First General Zagreb Index of Generalized $F$-sum Graphs

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The first general Zagreb (FGZ) index (also known as the general zeroth-order Randić index) of a graph $G$ can be defined as

$$M^c(G) = \sum_{uv \in E(G)} \left[ d^c(u) + d^c(v) \right],$$

where $c$ is a real number. As $M^c(G)$ is equal to the order and size of $G$ when $c = 0$ and $c = 1$, respectively, $c$ is usually assumed to be different from 0 to 1. In this paper, for every integer $c \geq 2$, the FGZ index $M^c$ is computed for the generalized $F$-sums graphs which are obtained by applying the different operations of subdivision and Cartesian product. The obtained results can be considered as the generalizations of the results appeared in (IEEE Access; 7 (2019) 47494–47502) and (IEEE Access 7 (2019) 105479–105488).

1. Introduction

Graph theory concepts are being utilized to model and study the several problems in different fields of science, including chemistry and computer science. A topological index (TI) of a (molecular) graph is a numeric quantity that remained unchanged under graph isomorphism [1,2]. Many topological indices have found applications in chemistry, especially in the quantitative structure-activity/property relationships studies; for detail, see [3–13].

Wiener index is the first TI introduced by Harry Wiener in 1947, when he was working on the boiling point of paraffin [14]. In 1972, Trinajstić and Gutman [15] obtained a formula concerning the total energy of $\pi$ electrons of molecules where the sum of square of valences of the vertices of a molecular structure was appeared. This sum is nowadays known as the first Zagreb index. In this paper, we are concerned with a generalized version of the first Zagreb index, known as the general first Zagreb index as well as the general zeroth-order Randić index.

There are several operations in graph theory such as product, complement, addition, switching, subdivision, and deletion. In many cases, graph operations may be helpful in finding graph quantities of more complicated graphs by considering the less complicated ones. In chemical graph theory, by using different graph operations, one can develop large molecular structures from the simple and basic structures. Recently, many classes of molecular structures are studied with the assistance of graph operations.

In 2007, Yan et al. [6] listed the five subdivision operations with the help of their vertices and edges. They also discussed the different features of Wiener index of graphs under these operations. After that, Eliasi and Taeri [16] introduced the $F_1$-sums graphs $\Gamma_1, \Gamma_2$ with the assistance of Cartesian product on graphs $F_1(\Gamma_1)$ and $\Gamma_2$, where $F_1(\Gamma_1)$ is obtained by applying the subdivision operations $S_1, R_1, Q_1$, and $T_1$. They also defined the Wiener indices of these resulting graphs $\Gamma_{1,S_1}, \Gamma_{1,R_1}, \Gamma_{1,Q_1}, \Gamma_{1,T_1}, \Gamma_2$. Later on, Deng et al. [17] calculated the 1st and 2nd Zagreb topological indices, and Imran and Akhtar [18] calculated the forgotten topological index of the $F_1$-sums graph. In 2019, Liu et al. [19] computed the first general Zagreb index of $F_1$-sums graphs.

Recently, Liu et al. [20] introduced the generalized version of the aforesaid subdivided operations of graphs denoted by $S_k, R_k, Q_k$, and $T_k$, where $k \geq 1$ is counting number. They also defined the generalized $F$-sums graphs using these generalized operations and calculated their 1st and 2nd Zagreb indices. In the present work, we compute the 1st general Zagreb index of the generalized $F$-sums graphs.
Definition 2. If $\Gamma$ be a connected graph, then the 1st and 2nd Zagreb topological indices as

\[
M_1(\Gamma) = \sum_{u \in V(\Gamma)} [d_1(u) + d_1(v)],
\]

\[
M_2(\Gamma) = \sum_{u \in V(\Gamma)} [d_2(u) + d_2(v)].
\]

These two descriptors of the graph were introduced by Trinajstí and Gutman [15]. Such type of TIs have been utilized to discuss the QSAR/QSPR of the different chemical structures such as chirality, complexity, hetero-system, ZE-isomers, $\pi$ electron energy, and branching [9, 10].

Definition 3. If $R$ is the real number, $\gamma \in R - \{0, 1\}$, and $\Gamma$ be a connected graph, so the 1st general Zagreb topological index is given as

\[
M^\gamma(\Gamma) = \sum_{u \in V(\Gamma)} [d_1^\gamma(u) + d_1^\gamma(v)].
\]

Definition 4. If $\Gamma_1$ and $\Gamma_2$ be two connected molecular structures, $F_k \in \{S_k, R_k, Q_k, T_k\}$ and $F_k(\Gamma_i)$ be a structure obtained after using $F_k$ on $\Gamma_i$ with bonds (edges) $E(F_k(\Gamma_i))$ and nodes (vertices) $V(F_k(\Gamma_i))$. So, the generalized F-sums graph $\Gamma_1 \ast F_k \Gamma_2$ is a structure with nodes:

\[
V(\Gamma_1 \ast F_k \Gamma_2) = V(F_k(\Gamma_1)) \times V(\Gamma_2),
\]

\[
= (V(\Gamma_1) \cup E(\Gamma_1)) \times V(\Gamma_2),
\]

in such a way two nodes $(a_1, b_1) \& (a_2, b_2)$ of $V(\Gamma_1 \ast F_k \Gamma_2)$ are adjacent if

(i) $a_1 = a_2 \in V(\Gamma_1) \& (b_1, b_2) \in E(\Gamma_2)$ or $[b_1 = b_2 \in V(\Gamma_2) \& (a_1, a_2) \in E(F_k(\Gamma_1))]$. For more details, see Figures 2 and 3.

