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Research Article

On the Existence of Three Positive Solutions for a Caputo Fractional Difference Equation

Lili Kong, Huiqin Chen, Luping Li, and Shugui Kang

School of Mathematics and Statistics, Shanxi Datong University, Datong, Shanxi 037009, China

Correspondence should be addressed to Huiqin Chen; dtdxchq@126.com

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In this paper, we introduce the application of three fixed point theorem by discussing the existence of three positive solutions for a class of Caputo fractional difference equation boundary value problem. We establish the condition of the existence of three positive solutions for this problem.

1. Introduction

After being proved to be a valuable tool in science and engineering fields, fractional difference equation has attracted attention of more and more scholars. And the existing results of positive solutions for boundary value problem of nonlinear fractional difference equations is the hot spot which has been discussed in recent years. So, a large number of scholars have devoted themselves to the study of fractional difference equations, such as [1–9].

At the same time, the fixed point theory (see [10–12]) has also been widely applied to study the fractional difference equations. After that, many authors obtained the existence of positive solutions for the fractional difference equations by using the fixed point theorem (see [13–22]). For example, Jiraporn Reunsumrit and Thanin Sitthiwirattham [20] considered the nonlinear discrete fractional boundary value problem of the form

$$\begin{cases} \triangle^{\alpha} x(t) + f(t+\alpha-1, x(t+\alpha-1)) = 0, t \in N_{0,T}, \\ \triangle x(\alpha-2) = 0, x(\alpha+T) = \lambda \Delta^{-\beta} x(\eta+\beta), \end{cases} \tag{1}$$

where $1 < \alpha \leq 2, 0 < \beta \leq 1$ and $f\colon N_{\alpha-2,\alpha+T} \times R \longrightarrow R$ is a continuous function. The authors employed some fixed-point theorems to obtain the existence, uniqueness of solutions, and the existence of positive solutions.

Reunsumrit and Sitthiwirattham [21] studied the three-point fractional sum boundary value problem of the form

$$\begin{cases} \triangle_C^{\alpha} u(t) + a(t+\alpha-1) f(u(\theta(t+\alpha-1))) = 0, t \in N_{0,T}, \\ u(\alpha-3) = \Delta^2 u(\alpha-3) = 0, \\ u(T+\alpha) = \lambda \Delta^{-\beta} u(\eta+\beta), \end{cases} \tag{2}$$

where $2 < \alpha \leq 3, 0 < \beta \leq 1$ and $f\colon [0,+\infty) \longrightarrow [0,+\infty)$ is a continuous function. The authors employed Guo-Krasnoselskii's fixed-point theorem to obtain the existence of at least one positive solution.

Based on the above research results, this article considers the existence of three positive solutions for the nonlinear fractional difference equation boundary value problem

$$\begin{cases} \triangle_C^{\nu} u(t) = -f(t+\nu-1, u(t+\nu-1)), \\ u(\nu-3) = \triangle u(b+\nu) = \triangle^2 u(\nu-3) = 0, \end{cases} \tag{3}$$

where $t \in \mathbb{N}_0^{b+1}$, $b \geq 5$ is an integer. $f\colon \mathbb{N}_{\nu-2}^{b+\nu} \times [0,+\infty) \longrightarrow [0,+\infty)$ is continuous and f is not identically zero, $2 < \nu \leq 3$, and $\triangle_C^{\nu} u(t)$ is the standard Caputo difference. Our analysis relies on Leggett-Williams fixed-point theorem to obtain sufficient conditions of the existence of three positive solutions for Caputo fractional boundary value problem (3). Chen et al. [22] considered the existence of positive solutions for (3). In this article, the authors obtained the existence of one or two positive solutions by means of the cone theoretic fixed-point theorems. Compared with [22], the application of Leggett-Williams fixed-point theorem makes our proving process simpler and the number of solutions increased. The research in this article shows that employing the Leggett-Williams fixed-point theorem to prove the existence of positive solutions for the fractional difference equation can get better results.

In the remainder of this paper, we will present basic definitions and some lemmas in order to prove our main results in Section 2. In Section 3, we establish some results for the existence of three positive solutions to problem (3). And some examples to corroborate our results are given in Section 4.

