Hindawi
Discrete Dynamics in Nature and Society
Volume 2020, Article ID 3970903, 8 pages
https://doi.org/10.1155/2020/3970903

Research Article

The Iterative Positive Solution for a System of Fractional q-Difference Equations with Four-Point Boundary Conditions

Chuanzhi Bai and Dandan Yang

Department of Mathematics, Huaiyin Normal University, Huai'an, Jiangsu 223300, China

Correspondence should be addressed to Dandan Yang; ydd@hytc.edu.cn

Received 31 December 2019; Accepted 4 February 2020; Published 20 March 2020

Guest Editor: Fahd Jarad

In this work, we investigate the following system of fractional q-difference equations with four-point boundary problems: $\{D_q^\alpha u(t) + f(t, v(t)) = 0, 0 < t < 1; D_q^\beta v(t) + g(t, u(t)) = 0, 0 < t < 1; u(0) = 0, u(1) = c_1 u(\eta_1)$; and $v(0) = 0, v(1) = c_2 u(\eta_2)$, where D_q^α and D_q^β are the fractional Riemann–Liouville q-derivative of order α and β, respectively, $0 < q < 1$, $1 < \beta \le \alpha \le 2$, $0 < \eta_1, \eta_2 < 1$, $0 < \gamma_1 \eta_1^{\alpha-1} < 1$, and $0 < \gamma_2 \eta_2^{\beta-1} < 1$. By virtue of monotone iterative approach, the iterative positive solutions are obtained. An example to illustrate our result is given.

1. Introduction

In [1, 2], Jackson studied the q-difference calculus firstly; since then, many authors have investigated this subject duo to applications of the q-difference calculus in quantum mechanics, particle physics, hypergeometric series, and complex analysis [3, 4]. The extension of q-difference calculus is the fractional q-difference calculus, which was originally investigated by Al-Salam [5] and Agarwal [6]. In the past decade, in many works concerning nonlinear fractional q-difference boundary value problem, the results of the existence and the uniqueness of solutions have been given. In [7], Ferreira considered the existence of positive solutions to the nonlinear fractional q-difference equation:

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, 1 < \alpha \le 2, \\ u(0) = u(1) = 0. \end{cases} \tag{1}$$

In [8], Ferreira studied the existence of positive solutions to the nonlinear fractional q-difference equation:

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, 2 < \alpha \le 3, \\ u(0) = D_q u(0) = 0, & D_q u(1) = \beta \ge 0. \end{cases} \tag{2}$$

By using a fixed-point theorem in partially ordered sets, Garzi and Agheli [9] studied the existence and uniqueness of a positive and nondecreasing solution to the fractional q-difference equation:

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t)) = 0, & 0 < t < 1, 3 < \alpha \le 4, \\ u(0) = D_q u(0) = D_q^2 u(0) = 0, & D_q^2 u(1) = \beta D_q^2 u(\eta), \end{cases} \tag{3}$$

where $0 < \eta < 1$ and $1 - \beta\eta^{\alpha-3} > 0$.

In [10], Guo and Kang obtained the existence and uniqueness of a positive solution for the fractional q-difference equation of the form

$$\begin{cases} D_q^\alpha u(t) + f(t, u(t), u(t)) + g(t, u(t)) = 0, & 0 < t < 1, 1 < \alpha \le 2, \\ u(0) = 0, & u(1) = \beta u(\eta), \end{cases} \tag{4}$$

by virtue of fixed-point theorems for the mixed monotone operator. Here, $1 < \alpha \le 2$ and $0 < \beta\eta^{\alpha-1} < 1$.

Recently, by using the monotone iterative approach, in [11], Wang investigated the iterative positive solutions of the following fractional q-difference equations with three-point boundary conditions:

$$\begin{cases} D_q^\alpha u(x) + \lambda h(x) f(u(x)) = 0, & 0 < x < 1, 2 < \alpha \leq 3, \\ u(0) = D_q u(0) = D_q u(1) = 0. \end{cases} \tag{5}$$

It should be noted that the existence of positive solutions of problem (5) had been studied by Li et al. [12] by means of a fixed-point theorem in cones. The novel idea of [11] is to find the positive solution.

Motivated by the above mentioned works, in this paper, we consider the following system of fractional q-difference equations with four-point boundary conditions:

$$\begin{cases} D_q^\alpha u(t) + f(t, v(t)) = 0, & 0 < t < 1, \\ D_q^\beta v(t) + g(t, u(t)) = 0, & 0 < t < 1, \\ u(0) = 0, \\ u(1) = \gamma_1 u(\eta_1), \\ v(0) = 0, \\ v(1) = \gamma_2 u(\eta_2), \end{cases} \tag{6}$$

where D_q^α and D_q^β are the fractional Riemann–Liouville q-derivative of order α and β, respectively, $0 < q < 1$, $1 < \beta \leq \alpha \leq 2$, $0 < \eta_1, \eta_2 < 1$, $0 < \gamma_1 \eta_1^{\alpha-1} < 1$, and $0 < \gamma_2 \eta_2^{\beta-1} < 1$.