Lemma 1. For $F_k \in \{S_k, R_k, Q_k, T_k\}$ and $(x, y) \in \Gamma_1 \ast F_k \Gamma_2$, the degree of $(x, y)$ in $\Gamma_1 \ast F_k \Gamma_2$ is

(i) $d^ch(x,y) = \begin{cases} d_{r_1}(x) + d_{r_1}(y), & \text{if } x \in V(\Gamma_1) \land y \in V(\Gamma_2), \\ 2, & \text{if } x \in V(S_k(\Gamma_1)) \land y \in V(\Gamma_2), \end{cases}$

(ii) $d^{ch}_c(x,y) = \begin{cases} d_{r_2}(x) + d_{r_2}(y), & \text{if } x \in V(\Gamma_1) \land y \in V(\Gamma_2), \\ 2, & \text{if } x \in V(S_k(\Gamma_1)) \land y \in V(\Gamma_2), \end{cases}$

(iii) $d^{ch}_r(x,y) = \begin{cases} d_{r_1}(x) + d_{r_2}(y), & \text{if } x \in V(\Gamma_1) \land y \in V(\Gamma_2), \\ d_{r_2}(x), & \text{if } x \in V(Q_k(\Gamma_1)) \land y \in V(\Gamma_2), \end{cases}$

(iv) $d^{ch}_t(x,y) = \begin{cases} d_{r_2}(x) + d_{r_2}(y), & \text{if } x \in V(\Gamma_1) \land y \in V(\Gamma_2), \\ d_{r_1}(x), & \text{if } x \in V(T_k(\Gamma_1)) \land y \in V(\Gamma_2). \end{cases}$
3. Main Results

The main results of FGZ index of the generalized $F$-sum graphs are presented in this section.

**Theorem 1.** Let $\Gamma_1$ and $\Gamma_2$ be two simple graphs and $\gamma \in N - \{0, 1\}$. The FGZ index of the generalized $S$-sum graph $\Gamma_1 + S_{\gamma} \Gamma_2$ is

$$M'(\Gamma_1 + S_{\gamma} \Gamma_2) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} M_{\Gamma_1}^{i-1} \left(M_{\Gamma_2}^{i+1} + n_{\Gamma_1} M_{S_{\gamma} \Gamma_1}^{i} \right)$$

$$+ \sum_{i=1}^{\alpha} \binom{\alpha}{i} M_{\Gamma_1}^{i-1} M_{\Gamma_2}^{i+1} + 2^{\alpha+1} (k-1) \eta_{\gamma} \partial_{\gamma},$$

where $N$ is the set of natural numbers and $\alpha = \gamma - 1$. 

![Figure 1](image1.png)

**Figure 1:** (a) $\Gamma$, (b) $S_2(\Gamma)$, (c) $R_2(\Gamma)$, (d) $Q_2(\Gamma)$, and (e) $T_2(\Gamma)$.

![Figure 2](image2.png)

**Figure 2:** (a) $\Gamma_1 \cong P_3$. (b) $\Gamma_2 \cong P_2$. (c) $\Gamma_1 + S_2 \Gamma_2$. (d) $\Gamma_1 + R_2 \Gamma_2$. 

Proof. Let

\[ M^\prime(\Gamma_1 + S \Gamma_2) = \sum_{(a,b) \in V(\Gamma_1 + S \Gamma_2)} d_{1,s_{1}}^{a}(a,b). \]  

(7)

For \( \alpha = \gamma - 1 \), then the above equation is considered as

\[ M^\prime(\Gamma_1 + S \Gamma_2) = \sum_{(a,b) \in V(\Gamma_1 + S \Gamma_2)} \left[ d_{1,s_{1}}^{a}(a,b) + d_{1,s_{1}}^{b}(c,d) \right] \]

\[ = \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_1)} \left[ d_{1,s_{1}}^{a}(a,b) + d_{1,s_{1}}^{b}(a,d) \right] - \sum_{b \in V(\Gamma_1)} \sum_{c \in E(\Gamma_1)} \left[ d_{1,s_{1}}^{a}(a,b) + d_{1,s_{1}}^{b}(b,c) \right] + \sum_{b \in V(\Gamma_1)} \sum_{c \in E(\Gamma_1)} \left[ d_{1,s_{1}}^{a}(a,b) + d_{1,s_{1}}^{b}(b,c) \right]. \]

(8)

For every vertex \( a \in V(\Gamma_1) \) and edge \( b \in E(\Gamma_2) \), then 1st term of (8) will be

\[ \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_2)} \left[ d_{1,s_{1}}^{a}(a,b) + d_{1,s_{1}}^{b}(a,d) \right] \]

\[ = \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) d_{1,s_{1}}^{a,i}(a) d_{1}^{b}(b) + \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) d_{1,s_{1}}^{a,i}(a) d_{1}^{b}(d) \right] \]

\[ = \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) d_{1,s_{1}}^{a,i}(a) \right] \left[ d_{1}^{b}(b) + d_{1}^{b}(d) \right] \]

\[ = \sum_{a \in V(\Gamma_1)} \sum_{i=0}^{\alpha} \left( \begin{array}{c} \alpha \\ i \end{array} \right) d_{1,s_{1}}^{a,i}(a) \left( M_{1}^{(\Gamma_2)} + M_{1}^{(\Gamma_2)} \right) \]

