Research Article

Strong Consistency of Estimators in a Partially Linear Model with Asymptotically Almost Negatively Associated Errors

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This paper studies a heteroscedastic partially linear regression model in which the errors are asymptotically almost negatively associated (AANA, in short) random variables with not necessarily identical distribution and zero mean. Under some mild conditions, we establish the strong consistency of least squares estimators, weighted least squares estimators, and the ultimate weighted least squares estimators for the unknown parameter, respectively. In addition, the strong consistency of the estimator for nonparametric component is also investigated. The results derived in the paper include the corresponding ones of independent random errors and some dependent random errors as special cases. At last, two simulations are carried out to study the numerical performance of the strong consistency for least squares estimators and weighted least squares estimators of the unknown parametric and nonparametric components in the model.

1. Introduction

Consider the following heteroscedastic partially linear regression model:

$$Y_i = x_i\beta + g(t_i) + \sigma_i e_i, \quad 1 \leq i \leq n, \quad (1)$$

where $\sigma_i^2 = f(u_i)$, (x_i, t_i, u_i) are known and nonrandom design points, β is an unknown parameter, $f(\cdot)$ and $g(\cdot)$ are unknown functions defined on a compact set A , and $\{e_i, 1 \leq i \leq n\}$ are random errors.

Model (1) belongs to a kind of model called partially linear model which was introduced by Engle et al. [1] to analyse the relationship between temperature and electricity usage. Since then, many statisticians pay attention to studying partially linear regression models. Under the case of independent random errors, Hu et al. [2] studied the asymptotic normality of DHD estimators in a partially linear model; Hu [3] established the strong consistency and mean consistency of the estimators for β and $g(\cdot)$ in model (1) with $\sigma_i^2 = \sigma^2$; Gao et al. [4] established the asymptotic normality for the least squares estimators and weighted least squares

estimators of β based on the family of nonparametric estimators for $g(\cdot)$ and $f(\cdot)$ in model (1); Chen et al. [5] investigated the strong consistency of the estimators in model (1); and so on. Under the case of dependent random errors, Zeng and Liu [6] studied the asymptotic properties of the estimators for parametric and nonparametric parts in a partially linear model with NSD errors; Wang et al. [7] established the mean consistency, complete consistency, and uniform complete consistency of the estimators in a partially linear model based on ϕ -mixing errors. Wang et al. [8] and Wu and Wang [9] discussed the moment consistency and strong consistency for least squares estimators and weighted least squares estimators of β and $f(\cdot)$ in a partially linear model with $\tilde{\rho}$ -mixing errors. Pan et al. [10] obtained the mean consistency and complete consistency of the estimators for β and $g(\cdot)$ in model (1) with $\sigma_i^2 = \sigma^2$ under L^q mixingale errors; Hu [11] obtained the mean consistency and complete consistency of the estimators for β and $g(\cdot)$ in model (1) with $\sigma_i^2 = \sigma^2$ under linear time series errors; Liang and Jing [12] studied the asymptotic normality of the least squares estimators and the weighted least squares estimators in model (1) with martingale difference errors and

linear process errors; Baek and Liang [13] investigated the strong consistency and asymptotic normality of the estimators in model (1) under negatively associated samples; Zhou et al. [14] derived the moment consistency of the estimators in model (1) with negatively associated errors; and so forth. For more studies on the asymptotic properties of the estimators in regression models, one can refer to [15, 16]. Asymptotically almost negatively associated sequences are widely used dependent sequences which include independent and negatively associated sequences as special cases. So, to study the limit

properties of asymptotically almost negatively associated sequences has more theoretical significance and application value. The concept of asymptotically almost negatively associated sequences of random variables was introduced by Chandra and Ghosal [17] as follows.

Definition 1. A sequence $\{X_n, n \geq 1\}$ of random variables is called asymptotically almost negatively associated (AANA, in short) if there exists a non-negative sequence $q(n) \rightarrow 0$ as $n \rightarrow \infty$ such that

$$\text{Cov}(f(X_n), g(X_{n+1}, X_{n+2}, \dots, X_{n+k})) \leq q(n) [\text{Var}(f(X_n)) \text{Var}(g(X_{n+1}, X_{n+2}, \dots, X_{n+k}))]^{1/2}, \quad (2)$$

for all $n, k \geq 1$ and for all coordinate-wise nondecreasing continuous functions f and g whenever the variances exist.

Chandra and Ghosal [17] pointed out that the family of AANA sequences contains negatively associated (NA, in short, see [18]) sequences (with $q(n) = 0, n \geq 1$) and some more sequences of random variables which are not much deviated from being negatively associated. Two examples of AANA sequences which are not NA were constructed by Chandra and Ghosal: $\xi_n = (1 + a_n^2)^{-1/2} (\eta_n + a_n \eta_{n+1})$ (see Chandra and Ghosal [17]) and $\zeta_n = \eta_n + a_n \eta_{n+1}$ (see [19]), where $\eta_1, \eta_2, \dots$ are independent and identically distributed $N(0, 1)$ random variables, $a_n > 0$, and $a_n \rightarrow 0$ as $n \rightarrow \infty$.

Many applications of AANA sequences have been found. Chandra and Ghosal [17] derived the Kolmogorov-type inequality and Marcinkiewicz-Zygmund type strong laws of large numbers. An [20] studied the complete moment convergence of weighted sums for processes under AANA assumptions. Wang et al. [21] investigated the large deviation and Marcinkiewicz type strong law of large numbers for AANA sequences. Ko et al. [22] established the Hájek-Rényi inequalities for AANA sequences. Shen and Wu [23] investigated the strong law of large numbers for AANA sequences. Yuan and An [24] provided some Rosenthal type inequalities for maximum partial sums of AANA sequences. Wang et al. [25] studied the complete convergence for weighted sums of arrays of rowwise AANA random variables. Xi et al. [26] investigated the L^p convergence and complete convergence for weighted sums of AANA random variables. Chen et al. [27] obtained the strong laws of large numbers for the weighted sums of AANA sequences; Hu and Zhang [28] obtained the strong consistency of M-estimator in the linear regression model with AANA errors; Zhang et al. [29] established the weak consistency of M-estimator in the linear regression model with AANA errors; and so on.

However, we have not found the studies on the strong consistency of the estimators for parametric and nonparametric components in model (1) with AANA random errors in the literature. In this paper, we will consider the estimation problem for model (1) under the assumption that the errors are AANA sequences of random variables with not necessarily identical distribution and zero mean. The strong consistency of least squares estimators, weighted least squares estimators, and the ultimate weighted least squares estimators for β is derived,

respectively, based on some mild conditions. In addition, the strong consistency of the estimators for $g(\cdot)$ and $f(\cdot)$ is also studied, respectively. These results extend and improve the corresponding ones for independent and identically distributed random errors and some dependent random errors.

The following concept of stochastic domination will be used in this work.

Definition 2. A sequence $\{X_n, n \geq 1\}$ of random variables is said to be stochastically dominated by a random variable X if there exists a positive constant C such that

$$P(|X_n| > x) \leq CP(|X| > x), \quad (3)$$

for all $x \geq 0$ and $n \geq 1$.

The remainder of this paper is organized as follows. The least squares estimators, weight least squares estimators, and ultimate weighted least squares estimators of β based on the family of nonparametric estimators for $g(\cdot)$ and $f(\cdot)$ and some assumptions are introduced in Section 2. The main results are given in Section 3. We give some preliminary lemmas in Section 4. We provide the proofs of the main results in Section 5. Two simulations are presented in Section 6.

Throughout this paper, let C, C_1 , and C_2 be positive constants whose values may vary at different places. $\xrightarrow{a.s.}$ stands for almost sure convergence, $f^+ = \max\{0, f\}$, and $f^- = \min\{0, f\}$.

2. Estimation and Assumptions

Assume that $\{Y_i, x_i, t_i \in A, u_i \in A, 1 \leq i \leq n\}$ satisfy model (1) and $W_{ni}(t) = W_{ni}(t; t_1, t_2, \dots, t_n)$ is a measurable weight function on the compact set A . Denote $\tilde{x}_i = x_i - \sum_{j=1}^n W_{nj}(t_i)x_j$, $\tilde{Y}_i = Y_i - \sum_{j=1}^n W_{nj}(t_i)Y_j$, $\gamma_i = (1/\sigma_i^2) \cdot (1 \leq i \leq n)$, $\tilde{S}_n^2 = \sum_{i=1}^n \tilde{x}_i^2$ and $\tilde{T}_n^2 = \sum_{i=1}^n \gamma_i \tilde{x}_i^2$.

For model (1), by $Ee_i = 0$, one can get that $g(t_i) = E(Y_i - x_i\beta)$ for $1 \leq i \leq n$. Hence, for any given β , we define an estimator of $g(\cdot)$ given by

$$g_n(t, \beta) = \sum_{i=1}^n W_{ni}(t) (Y_i - x_i\beta). \quad (4)$$

To estimate β , we seek to minimize

$$Q(\beta) = \sum_{i=1}^n (Y_i - x_i\beta - g_n(t, \beta))^2. \quad (5)$$

The minimum point of $Q(\beta)$ is found as

$$\hat{\beta}_{LS} = \sum_{i=1}^n \frac{\tilde{x}_i \tilde{Y}_i}{\tilde{S}_n^2}, \quad (6)$$

where $\hat{\beta}_{LS}$ is called a least squares (LS, in short) estimator of β . When the random errors are heteroscedastic, we modify $\hat{\beta}_{LS}$ to a weighted least squares (WLS, in short) estimator:

$$\hat{\beta}_{WLS} = \sum_{i=1}^n \frac{\gamma_i \tilde{x}_i \tilde{Y}_i}{\tilde{T}_n^2}. \quad (7)$$

When $Ee_i^2 = 1$, we have $f(u_i) = E(Y_i - x_i\beta - g(t_i))^2$. Hence, the estimator of $f(\cdot)$ can be defined by

$$\hat{f}_n(u) = \sum_{i=1}^n \hat{W}_{ni}(u) (\tilde{Y}_i - \tilde{x}_i \hat{\beta}_{LS})^2, \quad (8)$$

where $\hat{W}_{ni}(u) = \hat{W}_{ni}(u; u_1, u_2, \dots, u_n)$ is a measurable weight function on A . In general, one assumes that $\min_{1 \leq i \leq n} |\hat{f}_n(u_i)| > 0$. Therefore, the ultimate weighted least squares estimators (UWLS, in short) of β is

$$\hat{\beta}_{UWLS} = \sum_{i=1}^n \frac{\gamma_{ni} \tilde{x}_i \tilde{Y}_i}{\tilde{U}_n^2}, \quad (9)$$

where $\tilde{U}_n^2 = \sum_{i=1}^n \gamma_{ni} \tilde{x}_i^2$, $\gamma_{ni} = 1/\hat{f}_n(u_i)$, $1 \leq i \leq n$.

