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Research Article

On a Partial Boundary Value Condition of a Porous Medium Equation with Exponent Variable

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The initial-boundary value problem of a porous medium equation with a variable exponent is considered. Both the diffusion coefficient $a(x,t)$ and the variable exponent $p(x,t)$ depend on the time variable t, and this makes the partial boundary value condition not be expressed as the usual Dirichlet boundary value condition. In other words, the partial boundary value condition matching up with the equation is based on a submanifold of $\partial\Omega \times (0,T)$. By this innovation, the stability of weak solutions is proved.

1. Introduction

The porous medium equation with a constant exponent is widely used to model several real-life problems, and it has been extensively studied, one can refer to the survey books [1–6] and the references therein. The dynamics system of a partially nonlocal and inhomogeneous nonlinear medium has been considered in [7–9]. The case where the exponent of nonlinearity is not constant was proposed by Antontsev and Shmarev in [10], where the existence, uniqueness, and some properties of the solution in a bounded fixed domain were researched. By using the Galerkin finite element method, Duque et al. [11] proved the convergence of a fully discrete solution for this problem in a fixed domain. Based on one of the properties proved in [11] that the solution is with the finite speed of propagation, Duque et al. [12] considered the free boundary problem by using the moving mesh method to the porous medium. However, the moving mesh method was first introduced by Huang and Russell in [13].

In this paper, we consider the initial-boundary value problem of a generalized porous medium equation with a variable exponent:

$$u_t = \operatorname{div}\left(a(x,t)|u|^{m(x,t)}\nabla u\right) + \sum_{i=1}^{N} \frac{\partial b_i\left(u^{m(x,t)+1}\right)}{\partial x_i}, \quad (x,t) \in Q_T = \Omega \times (0,T), \tag{1}$$

where $m(x,t) > 0$ is a $C^1(\overline{Q_T})$ function, $a(x,t) \geq 0$ is a $C^1(\overline{Q_T})$ function, $b_i(s) \in C^1(\mathbb{R})$, and $\Omega \subset \mathbb{R}^N$ is a bounded domain with a smooth boundary $\partial\Omega$.

Equation (1) is a special case of the reaction-diffusion equation:

$$u_t = \operatorname{div}(a(u,x,t)\nabla u) + \operatorname{div}(\overrightarrow{b}(u,x,t)). \tag{2}$$

Because $a(\cdot,x,t)$ may degenerate on the boundary, how to impose a suitable boundary value condition to study the well posedness of weak solutions to equation (2) has attracted extensive attention and has been widely studied for a long time. In some details, though the initial value

$$u(x,0) = u_0(x), \quad x \in \Omega, \tag{3}$$

is always imposed, the Dirichlet boundary condition

$$u(x,t) = 0, (x,t) \in \partial\Omega \times (0,T), \tag{4}$$

may not be imposed or be imposed in a weaker sense than the traditional trace. One can refer to the references [14–19] for the details.

Naturally, besides the porous medium equation with variable exponents, the so-called electrorheological fluid equations with the form

$$u_t = \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) + f(x,t,u,\nabla u), \tag{5}$$

have been brought to the forefront by many more scholars. Since the beginning of this century, there are a great deal of papers devoting to the well-posedness problem, the intrinsic Harnack inequalities, the long-time behavior, and the Hölder regularity of weak solutions, one can refer to the literatures [20–31] and the references therein.

If $m(x,t) = m(x)$, then $a(x,t) = a(x)$ satisfies

$$a(x) > 0,\ x \in \Omega,\ a(x) = 0,\ x \in \partial\Omega, \tag{6}$$

$$\begin{gathered} 0 \le u_0 \in L^\infty(\Omega), \\ \sqrt{a(x)}\nabla u_0^{m(x)} \in L^2(\Omega). \end{gathered} \tag{7}$$

The existence and the stability of weak solutions to equation

$$\begin{gathered} u_t = \operatorname{div}\left(a(x)|u|^{m(x)}\nabla u\right) + \sum_{i=1}^{N} \frac{\partial b_i\left(u^{m(x)+1}\right)}{\partial x_i}, \\ (x,t) \in \Omega \times (0,T), \end{gathered} \tag{8}$$

has been studied in [32]. It is found that the degeneracy of $a(x)$ on boundary (6) may replace the usual boundary value condition (4). In other words, if $a(x)$ satisfies (6), the stability of weak solutions may be proved without the usual boundary value condition (4).

In this paper, we will use some ideas of [32] to study the well posedness of weak solutions to equation (1). Because both $a(x,t)$ and $p(x,t)$ are dependent on the time variable t, the problem becomes more difficult and the question of existence of such solutions is still open for equation (8), as well as for evolution p-Laplace equation with the exponent p depending on t [22]. Instead of condition (6), we only assume that $a(x,t) \ge 0$. By this assumption, we find out a partial boundary value condition matching up with the equation. Moreover, because both $a(x,t)$ and $p(x,t)$ are dependent on the time variable t, the partial boundary value condition cannot be expressed as the usual Dirichlet boundary value condition. In other words, the partial boundary value condition is based on a submanifold of $\partial\Omega \times (0,T)$. By this innovation, the stability of weak solutions is proved.

2. The Partial Boundary Value Condition and the Main Results

For any given $t \in [0,T)$ and small enough $\lambda > 0$, we set

$$\begin{aligned} \Omega_{\lambda t} &= \{x \in \Omega\colon a(x,t) > \lambda\}, \\ \Omega_{\lambda i1t} &= \left\{x \in \Omega\backslash\Omega_{\lambda t}\colon a_{x_i} < 0\right\}, \\ \Sigma_{1it} &= \lim_{\lambda \longrightarrow 0} \Omega_{\lambda i1t}, \quad i = 1,2,\ldots,N, \end{aligned} \tag{9}$$

$$\begin{aligned} \Omega_{\lambda i2t} &= \left\{x \in \Omega\backslash\Omega_{\lambda t}\colon a_{x_i} \ge 0\right\}, \\ \Sigma_{2it} &= \lim_{\lambda \longrightarrow 0} \Omega_{\lambda i2t}, \quad i = 1,2,\ldots,N. \end{aligned} \tag{10}$$