2. Background Materials and Preliminaries

For convenience, we first review some basic results about fractional sums and differences. For any $t \in \mathbb{N}_0^{b+1}$ and $\nu > 0$, we define

$$t^{\underline{\nu}} = \frac{\Gamma(t+1)}{\Gamma(t+1-\nu)}, \tag{4}$$

for which the right-hand side is defined. We appeal to the convention that if $t+1-\nu$ is a pole of the Gamma function and $t+1$ is not a pole, then $t^{\underline{\nu}} = 0$.

Definition 1. For $\nu > 0$ and a function f defined on $N_a := \{a, a+1, \cdots\}$, the ν-th fractional sum of f is defined by

$$\triangle^{-\nu} f(t) = \frac{1}{\Gamma(\nu)} \sum_{s=a}^{t-\nu} (t-s-1)^{\underline{\nu-1}} f(s), \tag{5}$$

where $t \in \{a+\nu, a+\nu+1, \ldots\} = \mathbb{N}_{a+\nu}$.

Definition 2. For $\nu > 0$ and a continuous function $f\colon (0,+\infty) \longrightarrow \mathbb{R}$ defined on N_a, the ν-th Caputo fractional difference of f is given by

$$\begin{aligned} \triangle_C^{\nu} f(t) &= \triangle^{-(n-\nu)} \triangle^n f(t) \\ &= \frac{1}{\Gamma(n-\nu)} \sum_{s=a}^{t-(n-\nu)} (t-s-1)^{\underline{n-\nu-1}} \triangle^n f(s), \end{aligned} \tag{6}$$

where $n-1 < \nu < n$. If $\nu = n$, then $\triangle_C^{\nu} f(t) = \triangle^n f(t)$.

Lemma 1 (see [13]). *Assume that $\nu > 0$ and f is defined on domains $\mathbb{N}_a$, then*

$$\triangle_{a+(n-\nu)}^{-\nu} \triangle_C^{\nu} f(t) = f(t) - \sum_{k=0}^{n-1} c_k (t-a)^{\underline{k}}, \tag{7}$$

where $c_k \in \mathbb{R}$, $k = 0, 1, \ldots n-1$ and $n-1 < \nu \le n$.

Lemma 2 (see [22]). *Let $2 < \nu \le 3$ and $g\colon \mathbb{N}_{\nu-2}^{b+\nu} \longrightarrow \mathbb{R}$ be given. Then the solution of the FBVP*

$$\begin{cases} \triangle_C^{\nu} u(t) = -g(t+\nu-1), \\ u(\nu-3) = \triangle u(b+\nu) = \triangle^2 u(\nu-3) = 0, \end{cases} \tag{8}$$

is given by

$$u(t) = \sum_{s=0}^{b+1} G(t,s) g(s+\nu-1), \tag{9}$$

where function $G\colon \mathbb{N}_{\nu-2}^{b+\nu} \times \mathbb{N}_0^{b+1} \longrightarrow \mathbb{R}$ is defined by

$$G(t,s) = \begin{cases} \dfrac{(\nu-1)(t-\nu+3)(b+\nu-s-1)^{\underline{\nu-2}} - (t-s-1)^{\underline{\nu-1}}}{\Gamma(\nu)}, & 0 \le s < t-\nu+1 \le b+1, \\[2ex] \dfrac{(\nu-1)(t-\nu+3)(b+\nu-s-1)^{\underline{\nu-2}}}{\Gamma(\nu)}, & 0 \le t-\nu+1 \le s \le b+1. \end{cases} \tag{10}$$

Here $G(t,s)$ is called the Green function of boundary value problem (8).

Remark 1. Notice that $G(\nu-3,s) = 0$, $G(t,b+2) = 0$. G could be extended to $\mathbb{N}_{\nu-3}^{b+\nu} \times \mathbb{N}_0^{b+2}$, so we only discuss $(t,s) \in \mathbb{N}_{\nu-2}^{b+\nu} \times \mathbb{N}_0^{b+1}$.

Lemma 3 (see [22]). *The Green function $G(t,s)$ defined by (10) satisfies*

(i) $G(t,s) > 0$, $(t,s) \in \mathbb{N}_{\nu-2}^{b+\nu} \times \mathbb{N}_0^{b+1}$.