From (4)–(9), we further define

$$\hat{g}_n(t) = \sum_{i=1}^n W_{ni}(t) (Y_i - x_i \hat{\beta}_{LS}), \quad (10)$$

$$\tilde{g}_n(t) = \sum_{i=1}^n W_{ni}(t) (Y_i - x_i \hat{\beta}_{WLS}), \quad (11)$$

$$\bar{g}_n(t) = \sum_{i=1}^n W_{ni}(t) (Y_i - x_i \hat{\beta}_{UWLS}). \quad (12)$$

To obtain our results, the following assumptions are sufficient.

(A₁)

- (i) $\lim_{n \rightarrow \infty} (\tilde{S}_n^2/n) = C$.
- (ii) $0 < m_0 \leq \min_{1 \leq i \leq n} f(u_i) \leq \max_{1 \leq i \leq n} f(u_i) \leq M_0 < \infty$.
- (iii) $g(\cdot)$ satisfies the first-order Lipschitz condition on compact subset A .
- (iv) $f(\cdot)$ and $g(\cdot)$ are continuous functions on compact subset A .
- (v) $\max_{1 \leq i \leq n} |\tilde{x}_i| = O(\sqrt{n}/\log n)$.

(A₂)

$$\max_{1 \leq i \leq n} \sum_{j=1}^n |W_{nj}(t_i)| = O(1), \quad \max_{1 \leq i, j \leq n} |W_{nj}(t_i)| = O(n^{-1/2}(\log n)^{-1}).$$

(A₃)

- (i) $\max_{1 \leq i \leq n} |\sum_{j=1}^n W_{nj}(t_i) - 1| = o(1)$.
- (ii) $\max_{1 \leq i \leq n} \sum_{j=1}^n |W_{nj}(t_i)| I(|t_i - t_j| > \delta) = o(1)$ for any $\delta > 0$.

(A₄)

- (i) $\max_{1 \leq i \leq n} |\sum_{j=1}^n W_{nj}(t_i) - 1| = O(n^{-1/4})$.
- (ii) $\max_{1 \leq i \leq n} \sum_{j=1}^n |W_{nj}(t_i)| I(|t_i - t_j| > \alpha n^{-1/4}) = O(n^{-1/4})$ for some $\alpha > 0$.

(A₅)

$\hat{W}_{ni}(\cdot)$ satisfies the assumptions (A₂) and (A₃) replacing t_i and W_{ni} by u_i and $\hat{W}_{ni}$, respectively.

(A₆)

$$\sum_{i=1}^{\infty} q(i) < \infty.$$

Remark 1. The assumptions (A₁) are used in Chen et al. [5], Baek and Liang [13], Zhou et al. [30], and so forth. From (i) and (ii) of (A₁), it follows that

$$\frac{\tilde{T}_n^2}{n} \leq C, \quad \tilde{S}_n^{-2} \sum_{i=1}^n |\tilde{x}_i| \leq C \text{ and } \tilde{T}_n^{-2} \sum_{i=1}^n |\gamma_i \tilde{x}_i| \leq C. \quad (13)$$

Remark 2. The following two weight functions satisfy the assumptions (A₂)–(A₄):

$$\begin{aligned} W'_{ni}(t) &= \frac{1}{H_n} \int_{S_{i-1}}^{S_i} K\left(\frac{t-s}{H_n}\right) ds, \\ W''_{ni}(t) &= \frac{K((t-T_i)/H_n)}{\sum_{j=1}^n K((t-T_j)/H_n)}, \end{aligned} \quad (14)$$

where $S_i = ((T_{i-1} + T_i)/2)$, $1 \leq i \leq n-1$, $S_0 = 0$, $S_n = 1$, $K(\cdot)$ is the Parzen–Rosenblatt kernel function (see [31]), and H_n is a bandwidth parameter.

3. Statement of Main Results

In this section, let $\{e_i, i \geq 1\}$ be an AANA sequence of random variables with zero mean and mixing coefficients $\{q(i), i \geq 1\}$, which is stochastically dominated by a random variable e .

Theorem 1. Suppose that (i), (ii), (iv), and (v) of (A₁), (A₂), (A₃), and (A₆) hold. If $Ee^2 < \infty$, then

$$\hat{\beta}_{LS} \xrightarrow{a.s.} \beta, \quad (15)$$

$$\hat{\beta}_{WLS} \xrightarrow{a.s.} \beta. \quad (16)$$

Remark 3. Since independent sequences are special AANA sequences (see Chandra and Ghosal [17]), Theorem 1 extends and improves the corresponding results of Chen et al. [5] for identically distributed independent random errors to the case of not necessarily identically distributed AANA setting.

Theorem 2. Suppose that the conditions of Theorem 1 are satisfied. Assume further that $\max_{1 \leq i \leq n} |\sum_{j=1}^n W_{nj}(t_i)x_j| = O(1)$; then,

$$\max_{1 \leq i \leq n} |\hat{g}_n(t_i) - g(t_i)| \xrightarrow{a.s.} 0, \quad (17)$$

$$\max_{1 \leq i \leq n} |\tilde{g}_n(t_i) - g(t_i)| \xrightarrow{a.s.} 0. \quad (18)$$

Remark 4. When $q(i) \equiv 0$, AANA sequences are reduced to NA sequences (see Chandra and Ghosal [17]). Therefore, Theorems 1 and 2 also hold for not necessarily identically distributed NA random errors.

Theorem 3. Let $Ee_i^2 = 1$. Suppose that (A_1) , (A_2) , and (A_4) – (A_6) hold. If $Ee^4 < \infty$, then

$$\hat{\beta}_{LS} - \beta = O(n^{-1/4}) \text{ a.s.}, \quad (19)$$

$$\max_{1 \leq i \leq n} |\hat{f}_n(u_i) - f(u_i)| \xrightarrow{a.s.} 0, \quad (20)$$

$$\hat{\beta}_{UWLS} \xrightarrow{a.s.} \beta. \quad (21)$$

Remark 5. As independent sequences are special AANA sequences, Theorem 3 extends and improves the corresponding results of Chen et al. [5] for identically distributed

independent random errors to the case of not necessarily identically distributed AANA random errors.

Theorem 4. Suppose that the conditions of Theorem 3 are satisfied. Assume further that $\max_{1 \leq i \leq n} |\sum_{j=1}^n W_{nj}(t_i)x_j| = O(1)$; then,

$$\max_{1 \leq i \leq n} |\bar{g}_n(t_i) - g(t_i)| \xrightarrow{a.s.} 0. \quad (22)$$

Remark 6. As NA sequences are special AANA sequences with $q(i) \equiv 0$, Theorems 3 and 4 also hold for not necessarily identically distributed NA random errors.

4. Preliminary Lemmas

To prove the results of this paper, the following lemmas are needed.

Lemma 1 (see [24]). If $\{X_i, i \geq 1\}$ is an AANA sequence with mixing coefficients $\{q(i), i \geq 1\}$, then $\{f_i(X_i), i \geq 1\}$ is still an AANA sequence with mixing coefficients $\{q(i), i \geq 1\}$, where $f_1, f_2, \dots$ are nondecreasing or nonincreasing functions.

Lemma 2 (see [28]). Let $\{X_i, i \geq 1\}$ be an AANA sequence of random variables with zero mean and mixing coefficients $\{q(i), i \geq 1\}$. If $|X_i| \leq b_i$, $t > 0$ and $t \max_{1 \leq i \leq n} b_i \leq 1$ a.s. Then,

$$P\left(\left|\sum_{i=1}^n X_i\right| \geq \varepsilon\right) \leq C \exp(-t\varepsilon) \sum_{i=1}^{n-1} q(i) + 2 \exp\left(-t\varepsilon + t^2 \sum_{i=1}^n EX_i^2\right), \quad (23)$$

for any $\varepsilon > 0$.

Lemma 3. Let $\{X_i, i \geq 1\}$ be an AANA sequence of random variables with zero mean and mixing coefficients $\{q(i), i \geq 1\}$. Assume that $|X_i| \leq b$ a.s. for $i \geq 1$ and $\sum_{i=1}^{\infty} q(i) < \infty$; then,

$$P\left(\left|\sum_{i=1}^n X_i\right| \geq \varepsilon\right) \leq C \exp\left\{\frac{-\varepsilon^2}{2(2 \sum_{i=1}^n EX_i^2 + b\varepsilon)}\right\}, \quad (24)$$

for any $\varepsilon > 0$.

Proof. From Lemma 2 and $\sum_{i=1}^{\infty} q(i) < \infty$, it follows that

$$P\left(\left|\sum_{i=1}^n X_i\right| \geq \varepsilon\right) \leq C \exp\left(-t\varepsilon + t^2 \sum_{i=1}^n EX_i^2\right). \quad (25)$$

Take $t = \varepsilon / (2 \sum_{i=1}^n \{EX_i^2\} + b\varepsilon)$; then, $bt \leq 1$. By (25), we derive that

$$\begin{aligned} P\left(\left|\sum_{i=1}^n X_i\right| \geq \varepsilon\right) &\leq C \exp\left(-t\varepsilon + t^2 \sum_{i=1}^n EX_i^2\right) = C \exp\left(-\frac{\varepsilon^2}{2 \sum_{i=1}^n EX_i^2 + b\varepsilon} + \frac{\varepsilon^2 \sum_{i=1}^n EX_i^2}{(2 \sum_{i=1}^n EX_i^2 + b\varepsilon)^2}\right) \\ &= C \exp\left(-\frac{\varepsilon^2}{2(2 \sum_{i=1}^n EX_i^2 + b\varepsilon)} \left(2 - \frac{2 \sum_{i=1}^n EX_i^2}{2 \sum_{i=1}^n EX_i^2 + b\varepsilon}\right)\right). \end{aligned} \quad (26)$$

Since $2 - 2 \sum_{i=1}^n EX_i^2 / (2 \sum_{i=1}^n EX_i^2 + b\varepsilon) \geq 1$, we have

$$P\left(\left|\sum_{i=1}^n X_i\right| \geq \varepsilon\right) \leq C \exp\left\{\frac{-\varepsilon^2}{2(2 \sum_{i=1}^n EX_i^2 + b\varepsilon)}\right\}. \quad (27)$$