The most important and essential improvement is that instead of the usual boundary value condition (4), the stability of weak solutions is proved based on a partial boundary value condition:

$$u(x,t) = 0,\ (x,t) \in \Sigma_1 \cup \Sigma_2, \tag{11}$$

where

$$\Sigma_2 = \{x \in \partial\Omega\colon a(x,t) \ne 0\}, \tag{12}$$

and for any given $i \in \{1,2,\ldots,N\}$, if $b_i'(s) \ge 0$,

$$\Sigma_1 = \left\{\bigcup_{i=1}^{N} \Sigma_{1it} \times (0,T)\right\}, \tag{13}$$

and if $b_i'(s) \le 0$,

$$\Sigma_1 = \left\{\bigcup_{i=1}^{N} \Sigma_{2it} \times (0,T)\right\}. \tag{14}$$

Thus, if for every $i \in \{1,2,\ldots,N\}$, either $b_i'(s) \ge 0$ or $b_i'(s) \le 0$, then one can deduce an expression Σ_1 from the above discussion. For example, $b_i(s) \ge 0$ when $1 \le i \le k$ and $b_i'(s) \le 0$ when $k+1 \le i \le N$; then,

$$\Sigma_1 = \left\{\bigcup_{i=1}^{k} \Sigma_{1it} \times (0,T)\right\} \cup \left\{\bigcup_{i=k+1}^{N} \Sigma_{2it} \times (0,T)\right\}. \tag{15}$$

The most characteristic out of the ordinary is that Σ_1 or Σ_2 is just a submanifold of $\partial\Omega \times (0,T)$, and it cannot be expressed as a cylinder with the form $\Gamma \times (0,T)$ and $\Gamma \subset \partial\Omega$.

Definition 1. If $u(x,t) \ge 0$ and satisfies

$$\begin{gathered} u \in L^\infty(Q_T), \\ \sqrt{a(x,0)}|u|^{m(x,0)}|\nabla u| \in L^\infty\left(0,T;L^2(\Omega)\right), \end{gathered} \tag{16}$$

and for $\forall\varphi \in C_0^1(Q_T)$,

$$\begin{aligned} &\iint_{Q_T}\left(-\frac{\partial\varphi}{\partial t}u + a(x,t)|u|^{m(x,t)}\nabla u\nabla\varphi\right)dx\,dt \\ &\quad + \sum_{i=1}^{N}\iint_{Q_T} b_i\left(u^{m(x,t)+1}\right)\varphi_{x_i}(x,t)\,dx\,dt = 0, \end{aligned} \tag{17}$$

and then $u(x,t)$ is said to be a weak solution of equation (1) with the initial value (3) in the sense

$$\lim_{t\longrightarrow 0}\int_{\Omega}\left|u(x,t)-u_0(x)\right|\mathrm{d}x=0. \tag{18}$$

Moreover, if $u(x,t)$ satisfies (4) or (3) in the sense of the trace in addition, then it is said to be a weak solution of the initial-boundary value problem of equation (1).

Theorem 1. *If $m(x,t)>0$ is a $C^1(\overline{Q_T})$ function, $b_i(s)$ satisfies*

$$\left|b_i(s_1)-b_i(s_2)\right|\le c\left|s_1-s_2\right|,\quad i=1,2,\ldots,N. \tag{19}$$

If $u_0(x)\ge 0$ satisfies (16), then equation (1) with initial value (3) has a nonnegative solution.

Theorem 2. *Let $m(x,t)>0$ be a $C^1(\overline{Q_T})$ function, then $b_i(s)\in C^1(\mathbb{R})$ satisfy (19):*

$$\int_{\Omega}a^{-1}(x,t)\mathrm{d}x\le c(T). \tag{20}$$

Then, the initial-boundary value problem (1), (3), and (4) has a uniqueness solution.

Theorem 3. *Let $u(x,t)$ and $v(x,t)$ be two solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$, respectively, and with a partial boundary value condition*

$$u(x,t)=v(x,t)=0,\quad (x,t)\in\Sigma_1\cup\Sigma_2. \tag{21}$$

It is supposed that, for every $i\in\{1,2,\ldots,N\}$, either $b_i'(s)\ge 0$ or $b_i'(s)\le 0$, $a(x,t)$ satisfies

$$\frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_{\lambda t}}a(x,t)|\nabla a|^2\mathrm{d}x\right)^{1/2}\le c(T), \tag{22}$$

and $u(x,t)$ and $v(x,t)$ satisfy

$$\begin{aligned}&\int_{\Omega}a(x,t)[1+(m(x,t)+1)\log u]^2|\nabla m|^2\mathrm{d}x\le c(T),\\&\int_{\Omega}a(x,t)[1+(m(x,t)+1)\log v]^2|\nabla m|^2\mathrm{d}x\le c(T).\end{aligned} \tag{23}$$

Then,

$$\int_{\Omega}|u(x,t)-v(x,t)|\mathrm{d}x\le c(T)\int_{\Omega}\left|u_0(x)-v_0(x)\right|\mathrm{d}x. \tag{24}$$

Hereinafter, the constant $c(T)$ represents a constant which depends on T.

At the end of this section, we would like to suggest that if $m(x,t)=m$ is a constant, then condition (22) in Theorem 3 is naturally true.

3. The Proof of Theorem 1

First, we suppose that $u_0\in C_0^{\infty}(\Omega)$ and $0\le u_0\le M$ and consider the following regularized problem:

$$\begin{cases}u_{nt}=\operatorname{div}\left(\left(a(x,t)+\dfrac{1}{n}\right)\left(\left|u_n\right|^{m(x,t)}+\dfrac{1}{n}\right)\nabla u\right)+\displaystyle\sum_{i=1}^{N}\frac{\partial b_i\left(u_n^{m(x,t)+1}\right)}{\partial x_i},\ (x,t), & (x,t)\in Q_T,\\[2ex] u_n(x,t)=\dfrac{1}{n}, & (x,t)\in\partial\Omega\times(0,T),\\[2ex] u_n(x,0)=u_{0n}(x)=u_0(x)+\dfrac{1}{n}, & x\in\Omega.\end{cases} \tag{25}$$

According to the standard parabolic equation theory, there is a weak solution $u_n\in L^{\infty}(Q_T)$ satisfying

$$\begin{aligned}&\left(a(x,t)+\frac{1}{n}\right)^{1/2}\left(\left|u_n\right|^{m(x,t)}+\frac{1}{n}\right)^{1/2}\nabla u\in L^2(Q_T),\\&\frac{1}{n}\le u_n(x,t)\le\left\|u_0\right\|_{L^{\infty}(\Omega)}+\frac{1}{n},\quad (x,t)\in Q_T.\end{aligned} \tag{26}$$

Moreover, by comparison theorem, we clearly have

$$u_{n+1}(x,t)\le u_n(x,t), \tag{27}$$

which yields

$$u(x,t)=\lim_{n\longrightarrow\infty}u_n(x,t), \tag{28}$$

$$|u(x,t)|\le M+1. \tag{29}$$

In what follows, we are able to prove that the limit function u is a weak solution of (1) with the initial value (3).