By using the monotone iterative approach, in this paper, we will construct two convergent monotone iterative schemes for seeking one coupled positive solution and obtain the coupled positive solution of problem (6). To the best of our knowledge, there is no paper to study the iterative coupled positive solutions for the coupled system of fractional q-difference boundary value problems. It is noted that we may investigate the approximate solutions of problem (6) by numerical approximation algorithms, which will be presented as another paper. For the latest development of numerical approximation algorithms of some boundary value problems, see [13–17] and the references therein.

2. Preliminaries

Let $q \in (0, 1)$, the q-derivative of a function f is defined by

$$\begin{aligned} \left(D_q f\right)(x) &= \frac{f(qx) - f(x)}{(q-1)x}, \\ \left(D_q f\right)(0) &= \lim_{x \longrightarrow 0} \left(D_q f\right)(x), \end{aligned} \tag{7}$$

and q-derivatives of higher order by

$$\begin{aligned} \left(D_q^0 f\right)(x) &= f(x), \\ \left(D_q^n f\right)(x) &= D_q\left(D_q^{n-1} f\right)(x), \quad n \in N. \end{aligned} \tag{8}$$

The q-integral of a function f defined in the interval $[0, b]$ is given by

$$\left(I_q f\right)(x) = \int_0^x f(s) \mathrm{d}_q s = x(1-q) \sum_{k=0}^{\infty} f\left(x q^k\right) q^k, \quad x \in [0, b]. \tag{9}$$

Similar to the derivatives, the operator I_q^n is given by

$$\begin{aligned} \left(I_q^0 f\right)(x) &= f(x), \\ \left(I_q^n f\right)(x) &= I_q\left(I_q^{n-1} f\right)(x), \quad n \in N. \end{aligned} \tag{10}$$

Define

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad a \in \mathbb{R}. \tag{11}$$

The q-analogue of the power function $(a-b)^n$ with $n \in \mathbb{N}_0$ is

$$\begin{aligned} (a-b)^0 &= 1, \\ (a-b)^n &= \prod_{k=0}^{n-1} \left(a - b q^k\right), \quad n \in \mathbb{N}, a, b \in \mathbb{R}. \end{aligned} \tag{12}$$

Moreover, if $\alpha \in \mathbb{R}$, then

$$(a-b)^{(\alpha)} = a^\alpha \prod_{n=0}^{\infty} \frac{a - bq^n}{a - bq^{\alpha+n}}. \tag{13}$$

Remark 1. If $b = 0$, then $a^{(\alpha)} = a^\alpha$. If $\alpha > 0$ and $a \leq b \leq t$, then $(t-a)^{(\alpha)} \geq (t-b)^{(\alpha)}$.

The q-gamma function [18] is defined by

$$\Gamma_q(x) = \frac{(1-q)^{(x-1)}}{(1-q)^{x-1}}, \quad x \in \mathbb{R} \backslash \{0, -1, -2, \ldots\}, \tag{14}$$

and satisfies $\Gamma_q(x+1) = [x]_q \Gamma_q(x)$.

Definition 1. We say (u_*, v_*) is a solution of system (6), if (u_*, v_*) satisfies the first and second equations of (6) and boundary conditions of (6).

Definition 2 (see [19]). Let $\alpha > 0$ and f be a function defined on $[0, 1]$. The fractional q-integral of the Riemann–Liouville type is

$$\left(I_q^\alpha f\right)(x) = \frac{1}{\Gamma_q(\alpha)} \int_0^x (x - qt)^{(\alpha-1)} f(t) \mathrm{d}_q t, \quad x \in [0, 1]. \tag{15}$$

Definition 3 (see [19]). The fractional q-derivative of the Riemann–Liouville type is defined by

$$\left(D_q^\alpha f\right)(x) = \left(D_q^n I_q^{n-\alpha} f\right)(x), \quad \alpha > 0, \tag{16}$$

where n is the smallest integer greater than or equal to α.

Lemma 1 (see [19]). *Let $\alpha, \beta \geq 0$ and f be a function defined on $[0, 1]$. Then, the following formulas hold:*

(1) $(I_q^\beta I_q^\alpha f)(x) = (I_q^{\alpha+\beta} f)(x)$,

(2) $(D_q^\alpha I_q^\alpha f)(x) = f(x)$.

