Research Article

Global Dynamical Properties of Rational Higher-Order System of Difference Equations

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In this paper, global dynamical properties of rational higher-order system are explored in the interior of $\mathbb{R}_+^3$. It is explored that under certain parametric conditions, the discrete-time system has at most eight equilibria. By the method of linearization, local dynamics has been explored. It is explored that positive solution of the system is bounded, and moreover fixed point P_{000} is globally stable if $(\alpha_1/\alpha_2) < 1$, $(\alpha_4/\alpha_5) < 1$, $(\alpha_7/\alpha_8) < 1$. It is also investigated that the positive solution of the system under consideration converges to P_{000} . Lastly, theoretical results are confirmed by numerical simulation. The presented work is significantly extended and improves current results in the literature.

1. Introduction

It is a well-known fact that difference equations arise naturally as discrete analogues and as numerical solutions of differential as well as delay differential equations having applications in many fields like physics, biology, economy, and ecology. Recently, a lot of studies have been conducted concerning the global dynamics of difference equations and their systems [3–19]. It is really not easy to understand global dynamics of difference equations along their systems; particularly, investigating the global behavior of higher-order equations is a challenging job in recent years. Therefore, investigating the global dynamics of such difference equations along their systems is worth further consideration. For illustration, Gibbons et al. [20] explored global dynamics of the following equation:

$$x_{n+1} = \frac{\alpha_1 + \alpha_2 x_{n-1}}{\alpha_3 + x_n}, \quad (1)$$

where α_ℓ ($\ell = 1, 2, 3$) and $x_{-\ell}$ ($\ell = 1, 0$) are positive constants. Çinar [21] investigated the dynamics of the following equations:

$$x_{n+1} = \frac{x_{n-1}}{1 + \prod_{i=0}^1 x_{n-i}}, \quad (2)$$

where $x_{-\ell}$ ($\ell = 1, 0$) are positive constants. Shojaei et al. [22] investigated global dynamics of the following difference equations:

$$x_{n+1} = \frac{\alpha_1 x_{n-2}}{\alpha_2 + \alpha_3 \prod_{i=0}^2 x_{n-i}}, \quad (3)$$

where α_ℓ ($\ell = 1, 2, 3$) and $x_{-\ell}$ ($\ell = 2, 1, 0$) are positive constants. Bajo and Liz [23] investigated the dynamics of the following difference equation:

$$x_{n+1} = \frac{x_{n-1}}{\alpha_1 + \alpha_2 \prod_{i=0}^1 x_{n-i}}, \quad (4)$$

where α_ℓ ($\ell = 1, 2$) and $x_{-\ell}$ ($\ell = 1, 0$) are positive constants. Zhang et al. [24] have extended the work explored by numerous authors [21–23] to investigate the dynamics of the following rational system:

$$\begin{aligned} x_{n+1} &= \frac{x_{n-2}}{\alpha_2 + \prod_{i=0}^2 y_{n-i}}, \\ y_{n+1} &= \frac{y_{n-1}}{\alpha_1 + \prod_{i=0}^2 x_{n-i}}, \end{aligned} \quad (5)$$

where α_ℓ ($\ell = 1, 2$) and $x_\ell, y_{-\ell}$ ($\ell = 2, 1, 0$) are positive constants. Recently, Qureshi and Khan [25] investigated the global dynamics of the following rational system, which is extension of the work [21–24]:

$$\begin{aligned} x_{n+1} &= \frac{\alpha_1 x_{n-1}}{\alpha_2 + \alpha_3 \prod_{i=0}^1 y_{n-i}}, \\ y_{n+1} &= \frac{\alpha_4 y_{n-1}}{\alpha_5 + \alpha_6 \prod_{i=0}^1 z_{n-i}}, \\ z_{n+1} &= \frac{\alpha_7 z_{n-1}}{\alpha_8 + \alpha_9 \prod_{i=0}^1 x_{n-i}}, \end{aligned} \quad (6)$$

where α_ℓ ($\ell = 1, 2, \dots, 9$) and $x_{-\ell}, y_{-\ell}, z_{-\ell}$ ($\ell = 1, 0$) are positive constants. Inspired from aforesaid studies, we will extend the work studied by numerous authors [21–25] to investigate global dynamics of the following $3k + 3$ -dimensional system:

$$\begin{aligned} x_{n+1} &= \frac{\alpha_1 x_{n-k}}{\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i}}, \\ y_{n+1} &= \frac{\alpha_4 y_{n-k}}{\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i}}, \\ z_{n+1} &= \frac{\alpha_7 z_{n-k}}{\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i}}, \end{aligned} \quad (7)$$

where α_ℓ ($\ell = 1, 2, \dots, 9$) and $x_{-\ell}, y_{-\ell}, z_{-\ell}$ ($\ell = k, -k+1, \dots, 1, 0$) are positive constants.

The organization of this paper is as follows. In Section 2, existence of equilibria in $\mathbb{R}_+^3$ and corresponding linearized form are investigated. Section 3 deals with the study of local dynamics about equilibrium points. Boundedness of positive solution for the discrete-time system is studied in Section 4. Further, global dynamics about P_{000} is explored in Section 5. In Section 6, we studied the rate of convergence which converges to P_{000} of the system. Theoretical results are numerically verified in Section 7, while concluding remarks are given in Section 8.

2. Equilibria and Linearized Form of System (7)

The existence of equilibrium solution in the interior of $\mathbb{R}_+^3$ and linearized form about $\Lambda = (x, y, z)$ of system (7) are investigated in this section. So, existence of equilibrium solution can be summarized as the following lemma.

Lemma 1. *In the interior of $\mathbb{R}_+^3$, discrete-time system (7) has at most eight equilibria. More precisely,*

