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Research Article

On the Complex Oscillation of Meromorphic Solutions of Nonhomogeneous Linear Differential Equations with Meromorphic Coefficients

Chuang-Xin Chen[1], **Ning Cui**[2] **and Zong-Xuan Chen**[3]

[1]College of Computational Sciences, Zhongkai University of Agriculture and Engineering, Guangzhou 510225, China
[2]College of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou 450002, China
[3]School of Mathematical Sciences, South China Normal University, Guangzhou 510631, China

Correspondence should be addressed to Ning Cui; cuining588@126.com

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In this paper, we study the higher order differential equation $f^{(k)} + Bf = H$, where B is a rational function, having a pole at ∞ of order $n > 0$, and $H \equiv 0$ is a meromorphic function with finite order, and obtain some properties related to the order and zeros of its meromorphic solutions.

1. Introduction and Results

In this paper, a meromorphic function means a function that is meromorphic in the whole complex plane $\mathbb{C}$. We assume that the reader is familiar with the fundamental results and the standard notations of Nevanlinna's value distribution theory of meromorphic functions (see [1, 2]) and use the same notations as in [3]. In addition, we use the notations $\lambda(f)$ and $\overline{\lambda}(f)$ to denote the exponents of convergence of the zeros and distinct zeros of a meromorphic function $f(z)$, respectively, and $\sigma(f)$ to denote the growth order of f.

In 1993, Chen [4] obtained the following theorems.

Theorem 1. *Let A be a rational function with n-th order pole at ∞, and let E be a transcendental meromorphic function with $\sigma(E) = \beta < (n+k)/k$. If all solutions of the differential equation*

$$f^{(k)} + Af = E, \ k \geq 2, \tag{1}$$

are meromorphic, then every solution $f(z)$ of (1) satisfies

$$\overline{\lambda}(f) = \lambda(f) = \sigma(f) = \frac{(n+k)}{k}, \quad \lambda\left(\frac{1}{f}\right) = \lambda\left(\frac{1}{E}\right), \tag{2}$$

with at most one exceptional solution f_0 satisfying $\beta \leq \sigma(f_0) < (n+k)/k$.

Theorem 2. *Let A be a rational function with n-th order pole at ∞, and let E be a transcendental meromorphic function with $(n+k)/k < \sigma(E) = \beta < \infty$. If all solutions $f(z)$ of (1) are meromorphic, then*

(a) $\sigma(f) = \beta, \lambda(1/f) = \lambda(1/E)$.
(b) If $\beta = \lambda(E) > \lambda(1/E)$, then $\lambda(f) = \beta$.
(c) If $\beta > \max\{\lambda(E), \lambda(1/E)\}$, then $f(z)$ satisfies

$$\overline{\lambda}(f) = \lambda(f) = \sigma(f) = \beta, \tag{3}$$

with at most one exceptional solution f_0 satisfying $\sigma(f_0) < \beta$.

We all know from the fundamental theory of complex differential equation that all solutions of linear differential equation with entire coefficients are entire functions, but a solution of linear differential equation with meromorphic coefficients is not always a meromorphic function. For example, $f(z) = \exp\{1/z\}$ is a solution of the equation $f'' - (1/z^4 + 2/z^3)f = 0$, but $f(z)$ is not a meromorphic function. Hence, the condition "all solutions $f(z)$ of (1) are meromorphic" in Theorems 1 and 2 is very rigorous. A natural question to ask is whether the condition "all solutions $f(z)$ of (1) are meromorphic" in Theorems 1 and 2 can be omitted? In this paper, we consider the above problem and give a positive answer by proving the following theorems.

Theorem 3. *Let B be a rational function with n-th order pole at ∞, and let $H(\not\equiv 0)$ be a meromorphic function with $\sigma(H) = \beta$. If the differential equation*

$$f^{(k)} + Bf = H, \tag{4}$$

has a meromorphic solution, then

(a) *If $\beta < (n+k)/k$, then all meromorphic solutions $f(z)$ of (4) satisfy*

$$\overline{\lambda}(f) = \lambda(f) = \sigma(f) = \frac{(n+k)}{k}, \tag{5}$$

with at most one exceptional solution f_0 satisfying $\sigma(f_0) = \sigma(H) < (n+k)/k$.

(b) *If $\beta = (n+k)/k$, then all meromorphic solutions $f(z)$ of (4) satisfy $\sigma(f) = (n+k)/k$ and*

$$\max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} \geq \max\left\{\lambda(H), \lambda\left(\frac{1}{H}\right)\right\}. \tag{6}$$

Theorem 4. *Let B be a rational function with n-th order pole at ∞, and let $H(\not\equiv 0)$ be a meromorphic function with $((n+k)/k) < \sigma(H) = \beta < \infty$. If differential equation (4) has meromorphic solution, then*