This completes the proof of Lemma 3. $\square$

Lemma 4 (see [21]). Let $\{X_i, i \geq 1\}$ be an AANA sequence of random variables with zero mean, mixing coefficients $\{q(i), i \geq 1\}$, and $\sum_{i=1}^{\infty} q^2(i) < \infty$. If $\sum_{i=1}^{\infty} \text{Var}(X_i) < \infty$, then

$$\sum_{i=1}^{\infty} (X_i - EX_i) \text{converges a.s.} \quad (28)$$

Lemma 5. Let $\{X_i, i \geq 1\}$ be an AANA sequence of random variables with mixing coefficients $\{q(i), i \geq 1\}$ and $\sum_{i=1}^{\infty} q^2(i) < \infty$. Denote $X_i^c = X_i I(|X_i| \leq c)$ for some $c > 0$. Suppose that

$$\sum_{i=1}^{\infty} P(|X_i| > c) < \infty, \quad (29)$$

$$\sum_{i=1}^{\infty} E(X_i^c) \text{converges,} \quad (30)$$

Then,

$$\sum_{i=1}^{\infty} \text{Var}(X_i^c) < \infty. \quad (31)$$

$$\sum_{i=1}^{\infty} X_i \text{converges a.s.} \quad (32)$$

Proof. Denote $\bar{X}_i^c = X_i I(|X_i| \leq c) + cI(X_i > c) - cI(X_i < -c)$. By Lemma 1, we know that $\{\bar{X}_i^c, i \geq 1\}$ is still an AANA sequence of random variables with mixing coefficients $\{q(i), i \geq 1\}$. By (29) and (30), we have

$$\sum_{i=1}^{\infty} E(\bar{X}_i^c) = \sum_{i=1}^{\infty} EX_i^c - c \sum_{i=1}^{\infty} P(X_i < -c) + c \sum_{i=1}^{\infty} P(X_i > c) \text{converges.} \quad (33)$$

Note that

$$\begin{aligned} E(\bar{X}_i^c)^2 &= E(X_i^c)^2 + c^2 P(|X_i| > c), \\ (E\bar{X}_i^c)^2 &= (EX_i^c)^2 + c^2 P(|X_i| > c) + 2cP(X_i > c)EX_i^c - 2cP(X_i < -c)EX_i^c - 2c^2 P(X_i > c)P(X_i < -c). \end{aligned} \quad (34)$$

Hence,

$$\text{Var}(\bar{X}_i^c) = E(\bar{X}_i^c)^2 - (E\bar{X}_i^c)^2 = \text{Var}(X_i^c) + Q_i(c), \quad (35)$$

where $Q_i(c) = 2cP(X_i < -c)EX_i^c + 2c^2 P(X_i > c)P(X_i < -c) - 2cP(X_i > c)EX_i^c$.

By (30), we obtain that

$$|EX_i^c| \leq C < \infty \text{ a.s.} \quad (36)$$

Hence,

$$|Q_i(c)| \leq 2c(C + c)P(|X_i| > c). \quad (37)$$

Thus, it follows from (29) and (31) that

$$\sum_{i=1}^{\infty} \text{Var}(\bar{X}_i^c) < \infty. \quad (38)$$

By Lemma 4, we derive that

$$\sum_{i=1}^{\infty} (\bar{X}_i^c - E\bar{X}_i^c) \text{converges a.s.} \quad (39)$$

Hence, it follows from (33) that

$$-\infty < \sum_{i=1}^{\infty} \bar{X}_i^c < \infty \text{ a.s.} \quad (40)$$

By (29), we obtain that

$$\sum_{i=1}^{\infty} P(X_i \neq \bar{X}_i^c) = \sum_{i=1}^{\infty} P(|X_i| > c) < \infty. \quad (41)$$

Hence, it follows from Borel–Cantelli lemma that

$$X_i = \bar{X}_i^c \text{ a.s.} \quad (42)$$

Therefore, (32) follows from (40) and (42).

This completes the proof of Lemma 5. $\square$

Lemma 6. Let $\{X_i, i \geq 1\}$ be an AANA sequence of random variables with mixing coefficients $\{q(i), i \geq 1\}$ and $\sum_{i=1}^{\infty} q^2(i) < \infty$, which is stochastically dominated by a random variable X . For any $\varepsilon > 0$, denote

$$\begin{aligned}
X'_i &= -\varepsilon i^{(1/p)} I(X_i < -\varepsilon i^{(1/p)}) + X_i I(|X_i| \leq \varepsilon i^{(1/p)}) + \varepsilon i^{(1/p)} I(X_i > \varepsilon i^{(1/p)}), \\
X''_i &= X_i - X'_i = (X_i + \varepsilon i^{(1/p)}) I(X_i < -\varepsilon i^{(1/p)}) + (X_i - \varepsilon i^{(1/p)}) I(X_i > \varepsilon i^{(1/p)}).
\end{aligned} \tag{43}$$

If $E|X|^p < \infty$ for some $p > 0$, then

$$\sum_{i=1}^{\infty} |X''_i| < \infty \text{ a.s.} \tag{44}$$

Proof. By Lemma 1, we know that $\{X'_i, i \geq 1\}$ and $\{X''_i, i \geq 1\}$ are still AANA random variables with mixing coefficients $\{q(i), i \geq 1\}$. Denote

$$X''_i = X_i - X'_i = (X_i + \varepsilon i^{(1/p)}) I(X_i < -\varepsilon i^{(1/p)}) + (X_i - \varepsilon i^{(1/p)}) I(X_i > \varepsilon i^{(1/p)}) \triangleq X''^-_i + X''^+_i, \tag{45}$$

and then $|X''_i| = X''^+_i - X''^-_i$. By Lemma 4, we know that $\{X''^+_i, i \geq 1\}$ and $\{X''^-_i, i \geq 1\}$ are still AANA sequences of random variables with mixing coefficients $\{q(i), i \geq 1\}$. Since

X_i is stochastically dominated by a random variable X , for some fixed $d > 0$, we have

$$P(|X''^+_i| > d) = P(X_i - \varepsilon i^{(1/p)} > d) \leq P(X_i > \varepsilon i^{(1/p)}) \leq P(|X|^p > \varepsilon^p i) \leq CP(|X|^p > \varepsilon^p i). \tag{46}$$

Hence,

$$\sum_{i=1}^{\infty} P(|X''^+_i| > d) \leq C \sum_{i=1}^{\infty} P(|X|^p > \varepsilon^p i). \tag{47}$$

Denote $A_i = \{|X|^p > \varepsilon^p i\}$ for $i \geq 1$ and $A_0 = \{|X|^p \geq 0\}$. Then,

$$0 \leq \varepsilon^p \sum_{i=1}^n P(A_i) - nP(A_n)\varepsilon^p = \sum_{i=1}^n \varepsilon^p (i-1)(P(A_{i-1}) - P(A_i)) \leq \sum_{i=1}^{\infty} E|X|^p I_{A_{i-1}-A_i} = E|X|^p < \infty. \tag{48}$$

Since $\lim_{n \rightarrow \infty} nP(A_n)\varepsilon^p \leq \lim_{n \rightarrow \infty} E|X|^p I(|X|^p > \varepsilon^p n) = 0$, we have

$$\sum_{i=1}^{\infty} P(A_i) < \infty. \tag{49}$$

$$\sum_{i=1}^{\infty} P(|X''^+_i| > d) \leq C \sum_{i=1}^{\infty} P(|X|^p > \varepsilon^p i) = C \sum_{i=1}^{\infty} P(A_i) < \infty. \tag{50}$$

Denote

Thus,

$$\begin{aligned}
(|X''^+_i|)_d &\triangleq X''^+_i I(|X''^+_i| \leq d) = (X_i - \varepsilon i^{(1/p)}) I(X_i > \varepsilon i^{(1/p)}) I(X_i - \varepsilon i^{(1/p)} \leq d) \\
&= (X_i - \varepsilon i^{(1/p)}) I(0 < X_i - \varepsilon i^{(1/p)} \leq d).
\end{aligned} \tag{51}$$

Then,

$$\begin{aligned} \sum_{i=1}^{\infty} E(X''_i)_d &\leq \sum_{i=1}^{\infty} d \cdot EI(0 < X_i - \varepsilon i^{(1/p)} \leq d) \leq \sum_{i=1}^{\infty} d \cdot EI(|X_i| > \varepsilon i^{(1/p)}) \\ &\leq d \sum_{i=1}^{\infty} P(|X_i|^p > \varepsilon^p i) \leq C \sum_{i=1}^{\infty} P(|X|^p > \varepsilon^p i) < \infty, \end{aligned} \quad (52)$$

$$\begin{aligned} \text{Var} \sum_{i=1}^{\infty} (X''_i)_d &\leq \sum_{i=1}^{\infty} [E(X''_i)_d]^2 = \sum_{i=1}^{\infty} E(X_i - \varepsilon i^{(1/p)})^2 I(0 < X_i - \varepsilon i^{(1/p)} \leq d) \leq d^2 \sum_{i=1}^{\infty} E(I(|X_i| > \varepsilon i^{(1/p)})) \\ &\leq d^2 \sum_{i=1}^{\infty} P(|X_i|^p > \varepsilon^p i), \\ &\leq C \sum_{i=1}^{\infty} P(|X|^p > \varepsilon^p i) < \infty. \end{aligned} \quad (53)$$

Hence, it follows from (50), (52), (53), and Lemma 5 that

$$\sum_{i=1}^{\infty} X''_i \text{ converges a.s.} \quad (54)$$

Similar to the proof of (54), we can get that

$$\sum_{i=1}^{\infty} X''_i \text{ converges a.s.} \quad (55)$$

Therefore,

$$\sum_{i=1}^{\infty} |X''_i| = \sum_{i=1}^{\infty} (X''_i^+ - X''_i^-) = \sum_{i=1}^{\infty} X''_i^+ - \sum_{i=1}^{\infty} X''_i^- < \infty \text{ a.s.} \quad (56)$$

This completes the proof of Lemma 6. $\square$

Lemma 7 (see [32]). *Let $\{X_n, n \geq 1\}$ be a sequence of random variables which is stochastically dominated by a random variable X . For any $a > 0$ and $\beta > 0$, the following two statements hold:*

$$\begin{aligned} E|X_n|^\beta I(|X_n| \leq a) &\leq C_1 [E|X|^\beta I(|X| \leq a) + a^\beta P(|X| > a)], \\ E|X_n|^\beta I(|X_n| > a) &\leq C_2 E|X|^\beta I(|X| > a), \end{aligned} \quad (57)$$

where C_1 and C_2 are positive constants. Thus,

$$E|X_n|^\beta \leq CE|X|^\beta, \quad (58)$$

where C is a positive constant.

