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Research Article

Chaos on Discrete Neural Network Loops with Self-Feedback

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In this paper, the complex dynamical behaviors in a discrete neural network loop with self-feedback are studied. Specifically, an invariant closed set of the system of neural network loops is built and the subsystem restricted on this invariant closed set is topologically conjugate to a two-sided symbolic dynamical system which has two symbols. In the end, some illustrative numerical examples are given to demonstrate our theoretical results.

1. Introduction

In recent years, researchers have found various chaotic phenomena in the nervous system and that chaotic neural networks play an important role in neural activities. Chaos in neural networks systems have been applied to all kinds of practical problems such as combinatorial optimizations, associative recognition memory, deep learning, and biotechnology (see [1–5]). In fact, some nervous systems consist of large-scale and complex nonlinear dynamics. At present, neuroscience has provided abundant evidence to prove that the central nervous system has complex nonlinear dynamic behavior at all levels [6]. So how to analyze the dynamical behavior of neural networks plays an important role in practical applications. In order to obtain a deep and clear understanding of complex neural networks, there are increasing studies on bifurcations and chaotic behaviors of neural network systems [7].

Recently, Huang and Zou in [8] showed the discrete network system consisting of two identical neurons with a uniform delay demonstrates snapback repeller chaotic behaviors near an equilibrium point. For the Hopfield networks with two different neurons [9–11], the conditions that the systems exhibit chaos are obtained. In [12], Wu et al. analyzed the chaotic behaviors of the parameterized discrete dynamics of recurrent m-neuron networks evoked by external inputs and obtained some conditions which the subsystem is topologically conjugate to symbolic dynamical system. In this paper, we will devote to analysis of the chaotic behaviors of the following discrete neural network loops with multiple delays and self-feedback:

$$\begin{cases} x_1(n+1) = \beta_1 x_1(n) + \alpha_{11} f_1(x_1(n-k_{11})) + \alpha_{1m} f_m(x_m(n-k_{1m})), \\ x_2(n+1) = \beta_2 x_2(n) + \alpha_{21} f_1(x_1(n-k_{21})) + \alpha_{22} f_2(x_2(n-k_{22})), \\ \vdots \\ x_m(n+1) = \beta_m x_m(n) + \alpha_{mm-1} f_{m-1}(x_m(n-k_{mm-1})) + \alpha_{mm} f_m(x_m(n-k_{mm})), \end{cases} \quad n \in \mathbb{Z},\ k_{ij} \geq 1, \tag{1}$$

where $n \in \mathbb{Z}$, for $i = 1, 2, \ldots, m$, $\beta_i \in (0, 1)$ is the internal decay rate of the neurons, α_{ij} is the self-feedback strength or the connection strength of the ith neuron to the next neuron, and the transmission delay $k_{ij} \geq 1$ is a positive integer.

For the case of the neural network with m-identical neurons, Cheng constructed a snapback repeller in [13] and then justified chaos in neural networks. When the discrete neural network with m-different neurons has multiple time delays and self-feedback, it is challenging to rigorously analyze the dynamical behaviors. In this paper, we consider the chaotic behaviors of model (1). To this end, we first rewrite the model (1) as a system of difference equations without delay by a novel way. Especially, this transformation requires a little skill. Then, we find an invariant set for the transformed system by projection and show that the system restricted on this set is topologically conjugate to the full shift map on the symbolic dynamical system. This implies that the system has chaotic behaviors. The obtained results extend the related ones in [10, 11, 13]. Also, we provide some numerical simulations to verify the theoretical results.

2. Invariant Subsystem of Model (1)

Let l_∞ denote the Banach space of bounded sequences of real numbers with the supremum norm defined on it. The norm is denoted by $\|\cdot\|$. Let $\sigma: l_\infty \longrightarrow l_\infty$ be shift map defined by $(\sigma\xi)_n = (\xi)_{n+1}$, $n \in \mathbb{Z}$, for $\xi = (\ldots, \xi_{-n}, \ldots, \xi_{-1}, \tilde{\xi}_0, \xi_1, \ldots, \xi_n, \ldots) \in l_\infty$. That is,

$$\begin{aligned} &\sigma\left(\ldots, \xi_{-n}, \ldots, \xi_{-1}, \tilde{\xi}_0, \xi_1, \ldots, \xi_n, \ldots\right) \\ &\quad = \left(\ldots, \xi_{-n+1}, \ldots, \xi_0, \tilde{\xi}_1, \xi_2, \ldots, \xi_{n+1}, \ldots\right). \end{aligned} \tag{2}$$

Clearly, the shift map σ on l_∞ is continuously invertible, and its inverse σ^{-1} is being defined by $(\sigma^{-1}\xi)_n = \xi_{n-1}$, $n \in \mathbb{Z}$. The ith iterate of σ, $\overbrace{\sigma \circ \sigma \cdot \circ \sigma}^{i\text{ times}}$, is denoted as σ^i. Let $\Sigma_k = \{(\ldots i_{-1} i_0 i_1 \ldots) \mid i_n \in \{1, 2, \ldots, k\}, n \in \mathbb{Z}\}$ denote a symbolic space with k symbols. Endowing it with the metric

$$\begin{aligned} &d(s, t) = \max\left\{2^{-|n|} \mid t_n \neq s_n, n \in \mathbb{Z}\right\}, t = (\ldots t_{-1} t_0 t_1 \ldots), \\ &\qquad s = (\ldots s_{-1} s_0 s_1 \ldots) \in \Sigma_k, \end{aligned} \tag{3}$$

Σ_k becomes a compact and totally disconnected metric space. The shift map $\sigma: \Sigma_k \longrightarrow \Sigma_k$ is defined by $(\sigma t)_n = t_{n+1}$. Then, (Σ_k, σ) is a two-sided symbolic system. To proceed, let $m, l \geq 2, i, j$ be positive integers.

Lemma 1. *Let $q \leq m$ be a positive integer. $a_{q_1}, a_{q_2}, \ldots, a_{q_l}$ are l different real numbers with $l \geq 2$ and a_i is a real number with $1 \leq i \neq q \leq m$. $\Lambda = \left\{(\xi_n) \in l_\infty \mid \xi_{mn+i} = a_i, \xi_{mn+q} = a_{q_j}\right\}$ be a subset of l_∞. Then, (Λ, σ^m) is topological conjugate to (Σ_l, σ).*

Proof. Define $g: \Lambda \longrightarrow \Sigma_l$ by $g(\xi) = (\ldots, \xi_{-mn+q}, \ldots, \xi_{-m+q}, \xi_q, \ldots, \xi_{mn+q}, \ldots)$, for $\xi = (\xi_n) \in \Lambda$. In fact, $g(\xi)$ is defined by deleting the elements whose indexes are congruent i modulo m in W, where $1 \leq i \neq q \leq m$. It is not difficult to see that g is a homeomorphism. By definition of g, we have $g \circ \sigma^m = \sigma \circ g$. So (Λ, σ^m) and (Σ_l, σ) are topological conjugacy. □

Lemma 2 (see [14]). *Let X and Y be Banach spaces, L is an invertible linear map from X to Y, and S is a bounded linear map from X to Y. If $\|S\| < \|L^{-1}\|^{-1}$, then $L + S$ is an invertible linear map from X to Y.*

Lemma 3 (see [15]). *Let (Λ, d) be a metric space, Y and X be Banach spaces, and $U \subset \Lambda \times Y$ be open. Suppose that $F: U \longrightarrow X$ is a continuous map and that there exists a point $(\lambda_0, y_0) \in U$ with the following conditions:*

(i) $F(\lambda_0, y_0) = 0$.