(a) *$\sigma(f) = \beta$.*

(b) *If $\beta = \lambda(H) > \lambda(1/H)$, then $\lambda(f) = \beta$.*

(c) *If $\beta > \max\{\lambda(H), \lambda(1/H)\}$, then all meromorphic solutions $f(z)$ of (4) satisfy*

$$\overline{\lambda}(f) = \lambda(f) = \sigma(f) = \beta, \tag{7}$$

with at most one exceptional solution f_0 satisfying

$$\max\left\{\lambda(f_0), \lambda\left(\frac{1}{f_0}\right)\right\} \geq \max\left\{\lambda(H), \lambda\left(\frac{1}{H}\right)\right\}. \tag{8}$$

2. Lemmas for the Proof of Theorems

Lemma 1 (see [5]). *Let H be a meromorphic function and $\sigma(H) = \beta < \infty$. Then, for any $\varepsilon > 0$, there exists a set $E_1 \subset (1, \infty)$ that has finite linear measure and finite logarithmic measure, such that*

$$|H(z)| \leq \exp\left\{r^{\beta+\varepsilon}\right\}, \tag{9}$$

holds for all z satisfying $|z| = r \notin E_1 \cup [0,1]$.

Lemma 2 (see [5]). *Let $g(z)$ be a transcendental entire function and $\sigma(g) = \alpha < \infty$. Then, there exists a set $E_2 \subset (1, \infty)$ that has infinite logarithmic measure, such that*

$$\lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log\log M(r,g)}{\log r} = \lim_{\substack{r\to\infty \\ r\in E_2}} \frac{\log \nu_g(r)}{\log r} = \alpha, \tag{10}$$

where $\nu_g(r)$ is the central index of $g(z)$.

Lemma 3 (see [4]). *Let B be a rational function with n-th order pole at ∞. If $f(z)(\not\equiv 0)$ is a meromorphic solution of the differential equation*

$$f^{(k)} + Bf = 0, \tag{11}$$

then $\sigma(f) = ((n+k)/k)$.

Lemma 4 (see [4]). *Let $E(\not\equiv 0)$ be a meromorphic function, and let b_{k-j} $(j = 1, 2, \ldots, k)$ be some rational functions. If $f(z)$ is a meromorphic solution of the equation*

$$f^{(k)} + b_{k-1}f^{(k-1)} + \cdots + b_0 f = E, \tag{12}$$

satisfying $\sigma(E) < \sigma(f) < \infty$, then $\overline{\lambda}(f) = \lambda(f) = \sigma(f)$.

Lemma 5 (see [6]). *Let $u_1(z)$ be a meromorphic function with $\sigma(u_1) = \beta < \infty$, and let $\varepsilon > 0$ be a given constant. Then, there exists a set $E_3 \subset (1, \infty)$ that has finite logarithmic measure, such that*

$$\left|\frac{u_1^{(k)}(z)}{u_1^{(j)}(z)}\right| \leq |z|^{(k-j)(\beta-1+\varepsilon)}, \quad j = 0, 1, \ldots, k = j+1, j+2, \ldots, \tag{13}$$

holds for all z satisfying $|z| = r \notin E_3 \cup [0,1]$.

Lemma 6. *Let $b_0, b_1, \ldots, b_{k-1}, H(\not\equiv 0)$ be some meromorphic functions and $\sigma(H) = \beta < \infty$. For $b_0, b_1, \ldots, b_{k-1}$, there exists a set $E_4 \subset (1, \infty)$ that has finite logarithmic measure, such that for all z satisfying $|z| = r \notin E_4 \cup [0,1]$, we have*

$$\left|b_j(z)\right| \leq |z|^{c_j}, \quad j = 0, 1, \ldots, k-1, \tag{14}$$

where c_j $(j = 0, 1, \ldots, k-1)$ are positive constants. If there is an entire function $g(z)$ satisfying the equation

$$g^{(k)} + b_{k-1}g^{(k-1)} + \cdots + b_0 g = H, \tag{15}$$

then $\sigma(g) < \infty$.

Proof. Suppose that $\sigma(g) = \infty$, $\mu(r)$ is the maximum term of $g(z)$ when $|z| = r$, and $\nu(r)$ is the central index of $g(z)$.

Using a similar method as in the proof of Lemma 2 in [5], we know that there exists a subset $E_2 \subset (1, \infty)$ with infinite logarithmic measure, such that

$$\lim_{\substack{r\to\infty\\ r\in E_2}} \frac{\log \nu(r/2)}{\log(r/2)} = \infty. \tag{16}$$

Since $\nu(r)$ is a step function of r, without loss of generality, we suppose that $0 = t_0 < t_1 < t_2 < \cdots$ are discontinuity points of $\nu(r)$ and $\mu(t) = |a_{\nu(t)}|t^{\nu(t)}$ has a fixed central index $\nu(t) = m$ when $t \in (t_j, t_{j+1})$. Hence, for $t \in (t_j, t_{j+1})$, we have

$$\mu'(r) = m|a_m|t^{m-1} = \frac{\mu(t)\nu(t)}{t}, \tag{17}$$

outside of finite points in $[0, r)$. Since $\mu(t)$ is a continuous function, for $r > 2$, we have

$$\begin{aligned}\log\mu(r) - \log\mu(1) &= \int_1^r \left[\frac{\mu'(t)}{\mu(t)}\right]dt \\ &= \int_1^r \left[\frac{\nu(t)}{t}\right]dt > \int_{(r/2)}^r \left[\frac{\nu(t)}{t}\right]dt \\ &\geq \nu\left(\frac{r}{2}\right)\log 2.\end{aligned} \tag{18}$$