Multiplying both sides of the first equation in (25) by $\phi=(m(x,t)+2)\left(u_n^{m(x,t)+1}-(1/n)^{m(x,t)+1}\right)$ and integrating it over Q_T, we have

$$\iint_{Q_T} u_{nt}\left(u_n^{m(x)+1}-\left(\frac{1}{n}\right)^{m(x,t)+1}\right)(m(x,t)+2)\mathrm{d}x\,\mathrm{d}t$$
$$=\iint_{Q_T}\operatorname{div}\left[\left(a(x,t)+\frac{1}{n}\right)\left(|u_n|^{m(x,t)}+\frac{1}{n}\right)\nabla u\right]\left(u_n^{m(x,t)+1}-\frac{1}{n^{m(x,t)+1}}\right)(m(x,t)+2)\mathrm{d}x\,\mathrm{d}t$$
$$+\sum_{i=1}^{N}\iint_{Q_T}\frac{\partial b_i(u_n^{m(x,t)+1})}{\partial x_i}\left(u_n^{m(x,t)+1}-\left(\frac{1}{n}\right)^{m(x)+1}\right)(m(x,t)+2)\mathrm{d}x\,\mathrm{d}t. \tag{30}$$

For the left-hand side of (29),

$$\iint_{Q_t} u_{nt}\left(u_n^{m(x,t)+1}-\left(\frac{1}{n}\right)^{m(x,t)+1}\right)(m(x,t)+2)\mathrm{d}x\,\mathrm{d}t$$
$$=\int_\Omega\left[u_n^{m(x,t)+2}(x,t)-u_n^{m(x)+2}(x,0)\right]\mathrm{d}x$$
$$-\int_\Omega\left(\frac{1}{n}\right)^{m(x,t)+1}\left[u_n(x,t)-u_n(x,0)\right](m(x,t)+2)\mathrm{d}x$$
$$+\iint_{Q_T} u_n\frac{\partial}{\partial t}\left[\left(\frac{1}{n}\right)^{m(x,t)+1}(m(x,t)+2)\right]\mathrm{d}x\,\mathrm{d}t. \tag{31}$$

For the first term of the right-hand side of (29), because

$$\lim_{n\longrightarrow\infty}\left(a(x,t)+\frac{1}{n}\right)\left(|u_n|^{m(x,t)}+\frac{1}{n}\right)\left[(m(x,t)+1)u_n^{m(x,t)}\right]^{-1}|\nabla m|^2$$
$$\cdot\left|\log u_n u_n^{m(x,t)+1}-\left(\frac{1}{n}\right)^{m(x,t)+1}\log n\right|$$
$$=a(x,t)(m(x,t)+1)^{-1}|\nabla m(x,t)|^2u^{m(x,t)+1}|\log u|<\infty,$$
$$\iint_{Q_T}\left(a(x,t)+\frac{1}{n}\right)\left(|u_n|^{m(x,t)}+\frac{1}{n}\right)\left|\left(u_n^{m(x,t)+1}-\frac{1}{n^{m(x,t)}+1}\right)\nabla m\right|^2\mathrm{d}x\,\mathrm{d}t<\infty, \tag{32}$$

by a complicated calculation and using the Young inequality, we can deduce that

$$\iint_{Q_T}\operatorname{div}\left[\left(a(x,t)+\frac{1}{n}\right)\left(|u_n|^{m(x,t)}+\frac{1}{n}\right)\nabla u_n\right]\left(u_n^{m(x,t)+1}-\frac{1}{n^{m(x,t)+1}}\right)(m(x,t)+2)\mathrm{d}x\,\mathrm{d}t$$
$$\le -c\iint_{Q_T}\left(a(x,t)+\frac{1}{n}\right)\left(|u_n|^{m(x,t)}+\frac{1}{n}\right)(m(x,t)+1)u_n^{m(x,t)}|\nabla u_n|^2\mathrm{d}x\,\mathrm{d}t+c. \tag{33}$$

For the second term of the right-hand side of (29), because

$$\begin{aligned}
&\iint_{Q_T} \frac{\partial b_i(u_n^{m(x,t)+1})}{\partial x_i} u_n^{m(x,t)+1}(m(x,t)+2)\,dx\,dt \\
&= -\iint_{Q_T} b_i\left(u_n^{m(x,t)+1}\right)\frac{\partial}{\partial x_i}\left(u_n^{m(x)+1}(m(x,t)+2)\right)dx\,dt \\
&= -\iint_{Q_T}\left[\frac{\partial}{\partial x_i}\int_{(1/n)^{m(x,t)+1}}^{u_n^{m(x,t)+1}} b_i(s)\,ds + b_i\left(n^{-m(x,t)-1}\right)\left(\frac{1}{n}\right)^{m(x,t)+1}\log n m_{x_i}(x,t)\right](m(x,t)+2)\,dx\,dt \\
&\quad - \iint_{Q_T} b_i\left(u_n^{m(x,t)+1}\right)u_n^{m(x,t)+1} m_{x_i}\,dx\,dt \\
&= -\iint_{Q_T} b_i\left(n^{-m(x,t)-1}\right)\left(\frac{1}{n}\right)^{m(x,t)+1}\log n m_{x_i}(x,t)\,dx\,dt - \iint_{Q_T} b_i\left(u_n^{m(x,t)+1}\right)u_n^{m(x,t)+1} m_{x_i}\,dx\,dt,
\end{aligned} \tag{34}$$

we can deduce that

$$\left|\sum_{i=1}^{N}\iint_{Q_T}\frac{\partial b_i(u_n^{m(x,t)+1})}{\partial x_i}\left(u_n^{m(x,t)+1}-\left(\frac{1}{n}\right)^{m(x,t)+1}\right)dx\,dt\right| \le c. \tag{35}$$

From (28)–(34), we extrapolate

$$\iint_{Q_T}\left(a(x,t)+\frac{1}{n}\right)\left(|u_n|^{m(x,t)}+\frac{1}{n}\right)u_n^{m(x,t)}|\nabla u_n|^2\,dx\,dt \le c. \tag{36}$$

Accordingly, there is $\vec{\zeta} \in L^2(Q_T)$ such that

$$\left[\left(a(x,t)+\frac{1}{n}\right)\left(|u_n|^{m(x,t)}+\frac{1}{n}\right)u_n^{m(x,t)}\right]^{1/2}\nabla u_n \rightharpoonup \vec{\zeta}, \tag{37}$$

weakly in $L^2(Q_T)$. We now can prove

$$\vec{\zeta} = a(x,t)^{1/2}|u|^{m(x,t)}\nabla u, \tag{38}$$

as in a similar way as that in [32].