- (i) $\forall \alpha_\ell$ ($\ell = 1, 2, \dots, 9$), $P_{000}(0, 0, 0)$ is the unique boundary point of discrete-time system (7).
- (ii) $P_{x0z}(\sqrt{[k+1]}(\alpha_7 - \alpha_8/\alpha_9), 0, \sqrt{[k+1]}(\alpha_4 - \alpha_5/\alpha_6))$ is the boundary equilibrium point of system (7) if $\alpha_1 = \alpha_2$, $\alpha_4 > \alpha_5$ and $\alpha_7 > \alpha_8$.
- (iii) $P_{xy0}(\sqrt{[k+1]}(\alpha_7 - \alpha_8/\alpha_9), \sqrt{[k+1]}(\alpha_1 - \alpha_2/\alpha_3), 0)$ is the boundary equilibrium point of system (7) if $\alpha_1 > \alpha_2$, $\alpha_4 = \alpha_5$ and $\alpha_7 > \alpha_8$.
- (iv) $P_{0yz}(0, \sqrt{[k+1]}(\alpha_1 - \alpha_2/\alpha_3), \sqrt{[k+1]}(\alpha_4 - \alpha_5/\alpha_6))$ is the boundary equilibrium point of system (7) if $\alpha_1 > \alpha_2$, $\alpha_4 > \alpha_5$ and $\alpha_7 = \alpha_8$.
- (v) $P_{x00}(\sqrt{[k+1]}(\alpha_7 - \alpha_8/\alpha_9), 0, 0)$ is the boundary equilibrium point of discrete-time system (7) if $\alpha_1 = \alpha_2$, $\alpha_4 = \alpha_5$ and $\alpha_7 > \alpha_8$.
- (vi) $P_{00z}(0, 0, \sqrt{[k+1]}(\alpha_4 - \alpha_5/\alpha_6))$ is the boundary equilibrium point of system (7) if $\alpha_1 = \alpha_2$, $\alpha_4 > \alpha_5$ and $\alpha_7 = \alpha_8$.
- (vii) $P_{0y0}(0, \sqrt{[k+1]}(\alpha_1 - \alpha_2/\alpha_3), 0)$ is the boundary equilibrium point of discrete-time system (7) if $\alpha_1 > \alpha_2$, $\alpha_4 = \alpha_5$ and $\alpha_7 = \alpha_8$.
- (viii) $P_{xyz}(\sqrt{[k+1]}(\alpha_7 - \alpha_8/\alpha_9), \sqrt{[k+1]}(\alpha_1 - \alpha_2/\alpha_3), \sqrt{[k+1]}(\alpha_4 - \alpha_5/\alpha_6))$ is the unique positive equilibrium point of system (7) if $\alpha_1 > \alpha_2$, $\alpha_4 > \alpha_5$ and $\alpha_7 > \alpha_8$.

Hereafter, we establish the corresponding linearized form of (7). For this, one has the following map in order to construct the corresponding linearized form:

$$\begin{aligned} &(\beta, \beta_n, \dots, \beta_{n-k}, \gamma, \gamma_n, \dots, \gamma_{n-k}, \delta, \delta_n, \dots, \delta_{n-k}) \\ &\mapsto (x_{n+1}, x_n, \dots, x_{n-k+1}, y_{n+1}, y_n, \dots, y_{n-k+1}, z_{n+1}, z_n, \dots, z_{n-k+1}), \end{aligned} \quad (8)$$

where

$$\begin{aligned} \beta &= \frac{\alpha_1 x_{n-k}}{\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i}}, \\ \beta_n &= x_n, \dots, \beta_{n-k} = x_{n-k+1}, \\ \gamma &= \frac{\alpha_4 y_{n-k}}{\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i}}, \end{aligned} \quad (9)$$

$$\gamma_n = y_n, \dots, \gamma_{n-k} = y_{n-k+1},$$

$$\delta = \frac{\alpha_7 z_{n-k}}{\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i}},$$

$$\delta_n = z_n, \dots, \delta_{n-k} = z_{n-k+1}.$$

Finally, $J|_\Lambda$ about Λ under map (8) becomes

$$J|_{\Lambda} = \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\alpha_1}{\alpha_2 + \alpha_3 y^{k+1}} \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \frac{\alpha_7 \alpha_9 x^k z}{(\alpha_8 + \alpha_9 x^{k+1})^2} & \frac{\alpha_7 \alpha_9 x^k z}{(\alpha_8 + \alpha_9 x^{k+1})^2} & \dots & \frac{\alpha_7 \alpha_9 x^k z}{(\alpha_8 + \alpha_9 x^{k+1})^2} & \frac{\alpha_7 \alpha_9 x^k z}{(\alpha_8 + \alpha_9 x^{k+1})^2} \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \frac{\alpha_1 \alpha_3 x y^k}{(\alpha_2 + \alpha_3 y^{k+1})^2} & \frac{\alpha_1 \alpha_3 x y^k}{(\alpha_2 + \alpha_3 y^{k+1})^2} & \dots & \frac{\alpha_1 \alpha_3 x y^k}{(\alpha_2 + \alpha_3 y^{k+1})^2} & \frac{\alpha_1 \alpha_3 x y^k}{(\alpha_2 + \alpha_3 y^{k+1})^2} \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{\alpha_4}{\alpha_5 + \alpha_6 z^{k+1}} \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \frac{\alpha_4 \alpha_6 y z^k}{(\alpha_5 + \alpha_6 z^{k+1})^2} & \frac{\alpha_4 \alpha_6 y z^k}{(\alpha_5 + \alpha_6 z^{k+1})^2} & \dots & \frac{\alpha_4 \alpha_6 y z^k}{(\alpha_5 + \alpha_6 z^{k+1})^2} & \frac{\alpha_4 \alpha_6 y z^k}{(\alpha_5 + \alpha_6 z^{k+1})^2} \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \frac{\alpha_7}{\alpha_8 + \alpha_9 x^{k+1}} \\ 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \quad (10)$$

3. Local Dynamics about Equilibria

By Theorem 1.5 of [1], the detailed local stability analysis about boundary equilibria P_{000} , P_{x0z} , P_{xy0} , P_{0yz} , P_{x00} , P_{00z} , and P_{0y0} and the positive equilibrium point P_{xyz} will be investigated in this section.

3.1. Local Dynamics about Boundary Points

Theorem 1. P_{000} of system (7) is a sink if

$$\frac{\alpha_1}{\alpha_2} < 1, \frac{\alpha_4}{\alpha_5} < 1, \frac{\alpha_7}{\alpha_8} < 1. \quad (11)$$

Proof. About the equilibrium point P_{000} , (10) takes the following form:

$$J|_{P_{000}} = \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\alpha_1}{\alpha_2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{\alpha_4}{\alpha_5} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{\alpha_7}{\alpha_8} \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \end{pmatrix}. \quad (12)$$

Now, if $\gamma_1, \gamma_2, \dots, \gamma_{k+1}, \dots, \gamma_{2k+1}, \dots, \gamma_{3k+3}$ denotes characteristic roots of P_{000} , the diagonal matrix $\Omega = \text{diag}(\rho_1, \rho_2, \dots, \rho_{k+1}, \dots, \rho_{2k+1}, \dots, \rho_{3k+3})$, where

$$\rho_1 = \rho_{k+2} = \rho_{2k+3} = 1,$$

$$\rho_{1+\zeta} = \rho_{k+2+\zeta} = \rho_{2k+3+\zeta} = 1 - \zeta\varepsilon,$$

$$1 \leq \zeta \leq k,$$

$$0 < \varepsilon < 1,$$

(13)

$$0 < \varepsilon < \min \left\{ \frac{1}{k} \left(1 - \frac{\alpha_1}{\alpha_2} \right), \frac{1}{k} \left(1 - \frac{\alpha_4}{\alpha_5} \right), \frac{1}{k} \left(1 - \frac{\alpha_7}{\alpha_8} \right) \right\}. \quad (14)$$