(ii) $\max_{t \in \mathbb{N}_{\nu-2}^{b+\nu}} G(t,s) = G(b+\nu,s)$, $s \in \mathbb{N}_0^{b+1}$.

(iii) $\min_{(b+\nu/4) \le t \le (3(b+\nu)/4)} G(t,s) \ge (1/4)\max_{t \in \mathbb{N}_{\nu-2}^{b+\nu}} G(t,s) = (1/4) G(b+\nu,s)$, $s \in \mathbb{N}_0^{b+1}$.

Definition 3 (see [12]). If P is a cone of the real Banach space E, a mapping $\psi\colon P \longrightarrow [0,\infty)$ is continuous and with

$$\psi(tx + (1-t)y) \ge t\psi(x) + (1-t)\psi(y), \quad x, y \in P, t \in [0,1], \tag{11}$$

is called a nonnegative concave continuous functional ψ on P.

Assume that r, a, b are positive constants, we will employ the following notations

$$\begin{aligned} P_r &= \{u \in P\colon \|u\| < r\}, \\ \overline{P}_r &= \{u \in P\colon \|u\| \le r\}, \\ P(\psi, a, b) &= \{u \in P\colon \psi(u) \ge a, \|u\| \le b\}. \end{aligned} \tag{12}$$

Our existence criteria will be based on the following Leggett-Williams fixed-point theorem.

Lemma 4 (see [12]). *Let $E = (E, \|\cdot\|)$ be a Banach space, $P \subset E$ be a cone of E, and $c > 0$ be a constant. Suppose there exists a concave nonnegative continuous functional ψ on P with $\psi(u) \le \|u\|$ for $u \in \overline{P}_c$. Let A: $\overline{P}_c \longrightarrow \overline{P}_c$ be a completely continuous operator. Assume there are numbers d, a, and b with $0 < d < a < b \le c$ such that*

(i) The set $\{u \in P(\psi, a, b): \psi(u) > a\}$ is nonempty and $\psi(Au) > a$ for all $u \in P(\psi, a, b)$.
(ii) $\|Au\| < d$ for $u \in \overline{P}_d$.
(iii) $\psi(Au) > a$ for all $u \in P(\psi, a, c)$ with $\|Au\| > b$.

Then A has at least three fixed points u_1, u_2, and $u_3 \in \overline{P}_c$. Furthermore, we have

$$\max_{t\in[0,1]} u_1(t) < d, a < \min_{t\in[0,1]} u_2(t) < \max_{t\in[0,1]} u_2(t) < c \text{ and } d < \max_{t\in[0,1]} u_3(t) \le c, \min_{t\in[0,1]} u_3(t) < a. \tag{13}$$

3. Main Results

Set

$$\mathscr{B} = \left\{u\colon \mathbb{N}_{\nu-3}^{b+\nu} \longrightarrow \mathbb{R}, u(\nu-3) = \triangle u(b+\nu) = \triangle^2 u(\nu-3) = 0\right\}. \tag{14}$$

Then $\mathscr{B}$ is a Banach space with respect to the norm $\|u\| = \max_{t\in\mathbb{N}_{\nu-3}^{b+\nu}} |u(t)|$. We define a cone in $\mathscr{B}$ by

$$P = \left\{u \in \mathscr{B}\colon u(t) \ge 0, \min_{(b+\nu/4)\le t\le (3(b+\nu)/4)} u(t) \ge \frac{1}{4}\|u\|\right\}. \tag{15}$$

Now consider the operator T defined by

$$(\mathrm{Tu})(t) = \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, u(s+\nu-1)). \tag{16}$$

It is easy to see that $u = u(t)$ is a solution of the FBVP (3) if and only if $u = u(t)$ is a fixed point of T. We shall obtain conditions for the existence of three fixed points of T. First, we notice that T is a summation operator on a discrete finite set. Hence, T is trivially completely continuous. From (16),

$$\begin{aligned}\min_{((b+\nu)/4)\le t\le (3(b+\nu)/4)} (\mathrm{Tu})(t) &\ge \frac{1}{4}\sum_{s=0}^{b+1} G(b+\nu, s) \cdot f(s+\nu-1, u(s+\nu-1)) \\ &\ge \frac{1}{4}\max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} \sum_{s=0}^{b+1} G(t,s) \cdot f(s+\nu-1, u(s+\nu-1)) \\ &= \frac{1}{4}\|Tu\|,\end{aligned} \tag{17}$$

hence, $\mathrm{TP} \subset P$.