(ii) $DF_y(\lambda, y)$ is continuous at (λ_0, y_0), where $DF_y(\lambda, y)$ is Frechet partial derivative of $F(\lambda, y)$ with respect to y.

(iii) $DF_y(\lambda_0, y_0): Y \longrightarrow X$ is an invertible linear map.

Then, there exist open balls $B_{\delta_0}(y_0) = \{y: \|y - y_0\| < \delta_0\}$ and $B_{r_0}(\lambda_0) = \{\lambda: d(\lambda, \lambda_0) < r_0\}$, where $\delta_0 > 0, r_0 > 0$ such that, for any $\lambda \in B_{r_0}(\lambda_0)$, the equation $F(\lambda, y) = 0$ has a unique continuous solution $y = h(\lambda) \in B_{\delta_0}(y_0)$ with $h(\lambda_0) = y_0$.

For convenience, we set $i - 1 = m$ when $i - 1 = 0$. Let $\alpha = \alpha_{11}, C_{ij} = (\alpha_{ij}/\alpha)(i \in \{1, 2, \ldots, m\}, j = i - 1 \text{ or } i)$. Without losing generality, we may suppose that $k_{mm} \geq k_{1m} \geq k_{m-1m-1} \geq k_{mm-1} \cdots \geq k_{22} \geq k_{32} \geq k_{11} \geq k_{21}$. In the other cases, we can discuss it in a similar way. The activation functions $f_i (i = 1, \ldots, m)$ have the following conditions $(G1)$:

$(G1)$ For every $i \in \{1, 2, \ldots, m\}$, f_i is a continuously differentiable function from $\mathbb{R}$ to $\mathbb{R}$. f_1 has two distinct zero points $\hat{x}^{q1}, \hat{x}^{q2}$, satisfying $f_1(\hat{x}^{q1}) = f_1(\hat{x}^{q2}) = 0$, $f_1'(\hat{x}^{q1}) \neq 0$, and $f_1'(\hat{x}^{q2}) \neq 0$, and $f_i (i \in \{2, \ldots, m\})$ has a zero point $\overline{x}^i$, satisfying $f_i(\overline{x}^i) = 0$ and $f_i'(\overline{x}^i) \neq 0$.

Let $p_1 = mk_{11}$, $p_l = mk_{11} + \sum_{i=2}^{l} (m - i + 1)(k_{ii} - k_{i-1i-1})$, $2 \leq l \leq m$, $p = p_m + m$ and define $\eta(n) = (\eta_1(n), \ldots, \eta_p(n))$, where

$$\begin{cases} \eta_{mj+i}(n) = x_i(n - k_{ii} + j), \quad 0 \le j \le k_{11},\ 1 \le i \le m, \\ \eta_{p_1+j(m-1)+i}(n) = x_i(n - k_{ii} + k_{11} + j), \quad 1 \le j \le k_{22} - k_{11},\ 2 \le i \le m, \\ \eta_{p_2+j(m-2)+i}(n) = x_i(n - k_{ii} + k_{22} + j), \quad 1 \le j \le k_{33} - k_{22},\ 3 \le i \le m, \\ \vdots \\ \eta_{p_{m-1}+m+j}(n) = x_i(n - k_{ii} + k_{m-1m-1} + j), \quad 1 \le j \le k_{mm} - k_{m-1m-1},\ i = m. \end{cases} \quad \forall n \in \mathbb{Z}. \tag{4}$$

For any $1 \le i \le m$, there exists $1 \le l_i \le m-1$ such that $k_{l_i l_i} < k_{i-1i-1} - k_{ii-1} \le k_{l_i+1 l_i+1}$. Then, we transform system (1) into the discrete dynamical system without delays on $\mathbb{R}^p$:

$$\eta(n+1) = F_\alpha(\eta(n)), \quad n \in \mathbb{Z}, \tag{5}$$

where $F_\alpha\colon \mathbb{R}^p \longrightarrow \mathbb{R}^p$ is defined as

$$F_\alpha \begin{pmatrix} \eta_1(n) \\ \eta_2(n) \\ \vdots \\ \eta_{p_1+1}(n) \\ \eta_{p_1+2}(n) \\ \vdots \\ \eta_{p_i+i}(n) \\ \eta_{p_i+i+1}(n) \\ \vdots \\ \eta_{p-1}(n) \\ \eta_p(n) \end{pmatrix} = \begin{pmatrix} \eta_{m+1}(n) \\ \eta_{m+2}(n) \\ \vdots \\ \beta_1 \eta_{p_1+1}(n) + C_{11}\alpha f_{11}(\eta_1(n)) + C_{1m}\alpha f_{1m}\left(\eta_{p_{l_1}+\left(k_{mm}-k_{1m}-k_{l_1 l_1}\right)(m-l_1)+m}(n)\right) \\ \eta_{p_1+m+1}(n) \\ \vdots \\ \beta_i \eta_{p_i+i}(n) + C_{ii-1}\alpha f_{ii-1}\left(\eta_{\widetilde{p}_{l_i}}(n)\right) + C_{ii}\alpha f_{ii}(\eta_i(n)) \\ \eta_{p_i+m+1}(n) \\ \vdots \\ \eta_p(n) \\ \beta_m \eta_p(n) + C_{mm-1}\alpha f_{mm-1}\left(\eta_{\widetilde{p}_{l_m}}(n)\right) + C_{mm}\alpha f_{mm}((\eta_m)(n)) \end{pmatrix}, \tag{6}$$

where $\widetilde{p}_{l_i} = p_{l_i} + (k_{i-1i-1} - k_{ii-1} - k_{l_i l_i})(m - l_i) + i - 1$, $2 \le i \le m$.

To investigate chaos in System (1), we only consider the chaotic behavior of the system $(\mathbb{R}^p, F_\alpha)$. Next, by the projection approach, we are going to find the invariant set Λ_α of F_α such that the subsystem $(F_\alpha, \Lambda_\alpha)$ has chaotic behavior for α being sufficiently large.