The last but not the least, by that $b_i \in C^1(\mathbb{R})$, using (27), we have

$$\lim_{n\longrightarrow\infty} b_i\left(u_n^{m(x)+1}\right) = b_i\left(u^{m(x)+1}\right). \tag{39}$$

Letting $n \longrightarrow \infty$ in (30), by (37), (38), and (39), we know $u(x,t)$ satisfies (18).

Secondly, if u_0 satisfies only (16), we should consider equation (9) with the initial value $u_{0\varepsilon}$ which is the mollified function of u_0, from the above that there is a weak solution u_ε satisfying (18). Letting $\varepsilon \longrightarrow 0$, the limit function $u(x,t)$ is a solution of (1) satisfying (17) and (18), but generally is not continuous at $t = 0$ as in the case $u_0 \in C_0^\infty(\Omega)$.

Thirdly, the initial value (4) can be proved in a similar way as that when $m(x,t) = m-1$ is a constant, one can refer to [5] for the details. Thus, u is a solution of equation (1) with the initial value (4). Thus, Theorem 1 is proved.

4. The Proof of Theorem 2

One can see that, in Definition 1, there is not any definition on the general derivative u_t. At the beginning of this section, we first answer this question.

For any $t \in [0,T)$, the Banach space $V_t(\Omega)$ is defined by

$$\begin{aligned}
V_t(\Omega) &= \left\{u(x,t)\colon u(x,t) \in L^2(\Omega)\cap W_0^{1,1}(\Omega),\right. \\
&\qquad \left.\cdot\,|\nabla u(x,t)|^2 \in L^1(\Omega)\right\},
\end{aligned} \tag{40}$$

$$\|u\|_{V_t(\Omega)} = \|u\|_{2,\Omega} + \|\nabla u\|_{2,\Omega},$$

and $V_t'(\Omega)$ is denoted as its dual space. The Banach space $W(Q_T)$ is defined by

$$\begin{aligned}
W(Q_T) &= \left\{u\colon [0,T] \longrightarrow V_t(\Omega) \,\middle|\, u \in L^2(Q_T), |\nabla u|^2\right. \\
&\qquad \left.\in L^1(Q_T), u = 0 \text{ on } \Gamma = \partial\Omega\right\},
\end{aligned} \tag{41}$$

$$\|u\|_{W(Q_T)} = \|\nabla u\|_{2,Q_T} + \|u\|_{2,Q_T},$$

and $W'(Q_T)$ is denoted as its dual space. From [21], we have

$$w \in W'(Q_T) \Leftrightarrow \begin{cases} w = w_0 + \sum_{i=1}^{N} D_i w_i, \quad w_0 \in L^2(Q_T),\ w_i \in L^2(Q_T), \\ \forall \phi \in W(Q_T),\ \ll w, \phi \gg\ = \iint_{Q_T}\left(w_0\phi + \sum_i^N w_i D_i\phi\right)dx\,dt. \end{cases} \tag{42}$$

It is easy to prove the following lemmas, so we omit the details here.

Lemma 1. *If $u(x,t)$ is a weak solution of equation (1) with the initial value (3), then $u_t \in W'(Q_T)$.*

Lemma 2. *Suppose that $u \in W(Q_T)$ and $u_t \in W'(Q_T)$. For any continuous function $h(s)$, let $H(s) = \int_0^s h(s)ds$. For a.e. $t_1, t_2 \in (0,T)$, there holds*

$$\int_{t_1}^{t_2}\int_\Omega h(u)u_t \mathrm{d}x\,\mathrm{d}t = \left[\int_\Omega \left(H(u)(x,t_2) - H(u)(x,t_1)\right)\mathrm{d}x\right]. \tag{43}$$

Lemma 3. *If $\int_\Omega a(x,t)^{-1}\mathrm{d}x \le c(T)$, then $\int_\Omega |\nabla u^{m(x,t)+1}|\,\mathrm{d}x \le c(T)$. So, the weak solution of equation (1) $u(x,t)$ can be defined as the homogeneous boundary value condition $u|_{\partial\Omega} = 0$ in the sense of the trace.*

Theorem 4. *Let $u(x,t)$ and $v(x,t)$ be two solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$ respectively, and with*

$$u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,T). \tag{44}$$

If $\int_\Omega a(x,t)^{-1}(x)\mathrm{d}x \le c(T)$, then

$$\int_\Omega |u(x,t) - v(x,t)| \le c\int_\Omega \left|u_0(x) - v_0(x)\right|\mathrm{d}x. \tag{45}$$

Proof. For any given positive integer n, let $g_n(s) = \int_0^s h_n(\tau)\mathrm{d}\tau$ and $h_n(s) = 2n(1 - n|s|)_+$. Then, $h_n(s) \in C(\mathbb{R})$, and we have

$$\begin{aligned} &\lim_{n\to\infty} g_n(s) = \operatorname{sgn} s, \\ &\lim_{n\to\infty} s g_n'(s) = 0. \end{aligned} \tag{46}$$

Because $u(x,t) = v(x,t) = 0$ on the boundary $\partial\Omega \times [0,T)$, we choose $g_n(u^{m(x,t)+1} - v^{m(x,t)+1})$ as the test function and integrate over $Q_t = \Omega \times (0,t)$. Then,

$$\begin{aligned} &\int_0^t\int_\Omega g_n\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\frac{\partial(u-v)}{\partial t}\mathrm{d}x\,\mathrm{d}s \\ &+ \iint_{Q_t}\frac{a(x,s)}{m(x,s)+1}\left|\nabla u^{m(x,s)+1} - \nabla v^{m(x,s)+1}\right|^2 h_n\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\mathrm{d}x\,\mathrm{d}s \\ &- \iint_{Q_t}\frac{a(x,s)}{m(x,s)+1}\left(u^{m(x,s)+1}\log u - v^{m(x,s)+1}\log u\right) \\ &\cdot h_n\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\nabla m\nabla\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\mathrm{d}x\,\mathrm{d}s \\ &+ \sum_{i=1}^N \iint_{Q_t}\left[b_i\left(u^{m(x,s)+1}\right) - b_i\left(v^{m(x,s)+1}\right)\right]\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)_{x_i}\cdot h_n\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\mathrm{d}x\,\mathrm{d}s = 0. \end{aligned} \tag{47}$$