Lemma 2 (see [13]). *Let $\alpha > 0$ and n be a positive integer. Then, the following equality holds:*

$$\left(I_q^\alpha D_q^n f\right)(x) = \left(D_q^n I_q^\alpha f\right)(x) - \sum_{k=0}^{n-1} \frac{x^{\alpha-n+k}}{\Gamma_q(\alpha + k - n + 1)} \left(D_q^k f\right)(0). \tag{17}$$

By Lemmas 1 and 2, Guo and Kang in [10] obtained the following lemma.

Lemma 3. *For any $g \in C[0,1]$, the boundary value problem*

$$\begin{cases} D_q^{\alpha} u(t) + g(t) = 0, & 0 < t < 1, \\ u(0) = 0, & u(1) = \gamma_1 u(\eta_1), \end{cases} \tag{18}$$

has a unique solution:

$$u(t) = \int_0^1 G_1(t, qs) g(s) \mathrm{d}_q s, \tag{19}$$

where

$$G_1(t,qs) = \begin{cases} \dfrac{t^{\alpha-1}(1-qs)^{(\alpha-1)} - t^{\alpha-1}\gamma_1(\eta_1 - qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)}(1-\gamma_1\eta_1^{\alpha-1})}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})}, & 0 \le qs \le t \le 1, qs \le \eta_1, \\ \dfrac{t^{\alpha-1}(1-qs)^{(\alpha-1)} - (t-qs)^{(\alpha-1)}(1-\gamma_1\eta_1^{\alpha-1})}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})}, & 0 \le \eta_1 \le qs \le t \le 1, \\ \dfrac{t^{\alpha-1}(1-qs)^{(\alpha-1)} - t^{\alpha-1}\gamma_1(\eta - qs)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})}, & 0 \le t \le qs \le 1, \\ \dfrac{t^{\alpha-1}(1-qs)^{(\alpha-1)}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})}, & 0 \le t \le qs \le 1, \eta_1 \le qs, \end{cases} \tag{20}$$

is the Green function of BVP (18).

Similarly, we have the following.

Lemma 4. *For any $h \in C[0,1]$, the boundary value problem*

$$\begin{cases} D_q^{\beta} v(t) + h(t) = 0, & 0 < t < 1, \\ v(0) = 0, & v(1) = \gamma_2 v(\eta_2), \end{cases} \tag{21}$$

has a unique solution:

$$v(t) = \int_0^1 G_2(t, qs) h(s) \mathrm{d}_q s, \tag{22}$$

where

$$G_2(t,qs) = \begin{cases} \dfrac{t^{\beta-1}(1-qs)^{(\beta-1)} - t^{\beta-1}\gamma_2(\eta_2 - qs)^{(\beta-1)} - (t-qs)^{(\beta-1)}(1-\gamma_2\eta_2^{\beta-1})}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})}, & 0 \le qs \le t \le 1, qs \le \eta_2, \\ \dfrac{t^{\beta-1}(1-qs)^{(\beta-1)} - (t-qs)^{(\beta-1)}(1-\gamma_2\eta_2^{\beta-1})}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})}, & 0 \le \eta_2 \le qs \le t \le 1, \\ \dfrac{t^{\beta-1}(1-qs)^{(\beta-1)} - t^{\beta-1}\gamma_2(\eta_2 - qs)^{(\beta-1)}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})}, & 0 \le t \le qs \le 1, \\ \dfrac{t^{\beta-1}(1-qs)^{(\beta-1)}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})}, & 0 \le t \le qs \le 1, \eta_2 \le qs, \end{cases} \tag{23}$$

is the Green function of BVP (21).

Lemma 5. *(see [10]). For $G_1(t,qs)$ and $G_2(t,qs)$ defined as in Lemmas 3 and 4, respectively, we have*

(i) $G_1(t,qs)$ and $G_2(t,qs)$ are two continuous functions

(ii) $(M_1 qs(1-qs)^{(\alpha-1)}/\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1}))t^{\alpha-1} \le G_1(t, qs) \le ((1-qs)^{(\alpha-1)}/\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1}))t^{\alpha-1}$, $\forall 0 \le t, s \le 1$, where $0 < M_1 = \min\{1-\gamma_1\eta_1^{\alpha-1}, \gamma_1\eta_1^{\alpha-2}(1-\eta_1), \gamma_1\eta_1^{\alpha-1}\} < 1$

(iii) $(M_2 qs(1-qs)^{(\beta-1)}/\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1}))t^{\beta-1} \le G_2(t, qs) \le ((1-qs)^{(\beta-1)}/\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1}))t^{\beta-1}$, $\forall 0 \le t, s \le 1$, *where* $0 < M_2 = \min\left\{1-\gamma_2\eta_2^{\beta-1}, \gamma_2\eta_2^{\beta-2}(1-\eta_2), \gamma_2\eta_2^{\beta-1}\right\} < 1$