We consider a family of maps $\Phi(\lambda, \cdot)\colon l_\infty \longrightarrow l_\infty$ depending on a parameter $\lambda \in \mathbb{R}$, and the class of maps is defined by

$$\begin{cases} \Phi(\lambda,\xi)_{m(n+1)+1} = \lambda\left(-\xi_{m(n+1)+1} + \beta_1 \xi_{mn+1}\right) + C_{11} f_1\left(\xi_{(m(n-k_{11}))+1}\right) + C_{1m} f_m\left(\xi_{(m(n-k_{1m}))+m}\right), \\ \vdots \\ \Phi(\lambda,\xi)_{m(n+1)+i} = \lambda\left(-\xi_{m(n+1)+i} + \beta_i \xi_{mn+i}\right) + C_{ii-1} f_{i-1}\left(\xi_{m(n-k_{ii-1})+i-1}\right) + C_{ii} f_i\left(\xi_{m(n-k_{ii})+i}\right), \end{cases} \quad \forall \xi = (\xi_n) \in l_\infty,\ 2 \le i \le m. \tag{7}$$

It is easy to see that if $\xi = \{\xi_n\}_{n\in\mathbb{Z}} \in l_\infty$ satisfies $\Phi(1/\alpha, \xi) = 0$, then the sequence $\{x_1(n), x_2(n), \ldots, x_m(n)\}_{n\in\mathbb{Z}}$ with $x_i(n) = \xi_{mn+i}$ satisfies (1). On the contrary, if the sequence $\{x_1(n), x_2(n), \ldots, x_m(n)\}_{n\in\mathbb{Z}}$ satisfies (1), then $\xi = \{\xi_n\}_{n\in\mathbb{Z}} \in l_\infty$ with $\xi_{mn+i} = x_i(n)$ satisfies $\Phi(1/\alpha, \xi) = 0$.

Let

$$
\begin{aligned}
\Gamma &= \left\{ \xi = (\xi_n) \in l_\infty | \xi_{mn+i} = \overline{x}^i, \xi_{mn+1} = \widehat{x}^{q1} \text{ or } \widehat{x}^{q2}, \quad 2 \le i \le m,\ n \in \mathbb{Z} \right\}, \\
b_{11} &= |C_{11} \max\left\{ \left| f_1'\left(\widehat{x}^{q1}\right)\right|, \left| f_1'\left(\widehat{x}^{q1}\right)\right| \right\}, \\
b_{21} &= \left|C_{21}\right| \min\left\{ \left| f_1'\left(\widehat{x}^{q1}\right)\right|, \left| f_1'\left(\widehat{x}^{q2}\right)\right| \right\}, \\
b_{ii} &= \left| C_{ii} f_i'\left(\overline{x}^i\right)\right|, \ (i \in \{2, \ldots, m\}), \\
b_{ii-1} &= \left| C_{ii-1} f_{i-1}'\left(\overline{x}^i\right)\right|, \ (i \in \{1, 3, 4, \ldots, m\}), \\
b &\triangleq \frac{1}{\max\left\{ b_{11}^{-1}, b_{ii}^{-1} + b_{ii}^{-1} b_{ii-1} b_{i-1i-1}^{-1} + \cdots + b_{ii}^{-1} b_{ii-1} b_{i-1i-1}^{-1} \ldots b_{21} b_{11}^{-1}, \quad \forall 2 \le i \le m \right\}}.
\end{aligned}
\tag{8}
$$

Lemma 4. *Under the assumption (G1), if $b_{11} > b_{1m}$ and $b_{ii} > b_{1m} + b_{ii-1}\,(2 \le i \le m)$, then we have the following:*

(i) There exist positive real numbers r_0 and δ_0 such that, for any $\overline{\xi} \in \Gamma$ and $-r_0 \le \lambda \le r_0$, there exists a unique point $\xi(\lambda) \in B_{\delta_0}(\overline{\xi})$, satisfying $\Phi(\lambda, \xi(\lambda)) = 0$.

(ii) For every $0 < \delta < \delta_0$, there exists $0 < r < r_0$ such that, for any $-r \le \lambda \le r$ and $\overline{\xi} \in \Gamma$, there is a unique point $\xi(\lambda)$, satisfying $\|\xi(\lambda) - \overline{\xi}\| \le \delta$ and $\Phi(\lambda, \xi(\lambda)) = 0$.

$B_{\delta_0}(\overline{\xi})$ is the open ball in l_∞ centered at $\overline{\xi}$ with radius δ_0.

Proof. For a given sequence $\overline{\xi} \in \Gamma$, we have $\Phi(0, \overline{\xi}) = 0$. By the assumption (G1) and the definition of $\Phi(\lambda, \xi)$, this can ensure the continuous differentiability of $\Phi(\lambda, \xi)$. The Fréchet derivative of $\Phi(0, \xi)$ with respect to ξ at the point $(0, \overline{\xi})$ be denoted as $D\Phi_\xi(0, \overline{\xi})$ which is represented as

$$
\begin{cases}
\left(D\Phi_\xi(0,\overline{\xi})\xi\right)_{m(n+1)+1} = C_{11} f_1'\left(\overline{\xi}_{m(n-k_{11})+1}\right)\xi_{m(n-k_{11})+1} + C_{1m} f_m'\left(\overline{\xi}_{m(n-k_{1m})+m}\right)\xi_{m(n-k_{1m})+m}, \\
\left(D\Phi_\xi(0,\overline{\xi})\xi\right)_{m(n+1)+i} = C_{ii-1} f_{i-1}'\left(\overline{\xi}_{m(n-k_{ii-1})+i-1}\right)\xi_{m(n-k_{ii-1})+i-1} + C_{ii} f_i'\left(\overline{\xi}_{m(n-k_{ii})+i}\right)\xi_{m(n-k_{ii})+i},
\end{cases}
\quad n \in \mathbb{Z},\ 2 \le i \le m.
\tag{9}
$$

Firstly, we have to show the invertibility of $D\Phi_\xi(0, \overline{\xi})$. We denote that $D\Phi_\xi(0, \overline{\xi}) = L_1(\overline{\xi}) + L_2(\overline{\xi})$, where

$$
\begin{aligned}
&\begin{cases}
\left(L_1(\overline{\xi})\xi\right)_{m(n+1)+1} = C_{11} f_1'\left(\overline{\xi}_{m(n-k_{11})+1}\right)\xi_{m(n-k_{11})+1}, \\
\left(L_1(\overline{\xi})\xi\right)_{m(n+1)+i} = C_{ii-1} f_{i-1}'\left(\overline{\xi}_{m(n-k_{ii-1})+i-1}\right)\xi_{m(n-k_{ii-1})+i-1} + C_{ii} f_i'\left(\overline{\xi}_{m(n-k_{ii})+i}\right)\xi_{m(n-k_{ii})+i},
\end{cases}
\quad n \in \mathbb{Z},\ 2 \le i \le m, \\
&\begin{cases}
\left(L_2(\overline{\xi})\xi\right)_{m(n+1)+1} = C_{1m} f_m'\left(\overline{\xi}_{m(n-k_{1m})+m}\right)\xi_{m(n-k_{1m})+m}, \\
\left(L_2(\overline{\xi})\xi\right)_{m(n+1)+i} = 0,
\end{cases}
\quad n \in \mathbb{Z}.
\end{aligned}
\tag{10}
$$