By Cauchy's inequality, we have $\mu(r) \leq M(r, g)$. From the inequality above, we can get that

$$\nu\left(\frac{r}{2}\right)\log 2 \leq \log M(r, g) - \log\mu(1). \tag{19}$$

Choosing a sufficiently large α such that

$$\alpha > \max\{c_0, c_1, \ldots, c_{k-1}, \beta\} + k + 2. \tag{20}$$

By (16)–(19), we can see that for sufficiently large r, $r \in E_2$, we have

$$\nu(r) \geq \nu\left(\frac{r}{2}\right) \geq \left(\frac{r}{2}\right)^{\alpha} = d_2 r^{\alpha}, \tag{21}$$

$$M(r, g) > d_3 \exp\{d_4 r^{\alpha}\}, \tag{22}$$

where $d_2, d_3,$ and d_4 are the positive constants.

From Wiman–Valiron theory (see [7]), we can choose $|z| = r, |g(z)| = M(r, g)$, such that we have

$$\frac{g^{(j)}(z)}{g(z)} = \left(\frac{\nu(r)}{z}\right)^j (1 + o(1)), \quad j = 1, 2, \ldots, k, \quad r \notin E_4, \tag{23}$$

where E_4 is of finite logarithmic measure.

By Lemma 1, we can see that there exists a set $E_1 \subset [1, \infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \in [1, \infty) - E_1, r \longrightarrow \infty$, we have

$$|H(z)| \leq \exp\left\{r^{\beta+(1/2)}\right\}. \tag{24}$$

Now, we can choose sufficiently large $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4), |g(z)| = M(r, g)$, $E_2 - (E_1 \cup E_3 \cup E_4)$ is of infinite logarithmic measure; by (15) and (23), we have

$$\begin{aligned}&\left(\frac{\nu(r)}{z}\right)^k (1 + o(1)) + b_{k-1}\left(\frac{\nu(r)}{z}\right)^{k-1} \\ &\quad\cdot (1 + o(1)) + \cdots + b_0 = \frac{H(z)}{g(z)},\end{aligned} \tag{25}$$

that is,

$$\begin{aligned}&\nu(r)(1 + o(1)) + b_{k-1}\cdot z(1 + o(1)) + b_{k-2} \\ &\quad\cdot \frac{z^2}{\nu}(1 + o(1)) + \cdots + b_0 \cdot \frac{z^k}{\nu^{k-1}} = \frac{H}{g}\frac{z^k}{\nu^{k-1}}.\end{aligned} \tag{26}$$

By $\alpha > \beta$, (20)–(22), and (24), we have

$$\left|\frac{H}{g}\frac{z^k}{\nu^{k-1}}\right| \leq \frac{\exp\left\{r^{\beta+(1/2)}\right\}\cdot r^k}{d_3\exp\{d_4 r^{\alpha}\}\cdot (d_2 r^{\alpha})^{k-1}} \longrightarrow 0, \quad (r \longrightarrow \infty), \tag{27}$$

where $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4)$, $|g(z)| = M(r, g)$. For $(j = 1, 2, \ldots, k-1)$, we have

$$\left|\frac{b_{j-1}z^{k-j+1}}{\nu^{k-j}}\right| \leq \frac{r^{c_{j-1}+k-j+1}}{(d_2 r^{\alpha})^{k-j}} \longrightarrow 0, \tag{28}$$

holds for $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4), r \longrightarrow \infty$. Hence,

$$\begin{aligned}&\left|\frac{H}{g}\frac{z^k}{\nu^{k-1}} - b_{k-1}\cdot z(1 + o(1)) - b_{k-2}\right. \\ &\quad\left.\cdot \cdot \frac{z^2}{\nu}(1 + o(1)) - \cdots - b_0 \cdot \frac{z^k}{\nu^{k-1}}\right| = O(b_{k-1}\cdot z) \\ &\quad= O\left(r^{c_{k-1}+1}\right) = o\left(r^{c_{k-1}+k+1}\right).\end{aligned} \tag{29}$$

On the other hand, by (20) and (21), we can see that for sufficiently large r, $r \in E_2$, we have

$$|\nu(r)(1 + o(1))| > \frac{d_2}{2}r^{\alpha} > r^{c_{k-1}+k+1}. \tag{30}$$

By (29) and (30), we can know that (26) implies a contradiction. Hence, $\sigma(g) < \infty$. □

Lemma 7. *Let B be a rational function with n-th order pole at ∞, and let $H(\equiv 0)$ be a meromorphic function with $\sigma(H) = \beta$. If $f(z)$ is a meromorphic solution of (4), then*

(a) If $\beta < (n + k)/k$, then all meromorphic solutions $f(z)$ of (4) satisfy $\sigma(f) = (n + k)/k$, with at most one exceptional solution f_0 satisfying $\sigma(f_0) = \beta$.