3. Main Result

In this paper, we will employ the Banach space $C[0,1]$, equipped with norm $\|u\| = \sup_{t\in[0,1]}|u(t)|$ for each $u \in C[0,1]$. Define two cones P_1 and P_2 in $C[0,1]$ as follows:

$$\begin{aligned} P_1 &= \left\{u \in C[0,1] \mid \text{there exist two positive numbers } 0 < a_1 < 1 < b_1, \text{ such that } a_1 t^{\alpha-1} \le u(t) \le b_1 t^{\alpha-1}, t \in [0,1]\right\}, \\ P_2 &= \left\{v \in C[0,1] \mid \text{there exist two positive numbers } 0 < a_2 < 1 < b_2, \text{ such that } a_2 t^{\beta-1} \le v(t) \le b_2 t^{\beta-1}, t \in [0,1]\right\}. \end{aligned} \tag{24}$$

Now, we define the operators $T_i: C[0,1] \longrightarrow C[0,1]$ $(i = 1, 2)$ by

$$\begin{aligned} T_1 v(t) &= \int_0^1 G_1(t, qs) f(s, v(s)) \mathrm{d}_q s, \\ T_2 u(t) &= \int_0^1 G_2(t, qs) g(s, u(s)) \mathrm{d}_q s. \end{aligned} \tag{25}$$

From Lemmas 3 and 4, BVP (6) can be transformed into the following system of integral equations:

$$\begin{cases} u(t) = \displaystyle\int_0^1 G_1(t, qs) f(s, v(s)) \mathrm{d}_q s, \\ v(t) = \displaystyle\int_0^1 G_2(t, qs) g(s, u(s)) \mathrm{d}_q s, \end{cases} \tag{26}$$

By (26), we know that (u_*, v_*) is a solution of (6) if and only if $u_* = T_1 v_*$ and $v_* = T_2 u_*$.

In order to facilitate our investigation, we make the following assumptions:

(H1) $f \in C([0,1] \times [0,\infty), [0,\infty))$ is nondecreasing with respect to v, and there exists a positive constant $\sigma_1 > 1$, such that

$$f(t, rv) \ge r^{\sigma_1} f(t, v), \quad \forall t \in [0,1], v \in [0,+\infty), r \in (0,1]. \tag{27}$$

(H2) $g \in C([0,1] \times [0,\infty), [0,\infty))$ is nondecreasing with respect to u, and there exists a positive constant $0 < \sigma_2 < 1$, such that

$$g(t, ru) \ge r^{\sigma_2} g(t, u), \quad \forall t \in [0,1], u \in [0,+\infty), r \in (0,1]. \tag{28}$$

(H3) $0 < \int_0^1 (1-qs)^{(\alpha-1)} f(s,1) \mathrm{d}_q s < +\infty$.

(H4) $0 < \int_0^1 (1-qs)^{(\beta-1)} g(s,1) \mathrm{d}_q s < +\infty$.

Remark 2. The conditions (H1) and (H2) imply that, for $\forall r > 1$, we have $f(t, rv) \le r^{\sigma_1} f(t, v)$ and $g(t, ru) \le r^{\sigma_2} g(t, u)$.

Theorem 1. *Assume that conditions (H1)–(H4) hold and there exist two positive constants R_1 and R_2 such that*

$$\frac{1}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})} \int_0^1 (1-qs)^{(\alpha-1)} f(s,1) \mathrm{d}_q s \le R_1^{1-\sigma_1}, \tag{29}$$

$$\frac{1}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})} \int_0^1 (1-qs)^{(\beta-1)} g(s,1) \mathrm{d}_q s \le R_2^{1-\sigma_2}, \tag{30}$$

then the fractional q-difference system (6) has one positive solution (u^*, v^*), where $u^* \in P_1$ and $v^* \in P_2$. Moreover, for each $t \in [0,1]$, there exist constants $0 < m_i < 1 < n_i$ $(i = 1, 2)$, such that

$$\begin{aligned} u^*(t) &\in \left[m_1 t^{\alpha-1}, n_1 t^{\alpha-1}\right], \\ v^*(t) &\in \left[m_2 t^{\beta-1}, n_2 t^{\beta-1}\right], \end{aligned} \tag{31}$$

which can be obtained by monotone iterative schemes $\{u_n\}$ and $\{v_n\}$ generated by

$$\begin{aligned} u_n(t) &= \int_0^1 G_1(t, qs) f(s, v_{n-1}(s)) \mathrm{d}_q s, \\ v_n(t) &= \int_0^1 G_2(t, qs) g(s, u_{n-1}(s)) \mathrm{d}_q s. \end{aligned} \tag{32}$$

i.e., $\|u_n - u^*\| \longrightarrow 0$ and $\|v_n - v^*\| \longrightarrow 0$ as $n \longrightarrow \infty$.