Now,

$$\Omega J|_{P_{000}} \Omega^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & \rho_1 \rho_{k+1}^{-1} \frac{\alpha_1}{\alpha_2} & 0 & 0 & \dots & 0 & 0 \\ \rho_2 \rho_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \rho_{k+1} \rho_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \rho_{k+2} \rho_{2k+2}^{-1} \frac{\alpha_4}{\alpha_5} \\ 0 & 0 & \dots & 0 & 0 & \rho_{k+3} \rho_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \rho_{2k+2} \rho_{2k+1}^{-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \rho_{2k+3} \rho_{3k+3}^{-1} \frac{\alpha_7}{\alpha_8} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \rho_{3k+2} \rho_{3k+1}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \rho_{3k+3} \rho_{3k+2}^{-1} & 0 \end{pmatrix}. \quad (15)$$

So,

$$0 < \rho_{k+1} < \dots < \rho_2 < \rho_1,$$

$$0 < \rho_{2k+2} < \dots < \rho_{k+2}, \quad (16)$$

$$0 < \rho_{3k+3} < \dots < \rho_{2k+3}.$$

From (16), one obtains

$$\rho_2 \rho_1^{-1} < 1, \dots, \rho_{k+1} \rho_k^{-1} < 1,$$

$$\rho_{k+3} \rho_{k+2}^{-1} < 1, \dots, \rho_{2k+2} \rho_{2k+1}^{-1} < 1, \quad (17)$$

$$\rho_{2k+4} \rho_{2k+3}^{-1} < 1, \dots, \rho_{3k+3} \rho_{3k+2}^{-1} < 1.$$

From (13) and (14), one obtains

$$\rho_1 \rho_{k+1}^{-1} \frac{\alpha_1}{\alpha_2} = \frac{\alpha_1}{\alpha_2} \frac{1}{1 - k\varepsilon} < 1,$$

$$\rho_{k+2} \rho_{2k+2}^{-1} \frac{\alpha_4}{\alpha_5} = \frac{\alpha_4}{\alpha_5} \frac{1}{1 - k\varepsilon} < 1, \quad (18)$$

$$\rho_{2k+3} \rho_{3k+3}^{-1} \frac{\alpha_7}{\alpha_8} = \frac{\alpha_7}{\alpha_8} \frac{1}{1 - k\varepsilon} < 1.$$

In view of (17) and (18), one obtains

$$\begin{aligned} \max_{1 \leq m \leq 3k+3} |\rho_m| &= |\Omega J|_{P_{000}} \Omega^{-1} \\ &= \max \left\{ \rho_1 \rho_{k+1}^{-1} \frac{\alpha_1}{\alpha_2}, \rho_2 \rho_1^{-1}, \dots, \rho_{k+1} \rho_k^{-1}, \rho_{k+3} \rho_{k+2}^{-1}, \dots, \rho_{2k+2} \rho_{2k+1}^{-1}, \rho_{3k+2} \rho_{3k+1}^{-1}, \dots, \rho_{3k+3} \rho_{3k+2}^{-1}, \rho_{k+2} \rho_{2k+2}^{-1} \frac{\alpha_4}{\alpha_5}, \rho_{2k+3} \rho_{3k+3}^{-1} \frac{\alpha_7}{\alpha_8} \right\} < 1. \end{aligned} \quad (19)$$

From (19), one gets the required statement. $\square$

Theorem 2. P_{x0z} of (7) is unstable.

Proof. About the equilibrium point P_{x0z} , (10) takes the following form:

$$J|_{P_{x0z}} = \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\alpha_1}{\alpha_2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \Gamma & \Gamma & \dots & \Gamma & \Gamma & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}, \quad (20)$$

where

$$\Gamma = \frac{-\alpha_9}{\alpha_7} \sqrt{[k+1]} \frac{\alpha_4 - \alpha_5}{\alpha_6} \sqrt{\left[\frac{k}{k+1} \right]} \frac{\alpha_7 - \alpha_8}{\alpha_9}. \quad (21)$$

Now, if $\nu_1, \nu_2, \dots, \nu_{k+1}, \dots, \nu_{2k+1}, \dots, \nu_{3k+3}$ denotes characteristic roots of P_{x0z} , the diagonal matrix $\Omega = \text{diag}(\rho_1, \rho_2, \dots, \rho_{k+1}, \dots, \rho_{2k+1}, \dots, \rho_{3k+3})$, where (13) holds. Moreover,

$$\Psi = \begin{pmatrix} 0 & 0 & \dots & 0 & \rho_1 \rho_{k+1}^{-1} \frac{\alpha_1}{\alpha_2} & 0 & 0 & \dots & 0 & 0 \\ \rho_2 \rho_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \rho_{k+1} \rho_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \rho_{k+2} \rho_{2k+2}^{-1} \\ 0 & 0 & \dots & 0 & 0 & \rho_{k+3} \rho_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \rho_{2k+2} \rho_{2k+1}^{-1} & 0 \\ \rho_{2k+3} \rho_{k+1}^{-1} \Gamma & \rho_{2k+3} \rho_{k+2}^{-1} \Gamma & \dots & \rho_{2k+3} \rho_k^{-1} \Gamma & \rho_{2k+3} \rho_{k+1}^{-1} \Gamma & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}, \quad (22)$$

where $\Phi = \Omega J|_{P_{x0z}} \Omega^{-1}$. From (16) and (17), we obtain

$$\rho_1 \rho_{k+1}^{-1} \frac{\alpha_1}{\alpha_2} = \frac{\alpha_1}{\alpha_2} \frac{1}{1 - k\varepsilon} < 1. \quad (23)$$

From (23), one gets $\varepsilon < 1 - (\alpha_1/\alpha_2)$. But $\rho_{k+2} \rho_{2k+2}^{-1} = \rho_{2k+2}^{-1} = (1/1 - k\varepsilon) > 1$, and hence P_{x0z} of (7) is unstable.

Similarly, local dynamics about P_{xy0} and P_{0yz} of system under consideration can be summarized as follows. $\square$

Theorem 3. (i) P_{xy0} of system (7) is unstable.

(ii) P_{0yz} of system (7) is unstable.

Proof. Same as proof of Theorems 1 and 2. $\square$

Theorem 4. P_{x00} of system (7) is unstable.

Proof. About the equilibrium point P_{x00} , (10) takes the following form:

$$J|_{P_{x00}} = \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\alpha_1}{\alpha_2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{\alpha_4}{\alpha_5} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \quad (24)$$

