z. Fu, S. Bai, D. O'Regan, and J. Xu, "Nontrivial solutions for an integral boundary value problem involving Riemann–Liouville fractional derivatives," *Journal of Inequalities and Applications*, vol. 2019, no. 1, p. 16, 2019.

[17] J. Xu, Z. Wei, D. O'Regan, and Y. Cui, "Infinitely many solutions for fractional Schrödinger-Maxwell equations," *Journal of Applied Analysis and Computation*, vol. 9, no. 3, pp. 1165–1182, 2019.

[18] Z. Ali, A. Zada, and K. Shah, "Ulam stability to a toppled systems of nonlinear implicit fractional order boundary value problem," *Boundary Value Problems*, vol. 2018, no. 1, p. 175, 2018.

[19] Z. Ali, A. Zada, and K. Shah, "On Ulam's stability for a coupled systems of nonlinear implicit fractional differential equations," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 42, no. 5, pp. 2681–2699, 2019.

[20] Z. Ali, P. Kumam, K. Shah, and A. Zada, "Investigation of Ulam stability results of a coupled system of nonlinear implicit fractional differential equations," *Mathematics*, vol. 7, no. 4, p. 341, 2019.

[21] R. Rizwan, A. Zada, and X. Wang, "Stability analysis of non linear implicit fractional Langevin equation with non-instantaneous impulses," *Advances in Difference Equations*, vol. 2019, no. 1, p. 85, 2019.

[22] R. Shah and A. Zada, "A fixed point approach to the stability of a nonlinear volterra integrodiferential equation with delay," *Hacettepe Journal of Mathematics and Statistics*, vol. 47, no. 3, pp. 615–623, 2018.

[23] S. O. Shah and A. Zada, "Existence, uniqueness and stability of solution to mixed integral dynamic systems with instantaneous and noninstantaneous impulses on time scales," *Applied Mathematics and Computation*, vol. 359, pp. 202–213, 2019.

[24] J. Wang, K. Shah, and A. Ali, "Existence and Hyers–Ulam stability of fractional nonlinear impulsive switched coupled evolution equations," *Mathematical Methods in the Applied Sciences*, vol. 41, no. 7, pp. 1–11, 2018.

[25] X. Wang, M. Arif, and A. Zada, "β-Hyers-Ulam-Rassias stability of semilinear nonautonomous impulsive system," *Symmetry*, vol. 11, no. 2, p. 231, 2019.

[26] J. Wang, A. Zada, and W. Ali, "Ulam's-type stability of first-order impulsive differential equations with variable delay in quasi-banach spaces," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 19, no. 5, pp. 553–560, 2018.

[27] A. Zada and S. Ali, "Stability analysis of multi-point boundary value problem for sequential fractional differential equations with non-instantaneous impulses," *International Journal of Nonlinear Sciences and Numerical Simulation*, vol. 19, no. 7-8, pp. 763–774, 2018.

[28] A. Zada and S. Ali, "Stability of integral Caputo-type boundary value problem with noninstantaneous impulses," *International Journal of Applied and Computational Mathematics*, vol. 5, no. 3, p. 55, 2019.

[29] A. Zada, S. Ali, and Y. Li, "Ulam–type stability for a class of implicit fractional differential equations with non–instantaneous integral impulses and boundary condition," *Advances in Difference Equations*, vol. 2017, no. 1, p. 317, 2017.

[30] A. Zada, W. Ali, and S. Farina, "Hyers-Ulam stability of nonlinear differential equations with fractional integrable impulses," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 15, pp. 5502–5514, 2017.

[31] A. Zada, W. Ali, and C. Park, "Ulam's type stability of higher order nonlinear delay differential equations via integral inequality of Grönwall-Bellman-Bihari's type," *Applied Mathematics and Computation*, vol. 350, pp. 60–65, 2019.

[32] A. Zada, P. Wang, D. Lassoued, and T. Li, "Connections between Hyers-Ulam stability and uniform exponential stability of 2-periodic linear nonautonomous systems," *Advances in Difference Equations*, vol. 2017, no. 1, p. 192, 2017.

[33] A. Zada and S. O. Shah, "Hyers–Ulam stability of first–order non–linear delay differential equations with fractional integrable impulses," *Hacettepe Journal of Mathematics and Statistics*, vol. 47, no. 6, pp. 1196–1205, 2017.

[34] A. Zada, O. Shah, and R. Shah, "Hyers-Ulam stability of non-autonomous systems in terms of boundedness of Cauchy problems," *Applied Mathematics and Computation*, vol. 271, pp. 512–518, 2015.

[35] A. Zada, S. Shaleena, and T. Li, "Stability analysis of higher order nonlinear differential equations in β -normed spaces," *Mathematical Methods in the Applied Sciences*, vol. 42, no. 4, pp. 1151–1166, 2019.

[36] A. U. K. Niazi, J. Wei, M. U. Rehman, and P. Denghao, "Ulam-Hyers-Mittag-Leffler stability for nonlinear fractional neutral differential equations," *Sbornik: Mathematics*, vol. 209, no. 9, pp. 1337–1350, 2018.

[37] L. Fu, Y. Chen, and H. Yang, "Time-space fractional coupled generalized Zakharov-Kuznetsov equations set for rossby solitary waves in two-layer fluids," *Mathematics*, vol. 7, no. 1, p. 41, 2019.

[38] A. A. Kilbas, H. M. Srivastava, and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations: North–Holland Mathematics Studies*, Vol. 204, Elsevier Science B.V., Amsterdam, Netherlands, 2006.

[39] Z. Bai and H. Lü, "Positive solutions for boundary value problem of nonlinear fractional differential equation," *Journal of Mathematical Analysis and Applications*, vol. 311, no. 2, pp. 495–505, 2005.

[40] D. Otrocol and V. Ilea, "Ulam stability for a delay differential equation," *Open Mathematics*, vol. 11, no. 7, pp. 1296–1303, 2013.

[41] R. Hilfer, *Application of Fractional Calculus in Physics*, World Scientific, Singapore, 1999.

[42] M. W. Michalski, "Derivatives of noninteger order and their applications," *Dissertationes Mathematicae Academic of Sciences and Institute of Mathematics*, vol. 328, p. 47, 2018.

[43] I. A. Rus, "Ulam stabilities of ordinary differential equations in a Banach space," *Carpathian Journal of Mathematics*, vol. 26, pp. 103–107, 2010.