Proof. For any $v \in P_2$, we know that there exist two constants a_1 and b_1 with $0 < a_1 < 1 < b_1$ such that

$$a_1 t^{\beta-1} \le v(t) \le b_1 t^{\beta-1}, \quad t \in [0,1]. \tag{33}$$

From Lemma 5 and condition (H1), we obtain

$$
\begin{aligned}
T_1 v(t) &= \int_0^1 G_1(t, qs) f(s, v(s)) \mathrm{d}_q s \\
&\geq \frac{q M_1 t^{\alpha-1}}{\Gamma_q(\alpha)\left(1-\gamma_1 \eta_1^{\alpha-1}\right)} \int_0^1 s(1-qs)^{(\alpha-1)} f(s, v(s)) \mathrm{d}_q s \\
&\geq \frac{q M_1 t^{\alpha-1}}{\Gamma_q(\alpha)\left(1-\gamma_1 \eta_1^{\alpha-1}\right)} \int_0^1 s(1-qs)^{(\alpha-1)} f\left(s, a_1 s^{\beta-1}\right) \mathrm{d}_q s \\
&\geq \frac{q M_1 a_1^{\sigma_1} t^{\alpha-1}}{\Gamma_q(\alpha)\left(1-\gamma_1 \eta_1^{\alpha-1}\right)} \int_0^1 s(1-qs)^{(\alpha-1)} f\left(s, s^{\beta-1}\right) \mathrm{d}_q s \\
&\geq c_1 t^{\alpha-1}, \\
T_1 v(t) &\leq \frac{t^{\alpha-1}}{\Gamma_q(\alpha)\left(1-\gamma_1 \eta_1^{\alpha-1}\right)} \int_0^1 (1-qs)^{(\alpha-1)} f(s, v(s)) \mathrm{d}_q s \\
&\leq \frac{t^{\alpha-1}}{\Gamma_q(\alpha)\left(1-\gamma_1 \eta_1^{\alpha-1}\right)} \int_0^1 (1-qs)^{(\alpha-1)} f\left(s, b_1 s^{\beta-1}\right) \mathrm{d}_q s \\
&\leq \frac{b_1^{\sigma_1} t^{\alpha-1}}{\Gamma_q(\alpha)\left(1-\gamma_1 \eta_1^{\alpha-1}\right)} \int_0^1 (1-qs)^{(\alpha-1)} f\left(s, s^{\beta-1}\right) \mathrm{d}_q s \\
&\leq d_1 t^{\alpha-1},
\end{aligned}
\tag{34}
$$

where d_1 and c_1 are two positive constants satisfying

$$
\begin{aligned}
d_1 &> \max\left\{1, \frac{b_1^{\sigma_1}}{\Gamma_q(\alpha)\left(1-\gamma_1 \eta_1^{\alpha-1}\right)} \int_0^1 (1-qs)^{(\alpha-1)} f\left(s, s^{\beta-1}\right) \mathrm{d}_q s\right\}, \\
0 < c_1 &< \min\left\{1, \frac{q M_1 a_1^{\sigma_1}}{\Gamma_q(\alpha)\left(1-\gamma_1 \eta_1^{\alpha-1}\right)} \int_0^1 s(1-qs)^{(\alpha-1)} f\left(s, s^{\beta-1}\right) \mathrm{d}_q s\right\}.
\end{aligned}
\tag{35}
$$

Thus, T_1 maps P_2 into P_1. For each $u \in P_1$, there exist two constants a_2 and b_2 with $0 < a_2 < 1 < b_2$ such that

$$a_2 t^{\alpha-1} \leq u(t) \leq b_2 t^{\alpha-1}, \quad t \in [0,1]. \tag{36}$$

Similarly, by Lemma 5 and condition (H2), we can get that

$$c_2 t^{\beta-1} \leq T_2 u(t) \leq d_2 t^{\beta-1}, \tag{37}$$

where d_2 and c_2 are two positive constants satisfying

$$
\begin{aligned}
d_2 &> \max\left\{1, \frac{b_2^{\sigma_2}}{\Gamma_q(\beta)\left(1-\gamma_2 \eta_2^{\beta-1}\right)} \int_0^1 (1-qs)^{(\beta-1)} g\left(s, s^{\alpha-1}\right) \mathrm{d}_q s\right\}, \\
0 < c_2 &< \min\left\{1, \frac{q M_2 a_2^{\sigma_2}}{\Gamma_q(\beta)\left(1-\gamma_2 \eta_2^{\beta-1}\right)} \int_0^1 s(1-qs)^{(\beta-1)} g\left(s, s^{\alpha-1}\right) \mathrm{d}_q s\right\},
\end{aligned}
\tag{38}
$$

which implies that T_2 maps P_1 into P_2. On the other hand, the proof of completely continuous T_1 and T_2 are as the same as in [12], and we omit it here.