(9)
Since $|E(S_k(\Gamma_1))| = 2|E(\Gamma_1)|$. So, for every $b \in V(\Gamma_2)$ and $ac \in E(S_k(\Gamma_1))$ with $a \in V(\Gamma_1)$, and $c \in V(S_k(\Gamma_1)) - V(\Gamma_1)$; then the 2nd term of (8) is

\[
\sum_{b \in V(\Gamma_2)} \sum_{ac \in E(S_k(\Gamma_1))} \left[ d_{\Gamma_1 + S_k \Gamma_2}^a (a, b) + d_{\Gamma_1 + S_k \Gamma_2}^a (b, c) \right]
\]

\[
= \sum_{b \in V(\Gamma_2)} \sum_{ac \in E(S_k(\Gamma_1))} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d_{S_k(\Gamma_1)}^{\alpha-i} (a) d_{\Gamma_1}^i (b) + d_{S_k(\Gamma_1)}^{\alpha} (c) \right]
\]

\[
= \sum_{b \in V(\Gamma_2)} \sum_{ac \in E(S_k(\Gamma_1))} \left[ d_{\Gamma_1}^a (a) + d_{S_k(\Gamma_1)}^{\alpha} (c) \right] + \sum_{i=1}^{\alpha} \binom{\alpha}{i} d_{S_k(\Gamma_1)}^{\alpha-i} (a) d_{\Gamma_1}^i (b)
\]

\[
= n_2 \left[ M_{S_k \Gamma_1}^T \right] + \sum_{i=1}^{\alpha} \binom{\alpha}{i} \left[ M_{\Gamma_1}^i \right] \left[ M_{\Gamma_1}^{T-i} \right]
\]

and the 3rd term of equation (8) will be

\[
\sum_{b \in V(\Gamma_2)} \sum_{ac \in E(S_k(\Gamma_1))} \left[ d_{\Gamma_1 + S_k \Gamma_2}^a (a, b) + d_{\Gamma_1 + S_k \Gamma_2}^a (c, b) \right] = \sum_{b \in V(\Gamma_2)} \sum_{ac \in E(S_k(\Gamma_1))} [2^\alpha + 2^n].
\]

Since in this case $|E(S_k(\Gamma_1))| = (k - 1)\varepsilon_{\Gamma_1}$, we have

\[
= 2^{\alpha+1} (k - 1)n_2 \varepsilon_{\Gamma_1}.
\]

By using (9), (10), & (12) in (8), we get

\[
M^T(\Gamma_1 + S_k \Gamma_2) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left( M_{\Gamma_1}^{\alpha-i} \right) \left( M_{\Gamma_1}^{i+1} \right) + n_2 \varepsilon_{\Gamma_1} \left[ M_{\Gamma_1}^0 \right] + \sum_{i=1}^{\alpha} \binom{\alpha}{i} \left( M_{\Gamma_1}^{i} \right) M_{\Gamma_1}^{T-i} + 2^{\alpha+1} (k - 1)n_2 \varepsilon_{\Gamma_1}.
\]

Theorem 2. Let $\Gamma_1$ and $\Gamma_2$ be two simple graphs and $\gamma \in N - \{0, 1\}$. The FGZ index of the generalized R-sum $\Gamma_1 + R_\gamma \Gamma_2$ graph is

\[
M^T(\Gamma_1 + R_\gamma \Gamma_2) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} 2^{\gamma-i} M_{\Gamma_1}^{\alpha-i} M_{\Gamma_1}^{i+1} + 2 \sum_{i=0}^{\alpha} \binom{\alpha}{i} 2^{\gamma-i} M_{\Gamma_1}^{i} M_{\Gamma_1}^{T-i} + 2^{\alpha+1} (k - 1)n_2 \varepsilon_{\Gamma_1},
\]

where $N$ is the set of natural numbers and $\alpha = \gamma - 1$.

Proof. Then by definition, we have

\[
M^T(\Gamma_1 + R_\gamma \Gamma_2) = \sum_{(a,b) \in V(\Gamma_1 + R_\gamma \Gamma_2)} d_{\Gamma_1 + R_\gamma \Gamma_2}^a (a, b).
\]
\[
M^T(\Gamma_1 + R_1 \Gamma_2) = \sum_{(a,b)} \sum_{e \in \Gamma_1} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(c, d) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(c, d) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(c, d) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

(16)

For every vertex \( a \in V(\Gamma_1) \) & edge \( b \in E(\Gamma_2) \), then the 1st term of (16) is

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

\[
= \sum_{a \in V(\Gamma_1)} \sum_{b \in V(\Gamma_2)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in V(\Gamma_1)} \left[ d_{i_1}^{\alpha_{e-i}}(a, b) + d_{i_1}^{\alpha_{e-i}}(b, c) \right]
\]