(b) If $\beta \geq ((n + k)/k)$, then $\sigma(f) = \beta$.

Proof. To our aim, we will consider the following three cases.

Case 1. Suppose that $H(z) = B_0(z)\exp\{P(z)\}$, where $B_0(z)$ is a rational function and $P(z)$ is a polynomial and then $\deg P = \beta$. By Lemma 3 in [4], we can know that the conclusion holds.

Case 2. Suppose that $H(z)$ is a meromorphic function with infinitely many poles. Then, by (4), we have $\sigma(f) \geq \beta$.

(a) First, we will prove that if $\sigma(f) = \alpha > \beta$, then $\sigma(f) = ((n+k)/k)$.

Suppose that z' is a pole of $f(z)$ of order $m_1 (\geq 1)$ and B and H are all analytic at z'. Then, z' must be a pole of $f^{(k)} + Bf$ of order $m_1 + k$, which contradicts the fact that H is analytic at z'. Hence, all poles of f come from poles of B and H. Since B has only finitely many poles, we can see that all poles of f come from poles of H with finitely many exceptions. Suppose that z'' is a pole of H of order m_2 and B is analytic at z'', then z'' is a pole of $f(z)$ of order $m_2 - k$. Thus, we can know that $f(z)$ has infinitely many poles. From

$$\begin{aligned} n(r,f) &= n(r,H) - k\overline{n}(r,H) + O(1) \\ &\leq n(r,H) + O(1), \\ n(r,H) &= n(r,f) + k\overline{n}(r,f) + O(1) \\ &\leq (k+1)n(r,f) + O(1), \end{aligned} \tag{31}$$

we can see that $f(z)$ and H have the same exponent of convergence of poles, that is,

$$\lambda\left(\frac{1}{f}\right) = \lambda\left(\frac{1}{H}\right) \leq \beta. \tag{32}$$

By Hadamard factorization theory, we can set $f(z) = g(z)/[z^{k_2}u(z)] = g(z)/u_1(z)$, where k_2 is a nonnegative integer, $g(z)$ is an entire function, and $u_1(z) = z^{k_2} \cdot u(z)$, where $u(z)$ is the canonical product formed with the nonzero poles $\{z_j\colon j = 1,2,\cdots; |z_j| = r_j, 0 < r_1 \leq r_2 \leq \cdots\}$ of $f(z)$. $\lambda(u_1) = \sigma(u_1) = \lambda(1/f) \leq \beta$. By $\sigma(f) = \alpha > \beta$, we have $\sigma(g) = \sigma(f) = \alpha$.

By Lemma 1, we can choose ε, such that

$$0 < (k+1)\varepsilon < \min\left\{\alpha - \beta, \frac{(n+k)}{k-\beta}\right\}, \tag{33}$$

holds, and there exists a set $E_1 \subset (1,+\infty)$ of finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_1, r \longrightarrow \infty$, we have

$$\left|\frac{1}{u_1(z)}\right| \leq \exp\left\{r^{\beta+\varepsilon}\right\}. \tag{34}$$

By (32), we can know that $f(z)$ and H have the same exponent of convergence of poles. From the above proofs, we can see that except for finitely many exceptions, $f(z)$ and H have the same poles with the difference of their orders being k; hence, the above set E_1 concerning $f(z)$, for H, we still have for all z satisfying $|z| = r \notin [0,1] \cup E_1, r \longrightarrow \infty$,

$$|H(z)| \leq \exp\left\{r^{\beta+\varepsilon}\right\}, \tag{35}$$

holds. Substituting $f(z) = g(z)/u_1(z)$ into (4), we get

$$\frac{g^{(k)}}{g} + b_{k-1}\frac{g^{(k-1)}}{g} + \cdots + (b_0 + B) = \frac{u_1 H}{g}, \tag{36}$$

where b_{k-j} are some differential polynomials, with constant coefficients, in $u_1'/u_1, u_1''/u_1, \ldots, u_1^{(j)}/u_1$. By $\sigma(u_1) \leq \beta$ and Lemma 5, we can see that there exists a set $E_3 \subset (1,+\infty)$ with finite logarithmic measure, such that for all z satisfying $|z| = r \notin [0,1] \cup E_3, r \longrightarrow \infty$, we have

$$\left|\frac{u_1^{(j)}(z)}{u_1(z)}\right| \leq |z|^{j(\beta-1+\varepsilon)}, \quad j = 1,2,\ldots,k. \tag{37}$$

By (33), we have $k(\beta - 1 + \varepsilon) < n$. Hence, by (37), we can easily obtain

$$\begin{cases} \left|b_{k-j}\right| \leq c|z|^{j(\beta-1+\varepsilon)}, & j = 1,2,\ldots,k, \\ b_0 + B = dz^n(1 + o(1)), \end{cases} \tag{38}$$

where c and d are two nonzero constants and $|z| = r \in (1,\infty) - E_3, r \longrightarrow \infty$. Thus, by Lemma 6, (36) and (38), we have $\sigma(g) = \alpha < \infty$.