We will discuss the existence of three fixed points of T by using Lemma 4. Thus, the conditions for the existence of the three positive solutions of (3) are obtained. For this purpose, let the nonnegative concave continuous function ψ on P be defined by

$$\psi(u) = \min_{((b+\nu)/4)\le t\le (3(b+\nu)/4)} u(t). \tag{18}$$

Denote

$$\begin{aligned} f^0 &= \limsup_{u\longrightarrow 0} \max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} \frac{f(t,u)}{u}, \\ f^\infty &= \limsup_{u\longrightarrow +\infty} \max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} \frac{f(t,u)}{u}, \\ l &= \sum_{s=0}^{b+1} G(b+\nu, s), \\ m &= \sum_{s=[(b+\nu/4)-\nu+1]}^{[(3(b+\nu)/4)-\nu+1]} \frac{1}{4} G(b+\nu, s). \end{aligned} \tag{19}$$

Suppose that the function $f(t,u)$ satisfies the following condition

(C) $f(t,u)$ is a nonnegative continuous function on $[0,1] \times [0,+\infty)$ and there exists $t_n \longrightarrow 0$ such that $f(t_n, u(t_n)) > 0$, $n = 1, 2, \cdots$.

Theorem 1. *Assume condition (C) holds and there exist constants $0 < d < a$ such that*

(C1) $f(t,u) < (1/l)d$ for $(t,u) \in \mathbb{N}_{\nu-2}^{b+\nu} \times [0,d]$
(C2) $f(t,u) \ge (1/m)a$ for $(t,u) \in [(b+\nu)/4, 3(b+\nu)/4] \times [a,c]$, where $c > 4a$
(C3) $f(t,u) \le \kappa u + \beta$ for $(t,u) \in \mathbb{N}_{\nu-2}^{b+\nu} \times [0,+\infty)$, where κ, β are positive numbers

Then the boundary value problem (3) has at least three positive solutions u_1, u_2, and u_3.

Proof. Set $c > \max\{\beta l/(1-\kappa l), 4a\}$, then for $u \in \overline{P}_c$, from (C3), we have

$$\begin{aligned}\|\mathrm{Tu}\| &= \max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, u(s+\nu-1)) \\ &\le \sum_{s=0}^{b+1} G(b+\nu, s)(\kappa u(s+\nu-1) + \beta) \\ &\le (\kappa\|u\| + \beta) \sum_{s=0}^{b+1} G(b+\nu, s) = (\kappa\|u\| + \beta) l < c,\end{aligned} \tag{20}$$

namely, $\mathrm{Tu} \in P_c$. Therefore T: $\overline{P}_c \longrightarrow \overline{P}_c$ is a completely continuous operator. From (C1), we can get

$$\|Tu\| = \max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, u(s+\nu-1))$$
$$\leq \sum_{s=0}^{b+1} G(b+\nu,s) f(s+\nu-1, u(s+\nu-1)) \quad (21)$$
$$< \frac{1}{l} d \sum_{s=0}^{b+1} G(b+\nu,s) = d.$$

Therefore, assumption (ii) of Lemma 4 is satisfied.

We choose $\varphi_0 = 5a/2$ for $t \in [(b+\nu)/4, 3(b+\nu)/4]$, then $\varphi_0 \in \{u \in P(\psi,a,4a): \psi(u) > a\}$ which implies $\{u \in P(\psi,a,4a): \psi(u) > a\} \neq \emptyset$. Hence, if $u \in P(\psi,a,4a)$, then $a \leq u(t) \leq 4a$ for $(b+\nu)/4 \leq t \leq 3(b+\nu)/4$. Thus,

$$\psi(Tu) = \min_{(b+\nu/4)\leq t\leq(3(b+\nu)/4)} |(Tu)(t)|$$
$$= \min_{(b+\nu/4)\leq t\leq(3(b+\nu)/4)} \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, u(s+\nu-1))$$
$$> \sum_{s=[(b+\nu/4)-\nu+1]}^{[(3(b+\nu)/4)-\nu+1]} \frac{1}{4} G(b+\nu,s) f(s+\nu-1, u(s+\nu-1))$$
$$\geq \frac{1}{m} a \sum_{s=[(b+\nu/4)-\nu+1]}^{[(3(b+\nu)/4)-\nu+1]} \frac{1}{4} G(b+\nu,s) = a, \quad (22)$$

from which we see that $\psi(Tu) > a$ for all $u \in P(\psi,a,c)$. This shows that condition (i) of Lemma 4 is satisfied.