(17)
For every vertex \( b \in V(\Gamma_2) \) & edge \( ac \in E(R_k(\Gamma_1)) \), then the 2nd term of (16) will be

\[
\sum_{beV(\Gamma_1)} \left( \sum_{ac \in E(R_k(\Gamma_1))} \left[ d_{a,b}^{ac} (a,b) + d_{b,c}^{ac} (b,c) \right] \right)
\]

\[
= \sum_{beV(\Gamma_1)} \left( \sum_{ac \in E(R_k(\Gamma_1))} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d_{R_k(\Gamma_1)}^{a^i} (a) d_{\Gamma_1}^{i} (b) \right] \right)
\]

\[
= \sum_{ac \in E(R_k(\Gamma_1))} \left( \sum_{i=0}^{\alpha} \binom{\alpha}{i} d_{\Gamma_1}^{i} (b) \right)
\]

\[
= \sum_{ac \in E(R_k(\Gamma_1))} \sum_{i=0}^{\alpha} d_{\Gamma_1}^{i} (b) \left[ d_{R_k(\Gamma_1)}^{a^i} (a) + d_{R_k(\Gamma_1)}^{a^i} (c) \right]
\]

\[
= \sum_{ac \in E(R_k(\Gamma_1))} \sum_{i=0}^{\alpha} \binom{\alpha}{i} M_{\Gamma_1}^{i} \left[ d_{\Gamma_1}^{a^i} (2d_{\Gamma_1}^{a^i} (a)) + (2a^{a^i})^i (d_{\Gamma_1}^{a^i} (c)) \right]
\]

\[
(18)
\]

For every vertex \( b \in V(\Gamma_2) \) & edge \( ac \in E(R_k(\Gamma_1)) \), \( a \in V(\Gamma_1), c \in v(R_k(\Gamma_1)) - V(\Gamma_1) \). Since we have \( d_{R_k(\Gamma_1)}^{a^i} (a) = 2d_{\Gamma_1}^{a^i} (a) \forall a \in V(\Gamma_1) \) also \( d_{R_k(\Gamma_1)}^{a^i} (c) = 2 \forall c \in V(R_k(\Gamma_1)) - V(\Gamma_1) \). So the 3rd term of (16) will be

\[
= \sum_{beV(\Gamma_1)} \left( \sum_{ac \in E(R_k(\Gamma_1))} \left[ d_{a,b}^{ac} (a,b) + d_{b,c}^{ac} (b,c) \right] \right)
\]

\[
= \sum_{beV(\Gamma_1)} \left( \sum_{ac \in E(R_k(\Gamma_1))} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d_{R_k(\Gamma_1)}^{a^i} (a) d_{\Gamma_1}^{i} (b) + d_{R_k(\Gamma_1)}^{a^i} (c) \right] \right)
\]

\[
(19)
\]
Here $d_{\mathbb{R}(\Gamma_1)}^n = 2^n$ and $d_{\mathbb{R}(\Gamma_1)}^{\alpha-i}(a) = (2d_{\Gamma_1}(a))^{\alpha-i}$:

$$
= \sum_{\text{be}\in\text{V} (\Gamma_2)} \sum_{\text{ac}\in\text{V} (\Gamma_2)} \left[ \sum_{\text{be}\in\text{V} (\Gamma_2)} \left( 2d_{\Gamma_1}(a) \right)^{\alpha-i} d_{\Gamma_2}^i (b) + 2^n \right]
$$

$$
= \sum_{\text{be}\in\text{V} (\Gamma_2)} \sum_{\text{ac}\in\text{V} (\Gamma_2)} \left[ \sum_{\text{be}\in\text{V} (\Gamma_2)} \left( 2d_{\Gamma_1}(a) \right)^{\alpha-i} d_{\Gamma_2}^i (b) + 2^n \right]
$$

$$
= \sum_{i=0}^{\alpha} \left[ \sum_{\text{be}\in\text{V} (\Gamma_2)} \sum_{\text{ac}\in\text{V} (\Gamma_2)} \left( 2d_{\Gamma_1}(a) \right)^{\alpha-i} d_{\Gamma_2}^i (b) \right] + 2^n
$$

(20)

and the 4th term of (5) is

$$
\sum_{\text{be}\in\text{V} (\Gamma_2)} \sum_{\text{ac}\in\text{V} (\Gamma_2)} \left[ d_{\Gamma_1+\mathbb{R}(\Gamma_2)}^n (a, b) + d_{\Gamma_1+\mathbb{R}(\Gamma_2)}^n (c, b) \right] = \sum_{\text{be}\in\text{V} (\Gamma_2)} \sum_{\text{ac}\in\text{V} (\Gamma_2)} \left[ 2^n + 2^n \right].
$$

(21)

Since in this case $|E(S_k(\Gamma_1))| = (k-1)|e_{\Gamma_1}|$, we have

$$
2^{\alpha+1} (k-1) n_{\Gamma_1} e_{\Gamma_1}.
$$

(22)

Using (17), (18), (20), and (22) in (16), then we have

$$
\sum_{i=0}^{\alpha} \left[ \sum_{\text{be}\in\text{V} (\Gamma_2)} \sum_{\text{ac}\in\text{V} (\Gamma_2)} \left( 2d_{\Gamma_1}(a) \right)^{\alpha-i} d_{\Gamma_2}^i (b) \right] + 2^n e_{\Gamma_1} n_{\Gamma_2} + 2^{\alpha+1} (k-1) n_{\Gamma_2} e_{\Gamma_1}.
$$

(23)