By Lemma 2 and $\sigma(g) = \alpha < \infty$, we can see that there exists a set $E_2 \subset (1,+\infty)$ with infinite logarithmic measure, such that

$$\lim_{\substack{r\longrightarrow\infty \\ r\in E_2}} \frac{\log\log M(r,g)}{\log r} = \lim_{\substack{r\longrightarrow\infty \\ r\in E_2}} \frac{\log \nu(r)}{\log r} = \alpha, \tag{39}$$

where $\nu(r)$ is the central index of $g(z)$.

Choosing $|z| = r, |g(z)| = M(r,g)$, from Wiman–Valiron theory, we can know that (23) holds outside a set E_4 with finite logarithmic measure. For sufficiently large r, by (39), we have

$$M(r,g) \geq \exp\left\{r^{\alpha-\varepsilon}\right\}, \tag{40}$$

where $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4), |g(z)| = M(r,g)$. Since u_1 is entire, $\sigma(u_1) \leq \beta$, combining (35), (40), and $\beta + 2\varepsilon \leq \beta + (k+1)\varepsilon < \alpha$, we have

$$\left|\frac{u_1(z)\cdot H(z)}{g(z)}\right| = \frac{|u_1(z)\cdot H(z)|}{M(r,g)} \le \exp\left\{2r^{\beta+\varepsilon} - r^{\alpha-\varepsilon}\right\} \longrightarrow 0, \quad (r \longrightarrow \infty), \tag{41}$$

where $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4), |g(z)| = M(r,g)$. Choosing z such that $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4)$, $|g(z)| = M(r,g)$, since $E_2 - (E_1 \cup E_3 \cup E_4)$ is of infinite logarithmic measure, by (23), (36), (38), and (41), we can see that for $r \longrightarrow \infty$, we have

$$\left(\frac{\nu(r)}{z}\right)^k (1+o(1)) + O\left(r^{\beta-1+\varepsilon}\right)\left(\frac{\nu(r)}{z}\right)^{k-1} \cdot (1+o(1)) + \cdots + |d| r^n (1+o(1)) = \frac{u_1(z)H(z)}{g(z)} = o(1). \tag{42}$$

By (39), we can see that for $r \in E_2 - (E_1 \cup E_3 \cup E_4), r \longrightarrow \infty$, we have

$$\nu(r) = r^{\alpha+o(1)}. \tag{43}$$

By (43), we can see that the degree of terms of the left-hand side of (42) in r is

$$k(\alpha+o(1)-1), (\alpha+o(1)-1)(k-1) + (\beta-1+\varepsilon), \ldots, (\alpha+o(1)-1)+(k-1)(\beta-1+\varepsilon), n. \tag{44}$$

By (33), we have $(\beta-1+\varepsilon) < (\alpha+o(1)-1)$, which implies that

$$\begin{aligned} k(\alpha+o(1)-1) &> (\alpha+o(1)-1)(k-1) + (\beta-1+\varepsilon) \\ &> (\alpha+o(1)-1)(k-2) + 2(\beta-1+\varepsilon) \\ &> \cdots \\ &> (\alpha+o(1)-1) + (k-1)(\beta-1+\varepsilon). \end{aligned} \tag{45}$$

Thus, comparing the degree of terms of both sides of (42) in r, we obtain $k(\alpha-1) = n$, which implies $\alpha = (n+k)/k$, that is, $\sigma(f) = \sigma(g) = (n+k)/k$.

Now, we will prove that all meromorphic solutions $f(z)$ of (4) satisfy $\sigma(f) = (n+k)/k$, with at most one exceptional solution f_0 satisfying $\sigma(f_0) = \beta$.

Suppose that f_0 and $f_1 (f_1 \equiv f_0)$ are two meromorphic solutions of (4) and satisfy $\sigma(f_0) = \sigma(f_1) = \beta$. Then, $\sigma(f_0 - f_1) < ((n+k)/k)$. Since $f_0 - f_1$ is a meromorphic solution of homogeneous equation (11) corresponding to (4), by Lemma 3, we have $\sigma(f_0 - f_1) = (n+k)/k$, which is a contradiction. Hence, equation (4) has at most one exceptional solution f_0 satisfying $\sigma(f_0) = \beta$, and all other meromorphic solutions are satisfying $\sigma(f) = (n+k)/k$.

(b) Since $\sigma(f) \ge \beta \ge (n+k)/k$, we will just prove that $\sigma(f) = \alpha > \beta$ is false.

Suppose that $\alpha > \beta$, and set $f(z) = g(z)/u_1(z)$, where $g(z)$ and $u_1(z)$ have the same meanings as in (a). Then, using a similar method as in the proof of (a), we can see that for any given $\varepsilon (0 < (k+1)\varepsilon < \alpha - \beta)$, (34)–(37) hold. By $\beta \ge (n+k)/k$ and (37), we can see that for $|z| = r \in (1,\infty) - E_3, r \longrightarrow \infty$, we have

$$\begin{cases} \left|b_{k-j}\right| \le c|z|^{j(\beta-1+\varepsilon)}, & j = 1,2,\ldots,k, \\ \left|b_0 + B\right| = |d| z^n (1+o(1)), \end{cases} \tag{46}$$

where c and d are the two nonzero constants.