Lemma 8. *Let $\{X_i, i \geq 1\}$ be an AANA sequence of random variables with mixing coefficients $\{q(i), i \geq 1\}$ and $\sum_{i=1}^{\infty} q^2(i) < \infty$, which is stochastically dominated by a random variable X with $E|X|^p < \infty$ for $0 < p < 2$. Then,*

$$n^{(-1/p)} \sum_{i=1}^n (X_i - EX_i) \xrightarrow{a.s.} 0. \quad (59)$$

Proof. Denote

$$X'_n = -n^{(1/p)} I(X_n \leq -n^{(1/p)}) + X_n I(|X_n| < n^{(1/p)}) + n^{(1/p)} I(X_n \geq n^{(1/p)}), \quad (60)$$

and then we obtain by $E|X|^p < \infty$ that

$$\sum_{n=1}^{\infty} P(X_n \neq X'_n) = \sum_{n=1}^{\infty} P(|X_n| \geq n^{(1/p)}) \leq C \sum_{n=1}^{\infty} P(|X| \geq n^{(1/p)}) \leq CE|X|^p < \infty. \quad (61)$$

Hence, it follows by the Borel–Cantelli lemma that

$$X_n = X'_n \text{ a.s.} \quad (62)$$

Thus, to prove (59), it suffices to show that

$$n^{(-1/p)} \sum_{i=1}^n (X'_i - EX'_i) \xrightarrow{a.s.} 0. \quad (63)$$

By Lemma 1, we know that $\{X'_n - EX'_n, n \geq 1\}$ is still an AANA sequence of random variables. Hence, by Lemma 4 and Kronecker lemma, to prove (63), we only need to show that

$$\sum_{n=1}^{\infty} \text{Var} \left(\frac{X'_n}{n^{(1/p)}} \right) < \infty. \quad (64)$$

By the Markov inequality, Lemma 4, and $E|X|^p < \infty$, we have

$$\begin{aligned}
 \sum_{n=1}^{\infty} \text{Var}\left(\frac{X'_n}{n^{(1/p)}}\right) &\leq \sum_{n=1}^{\infty} \frac{E(X'_n)^2}{n^{(2/p)}}, \\
 &\leq C \sum_{n=1}^{\infty} \frac{n^{(2/p)} P(|X| \geq n^{(1/p)}) + EX^2 I(|X| < n^{(1/p)})}{n^{(2/p)}}, \\
 &\leq C + C \sum_{n=1}^{\infty} n^{(-2/p)} \sum_{k=1}^n EX^2 I(k-1 \leq |X|^p < k), \\
 &= C + C \sum_{k=1}^{\infty} \sum_{n=k}^{\infty} n^{(-2/p)} EX^2 I(k-1 \leq |X|^p < k), \\
 &\leq C + C \sum_{k=1}^{\infty} k^{(-2/p+1)} E|X|^p k^{(2-p)/p} I(k-1 \leq |X|^p < k), \\
 &= C + C \sum_{k=1}^{\infty} E|X|^p I(k-1 \leq |X|^p < k) < \infty.
 \end{aligned} \tag{65}$$

Therefore, (59) follows.

This completes the proof of Lemma 8. $\square$

5. Proofs of the Main Results

By (1), (6), (7), and (9), we derive that

$$\hat{\beta}_{\text{LS}} - \beta = \tilde{S}_n^{-2} \left(\sum_{i=1}^n \tilde{x}_i \tilde{\varepsilon}_i + \sum_{i=1}^n \tilde{x}_i \tilde{g}_i \right), \tag{66}$$

$$\hat{\beta}_{\text{WLS}} - \beta = \tilde{T}_n^{-2} \left(\sum_{i=1}^n \gamma_i \tilde{x}_i \tilde{\varepsilon}_i + \sum_{i=1}^n \gamma_i \tilde{x}_i \tilde{g}_i \right), \tag{67}$$

$$\hat{\beta}_{\text{UWLS}} - \beta = \tilde{T}_n^{-2} \left(\sum_{i=1}^n \gamma_{ni} \tilde{x}_i \tilde{\varepsilon}_i + \sum_{i=1}^n \gamma_{ni} \tilde{x}_i \tilde{g}_i \right), \tag{68}$$

where $\tilde{\varepsilon}_i = \varepsilon_i - \sum_{j=1}^n W_{nj}(t_i) \varepsilon_j$, $\tilde{g}_i = g(t_i) - \sum_{j=1}^n W_{nj}(t_i) g(t_j)$, and $\varepsilon_i = \sigma_i e_i$.

Proof of Theorem 1. We prove (16) first. By (67), we can get that

$$\begin{aligned}
 \hat{\beta}_{\text{WLS}} - \beta &= \tilde{T}_n^{-2} \left(\sum_{i=1}^n \gamma_i \tilde{x}_i \varepsilon_i - \sum_{i=1}^n \gamma_i \tilde{x}_i \left(\sum_{j=1}^n W_{nj}(t_i) \varepsilon_j \right) + \sum_{i=1}^n \gamma_i \tilde{x}_i \tilde{g}_i \right), \\
 &\triangleq \Delta_{1n} - \Delta_{2n} + \Delta_{3n}.
 \end{aligned} \tag{69}$$

Hence, to prove (16), it suffices to show that

$$\Delta_{in} \xrightarrow{a.s.} 0 \quad (i = 1, 2, 3). \tag{70}$$

Firstly, we prove

$$\Delta_{1n} \xrightarrow{a.s.} 0. \tag{71}$$

In view of (69), we have

$$\Delta_{1n} = \tilde{T}_n^{-2} \left(\sum_{i=1}^n \gamma_i \tilde{x}_i \sigma_i e_i \right) = \sum_{i=1}^n \left(\tilde{T}_n^{-2} \gamma_i \tilde{x}_i \sigma_i \right) e_i \triangleq \sum_{i=1}^n b_{ni} e_i. \tag{72}$$

From (ii) and (v) of (A_1) and (13), it follows that

$$\begin{aligned}
 \sum_{i=1}^n b_{ni}^2 &= \sum_{i=1}^n \frac{\gamma_i \tilde{x}_i \gamma_i \sigma_i^2}{\tilde{T}_n^4} \leq \frac{C}{\tilde{T}_n^2} \leq \frac{C}{n}, \\
 \max_{1 \leq i \leq n} |b_{ni}| &\leq \max_{1 \leq i \leq n} \left(\frac{|\tilde{x}_i|}{n^{(1/2)}} \right) \left(\max_{1 \leq i \leq n} \gamma_i \sigma_i \right) \left(\frac{n^{1/2}}{\tilde{T}_n^2} \right) \leq C n^{-(1/2)} (\log n)^{-1}.
 \end{aligned} \tag{73}$$

Denote

$$\begin{aligned}
 e'_i &= -\varepsilon i^{(1/2)} I(e_i < -\varepsilon i^{(1/2)}) + e_i I(|e_i| \leq \varepsilon i^{(1/2)}) + \varepsilon i^{(1/2)} I(e_i > \varepsilon i^{(1/2)}), \\
 e''_i &= e_i - e'_i = (e_i + \varepsilon i^{(1/2)}) I(e_i < -\varepsilon i^{(1/2)}) + (e_i - \varepsilon i^{(1/2)}) I(e_i > \varepsilon i^{(1/2)}),
 \end{aligned} \tag{74}$$

for any $\varepsilon > 0$. Write $e_{ni} = b_{ni} (e'_i - E(e'_i)) = b_{ni}^+ (e'_i - E(e'_i)) + b_{ni}^- (e'_i - E(e'_i)) \triangleq e_{ni}^{(1)} + e_{ni}^{(2)}$, $1 \leq i \leq n$; then, we know by Lemma 1 that $\{e_{ni}^{(1)}, 1 \leq i \leq n\}$ and $\{e_{ni}^{(2)}, 1 \leq i \leq n\}$ are still

AANA sequences with mixing coefficients $\{q(i), i \geq 1\}$ and zero mean. By (ii) and (iv) of (A_1) and Lemma 7, we have

$$\begin{aligned} \max_{1 \leq i \leq n} |e_{ni}^{(k)}| &\leq 2\varepsilon n^{(1/2)} \max_{1 \leq i \leq n} |b_{ni}^{\pm}| \leq 2\varepsilon n^{(1/2)} \max_{1 \leq i \leq n} |b_{ni}| \leq C\varepsilon (\log n)^{-1}, \\ \sum_{i=1}^n E(e_{ni}^{(k)})^2 &\leq \sum_{i=1}^n (b_{ni}^{\pm})^2 E(e'_i)^2 \leq \sum_{i=1}^n b_{ni}^2 E(e'_i)^2 \leq C \sum_{i=1}^n b_{ni}^2 E(e)^2 \leq C \sum_{i=1}^n b_{ni}^2 \leq Cn^{-1}, \end{aligned} \quad (75)$$

for $k = 1, 2$.