**Theorem 3.** Let $\Gamma_1$ and $\Gamma_2$ be two simple graphs and $\gamma \in N \setminus \{0,1\}$. The FGZ index of the generalized Q-sum $\Gamma_1 + Q_{\gamma} \Gamma_2$ graph is

$$
\sum_{\text{be}\in\text{V} (\Gamma_2)} \sum_{\text{ac}\in\text{V} (\Gamma_2)} \left[ (M_{\gamma}^{\alpha-i})(M_{\gamma}^i) + \sum_{i=0}^{\alpha} \left( M_{\gamma}^{\alpha-i} \right) \left( M_{\gamma}^i \right) + 2 n_{\Gamma_2} \sum_{i=0}^{\alpha} \left( d_{\Gamma_1}^{\alpha-i} (u) \right) \left( d_{\Gamma_1}^i (v) \right) \right]
$$

$$
+ 2 (k-1) n_{\Gamma_1} \sum_{uv \in E(\Gamma_1)} \left[ \left( d_{\Gamma_1}^{\alpha-i} (u) \right) \left( d_{\Gamma_1}^i (v) \right) \right]
$$

(24)

where $N$ is the set of natural numbers and $\alpha = \gamma - 1$.

**Proof.** Then by definition, we have...
\[ M^\alpha(\Gamma_1 + Q_k \Gamma_2) = \sum_{(a,b) \in V(\Gamma_1 + Q_k \Gamma_2)} d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(a,b) = \sum_{(a,b) \in V(\Gamma_1 + Q_k \Gamma_2) \cup (\Gamma_1 + Q_k \Gamma_2)} \left[ d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(a,b) + d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(a,d) \right] \]
\]
\[ = \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_2)} \left[ d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(a,b) + d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(a,d) \right] + \sum_{b \in V(\Gamma_2)} \sum_{a \in E(\Gamma_2)} \left[ d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(a,b) + d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(b,c) \right]. \tag{25} \]

For every vertex \( a \in V(\Gamma_1) \) & edge \( b d \in E(\Gamma_2) \), then the 1st term of (25) will be

\[ \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_2)} \left[ d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(a,b) + d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(a,d) \right] \]
\[ = \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{Q_k(\Gamma_1)}(a) d^i_{\Gamma_2}(b) + \sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{Q_k(\Gamma_1)}(a) d^i_{\Gamma_2}(d) \right] \]
\[ = \sum_{a \in V(\Gamma_1)} \sum_{b \in E(\Gamma_2)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_2}(a) \left( d^i_{\Gamma_2}(b) + d^i_{\Gamma_2}(d) \right) \right] \]
\[ = \sum_{a \in V(\Gamma_1)} \left[ \sum_{i=0}^{\alpha} \binom{\alpha}{i} d^{\alpha-i}_{\Gamma_2}(a) \right] \left( M^{i+1}_{\Gamma_2} \right) \]
\[ = \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left( M^{i+1}_{\Gamma_2} \right). \tag{26} \]

For every vertex \( b \in V(\Gamma_2) \) & edge \( ac \in E(Q_k(\Gamma_1)) \) \( a, c \in V(\Gamma_1) \), then the 2nd term of equation (25) will be

\[ \sum_{b \in V(\Gamma_2)} \sum_{a \in E(\Gamma_2)} \left[ d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(a,b) + d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(b,c) \right] \]
\[ = \sum_{b \in V(\Gamma_2)} \sum_{a \in E(\Gamma_1) \cup E(\Gamma_1 \setminus (Q_k(\Gamma_1)))} \left[ d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(a,b) + d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(b,c) \right] \]
\[ + \sum_{b \in V(\Gamma_2)} \sum_{a \in E(\Gamma_1) \cup E(\Gamma_1 \setminus (Q_k(\Gamma_1)))} \left[ d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(a,b) + d^\alpha_{\Gamma_1 + Q_k \Gamma_2}(b,c) \right]. \tag{27} \]
Now \( \forall b \in V (\Gamma_2) \) and \( \forall c \in V (\Gamma_2) \) if the vertex \( a, c \in V (Q_k (\Gamma_1)) \) then the first term of (27) will be

\[
\sum_{b \in V (\Gamma_2)} \sum_{ac \in E (Q_k (\Gamma_1))} \left[ d_{\Gamma_1 + Q_k \Gamma_2} (a, b) \right] + \left[ d_{\Gamma_1 + Q_k \Gamma_2} (b, c) \right]
\]

\[
= \sum_{b \in V (\Gamma_2)} \sum_{ac \in E (Q_k (\Gamma_1))} \left[ d_{Q_k (\Gamma_1)} (a) + d_{\Gamma_1} (b) \right] + \left[ d_{Q_k (\Gamma_1)} (c) \right]
\]

\[
= \sum_{b \in V (\Gamma_2)} \sum_{ac \in E (Q_k (\Gamma_1))} \left[ \sum_{i=0}^{n} \left( \alpha \right) d_{Q_k (\Gamma_1)} ^{a} (a) d_{\Gamma_1} ^{i} (b) + d_{Q_k (\Gamma_1)} ^{a} (c) \right]
\]

\[
= \sum_{i=0}^{n} \left( \alpha \right) M_{\Gamma_1} ^{i} M_{Q_k} ^{a} + n_{\Gamma_1} \sum_{ac \in E (Q_k (\Gamma_1))} \left[ d_{Q_k (\Gamma_1)} ^{a} (c) \right]
\]

\[
= \sum_{i=0}^{n} \left( \alpha \right) \left( M_{\Gamma_1} ^{i} M_{Q_k} ^{a} + 2 \sum_{ac \in E (\Gamma_1)} \left( d_{\Gamma_1} (u) + d_{\Gamma_1} (v) \right) \right)
\]

\[
= \sum_{i=0}^{n} \left( \alpha \right) M_{\Gamma_1} ^{i} M_{Q_k} ^{a} + 2n_{\Gamma_1} \sum_{i=0}^{n} \left( \alpha \right) \sum_{ac \in E (\Gamma_1)} \left( d_{\Gamma_1} ^{a} (u) d_{\Gamma_1} ^{i} (v) \right).
\]