By using (36) and (46) in conjunction with Lemma 6, we have $\sigma(g) = \alpha < \infty$. Continuing using a similar method as in the proof of (a), we can see (39)–(43) hold. Hence, for $|z| = r \in E_2 - (E_1 \cup E_3 \cup E_4)$, we can see that in the left-hand side of (42), only one term $(\nu(r)/z)^k (1+o(1))$ has the highest degree $k(\alpha-1)$ in r. It is impossible. Hence, $\sigma(f) = \beta$.

Case 3. Suppose that $H(z)$ is a meromorphic function with finite many poles and infinite many zeros. Then, we can use a similar method as in the proof of Case 2 to obtain our conclusions.

Using a similar method as in the proof of (a) in Lemma 7, we can obtain the following lemma. ☐

Lemma 8. *Let $\beta (>1)$ be a positive integer, and let B_{k-j} be some rational functions with n_{k-j}-th order pole at ∞, where $n_{k-j} = j(\beta-1) (j = 1,2,\ldots,k)$. $U \equiv 0$ is a meromorphic function, and $\sigma(U) < \beta$. If the differential equation*

$$h^{(k)} + B_{k-1}h^{(k-1)} + \cdots + B_0 h = U, \tag{47}$$

has meromorphic solution, then all meromorphic solutions $h(z)$ of (47) satisfy $\sigma(h) = \beta$, with at most one exceptional solution h_0 satisfying $\sigma(h_0) = \sigma(U)$.

Lemma 9. *Let $B_{k-j} (j = 1,2,\ldots,k), Q$ be rational functions and B_{k-j} has a pole of order $n_{k-j} (>0)$ at ∞, and P is a polynomial with $\deg P = \beta$. If $f(z)$ is a meromorphic solution of the equation*

$$f^{(k)} + B_{k-1}f^{(k-1)} + \cdots + B_0 f = Qe^P, \tag{48}$$

then $\sigma(f) < \infty$.

Proof. Without loss of generality, we suppose that $f(z)$ is transcendental. Otherwise, $f(z)$ is a rational function, and we have $\sigma(f)<\infty$.

Differentiating both sides of (48), we have

$$\begin{aligned}&f^{(k+1)}+\left(B_{k-1}-\frac{Q'}{Q}-P'\right)f^{(k)}\\&-\left[B_{k-1}{}'+B_{k-2}-B_{k-1}\left(\frac{Q'}{Q}+P'\right)\right]f^{(k-1)}\\&+\cdots+\left[B_1'+B_0-B_1\left(\frac{Q'}{Q}+P'\right)\right]f'\\&+\left[B_0'-B_0\left(\frac{Q'}{Q}+P'\right)\right]f=0,\end{aligned}\tag{49}$$

that is,

$$f^{(k+1)}+\sum_{j=0}^{k}\left[B_{k-j}{}'+B_{k-j-1}-B_{k-j}\left(\frac{Q'}{Q}+P'\right)\right]f^{(k-j)}=0,\tag{50}$$

where $B_k=1$ and $B_{-1}=0$.

Since the poles of $f(z)$ just appear at the poles of $B_{k-j}\,(j=1,2,\ldots,k)$, Q, $f(z)$ has finitely many poles. Using f_1 to denote the sum of the major part of all poles of f, $f_2=f-f_1$ is a transcendental entire function and $\sigma(f)=\sigma(f_2)$. Substituting $f=f_1+f_2$ into (50), we have

$$\begin{aligned}&f_2^{(k+1)}+\sum_{j=0}^{k}\left[B_{k-j}'+B_{k-j-1}-B_{k-j}\left(\frac{Q'}{Q}+P'\right)\right]f_2^{(k-j)}\\&=-\left\{f_1^{(k+1)}+\sum_{j=0}^{k}\left[B_{k-j}'+B_{k-j-1}-B_{k-j}\left(\frac{Q'}{Q}+P'\right)\right]f_1^{(k-j)}\right\}.\end{aligned}\tag{51}$$

Since the right-hand side of (51) is a rational function, its order is finite.

Suppose that $B_{k-j}'=a_{k-j}z^{n_{k-j}}(1+o(1))$ and $P'=bz^{\beta-1}(1+o(1))$, where a_{k-j} and b are some nonzero constants. Then,

$$\begin{aligned}&B_{k-j}'+B_{k-j-1}-B_{k-j}\left(\frac{Q'}{Q}+P'\right)=n_{k-j}a_{k-j}\cdot z^{n_{k-j}-1}\\&\cdot(1+o(1))+a_{k-j-1}\cdot z^{n_{k-j-1}}(1+o(1))-ba_{k-j}\\&\cdot z^{n_{k-j}+\beta-1}(1+o(1)),\end{aligned}\tag{52}$$

hence, there must exist some positive constants $c_{k-j}\,(j=0,1,\ldots,k)$ such that

$$\left|B_{k-j}'+B_{k-j-1}-B_{k-j}\left(\frac{Q'}{Q}+P'\right)\right|\le|z|^{c_{k-j}},\quad j=0,1,\ldots,k.\tag{53}$$