Hence, for any $\mu > 0$ and sufficiently large n , by Lemma 4, we have

$$\begin{aligned} P\left(\left|\sum_{i=1}^n e_{ni}\right| > \mu\right) &= P\left(\left|\sum_{i=1}^n (e_{ni}^{(1)} + e_{ni}^{(2)})\right| > \mu\right) \leq P\left(\left|\sum_{i=1}^n e_{ni}^{(1)}\right| > \frac{\mu}{2}\right) + P\left(\left|\sum_{i=1}^n e_{ni}^{(2)}\right| > \frac{\mu}{2}\right), \\ &\leq C \exp\left\{-\frac{\mu^2/4}{2[2C/n + C_1\varepsilon\mu(\log n)^{-1}]}\right\}, \\ &\leq C \exp\left\{-\frac{\mu^2}{C_2\varepsilon\mu(\log n)^{-1}}\right\}, \\ &\leq \frac{C}{n^2} \left(0 < \varepsilon < \frac{\mu}{2C_2}\right). \end{aligned} \quad (76)$$

Hence, it follows from the Borel–Cantelli lemma that

$$\sum_{i=1}^n e_{ni} \xrightarrow{a.s.} 0. \quad (77)$$

By Lemma 6, we obtain that

$$\sum_{i=1}^{\infty} |e''_i| < \infty \text{ a.s.} \quad (78)$$

By (73) and (78), we derive that

$$\left|\sum_{i=1}^n b_{ni} e''_i\right| \leq \sum_{i=1}^n |b_{ni} e''_i| \leq \left(\max_{1 \leq i \leq n} |b_{ni}|\right) \left(\sum_{i=1}^n |e''_i|\right) = O\left(n^{-\frac{1}{2}} (\log n)^{-1}\right) \text{ a.s.}, \quad (79)$$

$$\begin{aligned} \left|\sum_{i=1}^n b_{ni} E e''_i\right| &\leq \sum_{i=1}^n |b_{ni} E e''_i| \leq \left(\max_{1 \leq i \leq n} |b_{ni}|\right) \left(\sum_{i=1}^n E(|e_i| I(|e_i| > \varepsilon i^{(1/2)}))\right), \\ &\leq Cn^{-(1/2)} (\log n)^{-1} \sum_{i=1}^n \varepsilon^{-1} i^{-(1/2)} E e_i^2 I(|e_i| > \varepsilon i^{(1/2)}) \leq Cn^{-(1/2)} (\log n)^{-1} \sum_{i=1}^n i^{-(1/2)} \leq C (\log n)^{-1}. \end{aligned} \quad (80)$$

Hence, combining (77), (79), and (80), we have

$$\begin{aligned} |I_{1n}| &= \left|\sum_{i=1}^n b_{ni} (e'_i + e''_i)\right| = \left|\sum_{i=1}^n e_{ni} + \sum_{i=1}^n b_{ni} e''_i - \sum_{i=1}^n b_{ni} E e''_i\right|, \\ &\leq \left|\sum_{i=1}^n e_{ni}\right| + \left|\sum_{i=1}^n b_{ni} e''_i\right| + \left|\sum_{i=1}^n b_{ni} E e''_i\right| \xrightarrow{a.s.} 0. \end{aligned} \quad (81)$$

Thus, (71) follows.

Secondly, we prove

$$\Delta_{2n} \xrightarrow{a.s.} 0. \quad (82)$$

In view of (69), we have

$$\Delta_{2n} = \tilde{T}_n^{-2} \sum_{i=1}^n \gamma_i \tilde{x}_i \left(\sum_{j=1}^n W_{nj}(t_i) \varepsilon_j \right) = \sum_{j=1}^n \left(\sum_{i=1}^n \tilde{T}_n^{-2} \gamma_i \tilde{x}_i W_{nj}(t_i) \sigma_j \right) e_j \triangleq \sum_{j=1}^n b'_{nj} e_j. \quad (83)$$

From (ii) of (A_1) , (A_2) , and (13), it follows that

$$\begin{aligned} \sum_{j=1}^n b'^2_{nj} &\leq \sum_{j=1}^n \left(\max_{1 \leq j \leq n} \sigma_j \right)^2 \left(\max_{1 \leq i, j \leq n} |W_{nj}(t_i)| \right)^2 \left(\tilde{T}_n^{-2} \sum_{i=1}^n |\gamma_i \tilde{x}_i| \right)^2 \leq C (\log n)^{-2}, \\ \max_{1 \leq j \leq n} |b'_{nj}| &\leq \left(\max_{1 \leq j \leq n} \sigma_j \right) \left(\max_{1 \leq i, j \leq n} |W_{nj}(t_i)| \right) \left(\tilde{T}_n^{-2} \sum_{i=1}^n |\gamma_i \tilde{x}_i| \right) \leq C n^{-\frac{1}{2}} (\log n)^{-1}. \end{aligned} \quad (84)$$

Hence, similar to the proof (71), one can derive (82).

Finally, we prove

$$\Delta_{3n} \xrightarrow{a.s.} 0. \quad (85)$$

In view of (69), we have

$$\Delta_{3n} = \tilde{T}_n^{-2} \sum_{i=1}^n \gamma_i \tilde{x}_i \tilde{g}_i. \quad (86)$$

By (iv) of (A_1) , (A_3) , and $\max_{1 \leq i \leq n} \sum_{j=1}^n |W_{nj}(t_i)| = O(1)$, we have

$$\begin{aligned} \max_{1 \leq i \leq n} |\tilde{g}_i| &= \max_{1 \leq i \leq n} \left| g(t_i) - \sum_{j=1}^n W_{nj}(t_i) g(t_j) \right| \leq \max_{1 \leq i \leq n} |g(t_i)| \left| \sum_{j=1}^n W_{nj}(t_i) - 1 \right|, \\ &+ \max_{1 \leq i \leq n} \sum_{j=1}^n |W_{nj}(t_i)| |g(t_i) - g(t_j)| I(|t_i - t_j| > \delta) + \max_{1 \leq i \leq n} \sum_{j=1}^n |W_{nj}(t_i)| |g(t_i) - g(t_j)| I(|t_i - t_j| \leq \delta) \longrightarrow 0, \end{aligned} \quad (87)$$

as $n \longrightarrow \infty$.

Hence, it follows from (13) that

$$|\Delta_{3n}| \leq \tilde{T}_n^{-2} \sum_{i=1}^n |\gamma_i \tilde{x}_i \tilde{g}_i| \leq \left(\max_{1 \leq i \leq n} |\tilde{g}_i| \right) \left(\tilde{T}_n^{-2} \sum_{i=1}^n |\gamma_i \tilde{x}_i| \right) \longrightarrow 0 \text{ as } n \longrightarrow \infty. \quad (88)$$

Thus, (85) follows. Therefore, (16) follows from (71), (82), and (85).

The proof of (15) is similar to that of (16); thus, we omit the details here.

This completes the proof of Theorem 1. $\square$

Proof of Theorem 2. We prove (18) first. In view of (11), we have

$$\begin{aligned} \tilde{g}_n(t_i) - g(t_i) &= \sum_{j=1}^n W_{nj}(t_i) (x_j \beta + g(t_j) + \sigma_j e_j - x_j \hat{\beta}_{\text{WLS}}) - g(t_i) = \sum_{j=1}^n W_{nj}(t_i) x_j (\beta - \hat{\beta}_{\text{WLS}}) - \left[g(t_i) - \sum_{j=1}^n W_{nj}(t_i) g(t_j) \right] \\ &+ \sum_{j=1}^n W_{nj}(t_i) \sigma_j e_j. \end{aligned} \quad (89)$$

Hence,

$$\max_{1 \leq i \leq n} |\tilde{g}_n(t_i) - g(t_i)| \leq \max_{1 \leq i \leq n} \left(\left| \beta - \tilde{\beta}_{\text{WLS}} \right| \left| \sum_{j=1}^n W_{nj}(t_i) x_j \right| \right) + |\tilde{g}_i| + \left| \sum_{j=1}^n W_{nj}(t_i) \sigma_j e_j \right| \triangleq I_{1n} + I_{2n} + I_{3n}. \quad (90)$$

From (16) and $\max_{1 \leq i \leq n} \left| \sum_{j=1}^n W_{nj}(t_i) x_j \right| = O(1)$, it follows that

$$I_{1n} \xrightarrow{a.s.} 0. \quad (91)$$

By (87), we have

$$I_{2n} \xrightarrow{a.s.} 0. \quad (92)$$

Next, we will prove

$$I_{3n} \xrightarrow{a.s.} 0. \quad (93)$$

Denote

$$\sum_{j=1}^n W_{nj}(t_i) \sigma_j e_j \triangleq \sum_{j=1}^n c_{nj} e_j. \quad (94)$$

By (ii) of (A_1) and (A_2) , we derive that

$$\begin{aligned} \sum_{j=1}^n c_{nj}^2 &= \sum_{j=1}^n W_{nj}^2(t_i) \sigma_j^2 \leq C \max_{1 \leq j \leq n} |W_{nj}(t_i)| \sum_{j=1}^n |W_{nj}(t_i)|, \\ &\leq C n^{-\frac{1}{2}} (\log n)^{-1}, \\ \max_{1 \leq j \leq n} |c_{nj}| &\leq C n^{-\frac{1}{2}} (\log n)^{-1}. \end{aligned} \quad (95)$$

Similar to the proof of (71), we can get (93). Therefore, (18) follows from (90)–(93).

The proof of (17) is similar to that of (18); thus, we omit the details here.

This completes the proof of Theorem 2. $\square$

Proof of Theorem 3. Firstly, we prove (19). In view of (66), we have

$$\tilde{\beta}_{LS} - \beta = \tilde{S}_n^{-2} \left(\sum_{i=1}^n \tilde{x}_i \tilde{\varepsilon}_i - \sum_{i=1}^n \tilde{x}_i \left(\sum_{j=1}^n W_{nj}(t_j) \varepsilon_j \right) + \sum_{i=1}^n \tilde{x}_i \tilde{g}_i \right) \triangleq J_{1n} + J_{2n} + J_{3n}. \quad (96)$$

By (ii) of (A_1) , (A_4) , and $\max_{1 \leq i \leq n} \sum_{j=1}^n |W_{nj}(t_i)| = O(1)$, we have

$$\begin{aligned} \max_{1 \leq i \leq n} |\tilde{g}_i| &= \max_{1 \leq i \leq n} \left| g(t_i) - \sum_{j=1}^n W_{nj}(t_i) g(t_j) \right| \\ &\leq \max_{1 \leq i \leq n} |g(t_i)| \left| \sum_{j=1}^n W_{nj}(t_i) - 1 \right| + \max_{1 \leq i \leq n} \sum_{j=1}^n |W_{nj}(t_i)| |g(t_i) - g(t_j)| I(|t_i - t_j| > \alpha n^{-\frac{1}{4}}) \\ &\quad + \max_{1 \leq i \leq n} \sum_{j=1}^n |W_{nj}(t_i)| |g(t_i) - g(t_j)| I(|t_i - t_j| \leq \alpha n^{-1/4}), \\ &= O(n^{-1/4}). \end{aligned} \quad (97)$$

By (13), we obtain that

$$|J_{3n}| \leq \left(\max_{1 \leq i \leq n} |\tilde{g}_i| \right) \left(\sum_{i=1}^n \frac{|\tilde{x}_i|}{\tilde{S}_n^2} \right) = O(n^{-(1/4)}). \quad (98)$$