Let us analyse every term in (47). In the first place,

$$\iint_{Q_t}\frac{a(x,s)}{m(x,s)+1}\left|\nabla u^{m(x,s)+1} - \nabla v^{m(x,s)+1}\right|^2 h_n\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\mathrm{d}x\,\mathrm{d}t \ge 0, \tag{48}$$

$$\begin{aligned} &- \iint_{Q_t}\frac{a(x,s)}{m(x,s)+1}\left(u^{m(x,s)+1}\log u - v^{m(x,s)+1}\log u\right) \\ &\cdot h_n\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\nabla m\nabla\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\mathrm{d}x\,\mathrm{d}s \\ &\ge -\frac{1}{2}\iint_{Q_t}\frac{a(x,s)}{m(x,s)+1}\left|\nabla u^{m(x,s)+1} - \nabla v^{m(x,s)+1}\right|^2 h_n\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\mathrm{d}x\,\mathrm{d}s \\ &- \frac{1}{2}\iint_{Q_t}\frac{a(x,s)}{m(x,s)+1}\left[\left(u^{m(x,s)+1}\log u - v^{m(x,s)+1}\log u\right)h_n\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\right]^2\mathrm{d}x\,\mathrm{d}s. \end{aligned} \tag{49}$$

In the second place, we deal with the fourth term on the left-hand side of (47). For any given $t \in [0,T)$, we set

$$\begin{aligned} D_{nt} &= \left\{\Omega: \left|u^{m(x,t)+1} - v^{m(x,t)+1}\right| < \frac{1}{n}\right\}, \\ D_{0t} &= \{x \in \Omega: |u - v| = 0\}. \end{aligned} \tag{50}$$

Clearly,

$$\lim_{n\longrightarrow\infty} D_{nt} = D_{0t}. \tag{51}$$

Based on these denotations, we have

$$\begin{aligned}
&\left|\int_0^t \int_\Omega \left[b_i\left(u^{m(x,s)+1}\right) - b_i\left(v^{m(x,s)+1}\right)\right]\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)_{x_i} h_n\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right) dx\, ds\right| \\
&= \left|\int_0^t \int_{D_{ns}} \left[b_i\left(u^{m(x,s)+1}\right) - b_i\left(v^{m(x,s)+1}\right)\right]\right. \\
&\quad \left.\cdot h_n\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)_{x_i} dx\, ds\right| \\
&\le c \int_0^t \int_{D_{ns}} \left|\frac{b_i\left(u^{m(x,s)+1}\right) - b_i\left(v^{m(x,s)+1}\right)}{u^{m(x,s)+1} - v^{m(x,s)+1}}\right| \left|\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)_{x_i}\right| dx\, ds \\
&= c \int_0^t \int_{D_{ns}} \left|a(x,s)^{-(1/2)} \frac{b_i\left(u^{m(x,s)+1}\right) - b_i\left(v^{m(x,s)+1}\right)}{u^{m(x,s)+1} - v^{m(x,s)+1}}\right| \left|a(x,s)\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)_{x_i}\right| dx\, ds \\
&\le c \left[\int_0^s \int_{D_{ns}} \left(a(x,s)^{-(1/2)} \frac{b_i\left(u^{m(x,s)+1}\right) - b_i\left(v^{m(x,s)+1}\right)}{u^{m(x,s)+1} - v^{m(x,s)+1}}\right)^2 dx\right]^{1/2} \\
&\quad \cdot \left[\int_0^t \int_{D_{ns}} a(x,t)\left|\nabla\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\right|^2 dx\, ds\right]^{1/2}.
\end{aligned} \tag{52}$$

Because $\int_\Omega a(x,s)^{-1}(x) dx \le c(T)$,

$$\begin{aligned}
&\int_0^t \int_{D_{ns}} \left(a(x,s)^{-(1/2)} \frac{b_i\left(u^{m(x,s)+1}\right) - b_i\left(v^{m(x,s)+1}\right)}{u^{m(x,s)+1} - v^{m(x,s)+1}}\right)^2 dx\, ds \\
&\le c \int_{D_{ns}} a(x,s)^{-1} dx\, ds \\
&\le c(T).
\end{aligned} \tag{53}$$

If $D_{0s} = \{x \in \Omega: |u^{m(x,s)+1} - v^{m(x,s)+1}| = 0\}$ has 0 measure, then

$$\lim_{n\longrightarrow\infty} \int_0^t \int_{D_{ns}} a(x,s)^{-1} dx = \int_0^t \int_{D_{0s}} a(x,s)^{-1} dx\, ds = 0. \tag{54}$$

If $D_{0s} = \{x \in \Omega: |u^{m(x,s)+1} - v^{m(x,s)+1}| = 0\}$ has a positive measure, then

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \int_0^t \int_{D_{ns}} a(x,s)\left|\nabla\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\right|^2 dx\, ds \\
&= \int_0^t \int_{D_{0s}} a(x,s)\left|\nabla\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)\right|^2 dx\, ds \\
&= 0.
\end{aligned} \tag{55}$$

Thus, in both cases, we always have

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \int_0^t \int_\Omega \left(b_i\left(u^{m(x,s)+1}\right) - b_i\left(v^{m(x,s)+1}\right)\right) h_n \\
&\quad \cdot \left(u^{m(x,s)+1} - v^{m(x,s)+1}\right) \\
&\quad \cdot \left(u^{m(x,s)+1} - v^{m(x,s)+1}\right)_{x_i} dx\, ds \\
&= 0.
\end{aligned} \tag{56}$$

In the third place, for the first term on the left-hand side of (47), by Lemma 2,

$$\begin{aligned}
&\lim_{n\longrightarrow\infty} \int_0^t \int_\Omega g_n\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right) \frac{\partial(u-v)}{\partial t} dx\, ds \\
&= \int_0^t \int_\Omega \operatorname{sgn}\left(u^{m(x,s)+1} - v^{m(x,s)+1}\right) \frac{\partial(u-v)}{\partial t} dx\, ds \\
&= \int_0^t \int_\Omega \operatorname{sgn}(u-v) \frac{\partial(u-v)}{\partial s} ds \\
&= \int_0^t \frac{d}{dt} \left\|u - v_1\right\| ds.
\end{aligned} \tag{57}$$

Let $n \longrightarrow \infty$ in (47). Formulas (48)–(57) yield

$$\int_{\Omega}|u(x,t)-v(x,t)|\mathrm{d}x \le \int_{\Omega}\left|u_0(x)-v_0(x)\right|\mathrm{d}x. \qquad (58)$$