Now, if $\nu_1, \nu_2, \dots, \nu_{k+1}, \dots, \nu_{2k+1}, \dots, \nu_{3k+3}$ denotes characteristic roots of P_{x00} , the diagonal matrix $\Omega = \text{diag}(\rho_1, \rho_2, \dots, \rho_{k+1}, \dots, \rho_{2k+1}, \dots, \rho_{3k+3})$, where (13) holds. Moreover,

$$\Omega J|_{P_{s100}} \Omega^{-1} = \begin{pmatrix} 0 & 0 & \dots & 0 & \rho_1 \rho_{k+1}^{-1} \frac{\alpha_1}{\alpha_2} & 0 & 0 & \dots & 0 & 0 \\ \rho_2 \rho_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \rho_{k+1} \rho_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \rho_{k+2} \rho_{2k+2}^{-1} \frac{\alpha_4}{\alpha_5} \\ 0 & 0 & \dots & 0 & 0 & \rho_{k+3} \rho_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \rho_{2k+2} \rho_{2k+1}^{-1} & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ & & & & 0 & 0 & \dots & 0 & 0 & \\ & & & & 0 & 0 & \dots & 0 & 0 & \\ & & & & \vdots & \vdots & \ddots & \vdots & \vdots & \\ & & & & 0 & 0 & \dots & 0 & 0 & \\ & & & & 0 & 0 & \dots & 0 & 0 & \\ & & & & 0 & 0 & \dots & 0 & 0 & \\ & & & & \vdots & \vdots & \ddots & \vdots & \vdots & \\ & & & & 0 & 0 & \dots & 0 & 0 & \\ & & & & 0 & 0 & \dots & 0 & \rho_{2k+3} \rho_{3k+3}^{-1} & \\ & & & & \rho_{3k+2} \rho_{3k+1}^{-1} & 0 & \dots & 0 & 0 & \\ & & & & \vdots & \vdots & \ddots & \vdots & \vdots & \\ & & & & 0 & 0 & \dots & \rho_{3k+3} \rho_{3k+2}^{-1} & 0 & \end{pmatrix}.$$

From (16) and (17), we obtain

$$\rho_{2k+3}\rho_{3k+3}^{-1} = \rho_{3k+3}^{-1} = \frac{1}{1 - k\varepsilon} > 1, \quad (26)$$

which gives P_{x00} of (7) is unstable.

Similarly, local dynamics about P_{00z}, P_{0y0} of system under consideration can be summarized as follows. $\square$

Theorem 5. (i) P_{00z} of system (7) is unstable.

(ii) P_{0y0} of system (7) is unstable.

Proof. Same as proof of Theorems 1, 2, and 4. $\square$

3.2. Local Dynamics about Positive Point

Theorem 6. P_{xyz} of (7) is unstable.

Proof. About the equilibrium point P_{xyz} , (10) takes the following form:

$$J|_{P_{xyz}} = \begin{pmatrix} 0 & 0 & \dots & 0 & 1 & \Lambda_1 & \Lambda_1 & \dots & \Lambda_1 & \Lambda_1 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 & \Lambda_2 & \Lambda_2 & \dots & \Lambda_2 & \Lambda_2 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ \Gamma & \Gamma & \dots & \Gamma & \Gamma & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix}. \quad (27)$$

where

$$\Lambda_1 = \frac{\alpha_3}{\alpha_1} \sqrt{\left[\frac{k}{k+1} \right]} \frac{\alpha_1 - \alpha_2}{\alpha_3} \sqrt{[k+1]} \frac{\alpha_7 - \alpha_8}{\alpha_9}, \quad (28)$$

$$\Lambda_2 = -\frac{\alpha_6}{\alpha_4} \sqrt{[k+1]} \frac{\alpha_1 - \alpha_2}{\alpha_3} \sqrt{\left[\frac{k}{k+1} \right]} \frac{\alpha_4 - \alpha_5}{\alpha_6},$$

and Γ is depicted in (21). Moreover,

$$\chi = \begin{pmatrix} 0 & 0 & \dots & 0 & \rho_1 \rho_{k+1}^{-1} & \rho_1 \rho_{k+2}^{-1} \Lambda_1 & \rho_1 \rho_{k+3}^{-1} \Lambda_1 & \dots & \rho_1 \rho_{2k+1}^{-1} \Lambda_1 & \rho_1 \rho_{2k+2}^{-1} \Lambda_1 \\ \rho_2 \rho_1^{-1} & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \rho_{k+1} \rho_k^{-1} & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \rho_{k+2} \rho_{2k+2}^{-1} \\ 0 & 0 & \dots & 0 & 0 & \rho_{k+3} \rho_{k+2}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & \rho_{2k+2} \rho_{2k+1}^{-1} & 0 \\ \rho_{2k+3} \rho_1^{-1} \Gamma & \rho_{2k+3} \rho_2^{-1} \Gamma & \dots & \rho_{2k+3} \rho_k^{-1} \Gamma & \rho_{2k+3} \rho_{k+1}^{-1} \Gamma & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix} \quad (29)$$

$$\begin{pmatrix} 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ \rho_{k+2} \rho_{2k+3}^{-1} \Lambda_2 & \rho_{k+2} \rho_{2k+4}^{-1} \Lambda_2 & \dots & \rho_{k+2} \rho_{3k+2}^{-1} \Lambda_2 & \rho_{k+2} \rho_{3k+3}^{-1} \Lambda_2 \\ 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & \rho_{2k+3} \rho_{3k+3}^{-1} \\ \rho_{3k+2} \rho_{3k+1}^{-1} & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & \rho_{3k+3} \rho_{3k+2}^{-1} & 0 \end{pmatrix},$$

where $\chi = \Omega J|_{P_{xyz}} \Omega^{-1}$. From (16) and (17), we obtain

$$\begin{aligned} \rho_1 \rho_{k+1}^{-1} + \rho_1 (\rho_{k+2}^{-1} + \rho_{k+3}^{-1} + \cdots + \rho_{2k+1}^{-1} + \rho_{2k+2}^{-1}) \Lambda_1 &= \frac{1}{1-k\varepsilon} + \left(1 + \frac{1}{1-\varepsilon} + \cdots + \frac{1}{1-(k-1)\varepsilon} + \frac{1}{1-k\varepsilon}\right) \Lambda_1 > 1, \\ \rho_{k+2} \rho_{2k+2}^{-1} + \rho_{k+2} (\rho_{2k+3}^{-1} + \rho_{2k+4}^{-1} + \cdots + \rho_{3k+2}^{-1} + \rho_{3k+3}^{-1}) \Lambda_2 &= \frac{1}{1-k\varepsilon} + \left(1 + \frac{1}{1-\varepsilon} + \cdots + \frac{1}{1-(k-1)\varepsilon} + \frac{1}{1-k\varepsilon}\right) \Lambda_2 > 1, \\ \rho_{2k+3} \rho_{3k+3}^{-1} + \rho_{2k+3} (\rho_1^{-1} + \rho_2^{-1} + \cdots + \rho_k^{-1} + \rho_{k+1}^{-1}) \Gamma &= \frac{1}{1-k\varepsilon} + \left(1 + \frac{1}{1-\varepsilon} + \cdots + \frac{1}{1-(k-1)\varepsilon} + \frac{1}{1-k\varepsilon}\right) \Lambda_2 > 1. \end{aligned} \quad (30)$$

From (30), one can conclude that P_{xyz} of (7) is unstable. $\square$

Proof. For $v = 0$, (31)–(33) are true trivially. Suppose that (31)–(33) are true for $v = \sigma \geq 1$, that is,

4. Boundedness

The boundedness of positive solution of (7) is investigated in this section, as follows.