Let $P_i(R) = \{u \mid u \in P_i, \|u\| \le R\}\ (i = 1, 2)$. In the following, we will prove $T_1(P_2(R_1)) \subset P_1(R_1)$ and $T_2(P_1(R_2)) \subset P_2(R_2)$. In fact, for any $v \in P_2(R_1)$ and $u \in P_1(R_2)$, by conditions (29) and (30), we obtain

$$\begin{aligned}
T_1 v(t) &\le \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})}\int_0^1 (1-qs)^{(\alpha-1)} f(s, v(s))\,\mathrm{d}_q s \\
&\le \frac{t^{\alpha-1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})}\int_0^1 (1-qs)^{(\alpha-1)} f(s, R_1)\,\mathrm{d}_q s \\
&\le \frac{R_1^{\sigma_1}}{\Gamma_q(\alpha)(1-\gamma_1\eta_1^{\alpha-1})}\int_0^1 (1-qs)^{(\alpha-1)} f(s, 1)\,\mathrm{d}_q s \\
&\le R_1, \\
T_2 u(t) &\le \frac{t^{\beta-1}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})}\int_0^1 (1-qs)^{(\beta-1)} g(s, u(s))\,\mathrm{d}_q s \\
&\le \frac{t^{\beta-1}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})}\int_0^1 (1-qs)^{(\beta-1)} g(s, R_2)\,\mathrm{d}_q s \\
&\le \frac{R_2^{\sigma_2}}{\Gamma_q(\beta)(1-\gamma_2\eta_2^{\beta-1})}\int_0^1 (1-qs)^{(\beta-1)} g(s, 1)\,\mathrm{d}_q s \\
&\le R_2,
\end{aligned} \tag{39}$$

which implies that $\|T_1 v\| \le R_1$ and $\|T_2 u\| \le R_2$. So, $T_1(P_2(R_1)) \subset P_1(R_1)$ and $T_2(P_1(R_2)) \subset P_2(R_2)$.

Taking $e_1(t) = t^{\alpha-1}$ and $e_2(t) = t^{\beta-1}$, then $e_1 \in P_1$, $e_2 \in P_2$, $T_1(e_2) \in P_1$, and $T_2(e_1) \in P_2$. Thus, there exist constants $0 < m_i < 1 < n_i\ (i = 1, 2)$ such that

$$\begin{aligned}
m_1 t^{\alpha-1} &\le T_1 e_2(t) \le n_1 t^{\alpha-1}, \\
m_2 t^{\beta-1} &\le T_2 e_1(t) \le n_2 t^{\beta-1}.
\end{aligned} \tag{40}$$

Let l_1 and l_2 be two positive numbers satisfying $0 < l_1 < l_2 < 1$, $l_2 < l_1^{\sigma_2}$, and

$$\begin{aligned}
l_1 l_2^{-\sigma_1} &\le m_1, \\
l_2 l_1^{-\sigma_2} &\le m_2.
\end{aligned} \tag{41}$$

Set

$$\begin{aligned}
u_0(t) &= l_1 e_1(t), \\
v_0(t) &= l_2 e_2(t),
\end{aligned} \tag{42}$$

$$\begin{aligned}
u_n &= T_1 v_{n-1}, \\
v_n &= T_2 u_{n-1}, \\
n &= 1, 2, \ldots
\end{aligned} \tag{43}$$

Obviously, $u_0(t) \le v_0(t)$ by $\beta \le \alpha$ and $0 < l_1 < l_2 < 1$, $u_0 \in P_1(R_2)$ and $v_0 \in P_2(R_1)$. By (H1) and (H2), we have

$$\begin{aligned}
u_1(t) &= T_1 v_0(t) = \int_0^1 G_1(t, qs) f(s, v_0(s))\,\mathrm{d}_q s \\
&= \int_0^1 G_1(t, qs) f(s, l_2 e_2(s))\,\mathrm{d}_q s \\
&\ge l_2^{\sigma_1}\int_0^1 G_1(t, qs) f(s, e_2(s))\,\mathrm{d}_q s \\
&= l_2^{\sigma_1} T_1 e_2(t) \ge l_2^{\sigma_1} m_1 e_1(t) \ge l_1 e_1(t) = u_0(t), \\
v_1(t) &= T_2 u_0(t) = \int_0^1 G_2(t, qs) g(s, u_0(s))\,\mathrm{d}_q s \\
&= \int_0^1 G_2(t, qs) g(s, l_1 e_1(s))\,\mathrm{d}_q s \\
&\ge l_1^{\sigma_2}\int_0^1 G_2(t, qs) g(s, e_1(s))\,\mathrm{d}_q s \\
&= l_1^{\sigma_2} T_2 e_1(t) \ge l_1^{\sigma_2} m_2 e_2(t) \ge l_2 e_2(t) = v_0(t).
\end{aligned} \tag{44}$$