Now \( \forall b \in V (\Gamma_2) \) and \( \forall c \in V (\Gamma_2) \) if the vertex \( a, c \in V (Q_k (\Gamma_1)) \) then the second term of equation (27) splits into two parts for the vertices \( a \) and \( c \), then the equation will be
Using (26), (28), (29), and (30) in (25), we get the required result:

\[
\begin{align*}
&= \sum_{i=0}^{\alpha} \left( \binom{\alpha}{i} \sum_{i=0}^{\alpha} \binom{\alpha}{i} M_{i+1}^{t_1} M_{i}^{t_2} + \sum_{i=0}^{\alpha} \binom{\alpha}{i} M_{i+1}^{t_2} M_{i}^{t_1} \right) \\
&+ 2n_{t_2} \sum_{i=0}^{\alpha} \binom{\alpha}{i} \sum_{u \in V \Gamma_1} \left[ d_{i+1}^{T_1}(u) \cdot d_{i}^{T_2}(v) \right] \\
&+ n_{t_1} \sum_{u \in V \Gamma_1} \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left[ d_{i+1}^{T_1}(u) \cdot d_{i}^{T_2}(v) \right] \\
&+ 2n_{t_2} \sum_{u \in V \Gamma_1} \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left[ d_{i+1}^{T_2}(u) \cdot d_{i}^{T_1}(v) \right] \\
&+ (2k-1)n_{t_2} \sum_{u \in V \Gamma_1} \left[ d_{i}^{T_2}(u) + d_{i+1}^{T_1}(v) \right].
\end{align*}
\]

(31)

**Theorem 4.** Let $\Gamma_1$ and $\Gamma_2$ be two simple graphs. The FGZ index of the generalized T-sum graph $\Gamma_1^{\alpha} \Gamma_2^{\beta}$ is

\[
M^{\alpha}(\Gamma_1^{\alpha} \Gamma_2^{\beta}) = \sum_{i=0}^{\alpha} \binom{\alpha}{i} (2)^{\alpha-i} M_{i+1}^{t_1} M_{i}^{t_2} + \sum_{i=0}^{\alpha} \binom{\alpha}{i} M_{i+1}^{t_2} M_{i}^{t_1} (2)^{\alpha-i} M_{i+1}^{t_1} M_{i}^{t_2} + \sum_{i=0}^{\alpha} \binom{\alpha}{i} (2)^{\alpha-i} M_{i+1}^{t_2} M_{i}^{t_1} \\
+ 2n_{t_2} \sum_{i=0}^{\alpha} \binom{\alpha}{i} \sum_{u \in V \Gamma_1} \left[ d_{i+1}^{T_1}(u) \cdot d_{i}^{T_2}(v) \right] + n_{t_2} \\
+ n_{t_1} \sum_{u \in V \Gamma_1} \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left[ d_{i+1}^{T_1}(u) \cdot d_{i}^{T_2}(v) \right] + n_{t_1} \sum_{u \in V \Gamma_1} \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left[ d_{i+1}^{T_2}(u) \cdot d_{i}^{T_1}(v) \right] \\
+ 2n_{t_2} \sum_{u \in V \Gamma_1} \sum_{i=0}^{\alpha} \binom{\alpha}{i} \left[ d_{i}^{T_2}(u) + d_{i+1}^{T_1}(v) \right].
\]

(32)
where $\gamma \in N^+ - \{0, 1\}$ and $\alpha = \gamma - 1$.

Proof. Since we have $d_{\Gamma_1 + \Gamma_2}(a, b) = d_{\Gamma_1 + \Gamma_2}(a, b)$ for every vertex $a \in V(\Gamma_1)$ and $b \in V(\Gamma_2)$, also $d_{\Gamma_1 + \Gamma_2}(a, b) = d_{\Gamma_1 + \Gamma_2}(a, b)$ for every vertex $a \in \Gamma_1(T_k(\Gamma_1)) - V(\Gamma_1)$ and $b \in V(\Gamma_2)$, the result follows by the proof of Theorems 2 and 3.