Let

$$
\begin{aligned}
a_{ii-1}^l &= C_{i-l+1i-l} f_{i-l}'\left(\overline{\xi}_{m(n+k_{ii}-k_{ii-1}+\cdots+k_{i-l+1i-l+1}-k_{i-li-l-1}+i-l-1)}\right)(0 \le l \le i-1), \\
a_{ii}^l &= C_{i-li-l} f_{i-l}'\left(\overline{\xi}_{m(n+k_{ii}-k_{ii-1}+k_{i-1i-1}-\cdots+k_{i-li-l}+i-l)}\right)(0 \le l \le i-1).
\end{aligned}
\tag{11}
$$

It follows from (G1) that the linear operator $L_1(\overline{\xi})$ is invertible. By directing calculation, the inverse operator $L_1(\overline{\xi})^{-1}$ is

$$\begin{cases} \left(L_1(\overline{\xi})^{-1}\xi\right)_{m(n-1)+1} = \dfrac{1}{a_{11}^0}\xi_{m(n+k_{11})+1}, \\ \left(L_1(\overline{\xi})^{-1}\xi\right)_{m(n-1)+i} = \dfrac{1}{a_{ii}^0}\xi_{m(n+k_{ii})+i} - \dfrac{1}{a_{ii}^0}a_{ii-1}^0\dfrac{1}{a_{ii}^1}\xi_{m(n+k_{ii}-k_{ii-1}+k_{i-1i-1})+i-1}+, \quad n \in \mathbb{Z},\ 2 \le i \le m. \\ \cdots + (-1)^{i-1}\dfrac{1}{a_{ii}^0}a_{ii-1}^0\dfrac{1}{a_{ii}^1}, \ldots, a_{ii-1}^{i-2}\dfrac{1}{a_{ii}^{i-1}}\xi_{m(n+k_{ii}-k_{ii-1}+\cdots-k_{21}+k_{11})+1}. \end{cases} \tag{12}$$

Since $\overline{\xi} \in \Gamma$, $\overline{\xi}_{mn+1} = \widehat{x}^{q1}$ or $\widehat{x}^{q2}$, $\overline{\xi}_{mn+i} = \overline{x}^i$, $2 \le i \le m$. This implies that

$$\begin{gathered} \left\|L_1(\overline{\xi})^{-1}\right\| \le \frac{1}{b}, \\ \left\|L_2(\overline{\xi})\right\| = b_{1m}, \end{gathered} \tag{13}$$

so

$$\left\|L_1(\overline{\xi})^{-1}\right\|^{-1} > \left\|L_2(\overline{\xi})\right\|, \tag{14}$$

by the fact that $b_{11} > b_{1m}$ and $b_{ii} > b_{1m} + b_{ii-1}$ $(2 \le i \le m)$. This shows the invertibility of $D\Phi_{\xi}(0, \overline{\xi})$ by Lemma 2.

Therefore, according the implicit function theorem, there exist positive constants $r_{\overline{\xi}}, \delta_{\overline{\xi}}$ such that, for every $-r_{\overline{\xi}} \le \lambda \le r_{\overline{\xi}}$, there is a unique point $\xi = \xi(\lambda) \in B_{\delta_{\overline{\xi}}}(\overline{\xi})$ with $\Phi(\lambda, \xi(\lambda)) = 0$.

To complete the proof of (i), it only needs to prove that there exist two positive constants r_0, δ_0 which are independent of $\overline{\xi} \in \Gamma$ such that the conclusion is satisfied in (i). From the proof of the implicit function theorem, for the given $\overline{\xi} \in \Gamma$, the constants $r_{\overline{\xi}}$ and $\delta_{\overline{\xi}}$ above are chosen such that, for $-r_{\overline{\xi}} \le \lambda \le r_{\overline{\xi}}$ and $\xi \in B_{\delta_{\overline{\xi}}}(\overline{\xi})$, we have

$$\begin{gathered} \left\|\left(D\Phi_{\xi}(\lambda, \xi)\right) - \left(D\Phi_{\xi}(0, \overline{\xi})\right)\right\| \le \frac{1}{2M_{\overline{\xi}}}, \\ \|\Phi(\lambda, \overline{\xi})\| \le \frac{\delta_{\overline{\xi}}}{2M_{\overline{\xi}}}. \end{gathered} \tag{15}$$

Here, $M_{\overline{\xi}}$ is the constant such that $\|(D\Phi_{\xi}(0, \overline{\xi}))^{-1}\| \le M_{\overline{\xi}}$.

We now give the above estimates which are independent of $\overline{\xi} \in \Gamma$. Firstly, we have, for any $\overline{\xi} \in \Gamma$,

$$\left\|\left(D\Phi_{\xi}(0, \overline{\xi})\right)^{-1}\right\| \le \frac{1}{\left\|L_1(\overline{\xi})^{-1}\right\|^{-1} - \left\|L_2(\overline{\xi})\right\|} \le \frac{1}{b - b_{1m}} \triangleq M, \tag{16}$$

where b is given by (8). Secondly, by assumption $(G1)$, there exists δ_1 such that

$$\max\left\{\left|C_{11}\right|, \left|C_{21}\right|\right\}\left|f_1'(x) - f_1'\left(\widehat{x}^{q1}\right)\right| \le \frac{1}{8M}, \tag{17}$$

for $x \in B_{\delta_1}(\widehat{x}^{q1})$,

$$\max\left\{\left|C_{11}\right|, \left|C_{21}\right|\right\}\left|f_1'(x) - f_1'\left(\widehat{x}^{q2}\right)\right| \le \frac{1}{8M}, \tag{18}$$

for $x \in B_{\delta_1}(\widehat{x}^{q2})$, and

$$\max\left\{\left|c_{ii-1}\right|, \left|c_{i-1i-1}\right|\right\}\left|f_{i-1}'(x) - f_{i-1}'\left(\overline{x}^{i-1}\right)\right| \le \frac{1}{4M}, \tag{19}$$