Finally, for $u \in P(\psi,a,c)$ with $\|Tu\| > 4a$, we get

$$\psi(Tu) = \min_{((b+\nu)/4)\leq t\leq(3(b+\nu)/4)} |(Tu)(t)|$$
$$\geq \frac{1}{4} \sum_{s=0}^{b+1} G(b+\nu,s) f(s+\nu-1, u(s+\nu-1))$$
$$= \frac{1}{4} \sum_{s=0}^{b+1} \max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} G(t,s) f(s+\nu-1, u(s+\nu-1)) \quad (23)$$
$$\geq \frac{1}{4} \max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, u(s+\nu-1))$$
$$= \frac{1}{4} \|Tu\| > a,$$

this shows that condition (iii) of Lemma 4 is satisfied. By the use of Lemma 4, the boundary value problem (3) has at least three solutions u_1, u_2, and u_3. Take into account that condition (C) holds, we have $u_i(t) > 0, 0 < t < 1, i = 1,2,3$. The proof is completed. □

Theorem 2. *Assume condition (C) holds. There exist constants $0 < d < a < 4a < c$ such that (C1), (C2), and (C4) are satisfied, where*

(C4) $f(t,u) \leq (1/l)c$ for $(t,u) \in \mathbb{N}_{\nu-2}^{b+\nu} \times [0,c]$.

Then the boundary value problem (3) has at least three positive solutions u_1, u_2, and u_3 such that

$$\max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} u_1(t) < d,\ a < \min_{t\in[b+\nu/4,3(b+\nu)/4]} u_2(t) < \max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} u_2(t) < c,$$
$$d < \max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} u_3(t) \leq c,\ \min_{t\in[b+\nu/4,3(b+\nu)/4]} u_3(t) < a. \quad (24)$$

Proof. From (C4), we get

$$\|Tu\| = \max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} \sum_{s=0}^{b+1} G(t,s) f(s+\nu-1, u(s+\nu-1))$$
$$\leq \sum_{s=0}^{b+1} G(b+\nu,s) f(s+\nu-1, u(s+\nu-1)) \quad (25)$$
$$< \frac{1}{l} c \sum_{s=0}^{b+1} G(b+\nu,s) = c.$$

Therefore, $T: \overline{P}_c \longrightarrow \overline{P}_c$. The remainder of proof is essentially the same as that of Theorem 1 and is therefore omitted. By Lemma 4, the boundary value problem (3) has at least three positive solutions u_1, u_2, and u_3 satisfying

$$\max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} u_1(t) < d,\ a < \min_{t\in[b+\nu/4,3(b+\nu)/4]} u_2(t) < \max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} u_2(t) < c,$$
$$d < \max_{t\in\mathbb{N}_{\nu-2}^{b+\nu}} u_3(t) \leq c,\ \min_{t\in[b+\nu/4,3(b+\nu)/4]} u_3(t) < a. \quad (26)$$

The proof is complete. □

Theorem 3. *Assume condition (C) holds. There exist constants $0 < d < a$ such that (C1), (C2) are satisfied, and*

(C5) $f^{\infty} < 1/l$.

Then the boundary value problem (3) has at least three positive solutions.

Proof. By (C5), there exist $0 < \sigma < 1/l$ and $R > 0$, when $u \geq R$, we have

$$f(t,u) \leq \sigma u. \quad (27)$$

Set $M = \max_{(t,u)\in\mathbb{N}_{\nu-2}^{b+\nu}\times[0,R]} f(t,u)$, consequently

$$0 \leq f(t,u) \leq \sigma u + M, \quad 0 \leq u < +\infty. \quad (28)$$

This shows that condition (C3) of Theorem 1 is satisfied. By Theorem 1, the boundary value problem (3) has at least three positive solutions. The proof is completed. □

Theorem 4. *Assume there exist two positive constants a, c $(0 < 4a < c)$ such that conditions (C), (C2), and (C4) hold. And function $f(t,u)$ satisfies*

(C6) $f^0 < 1/l$.