Theorem 7. *If $\{\Omega_n\}_{n=-k}^\infty = \{(x_n, y_n, z_n)\}_{n=-k}^\infty$ is a positive solution of (7), then following holds for $v \geq 0$:*

$$0 < x_n \leq \begin{cases} \left(\frac{\alpha_1}{\alpha_2}\right)^{v+1} x_{-k}, & \text{if } n = (k+1)v + 1, \\ \left(\frac{\alpha_1}{\alpha_2}\right)^{v+1} x_{-k+1}, & \text{if } n = (k+1)v + 2, \\ \vdots \\ \left(\frac{\alpha_1}{\alpha_2}\right)^{v+1} x_{-1}, & \text{if } n = (k+1)v + k, \\ \left(\frac{\alpha_1}{\alpha_2}\right)^{v+1} x_0, & \text{if } n = (k+1)v + (k+1), \end{cases} \quad (31)$$

$$0 < y_n \leq \begin{cases} \left(\frac{\alpha_4}{\alpha_5}\right)^{v+1} y_{-k}, & \text{if } n = (k+1)v + 1, \\ \left(\frac{\alpha_4}{\alpha_5}\right)^{v+1} y_{-k+1}, & \text{if } n = (k+1)v + 2, \\ \vdots \\ \left(\frac{\alpha_4}{\alpha_5}\right)^{v+1} y_{-1}, & \text{if } n = (k+1)v + k, \\ \left(\frac{\alpha_4}{\alpha_5}\right)^{v+1} y_0, & \text{if } n = (k+1)v + (k+1), \end{cases} \quad (32)$$

$$0 < z_n \leq \begin{cases} \left(\frac{\alpha_7}{\alpha_8}\right)^{v+1} z_{-k}, & \text{if } n = (k+1)v + 1, \\ \left(\frac{\alpha_7}{\alpha_8}\right)^{v+1} z_{-k+1}, & \text{if } n = (k+1)v + 2, \\ \vdots \\ \left(\frac{\alpha_7}{\alpha_8}\right)^{v+1} z_{-1}, & \text{if } n = (k+1)v + k, \\ \left(\frac{\alpha_7}{\alpha_8}\right)^{v+1} z_0, & \text{if } n = (k+1)v + (k+1). \end{cases} \quad (33)$$

$$\begin{aligned} 0 < x_n &\leq \begin{cases} \left(\frac{\alpha_1}{\alpha_2}\right)^{\sigma+1} x_{-k}, & \text{if } n = (k+1)\sigma + 1, \\ \left(\frac{\alpha_1}{\alpha_2}\right)^{\sigma+1} x_{-k+1}, & \text{if } n = (k+1)\sigma + 2, \\ \vdots \\ \left(\frac{\alpha_1}{\alpha_2}\right)^{\sigma+1} x_{-1}, & \text{if } n = (k+1)\sigma + k, \\ \left(\frac{\alpha_1}{\alpha_2}\right)^{\sigma+1} x_0, & \text{if } n = (k+1)\sigma + (k+1), \end{cases} \\ 0 < y_n &\leq \begin{cases} \left(\frac{\alpha_4}{\alpha_5}\right)^{\sigma+1} y_{-k}, & \text{if } n = (k+1)\sigma + 1, \\ \left(\frac{\alpha_4}{\alpha_5}\right)^{\sigma+1} y_{-k+1}, & \text{if } n = (k+1)\sigma + 2, \\ \vdots \\ \left(\frac{\alpha_4}{\alpha_5}\right)^{\sigma+1} y_{-1}, & \text{if } n = (k+1)\sigma + k, \\ \left(\frac{\alpha_4}{\alpha_5}\right)^{\sigma+1} y_0, & \text{if } n = (k+1)\sigma + (k+1), \end{cases} \\ 0 < z_n &\leq \begin{cases} \left(\frac{\alpha_7}{\alpha_8}\right)^{\sigma+1} z_{-k}, & \text{if } n = (k+1)\sigma + 1, \\ \left(\frac{\alpha_7}{\alpha_8}\right)^{\sigma+1} z_{-k+1}, & \text{if } n = (k+1)\sigma + 2, \\ \vdots \\ \left(\frac{\alpha_7}{\alpha_8}\right)^{\sigma+1} z_{-1}, & \text{if } n = (k+1)\sigma + k, \\ \left(\frac{\alpha_7}{\alpha_8}\right)^{\sigma+1} z_0, & \text{if } n = (k+1)\sigma + (k+1). \end{cases} \end{aligned} \quad (34)$$

Finally for $v = \sigma + 1$ and using (7), one obtains

$$\begin{aligned} x_{(k+1)(\sigma+1)+1} &= \frac{\alpha_1 x_{(k+1)\sigma+1}}{\alpha_2 + \alpha_3 \prod_{i=0}^k y_{(k+1)\sigma+k+1-i}} \leq \frac{\alpha_1}{\alpha_2} x_{(k+1)\sigma+1} \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{\sigma+2} x_{-k}, \\ x_{(k+1)(\sigma+1)+2} &= \frac{\alpha_1 x_{(k+1)\sigma+2}}{\alpha_2 + \alpha_3 \prod_{i=0}^k y_{(k+1)\sigma+k+2-i}} \leq \frac{\alpha_1}{\alpha_2} x_{(k+1)\sigma+2} \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{\sigma+2} x_{-k+1}, \\ &\vdots \end{aligned} \quad (35)$$

$$\begin{aligned} x_{(k+1)(\sigma+1)+k} &= \frac{\alpha_1 x_{(k+1)\sigma+k}}{\alpha_2 + \alpha_3 \prod_{i=0}^k y_{(k+1)\sigma+2k-i}} \leq \frac{\alpha_1}{\alpha_2} x_{(k+1)\sigma+k} \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{\sigma+2} x_{-1}, \\ x_{(k+1)(\sigma+1)+k+1} &= \frac{\alpha_1 x_{(k+1)\sigma+k+1}}{\alpha_2 + \alpha_3 \prod_{i=0}^k y_{(k+1)\sigma+2k+1-i}} \leq \frac{\alpha_1}{\alpha_2} x_{(k+1)\sigma+k+1} \leq \left(\frac{\alpha_1}{\alpha_2}\right)^{\sigma+2} x_0, \\ y_{(k+1)(\sigma+1)+1} &= \frac{\alpha_4 y_{(k+1)\sigma+1}}{\alpha_5 + \alpha_6 \prod_{i=0}^k z_{(k+1)\sigma+k+1-i}} \leq \frac{\alpha_4}{\alpha_5} y_{(k+1)\sigma+1} \leq \left(\frac{\alpha_4}{\alpha_5}\right)^{\sigma+2} y_{-k}, \\ y_{(k+1)(\sigma+1)+2} &= \frac{\alpha_4 y_{(k+1)\sigma+2}}{\alpha_5 + \alpha_6 \prod_{i=0}^k z_{(k+1)\sigma+k+2-i}} \leq \frac{\alpha_4}{\alpha_5} y_{(k+1)\sigma+2} \leq \left(\frac{\alpha_4}{\alpha_5}\right)^{\sigma+2} y_{-k+1}, \\ &\vdots \end{aligned} \quad (36)$$