Note that

$$\begin{aligned} J_{1n} &= \sum_{i=1}^n \left(\tilde{S}_n^{-2} \tilde{x}_i \sigma_i \right) e_i \triangleq \sum_{i=1}^n d_{ni} e_i, \\ J_{2n} &= \sum_{j=1}^n \left(\tilde{S}_n^{-2} \sum_{i=1}^n \tilde{x}_i W_{nj}(t_i) \sigma_j \right) e_j \triangleq \sum_{j=1}^n \bar{d}_{nj} e_j. \end{aligned} \quad (99)$$

Denote

$$\begin{aligned}\bar{e}'_i &= -\varepsilon i^{(1/4)} I(e_i < -\varepsilon i^{(1/4)}) + e_i I(|e_i| \leq \varepsilon i^{(1/4)}) + \varepsilon i^{(1/4)} I(e_i > \varepsilon i^{(1/4)}), \\ \bar{e}''_i &= e_i - \bar{e}'_i = (e_i + \varepsilon i^{(1/4)}) I(e_i < -\varepsilon i^{(1/4)}) + (e_i - \varepsilon i^{(1/4)}) I(e_i > \varepsilon i^{(1/4)}),\end{aligned}\quad (100)$$

for any $\varepsilon > 0$. Write

$$\begin{aligned}e'_{ni} &= d_{ni}(\bar{e}'_i - E(\bar{e}'_i)) \\ &= d_{ni}^+(\bar{e}'_i - E(\bar{e}'_i)) + d_{ni}^-(\bar{e}'_i - E(\bar{e}'_i)) \triangleq e_{ni}'^{(1)} + e_{ni}'^{(2)},\end{aligned}\quad (101)$$

for $1 \leq i \leq n$, and

$$\bar{e}_{nj} = \bar{d}_{nj}(\bar{e}'_j - E(\bar{e}'_j)) = \bar{d}_{nj}^+(\bar{e}'_j - E(\bar{e}'_j)) + \bar{d}_{nj}^-(\bar{e}'_j - E(\bar{e}'_j)) \triangleq \bar{e}_{nj}^{(1)} + \bar{e}_{nj}^{(2)},\quad (102)$$

for $1 \leq j \leq n$.

Then, we know by Lemma 1 that $\{e_{ni}'^{(1)}, 1 \leq i \leq n\}$, $\{e_{ni}'^{(2)}, 1 \leq i \leq n\}$, $\{\bar{e}_{nj}^{(1)}, 1 \leq j \leq n\}$, and $\{\bar{e}_{nj}^{(2)}, 1 \leq j \leq n\}$ are still AANA sequences with mixing coefficients $\{q(i), i \geq 1\}$ and zero mean. Note that

$$\max_{1 \leq i \leq n} |d_{ni}| = \max_{1 \leq i \leq n} |\tilde{S}_n^{-2} \tilde{x}_i \sigma_i| \leq C \max_{1 \leq i \leq n} \left| \frac{\tilde{x}_i}{\sqrt{n}} \right| \left| \frac{\sqrt{n}}{\tilde{S}_n^2} \right| \leq C n^{-\frac{1}{2}} (\log n)^{-1},$$

$$\sum_{i=1}^n d_{ni}^2 \leq \frac{C}{n},$$

$$\max_{1 \leq j \leq n} |\bar{d}_{nj}| = \max_{1 \leq i, j \leq n} |W_{nj}(t_i)| \max_{1 \leq j \leq n} \tilde{S}_n^{-2} \sum_{i=1}^n |\tilde{x}_i| \leq C n^{-\frac{1}{2}} (\log n)^{-1},$$

$$\begin{aligned}\sum_{j=1}^n \bar{d}_{nj}^2 &\leq \sum_{j=1}^n \tilde{S}_n^{-4} \sigma_j^2 \left(\sum_{i=1}^n \tilde{x}_i^2 \right) \left(\sum_{i=1}^n W_{nj}^2(t_i) \right) \leq \tilde{S}_n^{-2} \sum_{i=1}^n \sum_{j=1}^n \sigma_j^2 W_{nj}^2(t_i), \\ &\leq \tilde{S}_n^{-2} \left(\max_{1 \leq i \leq n} \sigma_j^2 \right) \left(\max_{1 \leq i, j \leq n} |W_{nj}(t_i)| \right) \left(\sum_{i=1}^n \sum_{j=1}^n |W_{nj}(t_i)| \right), \\ &\leq C n^{-\frac{1}{2}} (\log n)^{-1}.\end{aligned}\quad (103)$$

By Lemma 7, we obtain that

$$E e_i^4 \leq C E e^4 < \infty. \quad (104)$$

Hence, similar to the proof of (78), we can easily derive that $\sum_{i=1}^{\infty} |e_i''| < \infty$ a.s.

Since

$$\begin{aligned}\left| \sum_{i=1}^n d_{ni} E(e_i'') \right| &\leq \left(\max_{1 \leq i \leq n} |d_{ni}| \right) \left(\sum_{i=1}^n E(|e_i| I(|e_i| > \varepsilon i^{(1/4)})) \right) \\ &\leq C n^{-\frac{1}{2}} (\log n)^{-1} \sum_{i=1}^n i^{-(3/4)} \varepsilon^{-3} E|e_i|^4 I(|e_i| > \varepsilon i^{(1/4)}), \\ &\leq C n^{-(1/2)} (\log n)^{-1} n^{(1/4)} = o(n^{-(1/4)}).\end{aligned}\quad (105)$$

Hence, similar to the proof of (71), we can easily obtain that

$$|J_{1n}| \leq \left| \sum_{i=1}^n e_{ni}' \right| + \left| \sum_{i=1}^n d_{ni} \bar{e}_{nj}'' \right| + \left| \sum_{i=1}^n d_{ni} E \bar{e}_{nj}'' \right| = o(n^{-\frac{1}{4}}) \text{ a.s.}, \quad (106)$$

$$|J_{2n}| \leq \left| \sum_{j=1}^n \bar{e}_{nj} \right| + \left| \sum_{j=1}^n \bar{d}_{nj} e_j'' \right| + \left| \sum_{j=1}^n \bar{d}_{nj} E e_j'' \right| = O(n^{-(1/4)}) \text{ a.s.} \quad (107)$$

Therefore, (19) follows from (96), (98), (106), and (107). Secondly, we prove (20). In view of (8), we have

$$\begin{aligned}
\max_{1 \leq j \leq n} |\hat{f}_n(u_j) - f(u_j)| &\leq \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \varepsilon_i^2 - f(u_j) \right| + 2 \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \varepsilon_i \left(\sum_{j=1}^n W_{nj}(t_i) \varepsilon_j \right) \right| \\
&\quad + \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \left(\sum_{j=1}^n W_{nj}(t_i) \varepsilon_j \right)^2 \right| + 2 \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \tilde{g}_i \tilde{\varepsilon}_i \right| \\
&\quad + 2 \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \tilde{x}_i \tilde{\varepsilon}_i \right| |\beta - \hat{\beta}_{LS}| + (\beta - \hat{\beta}_{LS})^2 \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \tilde{x}_i^2 \right| \\
&\quad + 2 |\beta - \hat{\beta}_{LS}| \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \tilde{x}_i \tilde{g}_i \right| + \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \tilde{g}_i^2 \right| \triangleq B_{1n} + B_{2n} + B_{3n} + B_{4n} + B_{5n} + B_{6n} + B_{7n} + B_{8n}.
\end{aligned} \tag{108}$$

Next, we will prove

$$B_{1n} \xrightarrow{a.s.} 0. \tag{109}$$

Since $e_i = e_i^+ + e_i^-$ and $e_i^2 = (e_i^+)^2 + (e_i^-)^2$, then $Ee_i^2 = E(e_i^+)^2 + E(e_i^-)^2 = 1$. We know by Lemma 1 that $\{(e_i^+)^2\}$ and $\{(e_i^-)^2\}$ are still AANA sequences with mixing coefficients $\{q(i), i \geq 1\}$. Observe that

$$\begin{aligned}
|B_{1n}| &\leq \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) f(u_i) ((e_i^+)^2 - E(e_i^+)^2) \right| + \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) f(u_i) ((e_i^-)^2 - E(e_i^-)^2) \right| \\
&\quad + \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) f(u_i) - f(u_j) \right| \triangleq B_{11n} + B_{12n} + B_{13n}.
\end{aligned} \tag{110}$$

Denote $\eta_i = (e_i^\pm)^2 - E(e_i^\pm)^2$, $1 \leq i \leq n$; then $E\eta_i = 0$, $E\eta_i^2 \leq Ee_i^4 \leq Ee^4 < \infty$, and $\{\eta_i\}$ is still an AANA sequence with mixing coefficients $\{q(i), i \geq 1\}$. Similar to the proof of (71), one can derive that

$$\begin{aligned}
B_{11n} &\xrightarrow{a.s.} 0, \\
B_{12n} &\xrightarrow{a.s.} 0.
\end{aligned} \tag{111}$$

By (iv) of (A_1) and (A_5) , we have

$$B_{13n} \xrightarrow{a.s.} 0. \tag{112}$$

Therefore, (109) follows.

Note that

$$|B_{2n}| \leq 2 \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \sigma_i e_i \right| \right) \left(\max_{1 \leq i \leq n} \left| \sum_{j=1}^n W_{nj}(t_i) \sigma_j e_j \right| \right). \tag{113}$$

Similar to the proof of (71), one can easily obtain that

$$\max_{1 \leq j \leq n} \left(\sum_{i=1}^n |\hat{W}_{ni}(u_j) \sigma_i e_i| \right) = O(1), \tag{114}$$

$$\max_{1 \leq i \leq n} \left| \sum_{j=1}^n W_{nj}(t_i) \sigma_j e_j \right| \xrightarrow{a.s.} 0. \tag{115}$$