Applying Lemma 6 on (51) and (53), we have $\sigma(f_2)<\infty$. Hence, $\sigma(f)<\infty$. □

Lemma 10. *Let $B_{k-j}\,(j=1,2,\ldots,k)$ be some rational functions with n_{k-j}-th order pole at ∞, and let $H\,(\equiv 0)$ be a meromorphic function with $\sigma(H)=\beta<\infty$. If the equation*

$$f^{(k)}+B_{k-1}f^{(k-1)}+\cdots+B_0f=H,\tag{54}$$

has meromorphic solution f, then

$$\begin{aligned}&\lambda\left(\frac{1}{f}\right)=\lambda\left(\frac{1}{H}\right),\overline{\lambda}\left(\frac{1}{f}\right)=\overline{\lambda}\left(\frac{1}{H}\right),\\&\max\left\{\lambda(f),\lambda\left(\frac{1}{f}\right)\right\}\ge\max\left\{\lambda(H),\lambda\left(\frac{1}{H}\right)\right\}.\end{aligned}\tag{55}$$

Proof. Since $B_{k-j}\,(j=1,2,\ldots,k)$ have only finitely many poles, we can see that if z_0 is not a pole of $B_{k-j}\,(j=1,2,\ldots,k)$, then z_0 is an α-th order pole of f if and only if z_0 is an $(\alpha+k)$-th order pole of H; hence, $\overline{\lambda}(1/f)=\overline{\lambda}(1/H)$. Since

$$\frac{\alpha+k}{2k}=\frac{\alpha}{2k}+\frac{1}{2}\le\alpha,\tag{56}$$

we have

$$\begin{aligned}&\frac{1}{2k}n(r,H)+O(1)\le n(r,f)\le n(r,H)+O(1),\\&\frac{1}{2k}N(r,H)+O(\log r)\le N(r,f)\le N(r,H)+O(\log r).\end{aligned}\tag{57}$$

Thus,

$$\lambda\left(\frac{1}{f}\right)=\lambda\left(\frac{1}{H}\right).\tag{58}$$

Now, we will prove that every meromorphic solution f of (54) satisfies $\sigma(f)<\infty$. So, we will consider the following three cases:

Case 1. Suppose that $H(z)=Q(z)\exp\{P(z)\}$, where $Q(z)$ is a rational function and $P(z)$ is a polynomial. Then, $\deg P=\beta$. Thus, by Lemma 9, we have $\sigma(f)<\infty$.

Case 2. Suppose that $H(z)$ is a meromorphic function with infinitely many poles. Then, by (54), we have $\sigma(f)\ge\beta$. If $\sigma(f)=\beta$, then $\sigma(f)<\infty$. Without loss of generality, we suppose that $\sigma(f)=\alpha>\beta$.

Set $f(z)=g(z)/u_1(z)$, where $g(z)$ and $u_1(z)$ have the same meanings as in the proof of (a) of Lemma 7. Using a similar method as in the proof of (a) of Lemma 7, we can get $\sigma(u_1)\le\beta$ and $\sigma(g)=\sigma(f)=\alpha$. Substituting $f(z)=g(z)/u_1(z)$ into (54), we have

$$\frac{g^{(k)}}{g}+b_{k-1}\frac{g^{(k-1)}}{g}+\cdots+b_0=\frac{u_1H}{g},\tag{59}$$

where b_{k-j} are some differential polynomials, with constant coefficients, in $u_1'/u_1, u_1''/u_1, \ldots, u_1^{(j)}/u_1$, and $B_{k-j}, \ldots, B_{k-1}$. Using a similar method as in the proof of (a) of Lemma 7, we know (37) holds obviously. Hence, we can easily obtain

$$\left|b_{k-j}\right| \leq c|z|^{j(\beta-1+\varepsilon)+n'}, \quad j = 1, 2, \ldots, k, \tag{60}$$

where c is a nonzero constant, ε satisfies $0 < \varepsilon < \alpha - \beta$, and $n' = \max\{n_0, n_1, \ldots, n_{k-1}\}$. By Lemma 6 and (59) and (60), we have $\sigma(g) = \alpha < \infty$. Hence, $\sigma(f) < \infty$.

Case 3. Suppose that $H(z)$ is a meromorphic function with finitely many poles and infinitely many zeros. Then, using a similar method as in the proof of Case 2, we can get $\sigma(f) < \infty$.