□

Corollary 1. *Theorem 2 is true.*

5. The Stability Based on the Partial Boundary Value Condition

Theorem 5. *Let $u(x,t)$ and $v(x,t)$ be two solutions of equation (1) with the initial values $u_0(x)$ and $v_0(x)$ respectively, and with a partial boundary value condition*

$$u(x,t)=v(x,t)=0, \quad (x,t)\in\Sigma_1\cup\Sigma_2, \qquad (59)$$

where

$$\Sigma_2=\{x\in\partial\Omega\colon a(x,t)\neq 0\}, \qquad (60)$$

and for any given $i\in\{1,2,\ldots,N\}$, if $b_i'(s)\ge 0$,

$$\Sigma_1=\left\{\bigcup_{i=1}^{N}\Sigma_{1it}\times(0,T)\right\}, \qquad (61)$$

and if $b_i'(s)\le 0$,

$$\Sigma_1=\left\{\bigcup_{i=1}^{N}\Sigma_{2it}\times(0,T)\right\}. \qquad (62)$$

It is supposed that

$$\frac{1}{\lambda}\left(\int_{\Omega\setminus\Omega_{\lambda t}}a(x,t)|\nabla a|^2dx\right)^{1/2}\le c(T), \qquad (63)$$

and $u(x,t)$ and $v(x,t)$ satisfy

$$\begin{aligned}&\int_{\Omega}a(x,t)\left[1+(m(x,t)+1)\log u\right]^2|\nabla m|^2\mathrm{d}x\le c(T),\\&\int_{\Omega}a(x,t)\left[1+(m(x,t)+1)\log v\right]^2|\nabla m|^2\mathrm{d}x\le c(T).\end{aligned} \qquad (64)$$

Then,

$$\int_{\Omega}|u(x,t)-v(x,t)|\mathrm{d}x\le c(T)\int_{\Omega}\left|u_0(x)-v_0(x)\right|\mathrm{d}x. \qquad (65)$$

Proof. According to the definition of weak solutions, for all $0\le\varphi\in C_0^1(Q_T)$, we have

$$\begin{aligned}&\iint_{Q_T}u_t\varphi(x,t)\mathrm{d}x\,\mathrm{d}t+\iint_{Q_T}\frac{a(x,t)}{m(x,t)+1}\nabla u^{m(x,t)+1}\nabla\varphi\mathrm{d}x\,\mathrm{d}t-\\&-\iint_{Q_T}\frac{a(x,t)}{m(x,t)+1}u^{m(x,t)+1}\log u\nabla m\nabla\varphi\mathrm{d}x\,\mathrm{d}t\\&+\sum_{i=1}^{N}\iint_{Q_T}b_i\left(u^{m(x,t)+1}\right)\varphi_{x_i}\mathrm{d}x\,\mathrm{d}t=0.\end{aligned} \qquad (66)$$

For any $t\in[0,T)$ and a small positive constant $\lambda>0$, based on

$$\Omega_{\lambda t}=\{x\in\Omega\colon a(x,t)>\lambda\}, \qquad (67)$$

we define that

$$\phi_{\lambda t}(x)=\begin{cases}1, & \text{if } x\in\Omega_{\lambda t},\\ \dfrac{a(x,t)}{\lambda}, & x\in\Omega\setminus\Omega_{\lambda t}.\end{cases} \qquad (68)$$

Let $\chi_{\tau,s}(t)$ be the characteristic function of $[\tau,s]\subset(0,T)$. Because

$$u(x,t)=0=v(x,t),x\in\{x\in\Omega\colon a(x,t)\neq 0\}, \qquad (69)$$

we can choose

$$\chi_{\tau,s}(t)\phi_{\lambda t}(x)g_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right), \qquad (70)$$

as the test function. Then,

$$\begin{aligned}&\int_{\tau}^{s}\int_{\Omega}\phi_{\lambda t}(x)g_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\frac{\partial(u-v)}{\partial t}\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_{\tau}^{s}\int_{\Omega}\frac{a(x,t)}{m(x,t)+1}\left(\nabla u^{m(x,t)+1}-\nabla v^{m(x,t)+1}\right)\cdot\phi_{\lambda t}(x)\nabla\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\cdot h_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\mathrm{d}x\,\mathrm{d}t\\&\quad+\int_{\tau}^{s}\int_{\Omega}\frac{a(x,t)}{m(x,t)+1}\left(\nabla u^{m(x,t)+1}-\nabla v^{m(x,t)+1}\right)\cdot\nabla\phi_{\lambda}(x,t)g_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\mathrm{d}x\,\mathrm{d}t\\&\quad-\int_{\tau}^{s}\int_{\Omega}\frac{a(x,t)}{m(x,t)+1}\left[u^{m(x,t)+1}\log u-v^{m(x,t)+1}\log v\right]\nabla m\cdot\phi_{\lambda t}(x)\nabla\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\cdot h_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\mathrm{d}x\,\mathrm{d}t\\&\quad-\int_{\tau}^{s}\int_{\Omega}\frac{a(x,t)}{m(x,t)+1}\left[u^{m(x,t)+1}\log u-v^{m(x,t)+1}\log v\right]\nabla m\cdot\nabla\phi_{\lambda t}(x)g_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\mathrm{d}x\,\mathrm{d}t\\&\quad+\sum_{i=1}^{N}\int_{\tau}^{s}\int_{\Omega}\left[b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right]\left[\phi_{\lambda x_i}(x)g_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)+\phi_{\lambda t}\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)_{x_i}h_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\right]\mathrm{d}x\,\mathrm{d}t\\&=0.\end{aligned} \qquad (71)$$

Firstly, we still have

$$\int_\tau^s \int_\Omega \frac{a(x,t)}{m(x,t)+1} \left(\nabla u^{m(x,t)+1} - \nabla v^{m(x,t)+1}\right) \cdot \phi_{\lambda t} \nabla\left(u^{m(x,t)+1} - v^{m(x,t)+1}\right) h_n\left(u^{m(x,t)+1} - v^{m(x,t)+1}\right) \mathrm{d}x\,\mathrm{d}t \geq 0. \tag{72}$$