Then the boundary vale problem (3) has at least three positive solutions.

Proof. In line with (C6), it is easy to see that there exists a positive constant $d < a$ such that for $\|u\| < d$, we have

$$f(t,u(t)) < \frac{1}{l}u. \tag{29}$$

Namely,

$$f(t,u(t)) < \frac{1}{l}d, \|u\| < d. \tag{30}$$

This implies that conditions of Theorem 2 are satisfied. By Theorem 2, the boundary vale problem (3) has at least three positive solutions. The proof is completed.

In the light of the proof of Theorem 3 and Theorem 4, we obtain one theorem as follows. □

Theorem 5. *Assume conditions (C), (C5), and (C6) hold. Suppose that there exists a positive constants a such that $f(t,u) \geq (1/m)a$ for $(t,u) \in \mathbb{N}_{\nu-2}^{b+\nu} \times [a, 4a]$. Then the boundary vale problem (3) has at least three positive solutions.*

4. Examples

This section, we present two examples to illustrate our results. Set $\nu = 33/16, b = 19$, by estimating, we then have $l \approx 304.4632$, $m \approx 33.5505$.

Example 1. We take

$$f(t,u) = \begin{cases} \dfrac{t}{40000} + \dfrac{1}{25}u^4, \ (t,u) \in \left[\dfrac{1}{16}, \dfrac{337}{16}\right] \times [0,1], \\[2ex] \dfrac{t}{40000} + \dfrac{3}{100} + \dfrac{1}{100}u, \ (t,u) \in \left[\dfrac{1}{16}, \dfrac{337}{16}\right] \times (1,+\infty). \end{cases} \tag{31}$$

There exist constants $d = 1/3$ and $a = 3/2$ such that

$$\begin{aligned} &f(t,u) = \frac{t}{40000} + \frac{1}{25}u^4 \leq 0.00102039 < \frac{1}{l}d, \ (t,u) \in \left[\frac{1}{16}, \frac{337}{16}\right] \times \left[0, \frac{1}{3}\right], \\ &f(t,u) = \frac{t}{40000} + \frac{3}{100} + \frac{1}{100}u \geq 0.045 > \frac{1}{m}a, \ (t,u) \in \left[\frac{337}{64}, \frac{1011}{64}\right] \times \left[\frac{3}{2}, \frac{15}{2}\right], \\ &f(t,u) \leq \frac{337}{2560000} + \frac{1}{25}u, \ (t,u) \in \left[\frac{1}{16}, \frac{337}{16}\right] \times [0,+\infty). \end{aligned} \tag{32}$$

All the conditions of Theorem 1 hold. Thus, this moment, by virtue of Theorem 1, we know that the boundary value problem (3) has three positive solutions.

Example 2. We take

$$f(t,u) = \begin{cases} 2000.002 \cdot 2^{-t}u^2, \ (t,u) \in \left[\dfrac{1}{16}, \dfrac{337}{16}\right] \times [0,1], \\[2ex] 2^{-t}\left[2000u^{1/2} + 0.002u\right], \ (t,u) \in \left[\dfrac{1}{16}, \dfrac{337}{16}\right] \times (1,+\infty). \end{cases} \tag{33}$$

There exist constants $d = 0.0000016$ and $a = 1.008$ such that

$$\begin{aligned} &f^{\infty} \leq 0.002 < \frac{1}{l}, \ (t,u) \in \left[\frac{1}{16}, \frac{337}{16}\right] \times [0,+\infty), \\ &f(t,u) \geq 0.030639 > \frac{1}{m}a, \ (t,u) \in \left[\frac{337}{64}, \frac{1011}{64}\right] \times \left[\frac{126}{125}, \frac{504}{125}\right], \\ &f(t,u) \leq 0.003064d \leq \frac{1}{l}d, \ (t,u) \in \left[\frac{1}{16}, \frac{337}{16}\right] \times [0,d]. \end{aligned} \tag{34}$$

All the conditions of Theorem 3 hold. Thus, in this case, by Theorem 3, we know that the boundary value problem (3) has three positive solutions.

Data Availability

No data were used in the study.

Conflicts of Interest

The authors declare that they have no conflicts of interests.

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