for $x \in B_{\delta_1}(\overline{x}^{i-1})$, $1 \le i \ne 2 \le m$. Note that

$$\begin{cases} \left(D\Phi_{\xi}(\lambda, \xi) - D\Phi_{\xi}(0, \overline{\xi})\xi\right)_{m(n+1)+1} \\ \quad = \lambda\left(-\xi_{m(n+1)+1} + \beta_1\xi_{m(n+1)+1}\right) \\ + C_{11}\left(f_1'\left(\xi_{m(n-k_{11})+1}\right) - f_1'\left(\overline{\xi}_{m(n-k_{11})+1}\right)\xi_{m(n-k_{11})+1}\right) \\ + C_{1m}\left(f_m'\left(\xi_{m(n-k_{1m})+m}\right) - f_m'\left(\overline{\xi}_{m(n-k_{1m})+m}\right)\xi_{m(n-k_{1m})+m}\right), \\ \left(D\Phi_{\xi}(\lambda, \xi) - D\Phi_{\xi}(0, \overline{\xi})\xi\right)_{m(n+1)+i} \\ \quad = \lambda\left(-\xi_{m(n+1)+i} + \beta_i\xi_{m(n+1)+i}\right) \\ + C_{ii-1}\left(f_{i-1}'\left(\xi_{m((n-k_{ii-1})+i-1}\right) - f_1'\left(\overline{\xi}_{m((n)-k_{ii-1})+i-1}\right) \right. \\ \quad \left. \xi_{m((n-k_{ii-1})+i-1}\right) \\ + C_{ii}\left(f_i'\left(\xi_{m(n-k_{ii})+i}\right) - f_i'\left(\overline{\xi}_{m(n-k_{ii})+i}\right)\right)\xi_{m(n-k_{ii})+i}, \quad 2 \le i \le m. \end{cases} \tag{20}$$

Taking $\delta_0 = \delta_1, r_1 = (1/4M(1+a))$, where $a \triangleq \max\{\beta_i | i = 1, 2, \ldots, m\}$, we have that, for $\overline{\xi} \in \Gamma$, $\xi \in l_{\infty}$ with $\|\xi - \overline{\xi}\| \le \delta_0$ and $|\lambda| \le r_1$:

$$\left\|D\Phi_{\xi}(\lambda, \xi) - D\Phi_{\xi}(0, \overline{\xi})\right\| \le |\lambda|(1+a) + \frac{1}{4M} \le \frac{1}{2M}. \tag{21}$$

On the contrary, let $r_2 = (\delta_0/2M(1+a))$, and it follows from the definition of $\Phi(\lambda, \cdot)$ that

$$\|\Phi(\lambda, \overline{\xi})\| \le |\lambda|(1+b) \le \frac{\delta_0}{2M}, \tag{22}$$

when $|\lambda| \le r_2$.

Finally, take $r_0 = \min\{r_1, r_2\}$ and then the constants r_0 and δ_0 satisfy (i).

For every $0 < \delta < \delta_0$, (ii) follows by taking $r = \min\{(1/4M(1+a)), (\delta/2M(1+a))\}(< r_0)$ and the proof of (i). □

3. Chaos in System (1)

In this section, we shall show that the system (1) exists chaotic behaviors. By Lemma 4, for sufficiently large $\alpha > 0$, we define the map T_α from Γ to l_∞ by

$$T_\alpha(\overline{\xi}) = \xi\left(\frac{1}{\alpha}\right), \tag{23}$$

where $\xi(1/\alpha)$ is the unique solution of $\Phi((1/\alpha), \xi) = 0$, satisfying $\|\xi(1/\alpha) - \overline{\xi}\| \leq \delta$. Then, we have the following proposition.

Proposition 1. *For sufficiently large $\alpha > 0$, let $\Gamma_\alpha = T_\alpha(\Gamma)$, then the map T_α and the shift map σ^m are commutative, i.e.,*

$$\sigma^m \circ T_\alpha = T_\alpha \circ \sigma^m. \tag{24}$$

Moreover, $\sigma^m(\Gamma_\alpha) = \Gamma_\alpha$.

Proof. Note that if ξ is a solution of $\Phi((1/\alpha), \xi) = 0$ so is $\sigma^m(\xi)$. Thus, for any $\overline{\xi} \in \Gamma, \sigma^m{}^\circ T_\alpha(\overline{\xi}) = \sigma^m(\xi(1/\alpha))$ is a solution of $\Phi((1/\alpha), \xi) = 0$. On the contrary, $\|\xi(1/\alpha) - \overline{\xi}\| \leq \delta$ by Lemma 4, which leads to $\|\sigma^m(T_\alpha(\overline{\xi}) - \sigma^m(\overline{\xi}))\| = \|\sigma^m(\xi(1/\alpha)) - \sigma^m(\overline{\xi})\| = \|\xi(1/\alpha) - \overline{\xi}\| \leq \delta$. Hence, by the uniqueness of $\xi(\lambda)$ in Lemma 4, we have $\sigma^m(T_\alpha(\overline{\xi})) = T_\alpha(\sigma^m(\overline{\xi}))$. Note that $\sigma^m(\Gamma) = \Gamma$, it follows that $\sigma^m(\Gamma_\alpha) = \Gamma_\alpha$.

For every $k \in \mathbb{Z}$, we define the projection $\Pi_k\colon\, l_\infty \longrightarrow \mathbb{R}^p$ by

$$\Pi_k(\xi) = \eta(k), \quad \forall \xi \in l_\infty, \tag{25}$$

where for $\xi = (\xi_n) \in l_\infty$, $\eta(k) = (\eta_1(k), \ldots, \eta_p(k)) \in \mathbb{R}^p$ is given by

$$\begin{aligned}
&\eta_{mj+i}(k) = \xi_{m(k-k_{ii}+j)+i}, \quad 0 \leq j \leq k_{11},\ 1 \leq i \leq m,\\
&\eta_{p_1+j(m-1)+i}(k) = \xi_{m(k-k_{ii}+k_{11}+j)+i}, \quad 1 \leq j \leq k_{22} - k_{11},\ 2 \leq i \leq m,\\
&\eta_{p_2+j(m-2)+i}(k) = \xi_{m(k-k_{ii}+k_{22}+j)+i}, \quad 1 \leq j \leq k_{33} - k_{22},\ 3 \leq i \leq m,\\
&\quad\vdots\\
&\eta_{p_{m-1}+m+j}(k) = \xi_{m(k-k_{mm}+k_{m-1m-1}+j)+m}, \quad 1 \leq j \leq k_{mm} - k_{m-1m-1}.
\end{aligned} \tag{26}$$

□

Proposition 2. *Let $\Lambda_\alpha = \Pi_0(\Gamma_\alpha)$, then Λ_α is invariant for F_α.*

Proof. For each $\eta(0) \in \Lambda_\alpha$, then there exists $\xi \in \Gamma_\alpha$ such that $\Pi_0(\xi) = \eta(0)$. Therefore,

$$F_\alpha(\eta(0)) = \eta(1) = \Pi_0(\sigma^m(\xi)) \in \Pi_0(\sigma^m(\Gamma_\alpha)) = \Pi_0(\Gamma_\alpha) = \Lambda_\alpha. \tag{27}$$

This proves $F_\alpha(\Lambda_\alpha) \subset \Lambda_\alpha$.