Thus, we can write f and H in the following form:

$$f = z^{m_1} \frac{E_1}{Q_1} e^{P_1}, \qquad H = z^{m_2} \frac{E_2}{Q_2} e^{P_2}, \tag{61}$$

where m_1 and m_2 are the integers, E_1 and E_2 are the canonical product formed with the nonzero zeros of f and H, Q_1 and Q_2 are the canonical product formed with the nonzero poles of f and H, and P_1 and P_2 are the polynomials such that $\deg P_1 \leq \sigma(f)$ and $\deg P_2 \leq \sigma(H)$. Substituting (61) into (54), we have

$$F(E_1, Q_1) = z^{m_2} \frac{E_2}{Q_2} e^{P_2 - P_1}, \tag{62}$$

where F is a rational function in E_1, Q_1, and its derivative, with constant coefficients. Comparing the growth order of both sides of (62) and noting that E_2 and Q_2 are canonical products, we have

$$\begin{aligned} \max\{\sigma(E_1), \sigma(Q_1)\} \geq \sigma(F) &= \sigma\left(z^{m_2} \frac{E_2}{Q_2} e^{P_2 - P_1}\right) \\ &= \max\{\sigma(E_2), \sigma(Q_2), \deg(P_2 - P_1)\} \\ &\geq \max\{\sigma(E_2), \sigma(Q_2)\}, \end{aligned} \tag{63}$$

that is,

$$\max\left\{\lambda(f), \lambda\left(\frac{1}{f}\right)\right\} \geq \max\left\{\lambda(H), \lambda\left(\frac{1}{H}\right)\right\}. \tag{64}$$

□

3. Proofs of Theorems

The proof of Theorem 3

(a) By Lemma 7, we can see that all meromorphic solutions $f(z)$ of equation (4) are satisfying $\sigma(f) = (n+k)/k$, with at most one exceptional meromorphic solution f_0 satisfying $\sigma(f_0) = \sigma(H) = \beta$. From Lemma 4, we can know that meromorphic solutions $f(z)$ of (4) with $\sigma(f) = (n+k)/k$ satisfy $\overline{\lambda}(f) = \lambda(f) = \sigma(f) = (n+k)/k$.

(b) By Lemmas 7 and 10, we get the required result.

The proof of Theorem 4

(a) By Lemma 7, we have $\sigma(f) = \beta$.

(b) If $\beta = \lambda(H) > \lambda(1/H)$, then, by Lemma 10, we have $\lambda(f) = \beta$.

(c) If $\beta > \max\{\lambda(H), \lambda(1/H)\}$, set $H = Ue^P$, where $U = z^k(V_1/V_2)$, k is an integer and V_1 and V_2 are the canonical product (or polynomial) formed with the nonzero zeros and nonzero poles of H, respectively. $\sigma(U) = \max\{\lambda(H), \lambda(1/H)\} < \beta$, and $P(z)$ is a polynomial with $\deg P(z) = \beta$. Setting $f = h \cdot e^P$, where h is a meromorphic function. Thus, f and h have the same zeros and poles. Substituting $f = h \cdot e^P$ and $H = Ue^P$ into (4), we have

$$h^{(k)} + b_{k-1}h^{(k-1)} + \cdots + b_0 h = U, \tag{65}$$

where

$$\begin{aligned} b_{k-1} &= kP', \\ b_{k-j} &= C_k^j (P')^j + H_{j-1}(P'), \quad (j = 2, \ldots, k-1), \\ b_0 &= C_k^k (P')^k + H_{k-1}(P') + B, \end{aligned} \tag{66}$$

where C_k^j are the binomial coefficients and $H_{j-1}(P')$ are the polynomials in P' and its derivatives, with constant coefficients and having degree $j-1$. It is easy to see the derivative of $H_{j-1}(P')$ with respect to z having the same form with $H_{j-1}(P')$. Since $\beta > (n+k)/k$, we know $\deg b_{k-j} = j(\beta - 1)$ $(j = 1, 2, \ldots, k)$. By $\sigma(U) < \beta$ and Lemma 8, we can see that all meromorphic solutions $h(z)$ of (65) satisfy $\sigma(f) = (n+k)/k$, with at most one exceptional meromorphic solution h_0 satisfying $\sigma(h_0) = \sigma(U)$. By Lemma 4, we have $\overline{\lambda}(h) = \lambda(h) = \sigma(h) = \beta$. From Lemma 10, we can know that h_0 satisfies

$$\sigma(h_0) \geq \max\left\{\lambda(h_0), \lambda\left(\frac{1}{h_0}\right)\right\} \geq \max\left\{\lambda(H), \lambda\left(\frac{1}{H}\right)\right\}. \tag{67}$$

Hence, (4) has at most one exceptional meromorphic solution $f_0 = h_0 e^P$ satisfying

$$\max\left\{\lambda(f_0), \lambda\left(\frac{1}{f_0}\right)\right\} \geq \max\left\{\lambda(H), \lambda\left(\frac{1}{H}\right)\right\}, \tag{68}$$

and all other meromorphic solutions $f = he^P$ satisfy $\overline{\lambda}(f) = \lambda(f) = \lambda(h) = \beta$.

4. Conclusion

Our paper investigates the nonhomogeneous linear differential equation $f^{(k)} + Bf = H$, where B is a rational function, having a pole at ∞ of order $n > 0$, and $H \equiv 0$ is a meromorphic function with finite order, and obtains some properties related to the order and zeros of its meromorphic solutions.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have drafted the manuscript and read and approved the final manuscript.

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