$$\begin{aligned} y_{(k+1)(\sigma+1)+k} &= \frac{\alpha_4 y_{(k+1)\sigma+k}}{\alpha_5 + \alpha_6 \prod_{i=0}^k z_{(k+1)\sigma+2k-i}} \leq \frac{\alpha_4}{\alpha_5} y_{(k+1)\sigma+k} \leq \left(\frac{\alpha_4}{\alpha_5}\right)^{\sigma+2} y_{-1}, \\ y_{(k+1)(\sigma+1)+k+1} &= \frac{\alpha_4 y_{(k+1)\sigma+k+1}}{\alpha_5 + \alpha_6 \prod_{i=0}^k z_{(k+1)\sigma+2k+1-i}} \leq \frac{\alpha_4}{\alpha_5} y_{(k+1)\sigma+k+1} \leq \left(\frac{\alpha_4}{\alpha_5}\right)^{\sigma+2} y_0, \\ z_{(k+1)(\sigma+1)+1} &= \frac{\alpha_7 z_{(k+1)\sigma+1}}{\alpha_8 + \alpha_9 \prod_{i=0}^k x_{(k+1)\sigma+k+1-i}} \leq \frac{\alpha_7}{\alpha_8} z_{(k+1)\sigma+1} \leq \left(\frac{\alpha_7}{\alpha_8}\right)^{\sigma+2} z_{-k}, \\ z_{(k+1)(\sigma+1)+2} &= \frac{\alpha_7 z_{(k+1)\sigma+2}}{\alpha_8 + \alpha_9 \prod_{i=0}^k x_{(k+1)\sigma+k+2-i}} \leq \frac{\alpha_7}{\alpha_8} z_{(k+1)\sigma+2} \leq \left(\frac{\alpha_7}{\alpha_8}\right)^{\sigma+2} z_{-k+1}, \\ &\vdots \end{aligned} \quad (37)$$

From (37)–(39), one has required result. $\square$

5. Global Dynamics about P_{000}

Lemma 2. If (11) holds, then $\{\Omega_n\}_{n=-k}^{\infty}$ of (7) is bounded.

Proof. It is consequence of Theorem 7. $\square$

Remark 1. From above theoretical studies, we conclude that no equilibria are locally stable except P_{000} , and hence in the following section we will investigate that it is globally asymptotically stable under certain parametric conditions.

Theorem 8. P_{000} of system (7) is globally asymptotically stable if conditions, which are depicted in (11), hold.

Proof. By Theorem 1, P_{000} of (7) is a sink and also $\{\Omega_n\}_{n=-k}^{\infty}$ is bounded by Lemma 2. In order to get the required statement, it is adequate to prove $\{(x_n, y_n, z_n)\}$ is nonincreasing. In view of (11), from (7), one gets $x_{n+1} < x_{n-k}$, which again gives $x_{(k+1)n+1} < x_{(k+1)n-k}$ and $x_{(k+1)n+k+2} < x_{(k+1)n+1}$. Thus, $\{(x_n)\}$ is decreasing because $\{x_{(k+1)n+1}\}, \{x_{(k+1)n+2}\}, \dots$

$\{x_{(k+1)n+k+1}\}$ are nonincreasing. Similarly, it is easy to investigate that $\{(y_n)\}$ and $\{(z_n)\}$ are also decreasing. This implies that $\lim_{n \rightarrow \infty} (x_n, y_n, z_n) = P_{000}$. $\square$

Lemma 3. *If*

$$\frac{\alpha_1}{\alpha_2} > 1, \frac{\alpha_4}{\alpha_5} > 1, \frac{\alpha_7}{\alpha_8} > 1, \quad (38)$$

then for $\{\Omega_n\}_{n=-k}^\infty$ of (7), the following holds:

- (i) $x_n \rightarrow \infty$ if $y_n \rightarrow 0$.
- (ii) $y_n \rightarrow \infty$ if $z_n \rightarrow 0$.
- (iii) $z_n \rightarrow \infty$ if $x_n \rightarrow 0$.

Lemma 4. *If (38) holds, then for $(\ell = k, -k+1, \dots, 1, 0)$, one gets the following invariant intervals:*

- (i) $(x_{-\ell}, y_{-\ell}, z_{-\ell}) \in (0, \sqrt{[k+1]} (\alpha_7 - \alpha_8/\alpha_9)) \times (0, \sqrt{[k+1]} (\alpha_1 - \alpha_2/\alpha_3)) \times (\sqrt{[k+1]} (\alpha_4 - \alpha_5/\alpha_6), \infty) \Rightarrow (x_n, y_n, z_n) \in (0, \sqrt{[k+1]} (\alpha_7 - \alpha_8/\alpha_9)) \times (0, \sqrt{[k+1]} (\alpha_1 - \alpha_2/\alpha_3)) \times (\sqrt{[k+1]} (\alpha_4 - \alpha_5/\alpha_6), \infty), \forall n \geq 1$.
- (ii) $(x_{-\ell}, y_{-\ell}, z_{-\ell}) \in (\sqrt{[k+1]} (\alpha_7 - \alpha_8/\alpha_9), \infty) \times (\sqrt{[k+1]} (\alpha_1 - \alpha_2/\alpha_3), \infty) \times (0, \sqrt{[k+1]} (\alpha_4 - \alpha_5/\alpha_6)) \Rightarrow (x_n, y_n, z_n) \in (\sqrt{[k+1]} (\alpha_7 - \alpha_8/\alpha_9), \infty) \times (\sqrt{[k+1]} (\alpha_1 - \alpha_2/\alpha_3), \infty) \times (0, \sqrt{[k+1]} (\alpha_4 - \alpha_5/\alpha_6)), \forall n \geq 1$.

Proof. This follows by induction. $\square$

6. Convergence Rate

Theorem 9. *If (11) is true, then error vector*

$$\mathcal{Q}_n = \begin{pmatrix} \mathcal{Q}_n^1 \\ \vdots \\ \mathcal{Q}_{n-k}^1 \\ \mathcal{Q}_n^2 \\ \vdots \\ \mathcal{Q}_{n-k}^2 \\ \mathcal{Q}_n^3 \\ \vdots \\ \mathcal{Q}_{n-k}^3 \end{pmatrix}, \quad (39)$$

of positive solution $\{\Omega_n\}_{n=-k}^\infty$ of (7) satisfies the following relations:

$$\lim_{n \rightarrow \infty} \sqrt{[n]} \|\mathcal{Q}_n\| = |\nu J|_{P_{000}}, \quad (40)$$

$$\lim_{n \rightarrow \infty} \frac{\|\mathcal{Q}_{n+1}\|}{\|\mathcal{Q}_n\|} = |\nu J|_{P_{000}},$$

where $|\nu J|_{P_{000}}$ is equivalent to modulus of one of the characteristic roots of $J|_{P_{000}}$ calculated at P_{000} .