Secondly, for the third term on the left-hand side of (71) by that $1/\lambda(\int_{\Omega\backslash\Omega_{\lambda t}} a(x,t)|\nabla a|^2 \mathrm{d}x)^{1/2} \leq c(T)$, we have

$$\begin{aligned}
&\left|\int_\tau^s \int_\Omega \frac{a(x,t)}{m(x,t)+1} \left(\nabla u^{m(x,t)+1} - \nabla v^{m(x,t)+1}\right) \cdot \nabla \phi_{\lambda t} g_n\left(u^{m(x,t)+1} - v^{m(x,t)+1}\right) \mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \leq \int_\tau^s \int_{\Omega\backslash\Omega_{\lambda t}} \left|\frac{a(x,t)}{m(x,t)+1} \left(\nabla u^{m(x,t)+1} - \nabla v^{m(x,t)+1}\right) \cdot \nabla \phi_{\lambda t} g_n\left(u^{m(x,t)+1} - v^{m(x,t)+1}\right)\right| \mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \int_\tau^s \int_{\Omega\backslash\Omega_{\lambda t}} \frac{a(x,t)}{m(x,t)+1} \left(\left|\nabla u^{m(x,t)+1}\right| + \left|\nabla v^{m(x,t)+1}\right|\right) \left|\nabla \phi_{\lambda t}\right| \mathrm{d}x\,\mathrm{d}t \\
&\quad \leq \int_\tau^s \frac{1}{\lambda} \int_{\Omega\backslash\Omega_{\lambda t}} \frac{a(x,t)}{m(x,t)+1} \left(\left|\nabla u^{m(x,t)+1}\right| + \left|\nabla v^{(m(x,t)+1}\right|\right) |\nabla a| \mathrm{d}x\,\mathrm{d}t \\
&\quad \leq c \int_\tau^s \left(\int_{\Omega\backslash\Omega_{\lambda t}} a(x,t) \left(\left|\nabla u^{m(x,t)+1}\right|^2 + \left|\nabla v^{m(x,t)+1}\right|^2\right) \mathrm{d}x\right)^{1/2} \cdot \frac{1}{\lambda} \left(\int_{\Omega\backslash\Omega_{\lambda t}} a(x,t) |\nabla a|^2 \mathrm{d}x\right)^{1/2} \mathrm{d}t \\
&\quad \longrightarrow 0,
\end{aligned} \tag{73}$$

as $\lambda \longrightarrow 0$. Thirdly, for the fourth term on the left-hand side of (71), we can show

$$\begin{aligned}
&\left|\int_\tau^s \int_\Omega \frac{a(x,t)}{m(x)+1} \left[u^{m(x,t)+1} \log u - v^{m(x,t)+1} \log v\right] \nabla m \cdot \phi_{\lambda t}(x) \nabla\left(u^{m(x,t)+1} - v^{m(x,t)+1}\right) h_n\left(u^{m(x,t)+1} - v^{m(x,t)+1}\right) \mathrm{d}x\,\mathrm{d}t\right| \\
&\quad \leq \int_\tau^s \left(\int_{D_{nt}} a(x,t) \left|\nabla u^{m(x,t)+1} - \nabla v^{m(x,t)+1}\right|^2 dx\right)^{1/2} \\
&\qquad \cdot c\left(\int_{D_{nt}} a(x,t) \left[u^{m(x,t)+1} \log u^{m(x,t)+1} - v^{m(x,t)+1} \log v^{m(x,t)+1}\right]^2 h_n\left(u^{m(x,t)+1} - v^{m(x,t)+1}\right)^2 |\nabla m|^2 dx\right)^{1/2} \mathrm{d}t \\
&\quad \longrightarrow 0,
\end{aligned} \tag{74}$$

as $\lambda \longrightarrow 0$. The corresponding details are given below.

If $D_{0t} = \{x \in \Omega\colon |u - v| = 0\}$ is with 0 measure, then

$$\begin{aligned}
&\lim_{n\longrightarrow 0}\int_\tau^s\int_{D_{nt}} a(x,t)\left|\nabla u^{m(x,t)+1}-\nabla v^{m(x,t)+1}\right|^2 dx\,dt\\
&\leq \int_\tau^s\int_{D_{0t}} a(x,t)\left(\left|\nabla u^{m(x,t)+1}\right|^2+\left|\nabla v^{m(x,t)+1}\right|^2\right)dx\,dt\\
&=0,
\end{aligned} \tag{75}$$

and by (64),

$$\int_\Omega a(x,t)[1+(m(x,t)+1)\log u]^2|\nabla m|^2 dx\leq c(T), \tag{76}$$

we obtain

$$\begin{aligned}
&\lim_{n\longrightarrow 0}\int_\tau^s\int_{D_{nt}} a(x,t)\left[u^{m(x,t)+1}\log u^{m(x,t)+1}-v^{m(x,t)+1}\log v^{m(x,t)+1}\right]^2\\
&\quad\cdot h_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)^2|\nabla m|^2 dx\,dt\Big|\\
&\leq \lim_{n\longrightarrow 0}\int_\tau^s\int_{D_{nt}} a(x,t)\frac{\left[u^{m(x,t)+1}\log u^{m(x,t)+1}-v^{m(x,t)+1}\log v^{m(x,t)+1}\right]^2}{\left|u^{m(x,t)+1}-v^{m(x,t)+1}\right|^2}|\nabla m|^2 dx\,dt\\
&\leq \int_\tau^s\int_{D_{0t}} a(x,t)[1+(m(x,t)+1)\log v]^2|\nabla m|^2 dx\\
&\leq c(T).
\end{aligned} \tag{77}$$

When $D_{0t}=\{x\in\Omega\colon |u-v|=0\}$ has a positive measure,

$$\begin{aligned}
&\lim_{n\longrightarrow 0}\int_\tau^s\int_{D_{nt}} a(x,t)\left|\nabla u^{m(x,t)+1}-\nabla v^{m(x,t)+1}\right|^2 dx\,dt\\
&\leq \int_\tau^s\int_{\Omega} a(x,t)\left(\left|\nabla u^{m(x,t)+1}\right|^2+\left|\nabla v^{m(x,t)+1}\right|^2\right)dx\,dt\\
&\leq c(T).
\end{aligned} \tag{78}$$

Also by (64),

$$\begin{aligned}
&\lim_{n\longrightarrow 0}\int_\tau^s\left|\int_{D_{nt}} a(x,t)\left[u^{m(x,t)+1}\log u^{m(x,t)+1}-v^{m(x,t)+1}\log v^{m(x,t)+1}\right]^2 h_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)^2|\nabla m|^2 dx\,dt\right|\\
&\leq \lim_{n\longrightarrow 0}\int_\tau^s\int_{D_{0t}} a(x,t)\frac{\left[u^{m(x,t)+1}\log u^{m(x,t)+1}-v^{m(x,t)+1}\log v^{m(x,t)+1}\right]^2}{\left|u^{m(x,t)+1}-v^{m(x,t)+1}\right|^2}|\nabla m|^2 dx\,dt\\
&=0.
\end{aligned} \tag{79}$$