From conditions (H1) and (H2), we know that T_1 and T_2 are two nondecreasing operators. Thus, by induction, we can obtain

$$\begin{aligned}
u_0 \le u_1 \le \cdots &\le u_n \le \cdots, \\
v_0 \le v_1 \le \cdots &\le v_n \le \cdots, \\
u_n &\in P_1(R_1), \\
v_n &\in P_2(R_2), \\
n &= 1, 2, \ldots.
\end{aligned} \tag{45}$$

By the compactness of the operators T_1 and T_2, we have that $\{u_n\}$ and $\{v_n\}$ are two sequentially compact sets. Therefore, there exist $u_* \in P_1(R_1)$ and $v_* \in P_2(R_2)$, such that u_n converges to u_* and v_n converges to v_* as $n \longrightarrow \infty$, respectively. Since the operators T_1 and T_2 are continuous, $u_n = T_1 v_{n-1}$ and $v_n = T_2 u_{n-1}$, and we obtain $u^* = T_1 v^*$ and $v^* = T_2 u^*$ as $n \longrightarrow \infty$, which implies that system (6) has a positive solution (u^*, v^*), and $u^* \in [m_1 t^{\alpha-1}, n_1 t^{\alpha-1}]$, $v^* \in [m_2 t^{\beta-1}, n_2 t^{\beta-1}]$, and $\forall t \in [0, 1]$, where m_i and n_i are constants and $0 < m_i < 1 < n_i\ (i = 1, 2)$, which can be achieved by the monotone scheme:

$$\begin{aligned}
u_n(t) &= \int_0^1 G_1(t, qs) f(s, v_{n-1}(s))\,\mathrm{d}_q s, \\
v_n(t) &= \int_0^1 G_2(t, qs) g(s, u_{n-1}(s))\,\mathrm{d}_q s,
\end{aligned} \tag{46}$$

with initial values $u_0(t)$ and $v_0(t)$ defined as in (42).

In the following, we give an example to illustrate the existence of positive solutions of BVP (6). □

Example 1. Consider the following system of fractional q-difference with boundary conditions:

$$\begin{cases} D_{(1/3)}^{(5/3)}u(t)+\frac{1}{8}tv^{(3/2)}(t)=0, & 0<t<1, \\ D_{(1/3)}^{(3/2)}v(t)+\sqrt{t}\left(u^{(1/4)}(t)+\frac{u^{(1/3)}(t)}{1+u^{(1/4)}(t)}\right)=0, & 0<t<1, \\ u(0)=0, & u(1)=u\left(\frac{3}{4}\right), \\ v(0)=0, & v(1)=\frac{5}{4}v\left(\frac{1}{2}\right), \end{cases} \tag{47}$$

where $q=(1/3)$, $\alpha=(5/3)$, $\beta=(3/2)$, $\eta_1=(3/4)$, $\eta_2=(1/2)$, $\gamma_1=1$, $\gamma_2=(5/4)$, and

$$f(t,v)=\frac{1}{8}tv^{(3/2)}, \qquad g(t,u)=\sqrt{t}\left(u^{(1/4)}+\frac{u^{(1/3)}}{1+u^{(1/4)}}\right), \tag{48}$$

Obviously, $f(t,v)$ and $g(t,u)$ are nondecreasing with respect to v and u, respectively, and

$$0<\gamma_1\eta_1^{\alpha-1}=\left(\frac{3}{4}\right)^{(2/3)}<1, \qquad 0<\gamma_2\eta_2^{\beta-1}=\frac{5}{4}\left(\frac{1}{2}\right)^{(1/2)}<1. \tag{49}$$

Choosing $\sigma_1=2>1$ and $\sigma_2=(1/3)<1$, we have

$$\begin{aligned} f(t,rv)&=\frac{1}{8}tr^{(3/2)}v^{(3/2)}\geq r^2 f(t,v), \quad \forall v\in[0,+\infty), r\in(0,1], \\ g(t,ru)&=\sqrt{t}\left(r^{(1/4)}u^{(1/4)}+\frac{r^{(1/3)}u^{(1/3)}}{1+r^{(1/4)}u^{(1/4)}}\right), \\ &\geq \sqrt{t}\left(r^{(1/3)}u^{(1/4)}+\frac{r^{(1/3)}u^{(1/3)}}{1+u^{(1/4)}}\right)=r^{(1/3)}g(t,u), \\ &\forall u\in[0,+\infty), r\in(0,1]. \end{aligned} \tag{50}$$

So, conditions (H1) and (H2) hold. Moreover, we can show that

$$\begin{aligned} 0&<\int_0^1\left(1-\frac{1}{3}s\right)^{(2/3)}f(s,1)d_qs=\frac{1}{8}\int_0^1\left(1-\frac{1}{3}s\right)^{(2/3)}sd_qs<\infty, \\ 0&<\int_0^1\left(1-\frac{1}{3}s\right)^{(1/2)}g(s,1)d_qs=\frac{3}{2}\int_0^1\left(1-\frac{1}{3}s\right)^{(1/2)}\sqrt{s}d_qs<\infty, \end{aligned} \tag{51}$$

which implies that (H3) and (H4) hold. Moreover, we know that there exist two positive constants R_1 and R_2 such that (29) and (30) hold, respectively. Thus, it follows from Theorem 1 that boundary value problem of fractional q-difference system (47) has one iterative positive solution (u^*,v^*) which can be obtained with the aid of monotone iterative sequences.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the manuscript.