\[
\begin{align*}
(i) \quad & M^N(\Gamma_1 + \Gamma_2) = \sum_{i=0}^{\infty} \left( \begin{array}{c} \alpha \\ i \end{array} \right) \left( M_{\Gamma_1}^i \right) \left( M_{\Gamma_2}^{\infty-i} \right) + n_{\Gamma_2} \cdot M_{\Gamma_2}^1 + \sum_{i=1}^{\infty} \left( \begin{array}{c} \alpha \\ i \end{array} \right) M_{\Gamma_2}^{\infty-i} M_{\Gamma_1}^i + 2^{\infty-i} (k-1)n_{\Gamma_2}e_{\Gamma_1}, \\
(ii) \quad & M^N(\Gamma_1 + \Gamma_2) = \sum_{i=0}^{\infty} \left( \begin{array}{c} \alpha \\ i \end{array} \right) 2^i M_{\Gamma_1}^i M_{\Gamma_2}^{\infty-i} + 2 \sum_{i=0}^{\infty} \left( \begin{array}{c} \alpha \\ i \end{array} \right) \left( 2^i M_{\Gamma_1}^i M_{\Gamma_2}^{\infty-i} \right) + 2^\infty e_{\Gamma_1} n_{\Gamma_2} + 2^{\infty-i} (k-1)n_{\Gamma_2}e_{\Gamma_1}, \\
(iii) \quad & M^N(\Gamma_1 + \Gamma_2) = \sum_{i=0}^{\infty} \left( \begin{array}{c} \alpha \\ i \end{array} \right) \left( M_{\Gamma_1}^i \right) \left( M_{\Gamma_2}^{\infty-i} \right) + \sum_{i=0}^{\infty} \left( \begin{array}{c} \alpha \\ i \end{array} \right) \left( M_{\Gamma_1}^{i+1} \right) \left( M_{\Gamma_2}^{\infty-i} \right) + 2n_{\Gamma_2} \left[ \sum_{i=0}^{\infty} \left( \begin{array}{c} \alpha \\ i \end{array} \right) \left( d_{\Gamma_1}(u) \cdot d_{\Gamma_1}^{\infty-i}(v) \right) \right] \\
& + n_{\Gamma_2} \sum_{uv \in E(\Gamma_1), uv \in E(\Gamma_1)} \left[ \sum_{i=0}^{\infty} \left( \begin{array}{c} \alpha \\ i \end{array} \right) d_{\Gamma_1}(u) \cdot d_{\Gamma_1}^{\infty-i}(v) \right] \right] \\
& + 2(k-1)n_{\Gamma_2} \sum_{uv \in E(\Gamma_1)} [d_{\Gamma_1}^u(u) + d_{\Gamma_1}^v(v)] \right]
\end{align*}
\]

\[ (33) \]

\[ \text{Theorem 5.} \quad \text{Assume that } \Gamma_1 \text{ and } \Gamma_2 \text{ are two simple graphs and } \alpha = \gamma - 1, \text{ where } \gamma \in \mathbb{R}^\infty - \{0, N^+\} \text{ and } \mathbb{R} \text{ is a set of real numbers. Then, the FGZ index of generalized F-sum graphs} \ (\Gamma_1 + \Gamma_2) \text{ are} \]
Proof. The above proof is similar as of Theorems 1–4.

Let \( \Gamma_1 \) be a negative integer, so from Theorem 5, Corollary 1 is obtained.

\[ \text{Corollary 1. Assume that } \Gamma_1 \in \mathbb{R} \land \Gamma_2 \in \mathbb{R} \text{ are two simple graphs and } \alpha = \gamma - 1, \text{ where } \gamma \text{ is a negative real number. The FGZ index of the generalized F-sums graphs } (\Gamma_1 + \gamma \Gamma_2, \Gamma_1 + \gamma \Gamma_2, \Gamma_1 + \gamma \Gamma_2, \text{ and } \Gamma_1 + \gamma \Gamma_2) \text{ are} \]

\[ \begin{align*}
(i) M^f(\Gamma_1 + \gamma \Gamma_2) &= \sum_{i=0}^{\infty} (-1)^i \left( \binom{\alpha + i - 1}{i} M_{\Gamma_1}^i M_{\Gamma_2}^{\alpha+i} + n_{\Gamma_1} M_{\Gamma_1}^i (\Gamma_1) + \sum_{i=0}^{\infty} (-1)^i \left( \binom{\alpha + i - 1}{i} M_{\Gamma_1}^i M_{\Gamma_2}^{\alpha+i} + 2^{\alpha+i} (k-1)n_{\Gamma_1} e_{\Gamma_1} \right) \\
(ii) M^f(\Gamma_1 + \gamma \Gamma_2) &= \sum_{i=0}^{\infty} (-1)^i \left( \binom{\alpha + i - 1}{i} 2^i M_{\Gamma_1}^i M_{\Gamma_2}^{\alpha+i} + 2 \left( \sum_{i=0}^{\infty} (-1)^i \left( \binom{\alpha + i - 1}{i} 2^i M_{\Gamma_1}^i M_{\Gamma_2}^{\alpha+i} + 2^{\alpha+i} (k-1)n_{\Gamma_1} e_{\Gamma_1} \right) \\
(iii) M^f(\Gamma_1 + \gamma \Gamma_2) &= \sum_{i=0}^{\infty} (-1)^i \left( \binom{\alpha + i - 1}{i} \left( M_{\Gamma_1}^i M_{\Gamma_2}^{\alpha+i} + \sum_{i=0}^{\infty} (-1)^i \left( \binom{\alpha + i - 1}{i} M_{\Gamma_1}^i M_{\Gamma_2}^{\alpha+i} + 2^{\alpha+i} (k-1)n_{\Gamma_1} e_{\Gamma_1} \right) \\
(iv) M^f(\Gamma_1 + \gamma \Gamma_2) &= \sum_{i=0}^{\infty} (-1)^i \left( \binom{\alpha + i - 1}{i} 2^i M_{\Gamma_1}^i M_{\Gamma_2}^{\alpha+i} + 2 \left( \sum_{i=0}^{\infty} (-1)^i \left( \binom{\alpha + i - 1}{i} 2^i M_{\Gamma_1}^i M_{\Gamma_2}^{\alpha+i} + 2^{\alpha+i} (k-1)n_{\Gamma_1} e_{\Gamma_1} \right) \\
\end{align*} \]

\[ \tag{34} \]

4. Applications

Now, we present some examples as applications of the obtained results Theorems 1–4. Also the numerical comparisons are presented in Tables 1–4, and the graphical representations are depicted in Figures 4–7.