Hence,

$$B_{2n} \xrightarrow{a.s.} 0. \tag{116}$$

Under the conditions of Theorem 3, by (13), (19), (97), (114), and (115), we derive that

$$|B_{3n}| \leq \left(\max_{1 \leq i \leq n} \left(\sum_{j=1}^n W_{nj}(t_i) \varepsilon_j \right)^2 \right) \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \right| \right) \xrightarrow{a.s.} 0, \tag{117}$$

$$\begin{aligned}
|B_{4n}| &\leq 2 \max_{1 \leq j \leq n} \left\{ \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \tilde{g}_i \varepsilon_i \right| + \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \tilde{g}_i \left(\sum_{k=1}^n W_{nk}(t_i) \varepsilon_k \right) \right| \right\} \leq 2 \left\{ \left(\max_{1 \leq i \leq n} |\tilde{g}_i| \right) \max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \varepsilon_i \right| \right. \\
&\quad \left. + \left(\max_{1 \leq i \leq n} |\tilde{g}_i| \right) \left(\max_{1 \leq i \leq n} \left| \sum_{k=1}^n W_{nk}(t_i) \varepsilon_k \right| \right) \left(\max_{1 \leq j \leq n} \left| \sum_{i=1}^n \hat{W}_{ni}(u_j) \right| \right) \right\} \xrightarrow{a.s.} 0,
\end{aligned} \tag{118}$$

$$|B_{5n}| \leq 2|\hat{\beta}_{LS} - \beta| \max_{1 \leq j \leq n} \left[\left(\sum_{i=1}^n \hat{W}_{mi}^2(u_j) \tilde{\varepsilon}_i^2 \right) \left(\sum_{i=1}^n \tilde{x}_i^2 \right) \right]^{(1/2)} \xrightarrow{a.s.} 0, \quad (119)$$

$$|B_{6n}| \leq (\hat{\beta}_{LS} - \beta)^2 \left(\max_{1 \leq i, j \leq n} |\hat{W}_{mi}(u_j)| \right) \left(\sum_{i=1}^n \tilde{x}_i^2 \right) \leq C(\log n)^{-1} \xrightarrow{a.s.} 0, \quad (120)$$

$$|B_{7n}| \leq 2|\hat{\beta}_{LS} - \beta| \left(\max_{1 \leq i \leq n} |\tilde{g}_i| \right) \left(\max_{1 \leq i, j \leq n} |\hat{W}_{mi}(u_j)| \right) \left(\sum_{i=1}^n |\tilde{x}_i| \right) \leq C(\log n)^{-1} \xrightarrow{a.s.} 0, \quad (121)$$

$$|B_{8n}| \leq \left(\max_{1 \leq i \leq n} \tilde{g}_i^2 \right) \left(\max_{1 \leq j \leq n} \sum_{i=1}^n |\hat{W}_{mi}(u_j)| \right) \xrightarrow{a.s.} 0. \quad (122)$$

Thus, (20) follows from (108), (109), and (116)–(122).

Lastly, we prove (21). By (20) and (ii) of (A_1) , we have, for sufficiently large n ,

$$0 < m_1 \leq \min_{1 \leq i \leq n} \hat{f}_n(u_i) \leq \max_{1 \leq i \leq n} \hat{f}_n(u_i) \leq M_1 < \infty, \quad (123)$$

It follows from (68) that

$$|\hat{\beta}_{UWLS} - \beta| = \frac{n}{\bar{U}_n^2} \left| \frac{1}{n} \sum_{i=1}^n \gamma_{mi} \tilde{x}_i \tilde{\varepsilon}_i \right| + \frac{1}{\bar{U}_n^2} \left| \sum_{i=1}^n \gamma_{mi} \tilde{x}_i \tilde{g}_i \right| \triangleq \frac{n}{\bar{U}_n^2} L_{1n} + L_{2n}. \quad (125)$$

By (87) and (125), one can obtain

$$\begin{aligned} \frac{\bar{U}_n^2}{n} &\leq C, \\ \frac{\sum_{i=1}^n |\gamma_{mi} \tilde{x}_i|}{\bar{U}_n^2} &\leq C. \end{aligned} \quad (124)$$

$$L_{2n} \leq \frac{\max_{1 \leq i \leq n} |\tilde{g}_i| \left| \sum_{i=1}^n \gamma_{mi} \tilde{x}_i \right|}{\bar{U}_n^2} \xrightarrow{a.s.} 0, \quad (126)$$

$$L_{1n} \leq \left| n^{-1} \sum_{i=1}^n (\gamma_{mi} - \gamma_i) \tilde{x}_i \tilde{\varepsilon}_i \right| + \left| n^{-1} \gamma_i \tilde{x}_i \tilde{\varepsilon}_i \right| \leq \left| n^{-1} \sum_{i=1}^n (\gamma_{mi} - \gamma_i) \tilde{x}_i \varepsilon_i \right| + \left| n^{-1} \sum_{i=1}^n (\gamma_{mi} - \gamma_i) \tilde{x}_i \sum_{j=1}^n W_{nj}(t_i) \varepsilon_j \right| + \left| n^{-1} \gamma_i \tilde{x}_i \tilde{\varepsilon}_i \right| \triangleq L_{11n} + L_{12n} + L_{13n}. \quad (127)$$

From (13), (20), (109), (115), and Lemma 8, it follows that

$$\begin{aligned} L_{11n} &\leq \left(\max_{1 \leq i \leq n} |\gamma_{mi} - \gamma_i| \sigma_i \right) \left(n^{-1} \sum_{i=1}^n |\tilde{x}_i| |e_i| \right) \leq \left(\max_{1 \leq i \leq n} \frac{|f(u_i) - \hat{f}_n(u_i)|}{\sqrt{f(u_i) \hat{f}_n(u_i)}} \right) \left[\left(\frac{1}{n} \sum_{i=1}^n \tilde{x}_i^2 \right) \left(\frac{1}{n} \sum_{i=1}^n e_i^2 \right) \right]^{(1/2)}, \\ &= \left(\max_{1 \leq i \leq n} \frac{|f(u_i) - \hat{f}_n(u_i)|}{\sqrt{f(u_i) \hat{f}_n(u_i)}} \right) \left[\left(\frac{1}{n} \sum_{i=1}^n \tilde{x}_i^2 \right) \left(\frac{1}{n} \sum_{i=1}^n (e_i^+)^2 + \frac{1}{n} \sum_{i=1}^n (e_i^-)^2 \right) \right]^{(1/2)} \xrightarrow{a.s.} \left(\max_{1 \leq i \leq n} \frac{|f(u_i) - \hat{f}_n(u_i)|}{\sqrt{f(u_i) \hat{f}_n(u_i)}} \right) \left[\left(\frac{1}{n} \sum_{i=1}^n \tilde{x}_i^2 \right) \left(\frac{1}{n} \sum_{i=1}^n (E(e_i^+)^2 + E(e_i^-)^2) \right) \right]^{(1/2)}, \\ &= \left(\max_{1 \leq i \leq n} \frac{|f(u_i) - \hat{f}_n(u_i)|}{\sqrt{f(u_i) \hat{f}_n(u_i)}} \right) \left[\left(\frac{1}{n} \sum_{i=1}^n \tilde{x}_i^2 \right) \cdot 1 \right]^{(1/2)} \xrightarrow{a.s.} 0, \end{aligned} \quad (128)$$

$$L_{12n} \leq \left(\max_{1 \leq i \leq n} \left| \sum_{j=1}^n W_{nj}(t_i) \varepsilon_j \right| \right) \left(\max_{1 \leq i \leq n} |\gamma_{mi} - \gamma_i| \left(n^{-1} \sum_{i=1}^n |\tilde{x}_i| \right) \right) \xrightarrow{a.s.} 0. \quad (129)$$

By (13), (69), (71), and (82), we have

$$L_{13n} = n^{-1} \tilde{T}_n^2 |\Delta_{1n} - \Delta_{2n}| \xrightarrow{a.s.} 0. \quad (130)$$

Hence, by (127)–(130), we have

$$\begin{aligned} \bar{g}_n(t_i) - g(t_i) &= \sum_{j=1}^n W_{nj}(t_i) (x_j \beta + g(t_j) + \sigma_j e_j - x_j \hat{\beta}_{UWLS}) - g(t_i) = \sum_{j=1}^n W_{nj}(t_i) x_j (\beta - \hat{\beta}_{UWLS}) - \left[g(t_i) - \sum_{j=1}^n W_{nj}(t_i) g(t_j) \right] \\ &\quad + \sum_{j=1}^n W_{nj}(t_i) \sigma_j e_j. \end{aligned} \quad (132)$$

Hence,

$$\max_{1 \leq i \leq n} |\bar{g}_n(t_i) - g(t_i)| \leq \max_{1 \leq i \leq n} \left(|\beta - \hat{\beta}_{UWLS}| \left| \sum_{j=1}^n W_{nj}(t_i) x_j \right| \right) + |\bar{g}_i| + \left| \sum_{j=1}^n W_{nj}(t_i) \sigma_j e_j \right|. \quad (133)$$

From (20) and $\max_{1 \leq i \leq n} |\sum_{j=1}^n W_{nj}(t_i) x_j| = O(1)$, it follows that

$$\max_{1 \leq i \leq n} \left(|\beta - \hat{\beta}_{UWLS}| \left| \sum_{j=1}^n W_{nj}(t_i) x_j \right| \right) \xrightarrow{a.s.} 0. \quad (134)$$

Therefore, (22) follows from (92), (93), and (134).

This completes the proof of Theorem 4. $\square$

6. Numerical Simulations

In this section, we will investigate the numerical performance of the strong consistency for the least squares estimators $\hat{\beta}_{LS}$ and $\hat{g}_n(t)$ and weighted least squares estimators $\hat{\beta}_{WLS}$ and $\hat{g}_n(t)$ with AANA random errors by two simulated examples.

Two AANA sequences are given as follows:

$$\begin{aligned} \text{Sequence1: } e_i &= (1 + a_i^2)^{-1/2} (\eta_i + a_i \eta_{i+1}), \\ \text{Sequence2: } \varepsilon_i &= \eta_i + a_i \eta_{i+1}, \end{aligned} \quad (135)$$

where $\eta_1, \eta_2, \dots$ are independent and identically distributed $N(0, 1)$ random variables and $a_i = (1/i^2)$. The two sequences have been proved to be AANA sequences but not NA sequences (see [17, 19]).

Next, we will apply the two sequences to the following two simulation examples, respectively.

6.1. Simulated Example 1. We will simulate a heteroscedastic partially linear model

$$Y_i = x_i \beta + g(t_i) + \sigma_i e_i, \quad (136)$$

$$L_{1n} \xrightarrow{a.s.} 0. \quad (131)$$

Therefore, (21) follows from (125), (126), and (131).

This completes the proof of Theorem 3. $\square$

Proof of Theorem 4. In view of (12), we have

where $\beta = 3.5$, $g(t) = \sin(2\pi t)$, $\sigma_i = (f(u_i))^{1/2} = (1 + 0.5 \sin(\pi u_i))^{1/2}$, $u_i = x_i = (-1)^i \cdot i/n$, $1 \leq i \leq n$, and the random errors are given by Sequence 1.