Proof. If $\{\Omega_n\}_{n=-k}^\infty$ is an arbitrary solution of (7) such that $\lim_{n \rightarrow \infty} x_n = x$, $\lim_{n \rightarrow \infty} y_n = y$ and $\lim_{n \rightarrow \infty} z_n = z$, then

$$\begin{aligned} x_{n+1} - x &= \frac{\alpha_1 x_{n-k}}{\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i}} - \frac{\alpha_1 x}{\alpha_2 + \alpha_3 y^{k+1}} \\ &= \frac{\alpha_1}{\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i}} (x_{n-k} - x) - \frac{\alpha_1 \alpha_3 x \prod_{i=1}^k y_{n-i}}{(\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i})(\alpha_2 + \alpha_3 y^{k+1})} (y_n - y) \\ &\quad - \frac{\alpha_1 \alpha_3 x y \prod_{i=2}^k y_{n-i}}{(\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i})(\alpha_2 + \alpha_3 y^{k+1})} (y_{n-1} - y) - \dots \\ &\quad - \frac{\alpha_1 \alpha_3 x y^{k-1} y_{n-k}}{(\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i})(\alpha_2 + \alpha_3 y^{k+1})} (y_{n-k+1} - y) \\ &\quad - \frac{\alpha_1 \alpha_3 x y^k}{(\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i})(\alpha_2 + \alpha_3 y^{k+1})} (y_{n-k} - y). \end{aligned} \quad (41)$$

Similarly from (7), we obtain

$$\begin{aligned}
y_{n+1} - y &= \frac{\alpha_4}{\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i}} (y_{n-k} - y) \\
&\quad - \frac{\alpha_4 \alpha_6 y \prod_{i=1}^k z_{n-i}}{(\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i})(\alpha_5 + \alpha_6 z^{k+1})} (z_n - z) \\
&\quad - \frac{\alpha_4 \alpha_6 y z \prod_{i=2}^k z_{n-i}}{(\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i})(\alpha_5 + \alpha_6 z^{k+1})} (z_{n-1} - z) - \dots \\
&\quad - \frac{\alpha_4 \alpha_6 y z^{k-1} z_{n-k}}{(\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i})(\alpha_5 + \alpha_6 z^{k+1})} (z_{n-k+1} - z) \\
&\quad - \frac{\alpha_4 \alpha_6 y z^k}{(\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i})(\alpha_5 + \alpha_6 z^{k+1})} (z_{n-k} - z), \\
z_{n+1} - z &= -\frac{\alpha_7 \alpha_9 z \prod_{i=1}^k x_{n-i}}{(\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i})(\alpha_8 + \alpha_9 z^{k+1})} (x_n - x) \\
&\quad - \frac{\alpha_7 \alpha_9 z x \prod_{i=2}^k x_{n-i}}{(\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i})(\alpha_8 + \alpha_9 z^{k+1})} (x_{n-1} - x) - \dots \\
&\quad - \frac{\alpha_7 \alpha_9 z x^{k-1} z_{n-k}}{(\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i})(\alpha_8 + \alpha_9 z^{k+1})} (x_{n-k+1} - x) \\
&\quad - \frac{\alpha_7 \alpha_9 z x^k}{(\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i})(\alpha_8 + \alpha_9 z^{k+1})} (x_{n-k} - x) \\
&\quad + \frac{\alpha_7}{\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i}} (z_{n-k} - z).
\end{aligned} \tag{42}$$

Now, set

$$\begin{aligned}
\varrho_n^1 &= x_n - x, \\
\varrho_n^2 &= y_n - y, \\
\varrho_n^3 &= z_n - z.
\end{aligned} \tag{43}$$

Utilizing (43) in (41) and (42), we get

$$\begin{aligned}
\varrho_{n+1}^1 &= A_n^{n-k} \varrho_{n-k}^1 + B_n^n \varrho_n^2 + B_n^{n-1} \varrho_{n-1}^2 + \dots + B_n^{n-k+1} \varrho_{n-k+1}^2 \\
&\quad + B_n^{n-k} \varrho_{n-k}^2, \\
\varrho_{n+1}^2 &= C_n^{n-k} \varrho_{n-k}^2 + D_n^n \varrho_n^3 + D_n^{n-1} \varrho_{n-1}^3 + \dots + D_n^{n-k+1} \varrho_{n-k+1}^3 \\
&\quad + D_n^{n-k} \varrho_{n-k}^3, \\
\varrho_{n+1}^3 &= E_n^n \varrho_n^1 + E_n^{n-1} \varrho_{n-1}^1 + \dots + E_n^{n-k+1} \varrho_{n-k+1}^1 + E_n^{n-k} \varrho_{n-k}^1 \\
&\quad + F_n^{n-k} \varrho_{n-k}^3,
\end{aligned} \tag{44}$$

where

$$\begin{aligned}
A_n^{n-k} &= \frac{\alpha_1}{\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i}}, \\
B_n^n &= -\frac{\alpha_1 \alpha_3 x \prod_{i=1}^k y_{n-i}}{(\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i})(\alpha_2 + \alpha_3 y^{k+1})}, \\
B_n^{n-1} &= -\frac{\alpha_1 \alpha_3 x y \prod_{i=2}^k y_{n-i}}{(\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i})(\alpha_2 + \alpha_3 y^{k+1})}, \\
&\vdots \\
B_n^{n-k+1} &= -\frac{\alpha_1 \alpha_3 x y^{k-1} y_{n-k}}{(\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i})(\alpha_2 + \alpha_3 y^{k+1})}, \\
B_n^{n-k} &= -\frac{\alpha_1 \alpha_3 x y^k}{(\alpha_2 + \alpha_3 \prod_{i=0}^k y_{n-i})(\alpha_2 + \alpha_3 y^{k+1})}, \\
C_n^{n-k} &= \frac{\alpha_4}{\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i}}, \\
D_n^n &= -\frac{\alpha_4 \alpha_6 y \prod_{i=1}^k z_{n-i}}{(\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i})(\alpha_5 + \alpha_6 z^{k+1})}, \\
D_n^{n-1} &= -\frac{\alpha_4 \alpha_6 y z \prod_{i=2}^k z_{n-i}}{(\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i})(\alpha_5 + \alpha_6 z^{k+1})}, \\
&\vdots \\
D_n^{n-k+1} &= -\frac{\alpha_4 \alpha_6 y z^{k-1} z_{n-k}}{(\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i})(\alpha_5 + \alpha_6 z^{k+1})}, \\
D_n^{n-k} &= -\frac{\alpha_4 \alpha_6 y z^k}{(\alpha_5 + \alpha_6 \prod_{i=0}^k z_{n-i})(\alpha_5 + \alpha_6 z^{k+1})}, \\
E_n^n &= -\frac{\alpha_7 \alpha_9 z \prod_{i=1}^k x_{n-i}}{(\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i})(\alpha_8 + \alpha_9 z^{k+1})}, \\
E_n^{n-1} &= -\frac{\alpha_7 \alpha_9 z x \prod_{i=2}^k x_{n-i}}{(\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i})(\alpha_8 + \alpha_9 z^{k+1})}, \\
&\vdots \\
E_n^{n-k+1} &= -\frac{\alpha_7 \alpha_9 z x^{k-1} z_{n-k}}{(\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i})(\alpha_8 + \alpha_9 z^{k+1})}, \\
E_n^{n-k} &= -\frac{\alpha_7 \alpha_9 z x^k}{(\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i})(\alpha_8 + \alpha_9 z^{k+1})}, \\
F_n^{n-k} &= \frac{\alpha_7}{\alpha_8 + \alpha_9 \prod_{i=0}^k x_{n-i}}.
\end{aligned} \tag{45}$$