On the contrary, by Proposition 1, we have $\sigma^m(\Gamma_\alpha) = \Gamma_\alpha$. Thus, there exists $\xi' \in \Gamma_\alpha$ such that $\xi = \sigma^m(\xi')$. Thus,

$$\begin{aligned}
\eta(0) = \Pi_0(\xi) = \Pi_0(\sigma^m(\xi')) = \eta'(1) = F_\alpha(\eta'(0))\\
= F_\alpha(\Pi_0(\xi')) \in F_\alpha(\Lambda_\alpha),
\end{aligned} \tag{28}$$

which shows that $\Lambda_\alpha \subset F_\alpha(\Lambda_\alpha)$. Therefore, $F_\alpha(\Lambda_\alpha) = \Lambda_\alpha$. □

Theorem 1. *Under the assumption of* (G1), *if $b_{11} > b_{1m}$ and $b_{ii} > b_{1m} + b_{ii-1}\ (2 \leq i \leq m)$, then there exists $\alpha_0 > 0$ such that, for any $\alpha > \alpha_0$, $(\Lambda_\alpha, F_\alpha)$ is topologically conjugate to the full shift map (Σ_2, σ), and therefore, the system is chaotic in the sense of Devaney.*

Proof. Note that (Γ, σ^m) is an invariant subsystem. By Lemma 1 and Proposition 1, we only need to prove that there is $\alpha_0 > 0$ such that, for any $\alpha > \alpha_0$, $(\Lambda_\alpha, F_\alpha)$ is topological conjugate to (Γ, σ^m).

Let $\Omega = \Pi_0(\Gamma)$, then Ω is a set in $\mathbb{R}^p$ consisting of $2^{k_{11}+1}$ elements, denoted by

$$\Omega = \{b_1, b_2, \ldots, b_{2^{k_{11}+1}}\}. \tag{29}$$

Let δ_0 and r_0 be given as in Lemma 4, and let $\delta \in (0, \delta_0)$ be small enough such that the family of closed balls $\{A_i = \overline{B}(b_i, \delta)\}_{i=1}^{2^{k_{11}+1}}$ in $\mathbb{R}^p$ is piecewise disjoint.

For the given δ and any $\overline{\xi} = (\overline{\xi}_n) \in \Gamma$, by (ii) in Lemma 4, there exists an $\alpha_0 = (1/r) > 0$ such that, for every $\alpha > \alpha_0$, there exists a unique $T_\alpha(\overline{\xi}) = \xi(1/\alpha)$ satisfying $\|\xi(1/\alpha) - \overline{\xi}\| \leq \delta$ and $\Phi((1/\alpha), \xi(1/\alpha)) = 0$. By the definition of the projections Π_k and Γ, we have $\Pi_k(\Gamma) = \Pi_0(\Gamma) = \Omega$. So we let

$$\begin{aligned}
S = \Big\{&s = (\ldots, s_{-1}, s_0, s_1, \ldots) | s_i \in \left\{1, 2, \ldots, 2^{k_{11}+1}\right\},\\
&\xi_{s_i} = \Pi_i(\overline{\xi}), \quad \text{for some } \overline{\xi} \in \Gamma\Big\}.
\end{aligned} \tag{30}$$

The set S is a subset of $\Sigma_{2^{k_{11}+1}}$. For every $s = (\ldots, s_{-1}, s_0, s_1, \ldots) \in S$, for all $i, j > 0$, we set

$$\begin{aligned}
V_{s_{-i}\ldots s_0\ldots s_j} &= F_\alpha^{-j}\left(A_{s_j}\right) \cap \cdots \cap A_{s_0} \cap \cdots \cap F_\alpha^i\left(A_{s_i}\right),\\
V_s &= \bigcap_{i>0, j>0} V_{s_{-i}\ldots s_0\ldots s_j}.
\end{aligned} \tag{31}$$

We claim the following:

(1) For every $s \in S$, V_s contains a unique point.

(2) $\cup_{s \in S} V_s = \Lambda_\alpha$.

In fact, for each $s \in S$, we note that

$$V_{s_{-i}\ldots s_0\ldots s_j} = \left\{\eta \in \mathbb{R}^p | F_\alpha^{-i}(\eta) \in A_{s_i}, \ldots, F_\alpha^j(\eta) \in A_{s_j}\right\}. \tag{32}$$

Therefore, there exists a unique $\overline{\xi} \in \Gamma$ such that, for all $i \in \mathbb{Z}$, $\Pi_i(\overline{\xi}) = \xi_{s_i} \in \Omega$. Then, by the definition of T_α and Lemma 4, there exists a unique $T_\alpha(\overline{\xi}) = \xi(1/\alpha)$, satisfying $\|\xi(1/\alpha) - \overline{\xi}\| \leq \delta$ and $\Phi((1/\alpha), \xi(1/\alpha)) = 0$. So $\{\Pi_n(\xi(1/\alpha)) = \eta(n)\}_{n\in\mathbb{Z}}$ is a bounded orbit of F_α, that is, $\eta(n) = F_\alpha^n(\eta(0)) \in A_{s_n}$, $\forall n \in \mathbb{Z}$. Therefore, $\eta(0) \in V_s$, which implies V_s is nonempty.

On the contrary, for any $\eta' \in V_s$, for all $n \in \mathbb{Z}$, there are $F_\alpha^n(\eta') \in A_{s_n}$. Thus, $\{F_\alpha^n(\eta')\}_{n\in\mathbb{Z}}$ is a bounded orbit of F_α. Then, there exists $\xi \in l_\infty$ such that $\Pi_n(\xi) = F_\alpha^n(\eta')$. So $\|\xi - \overline{\xi}\| \le \delta$, and $\Phi((1/\alpha), \xi) = 0$. Again by Lemma 4 (ii), there is $\xi = T_\alpha(\overline{\xi})$, and hence, $\eta' = \eta(0)$. Claim (1) holds.

For Claim (2), let $\eta \in \Lambda_\alpha$. Then, there exists a $\overline{\xi} \in \Gamma$ such that $\eta = \Pi_0(T_\alpha(\overline{\xi}))$ Let $s = (\ldots s_{-1}, s_0, s_1 \ldots) \in S$ be the corresponding sequence of $\overline{\xi}$. Similar to the above argument, we have $\eta \in V_s$. Therefore,

$$\Lambda_\alpha \subset \bigcup_{s\in S} V_s. \tag{33}$$

From Claim (1), each V_s contains a unique point which belongs to Λ_α, so the converse inclusion holds. This proves Claim (2).