In particular, we take the weight function $W_{ni}(\cdot)$ as the following nearest neighbor weight function (see [11]). Without loss of generality, let $A = [0, 1]$ and $t_i = (i/n)$ ($1 \leq i \leq n$). For each $t \in A$, we rewrite

$$|t_1 - t|, |t_2 - t|, \dots, |t_n - t|, \quad (137)$$

as follows:

$$|t_{R_1(t)} - t| \leq |t_{R_2(t)} - t| \leq \dots \leq |t_{R_n(t)} - t|. \quad (138)$$

Take $k_n = [n^{0.4}]$ and define the nearest neighbor weight function

$$W_{ni}(t) = \begin{cases} \frac{1}{k_n}, & \text{if } |t_i - t| \leq |t_{R_{k_n}} - t|, \\ 0, & \text{otherwise.} \end{cases} \quad (139)$$

The sample sizes are taken as $n = 200, 500, 1000, 1500, 2000$, and 2500 , respectively, and each case is repeated for 1000 times and the average values of $\hat{\beta}_{LS}$, $\hat{g}_n(t)$, $\hat{\beta}_{WLS}$, and $\hat{g}_n(t)$ are calculated as the estimators, respectively. Then, we calculate the corresponding absolute errors $|\hat{\beta}_{LS} - \beta|$, $|\hat{g}_n(t) - g(t)|$, $|\hat{\beta}_{WLS} - \beta|$, and $|\hat{g}_n(t) - g(t)|$ and the root mean square errors (RMSEs) of $\hat{\beta}_{LS}$, $\hat{g}_n(t)$, $\hat{\beta}_{WLS}$, and $\hat{g}_n(t)$, respectively. The above results are presented in Tables 1 and 2, and the curves of $g(t)$, $\hat{g}_n(t)$, and $\tilde{g}_n(t)$ are provided in Figures 1–2, respectively.

TABLE 1: The LS estimators of β and $g(t)$ and the absolute errors and RMSEs of $\hat{\beta}_{LS}$ and $\hat{g}_n(t)$ with $\beta = 3.5$ and $g(t) = \sin(2\pi t)$.

t	n	$\hat{\beta}_{LS}$	$ \hat{\beta}_{LS} - \beta $	RMSE($\hat{\beta}_{LS}$)	$\hat{g}_n(t)$	$ \hat{g}_n(t) - g(t) $	RMSE $\hat{g}_n(t)$
20/n	200	3.3890	0.1110	0.1308	0.5585	0.0293	0.0345
	500	3.5790	0.0790	0.0931	0.2278	0.0209	0.0246
	1000	3.5543	0.0543	0.0639	0.1108	0.0145	0.0171
	1500	3.5354	0.0354	0.0417	0.0749	0.0088	0.0104
	2000	3.4792	0.0208	0.0245	0.0577	0.0051	0.0060
	2500	3.5091	0.0091	0.0107	0.0478	0.0024	0.0029
40/n	200	3.6078	0.1078	0.1270	0.9172	0.0338	0.0398
	500	3.5761	0.0761	0.0897	0.4564	0.0253	0.0299
	1000	3.4486	0.0514	0.0606	0.2296	0.0191	0.0225
	1500	3.5326	0.0326	0.0385	0.1544	0.0124	0.0146
	2000	3.5179	0.0179	0.0211	0.1171	0.0083	0.0097
	2500	3.4936	0.0064	0.0075	0.0964	0.0039	0.0046

TABLE 2: The WLS estimators of β and $g(t)$ and the absolute errors and RMSEs of $\hat{\beta}_{WLS}$ and $\tilde{g}_n(t)$ with $\beta = 3.5$ and $g(t) = \sin(2\pi t)$.

t	n	$\hat{\beta}_{WLS}$	$ \hat{\beta}_{WLS} - \beta $	RMSE $\hat{\beta}_{WLS}$	$\tilde{g}_n(t)$	$ \tilde{g}_n(t) - g(t) $	RMSE $\tilde{g}_n(t)$
20/n	200	3.6075	0.1075	0.1266	0.5594	0.0284	0.0335
	500	3.4236	0.0764	0.0901	0.2285	0.0201	0.0237
	1000	3.5526	0.0526	0.0620	0.1113	0.0140	0.0165
	1500	3.4657	0.0343	0.0405	0.0752	0.0085	0.0100
	2000	3.4799	0.0201	0.0237	0.0578	0.0050	0.0059
	2500	3.5089	0.0089	0.0105	0.0479	0.0023	0.0027
40/n	200	3.3956	0.1044	0.1230	0.9183	0.0309	0.0386
	500	3.5738	0.0738	0.0869	0.4572	0.0232	0.0290
	1000	3.4512	0.0498	0.0587	0.2302	0.0174	0.0217
	1500	3.5316	0.0316	0.0372	0.1548	0.0113	0.0141
	2000	3.5174	0.0174	0.0205	0.1174	0.0075	0.0094
	2500	3.5061	0.0061	0.0072	0.0965	0.0036	0.0045

FIGURE 1: Curves of $g(t) = \sin(2\pi t)$ and $\hat{g}_n(t)$ with $\beta = 3.5$ and $n = 1500$. FIGURE 2: Curves of $g(t) = \sin(2\pi t)$ and $\tilde{g}_n(t)$ with $\beta = 3.5$ and $n = 1500$.

6.2. *Simulated Example 2.* We will simulate a heteroscedastic partially linear model

$$Y_i = x_i\beta + g(t_i) + \sigma_i\epsilon_i, \quad (140)$$

where $\beta = 2.5$, $g(t) = \cos(\pi t)$, $\sigma_i = (f(u_i))^{1/2} = (1 - 0.5 \sin(\pi u_i))^{1/2}$, $u_i = x_i = (-1)^i \cdot i/n$, $1 \leq i \leq n$, and the random errors are given by Sequence 2.

TABLE 3: The LS estimators of β and $g(t)$ and the absolute errors and RMSEs of $\hat{\beta}_{LS}$ and $\hat{g}_n(t)$ with $\beta = 2.5$ and $g(t) = \cos(\pi t)$.

t	n	$\hat{\beta}_{LS}$	$ \hat{\beta}_{LS} - \beta $	$\text{RMSE} \hat{\beta}_{LS}$	$\hat{g}_n(t)$	$ \hat{g}_n(t) - g(t) $	$\text{RMSE} \hat{g}_n(t)$
20/n	200	2.3926	0.1074	0.1265	0.9194	0.0317	0.0373
	500	2.5736	0.0736	0.0867	0.9689	0.0232	0.0274
	1000	2.5499	0.0499	0.0588	0.9812	0.0169	0.0199
	1500	2.5319	0.0319	0.0376	0.9880	0.0111	0.0131
	2000	2.4834	0.0166	0.0196	0.9931	0.0064	0.0075
	2500	2.5052	0.0052	0.0061	0.9969	0.0028	0.0032
40/n	200	2.6087	0.1087	0.1280	0.7761	0.0330	0.0388
	500	2.5766	0.0766	0.0903	0.9441	0.0245	0.0289
	1000	2.4482	0.0518	0.0611	0.9743	0.0178	0.0210
	1500	2.5331	0.0331	0.0390	0.9850	0.0114	0.0135
	2000	2.5183	0.0183	0.0216	0.9908	0.0072	0.0085
	2500	2.4932	0.0068	0.0080	0.9955	0.0033	0.0039

TABLE 4: The WLS estimators of β and $g(t)$ and the absolute errors and RMSEs of $\hat{\beta}_{WLS}$ and $\tilde{g}_n(t)$ with $\beta = 2.5$ and $g(t) = \cos(\pi t)$.

t	n	$\hat{\beta}_{WLS}$	$ \hat{\beta}_{WLS} - \beta $	$\text{RMSE} \hat{\beta}_{WLS}$	$\tilde{g}_n(t)$	$ \tilde{g}_n(t) - g(t) $	$\text{RMSE} \tilde{g}_n(t)$
20/n	200	2.3959	0.1041	0.1225	0.9203	0.0307	0.0362
	500	2.4287	0.0713	0.0839	0.9695	0.0226	0.0266
	1000	2.5483	0.0483	0.0569	0.9816	0.0164	0.0193
	1500	2.5310	0.0310	0.0364	0.9883	0.0108	0.0127
	2000	2.5161	0.0161	0.0190	0.9934	0.0061	0.0072
	2500	2.5051	0.0051	0.0060	0.9970	0.0027	0.0031
40/n	200	2.6053	0.1053	0.1238	0.7771	0.0319	0.0366
	500	2.4258	0.0742	0.0873	0.9448	0.0237	0.0279
	1000	2.5502	0.0502	0.0591	0.9748	0.0173	0.0203
	1500	2.5320	0.0320	0.0377	0.9854	0.0111	0.0131
	2000	2.4823	0.0177	0.0208	0.9910	0.0070	0.0082
	2500	2.4934	0.0066	0.0077	0.9956	0.0032	0.0037

FIGURE 3: Curves of $g(t) = \cos(\pi t)$ and $\hat{g}_n(t)$ with $\beta = 2.5$ and $n = 1500$. FIGURE 4: Curves of $g(t) = \cos(\pi t)$ and $\tilde{g}_n(t)$ with $\beta = 2.5$ and $n = 1500$.

By the same estimating methods as model (6), we obtain the estimators of β and $g_n(t)$ in model (140) under different sample size n , $|\hat{g}_n(t) - g(t)|$, $|\hat{\beta}_{WLS} - \beta|$, and $|\hat{g}_n(t) - g(t)|$ and the RMSE of $\hat{\beta}_{LS}$, $\hat{g}_n(t)$, $\hat{\beta}_{WLS}$, and $\hat{g}_n(t)$, respectively. The above results are provided in Tables 3 and 4, and the curves of $g(t)$, $\hat{g}_n(t)$, and $\tilde{g}_n(t)$ are presented in Figures 3 and 4, respectively.

It can be seen from Tables 1–4 that regardless of the values of t , the absolute errors of both the least squares estimators and weighted least squares estimators decrease gradually as the sample size n has good effects. The simulation results show the strong consistency of least squares estimators $\hat{\beta}_{LS}$ and $\hat{g}_n(t)$ and weighted least squares estimators $\hat{\beta}_{WLS}$ and $\tilde{g}_n(t)$ in model (1) with AANA random errors. The simulation results also show that the absolute error of weighted least squares estimator is smaller than that of least squares estimator. Moreover, consistency is the basic standard that all estimators should meet, and it is the necessary condition to measure whether an estimator is feasible. AANA sequences are widely used dependent sequences which include independent and NA sequences as special cases. Therefore, to study the consistency of estimators in regression models with AANA errors is of considerable significance.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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