Example 1. Let \( P_m \) and \( P_{n} \) be two simple graphs with \( m \geq 2 \) and \( n \geq 2 \). Then, we have

\[ \begin{align*}
1. M^f(P_m + \gamma P_n) &= \sum_{i=0}^{\gamma} C_i^{\gamma-i} \left[ 2^{\gamma-i} (m-2) + 2 \right] \left[ 2^{\gamma+i} (n-2) + 2 \right] + \sum_{i=0}^{\gamma} C_i^{\gamma-i} \left[ 2^{\gamma-i} (m-2) + 2 \right] \left[ 2^i (n-2) + 2 \right] \\
&+ n(2^\gamma(2m-3) + 2) + 4(k-1)n(m-1).
\end{align*} \]

From Figure 4, it is clear that the behavior of FGZ index of the generalized S-sum graph \( \Gamma_1 + \gamma \Gamma_2 \) at \( t = 2 \) is more better than \( t = 0 \) and \( t = 1 \):

\[ \begin{align*}
2. M^f(P_m + \gamma P_n) &= \sum_{i=0}^{\gamma} C_i^{\gamma-i} \left[ 2^{\gamma-i} (m-2) + 2 \right] \left[ 2^{\gamma+i} (n-2) + 2 \right] + 2 \sum_{i=0}^{\gamma} C_i^{\gamma-i} \left[ 2^{\gamma-i} (m-2) + 2 \right] \left[ 2^i (n-2) + 2 \right] \\
&+ 2(m-1)n + 4(k-1)n(m-1).
\end{align*} \]

\[ \tag{36} \]
Table 1: Numerical comparison for $M^t(P_{mxk} P_n)$.

<table>
<thead>
<tr>
<th>$[m,n,k]$</th>
<th>$T = 0$</th>
<th>$T = 1$</th>
<th>$T = 2$</th>
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</thead>
<tbody>
<tr>
<td>[1, 1, 1]</td>
<td>−4</td>
<td>−4</td>
<td>−13</td>
</tr>
<tr>
<td>[2, 2, 2]</td>
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<td>28</td>
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<td>[4, 4, 4]</td>
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<td>308</td>
<td>326</td>
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<td>631</td>
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<td>1036</td>
<td>1072</td>
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<td>1628</td>
<td>1673</td>
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<td>2404</td>
<td>2458</td>
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<tr>
<td>[10, 10, 10]</td>
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<td>4676</td>
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</table>

Table 2: Numerical comparison for $M^t(P_{mxk} P_n)$.

<table>
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<th>$T = 2$</th>
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<tbody>
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<td>[1, 1, 1]</td>
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</table>

Table 3: Numerical comparison for $M^t(P_{mxk} P_n)$.

<table>
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<tbody>
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</table>

Table 4: Numerical comparison for $M^t(P_{mxk} P_n)$.

<table>
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<th>$T = 2$</th>
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<tbody>
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<td>3</td>
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From Figure 5, it is clear that the behavior of FGZ index of the generalized $R$-sum graph $\Gamma_{1+R_{k}}\Gamma_{2}$ at $t = 0$ is more better than $t = 1$ and $t = 2$.
From Figure 6, it is clear that the behavior of FGZ index of the generalized $Q$-sum graph $\Gamma_1 + Qk \Gamma_2$ at $t = 2$ is more better than $t = 0$ and $t = 1$:

$$4. M^t(P_{m+T^l}P_n) = \sum_{t=0}^{n} C^{-1}_{t} 2^{t-1-t} \left[ 2^{r-1-t} (m - 2) + 2 \right] \left[ 2^{t+1} (n - 2) + 2 \right] + 2 \sum_{t=0}^{n} C^{-1}_{t} 2^{r-1-t} \left[ 2^{r-1-t} (m - 2) + 2 \right] \left[ 2^{t} (n - 2) + 2 \right]$$

From Figure 7, it is clear that the behavior of FGZ index of the generalized $T$-sum graph $\Gamma_1 + T_k \Gamma_2$ at $t = 0$ is more better than $t = 1$ and $t = 2$.

5. Conclusions

Now, we close our discussion with the following remarks:

(i) For positive integer $k$ and two graphs $\Gamma_1$ & $\Gamma_2$, we have computed FGZ index of the generalized $F$-sums graph $\Gamma_{1+f} \Gamma_2$, where generalized $F$-sums graphs are obtained by the different operations of subdivision and Cartesian product on $\Gamma_1$ & $\Gamma_2$.

(ii) The obtained results are also verified and illustrated for the particular classes of graphs.

(iii) The behavior of FGZ index is also analyzed with the help of numerical and graphical presentations.

(iv) However, the problem is still open to compute the different topological indices (degree and distance based) for the generalized $F$-sum graphs.

Data Availability

All the data are included within this paper. However, the reader may contact the corresponding author for more details of the data.

Conflicts of Interest

The authors have no conflicts of interest.

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