For every $\alpha > \alpha_0$, define a map $h: \Gamma \longrightarrow \Lambda_\alpha$ by $h = \Pi_0 \circ T_\alpha$. We claim that h is a conjugacy from σ^m to F_α. To prove this, we need to show that both h and h^{-1} are continuous and

$$h \circ \sigma^m = F_\alpha \circ h, \quad \text{on } \Gamma. \tag{34}$$

By Claim (2) and the definition of h, it is easy see that h is surjective. From Claim (1) and Lemma 4, it follows that h is injective. Therefore, h is bijective. Since h is a map from a compact metric space Γ to a Hausdorff space Λ_α, to prove that h is homeomorphic, we just need to show the continuity of h. Let the corresponding subindex sequence of $\overline{\xi} \in \Gamma$ be $\overline{s} = (\ldots s_{-1}, s_0, s_1 \ldots) \in S$. It follows from Claim (1) that

$$\lim_{i,j\longrightarrow+\infty} \operatorname{diam}\left(V_{s_{-i}\ldots s_0\ldots s_j}\right) = 0, \tag{35}$$

where $\operatorname{diam}(V_{s_{-i}\ldots s_0\ldots s_j})$ denotes the diameter of the set $V_{s_{-i}\ldots s_0\ldots s_j}$. Thus, for any $\varepsilon > 0$, there exists a positive integer n such that $\operatorname{diam}(V_{\overline{s}_{-n}\ldots\overline{s}_0\ldots\overline{s}_n}) < \varepsilon$. Take $\delta_1 = (1/2^{m(n+k_{mm}+1)})$. Then, for any $\widetilde{\xi} \in \Gamma$ with $\widetilde{d}(\widetilde{\xi}, \overline{\xi}) < \delta_1$, it follows that $\widetilde{\xi}$ agrees with $\overline{\xi}$ in those terms with lower indices from $i = -m(n + k_{mm} + 1)$ to $i = m(n + k_{mm} + 1)$. Let $\widehat{s}, \overline{s} \in S$ be the symbolic sequences corresponding to $\widehat{\xi}$ and $\overline{\xi}$, respectively. We have $\widehat{s}$ agrees with $\overline{s}$ in those terms with subscripts from $i = -n$ to $i = n + k_{mm} + 1$. Thus, $h(\widehat{\xi}), h(\overline{\xi}) \in V_{\overline{s}_{-n}\ldots\overline{s}_0\ldots\overline{s}_n}$ and $\|h(\widehat{\xi}) - h(\overline{\xi})\| < \varepsilon$. This shows the continuity of h. Hence, we conclude that h is a homeomorphism.

Finally, for any $\overline{\xi} \in \Gamma$, we have

$$h(\overline{\xi}) = \Pi_0 \circ T_\alpha(\overline{\xi}) = \eta(0) = \left(\eta_1(0), \ldots, \eta_q(0)\right)^T. \tag{36}$$

Thus,

$$\begin{aligned} F_\alpha(h(\overline{\xi})) &= \left(\eta_1(1), \eta_2(1), \ldots, \eta_q(1)\right)^T \\ &= \Pi_0 \circ \sigma^m \circ T_\alpha(\overline{\xi}) \text{ by } (1) \\ &= \Pi_0 \circ T_\alpha \circ \sigma^m(\overline{\xi}), \text{ by Proposition 1} \\ &= h \circ \sigma^m(\overline{\xi}). \end{aligned} \tag{37}$$

The Theorem 1 holds. ☐

4. Some Simulations

In this section, we will give some numerical simulation results to verify our theoretical results. We choose $\beta_1 = \beta_3 = (1/4), \beta_2 = \beta_4 = (3/4)3/4, f_1(t) = \sin\quad(t), f_2(t) = \tanh(t), f_3(t) = \cos(t), f_4(t) = \tanh(t), \alpha_{11} = 0.5\alpha, \alpha_{14} = \alpha, \alpha_{21} = -0.4\alpha, \alpha_{22} = \quad 2\alpha, k_{11} = 1, k_{21} = 2, k_{14} = 3, k_{22} = 4, k_{32} = 3, k_{33} = 1, k_{43} = 2$, and $k_{44} = 4$. In this case, system (1) becomes

$$\begin{cases} x_1(n+1) = \dfrac{1}{4}x_1(n) + 1.5\alpha\sin\left(x_1(n-1)\right) + \alpha\tanh\left(x_4(n-3)\right), \\ x_2(n+1) = \dfrac{3}{4}x_2(n) - 0.4\alpha\sin\left(x_1(n-2)\right) + 2\alpha\tanh\left(x_2(n-4)\right), \\ x_3(n+1) = \dfrac{1}{4}x_3(n) + \alpha\tanh\left(x_2(n-3)\right) + 1.5\alpha\cos\left(x_3(n-1)\right), \\ x_4(n+1) = \dfrac{3}{4}x_4(n) - 0.4\alpha\cos\left(x_3(n-2)\right) + 2\alpha\tanh\left(x_4(n-4)\right), \end{cases} \quad \forall n \ge 5. \tag{38}$$

In Figure 1, for every α value, the initial values were reset to $x_1(1) = -0.1, x_1(2) = 0.1, x_1(3) = 0.12, x_1(4) = -0.2, x_1(5) = 0.9, \quad x_2(1) = 0.11, x_2(2) = -0.2, x_2(3) = -0.1, x_2(4) = 0.2, x_2(5) = 0.1, x_3(1) = 0.12, x_3(2) = 0.15, x_3(3) = -0.2, x_3(4) = 0.22, x_3(5) = 1.1$, and $x_4(1) = -0.1, x_4(2) = -0.23, x_4(3) = -0.1, x_4(4) = 0.2, x_4\quad(5) = 0.11$. After 10^4 time steps being iterated, we plot the data consisting of 500 points for per α value. The plotting is for x_1, x_3 vs the parameter α. The bifurcation figures illustrate that the fixed point of x_1 loses stability and period bifurcation occurs when $\alpha \approx 0.95$, and the fixed point of x_3 loses stability and period bifurcation occurs when $\alpha \approx 1.1$. Making the bifurcation figures of the x_2 vs α and the x_4 vs α similar, they are omitted.

In Figure 2, we show the largest Lyapunov exponent diagram for $\alpha \in [0, 6]$. For every α value, the initial values were the same as Figure 1. From the simulation results in Figure 2, we can find that the largest Lyapunov exponent is negative when $\alpha \in (0, 1)$ and is positive when $\alpha > 2.8$. Thus,

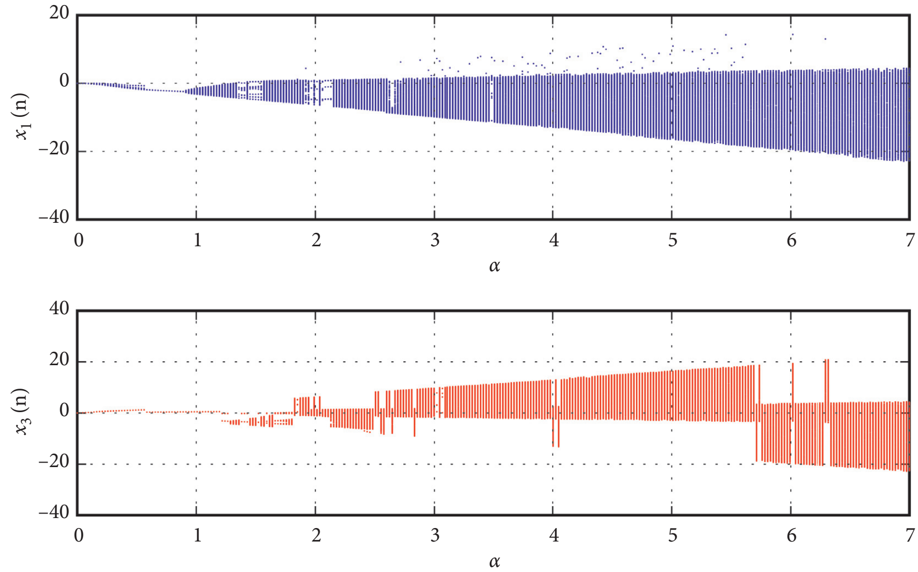

FIGURE 1: Bifurcation diagram.