From (45) and (46), we get

$$\begin{aligned}
\lim_{n \rightarrow \infty} A_n^{n-k} &= \frac{\alpha_1}{\alpha_2 + \alpha_3 y^{k+1}}, \\
\lim_{n \rightarrow \infty} B_n^n &= \lim_{n \rightarrow \infty} B_n^{n-1} = \dots = \lim_{n \rightarrow \infty} B_n^{n-k+1} = \lim_{n \rightarrow \infty} B_n^{n-k} = -\frac{\alpha_1 \alpha_3 x y^k}{(\alpha_2 + \alpha_3 y^{k+1})^2}, \\
\lim_{n \rightarrow \infty} C_n^{n-k} &= \frac{\alpha_4}{\alpha_5 + \alpha_6 z^{k+1}}, \\
\lim_{n \rightarrow \infty} D_n^n &= \lim_{n \rightarrow \infty} D_n^{n-1} = \dots = \lim_{n \rightarrow \infty} D_n^{n-k+1} = \lim_{n \rightarrow \infty} D_n^{n-k} = -\frac{\alpha_4 \alpha_6 y z^k}{(\alpha_5 + \alpha_6 z^{k+1})^2}, \\
\lim_{n \rightarrow \infty} E_n^n &= \lim_{n \rightarrow \infty} E_n^{n-1} = \dots = \lim_{n \rightarrow \infty} E_n^{n-k+1} = \lim_{n \rightarrow \infty} E_n^{n-k} = -\frac{\alpha_7 \alpha_9 x^k z}{(\alpha_8 + \alpha_9 x^{k+1})^2}, \\
\lim_{n \rightarrow \infty} F_n^{n-k} &= \frac{\alpha_7}{\alpha_8 + \alpha_9 x^{k+1}}.
\end{aligned} \tag{47}$$

So, we have the limiting system [26]:

$$\mathfrak{Q}_{n+1} = K \mathfrak{Q}_n, \tag{48}$$

where $\mathfrak{Q}_n$ is depicted in (39) and K is same as $J|_{\wedge}$ about $\wedge$. In particular, about P_{000} it becomes

$$\begin{pmatrix} \mathfrak{Q}_{n+1}^1 \\ \vdots \\ \mathfrak{Q}_{n+1}^1 \\ \mathfrak{Q}_{n+1}^2 \\ \vdots \\ \mathfrak{Q}_{n+1}^2 \\ \mathfrak{Q}_{n+1}^3 \\ \vdots \\ \mathfrak{Q}_{n+1}^3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & \dots & 0 & \frac{\alpha_1}{\alpha_2} & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{\alpha_4}{\alpha_5} & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & \frac{\alpha_7}{\alpha_8} \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 1 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 & \dots & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathfrak{Q}_n^1 \\ \vdots \\ \mathfrak{Q}_{n-k}^1 \\ \mathfrak{Q}_n^2 \\ \vdots \\ \mathfrak{Q}_{n-k}^2 \\ \mathfrak{Q}_n^3 \\ \vdots \\ \mathfrak{Q}_{n-k}^3 \end{pmatrix}, \tag{49}$$

which is same as linearized system of (7) about P_{000} . $\square$

7. Numerical Simulations

We will present numerous numerical examples, which represent different dynamical behaviors of system (7), in this section.

Example 1. Figure 1 indicates the dynamics of (7) about P_{000} if $k = 2$, α_ℓ ($\ell = 1, 2, \dots, 9$), respectively, are 14, 15, 3, 6, 5, 1, 11, 13, and 3 and initial values $x_{-\ell}$, $\ell_{-\ell}$, $z_{-\ell}$ ($\ell = 2, 1, 0$), respectively, are 0.07, 0.9, 0.02, 0.9, 0.9, 1.4, 4.9, 0.9, and 0.8. More precisely, Figures 1(a)–1(c) show that P_{000} of (7) is a sink while its corresponding global attractor is shown is Figure 1(d). Figure 1(d) clearly indicates that all

FIGURE 1: Dynamics of (7).

FIGURE 2: Dynamics of (7).

FIGURE 3: Dynamics of (7).

solutions go towards P_{000} ; this means that P_{000} is globally asymptotically stable. Hence, this simulation agrees with the obtained results.

Example 2. Figure 2 indicates the same dynamics of (7) about P_{000} if $k = 4$, α_ℓ ($\ell = 1, 2, \dots, 9$), respectively, are 14, 15, 33, 6, 5, 2, 11, 13, and 3 and initial values $x_{-\ell}$, $\ell_{-\ell}$, $z_{-\ell}$ ($\ell = 4, 3, 2, 1, 0$), respectively, are 0.07, 0.9, 0.07, 0.9, 0.02, 0.9, 0.9, 0.9, 0.9, 1.4, 4.9, 1.4, 4.9, 0.9, and 0.8.

Example 3. Figure 3 indicates the dynamics of (7) about P_{000} if $k = 6$, α_ℓ ($\ell = 1, 2, \dots, 9$), respectively, are 14, 15, 33, 6, 5, 2, 11, 13, and 3 and initial values $x_{-\ell}$, $\ell_{-\ell}$, $z_{-\ell}$ ($\ell = 6, 5, 4, 3, 2, 1, 0$), respectively, are 0.7, 1.9, 0.07, 0.9, 0.07, 0.9, 0.02, 1.2, 2.9, 0.9, 1.4, 2.9, 0.29, 4.9, 1.4, 4.9, 0.9, and 0.8.

8. Concluding Remarks

The presented work is a natural extension of [21–25]. In this work, we have investigated the global dynamics of higher-order rational system (7) which is our main contribution to dynamical system. We explored that system (7) has eight equilibria P_{000} , P_{x0z} , P_{xy0} , P_{0yz} , P_{x00} , P_{00z} , P_{0y0} , and P_{xyz} under certain conditions on parameters α_ℓ ($\ell = 1, 2, \dots, 9$). We investigated that only P_{000} is locally stable under the condition, which is depicted in (11), while the rest of the equilibria P_{x0z} , P_{xy0} , P_{0yz} , P_{x00} , P_{00z} , P_{0y0} , and P_{xyz} are locally unstable. Further, we have studied the global dynamics,

boundedness, and rate of convergence of positive solution of (7). Finally, we have verified theoretical results numerically [2].

Data Availability

All the data utilized in this article have been included, and the sources from where they were adopted were cited accordingly.

Conflicts of Interest

The authors declare that they have no conflicts of interest regarding the publication of this paper.

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