Fourthly, when $b_i'(s)\geq 0$, using the partial boundary value condition (59),

$$\begin{aligned}
&-\lim_{\lambda\longrightarrow 0}\int_\tau^s \lim_{n\longrightarrow\infty}\int_\Omega \left[b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right]\phi_{\lambda t x_i}(x) g_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\lim_{\lambda\longrightarrow 0}\int_\tau^s\int_\Omega \left[b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right]\phi_{\lambda t x_i}(x)\,\mathrm{sign}\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\lim_{\lambda\longrightarrow 0}\int_\tau^s\int_\Omega \left[b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right]\phi_{\lambda t x_i}(x)\,\mathrm{sign}\left(b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\lim_{\lambda\longrightarrow 0}\int_\tau^s\int_{\Omega\setminus\Omega_{\lambda t}} \left|b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right|\frac{a_{x_i}}{\lambda}\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\lim_{\lambda\longrightarrow 0}\int_\tau^s\int_{\Omega\setminus\Omega_{\lambda t}} \left|u^{m(x,t)+1}-v^{m(x,t)+1}\right|b_i'(\xi)\frac{a_{x_i}}{\lambda}\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\lim_{\lambda\longrightarrow 0}\int_\tau^s\frac{1}{\lambda}\int_{\Omega_{\lambda i1t}}\left(-c_i a_{x_i}\right)\left|u^{m(x,t)+1}-v^{m(x,t)+1}\right|\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_\tau^s\int_{\Sigma_{i1t}}\left(-c_i a_{x_i}\right)\left|u^{m(x,t)+1}-v^{m(x,t)+1}\right|\mathrm{d}\Sigma\,\mathrm{d}t\\
&\quad=0.
\end{aligned}\tag{80}$$

When $b_i'(s)\le 0$, using the partial boundary value condition (59), we have

$$\begin{aligned}
&-\lim_{\lambda\longrightarrow 0}\lim_{n\longrightarrow\infty}\int_\tau^s\int_\Omega \left[b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right]\phi_{\lambda t x_i}(x) g_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\lim_{\lambda\longrightarrow 0}\int_\tau^s\int_\Omega \left[b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right]\phi_{\lambda t x_i}(x)\,\mathrm{sign}\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\lim_{\lambda\longrightarrow 0}\int_\tau^s\int_\Omega \left[b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right]\phi_{\lambda t x_i}(x)\,\mathrm{sign}\left(b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right)\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\lim_{\lambda\longrightarrow 0}\int_\tau^s\int_{\Omega\setminus\Omega_{\lambda t}} \left|b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right|\frac{a_{x_i}}{\lambda}\mathrm{d}x\,\mathrm{d}t\\
&\quad=-\lim_{\lambda\longrightarrow 0}\int_\tau^s\int_{\Omega\setminus\Omega_{\lambda t}} \left|u^{m(x,t)+1}-v^{m(x,t)+1}\right|b_i'(\xi)\frac{a_{x_i}}{\lambda}\mathrm{d}x\,\mathrm{d}t\\
&\quad\le\lim_{\lambda\longrightarrow 0}\int_\tau^s\frac{1}{\lambda}\int_{\Omega_{\lambda i2t}}\left(-c_i a_{x_i}\right)\left|u^{m(x,t)+1}-v^{m(x,t)+1}\right|\mathrm{d}x\,\mathrm{d}t\\
&\quad=\int_\tau^s\int_{\Sigma_{i2t}}\left(-c_i a_{x_i}\right)\left|u^{m(x,t)+1}-v^{m(x,t)+1}\right|\mathrm{d}\Sigma\,\mathrm{d}t\\
&\quad=0.
\end{aligned}\tag{81}$$

The last but not the least, as in the proof of Theorem 2, we can show that

$$\begin{aligned}&\lim_{\lambda\longrightarrow 0}\lim_{n\longrightarrow\infty}\int_{\tau}^{s}\int_{\Omega}\left[b_i\left(u^{m(x,t)+1}\right)-b_i\left(v^{m(x,t)+1}\right)\right]\phi_{\lambda t}\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)_{x_i}\\&\quad\cdot h_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\mathrm{d}x\,\mathrm{d}t\\&=0.\end{aligned}\tag{82}$$

Then,

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\lim_{\lambda\longrightarrow 0}\int_{\tau}^{s}\int_{\Omega}g_n\left(u^{m(x,t)+1}-v^{m(x,t)+1}\right)\phi_{\lambda t}\frac{\partial(u-v)}{\partial t}\mathrm{d}x\,\mathrm{d}t\\&\quad=\int_{\Omega}|u(x,s)-v(x,s)|\mathrm{d}x-\int_{\Omega}|u(x,\tau)-v(x,\tau)|\mathrm{d}x.\end{aligned}\tag{83}$$

Letting $\lambda\longrightarrow 0$ and $n\longrightarrow\infty$ in (73), by (74) and (80)–(85), we obtain

$$\int_{\Omega}|u(x,s)-v(x,s)|\mathrm{d}x\le c\int_{\Omega}|u(x,\tau)-v(x,\tau)|\mathrm{d}x.\tag{84}$$

By the arbitraries of τ, we have

$$\int_{\Omega}|u(x,t)-v(x,t)|\mathrm{d}x\le\int_{\Omega}\left|u_0(x)-v_0(x)\right|\mathrm{d}x.\tag{85}$$

□

Proof of Theorem 2. Because for every $i\in\{1,2,\ldots,N\}$, either $b_i'(s)\ge 0$ or $b_i'(s)\le 0$, by checking the process of the proof of (80) or (81), we easily obtain Theorem 3. □

6. Conclusion

In this paper, we consider the initial-boundary value problem of a generalized porous medium equation with a variable exponent. Different from the previous related works, both the diffusion coefficient $a(x,t)$ and variable exponent $p(x,t)$ are dependent on the time variable t, we find out a partial boundary value condition matching up with the equation. The most important innovation is that the partial boundary value condition matching up with the equation is based on a submanifold of $\partial\Omega\times(0,T)$. However, because there is an additional condition (23) imposed, Theorem 3 has not answered the problem globally. In other words, how to obtain the same conclusion as that in Theorem 3 without condition (23) is remained to be studied in the future.

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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