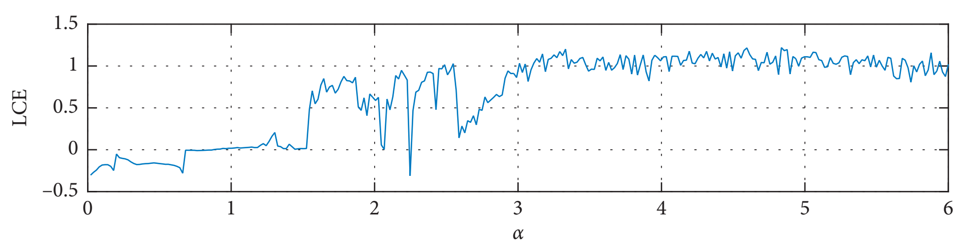

FIGURE 2: Largest Lyapunov exponents diagram.

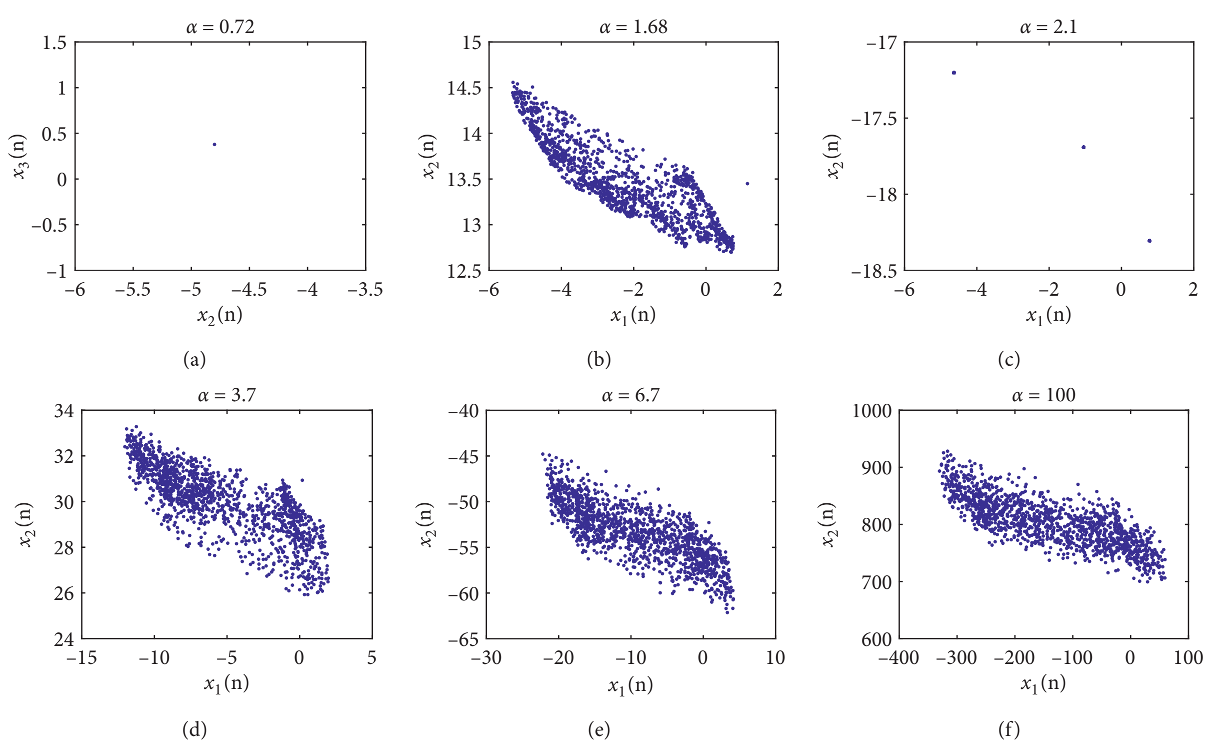

FIGURE 3: Continued.

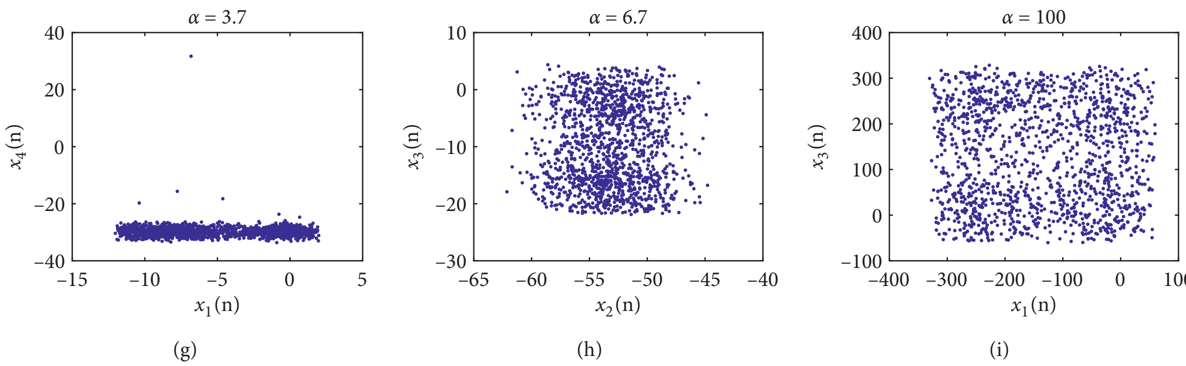

Figure 3: Chaos diagram.

the figures illustrate that the system (38) has chaotic behaviors when α is large enough.

In Figure 3, we show the chaotic figures. For each α value, after 6×10^5 time steps being iterated, plot the 6000 data points. The figure illustrates that there are no chaos for small α (e.g., $\alpha = 0.72, 2.1$) and chaotic behavior occurs when α is larger (e.g., $\alpha = 1.68, 3.0, 6.7, 100$). Those numerical simulations support the theoretical results in Section 2.

5. Conclusion

In this paper, the chaos of a discrete neural network loops with self-feedback is studied. The discrete neural network loops with multiple delays and self-feedback can demonstrate chaotic behavior when the interconnection strengths are large enough. Numerical simulations support the theoretical results. The theoretical results are to provide some new methods for the design of chaotic neural networks.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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