Acknowledgments

This work was supported by the Natural Science Foundation of China (11571136).

References

[1] F. Jackon, "On q-functions and a certain difference operator," *Transactions of the Royal Society of Edinburgh Earth Sciences*, vol. 46, pp. 253–281, 1908.

[2] F. Jackon, "On q-definite integrals," *The Quarterly Journal of Pure and Applied Mathematics*, vol. 41, pp. 193–203, 1910.

[3] T. Ernst, *A Comprehensive Treatment of q-Calculus*, Springer, Basel, Switzerland, 2012.

[4] B. Ahmad, S. Ntouyas, and J. Tariboon, *Quautum Calculus: New Concepts, Impulsive IVPs and BVPs, Inequalities*, World Scientific, Singapore, 2016.

[5] W. A. Al-Salam, "Some fractional q-integrals and q-derivatives," *Proceedings of the Edinburgh Mathematical Society*, vol. 15, no. 2, pp. 135–140, 1966.

[6] R. P. Agarwal, "Certain fractional q-integrals and q-derivatives," *Mathematical Proceedings of the Cambridge Philosophical Society*, vol. 66, no. 2, pp. 365–370, 1969.

[7] R. Ferreira, "Nontrivial solutions for fractional q-difference boundary value problems," *Electronic Journal of Qualitative Theory of Differential Equations*, vol. 70, p. 2010, 2010.

[8] R. A. C. Ferreira, "Positive solutions for a class of boundary value problems with fractional q-differences," *Computers & Mathematics with Applications*, vol. 61, no. 2, pp. 367–373, 2011.

[9] R. Garzi and B. Agheli, "Existence results to positive solution of fractional BVP with q-derivatives," *Journal of Applied*

Mathematics and Computing, vol. 55, no. 1-2, pp. 353–367, 2017.

[10] F. Guo and S. Kang, "Positive solutions for a class of fractional boundary value problem with q-derivatives," *Mediterranean Journal of Mathematics*, vol. 113, pp. 1–16, 2019.

[11] G. Wang, "Twin iterative positive solutions of fractional q-difference Schrödinger equations," *Applied Mathematics Letters*, vol. 76, pp. 103–109, 2018.

[12] X. Li, Z. Han, and X. Li, "Boundary value problems of fractional q-difference Schröinger equations," *Applied Mathematics Letters*, vol. 46, pp. 100–105, 2015.

[13] G. Kanwal, A. Ghaffar, M. M. Hafeezullah, S. A. Manan, M. Rizwan, and G. Rahman, "Numerical solution of 2-point boundary value problem by subdivision scheme," *Communications in Mathematics and Applications*, vol. 10, no. 1, pp. 19–29, 2019.

[14] S. A. Manan, A. Ghaffar, M. Rizwan, G. Rahman, and G. Kanwal, "A subdivision approach to the approximate solution of 3rd order boundary value problem," *Communications in Mathematics and Applications*, vol. 9, no. 4, pp. 499–512, 2018.

[15] A. Khalid, M. N. Naeem, Z. Ullah et al., "Numerical solution of the boundary value problems arising in magnetic fields and cylindrical shells," *Mathematics*, vol. 7, no. 508, pp. 1–20, 2019.

[16] A. Khalid, M. N. Naeem, P. Agarwal, A. Ghaffar, Z. Ullah, and S. Jain, "Numerical approximation for the solution of linear sixth order boundary value problems by cubic B-spline," *Advances in Difference Equations*, vol. 2019, no. 492, pp. 1–16, 2019.

[17] U. Filobello-Nino, H. Vazquez-Leal, M. A. Fariborzi Araghi et al., "A novel distribution and optimization procedure of boundary conditions to enhance the classical perturbation method applied to solve some relevant heat problems," *Discrete Dynamics in Nature and Society*, vol. 2020, Article ID 1303701, 12 pages, 2020.

[18] P. M. Rajković, S. D. Marinković, and M. S. Stanković, "Fractional integrals and derivatives in q-calculus," *Applicable Analysis and Discrete Mathematics*, vol. 1, pp. 311–323, 2007.

[19] M. Annaby and Z. Mansour, "q-fractional calculus and equations,"*Lecture Notes in Mathematics,* vol. 2056, Springer, Berlin, Germany, 2012.