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Research Article

Unicity of Meromorphic Solutions of the Pielou Logistic Equation

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This paper mainly considers the unicity of meromorphic solutions of the Pielou logistic equation $y(z+1) = ((R(z)y(z))/(Q(z)+P(z)y(z)))$, where $P(z), Q(z)$, and $R(z)$ are nonzero polynomials. It shows that the finite order transcendental meromorphic solution of the Pielou logistic equation is mainly determined by its poles and 1-value points. Examples are given for the sharpness of our result.

1. Introduction

For a meromorphic function $f(z)$, we use standard notations of the Nevanlinna theory, such as $T(r,f)$, $m(r,f)$, and $N(r,f)$ (see, e.g., [1–3]). Let $S(r,f)$ denote any quantity that satisfies $S(r,f) = o(T(r,f))$ as $r \longrightarrow \infty$ possibly outside of an exceptional set of finite logarithmic measure. And we define the order of growth of $f(z)$ by

$$\rho(f) := \limsup_{r \longrightarrow \infty} \frac{\log T(r,f)}{\log r}. \quad (1)$$

Also we know that the unicity of solutions of a given equation is always one of its most essential properties. This paper is to discuss the unicity of meromorphic solutions of the Pielou logistic equation

$$y(z+1) = \frac{R(z)y(z)}{Q(z)+P(z)y(z)}, \quad (2)$$

where $P(z), Q(z)$, and $R(z)$ are nonzero polynomials. Equation (2) is an important equation generalized from the famous Verhulst-Pearl equation, which is the most popular continuous model of growth of a population:

$$x'(t) = x(t)[a - bx(t)], \quad a, b > 0. \quad (3)$$

By denoting $f(z) = 1/y(z)$, we can get from (2) that

$$R(z)f(z+1) - Q(z)f(z) = P(z), \quad (4)$$

which is a linear difference equation.

On the growth, zeros, and poles of meromorphic solutions of (2) and (4), Chen proved numbers of significant results in [4]. Then, Cui and Chen [5, 6] began to consider the unicity of meromorphic solutions concerning their zeros, 1-value points, and poles and proved.

Theorem 1 (see [5]). *Let $f(z)$ be a finite order transcendental meromorphic solution of the equation*

$$P_1(z)f(z+1) + P_2(z)f(z) = 0, \quad (5)$$

where $P_1(z)$ and $P_2(z)$ are nonzero polynomials such that $P_1(z) + P_2(z) \equiv 0$. If a meromorphic function $g(z)$ shares $0, 1, \infty$ CM with $f(z)$, then either $f(z) \equiv g(z)$ or $f(z)g(z) \equiv 1$.

Theorem 2 (see [6]). *Let $f(z)$ be a finite order transcendental meromorphic solution of the equation*

$$P_1(z)f(z+1) + P_2(z)f(z) = P_3(z), \quad (6)$$

where $P_1(z), P_2(z)$, and $P_3(z)$ are nonzero polynomials such that $P_1(z) + P_2(z) \equiv 0$. If a meromorphic function $g(z)$ shares $0, 1, \infty$ CM with $f(z)$, then one of the following cases holds:

(i) $f(z) \equiv g(z)$,

(ii) $f(z) + g(z) = f(z)g(z)$,

(iii) *There exists a polynomial* $\beta(z) = az + b_0$ *and a constant* a_0 *satisfying* $e^{a_0} \neq e^{b_0}$ *such that*

$$f(z) = \frac{1 - e^{\beta(z)}}{e^{\beta(z)}\left(e^{a_0 - b_0} - 1\right)},$$
$$g(z) = \frac{1 - e^{\beta(z)}}{1 - e^{b_0 - a_0}}, \tag{7}$$

where $a_0 \neq 0$, b_0 *are constants.*

Here and in the following, $f(z)$ and $g(z)$ are said to share the value a CM (IM), provided that $f(z) - a$ and $g(z) - a$ have the same zeros counting multiplicities (ignoring multiplicities). And $f(z)$ and $g(z)$ are said to share the value ∞ CM (IM), provided that $f(z)$ and $g(z)$ have the same poles with the same multiplicities (ignoring multiplicities).

Cui and Chen's work is a natural product of generalization work (see, e.g., [1, 3, 7–11]) on the famous Nevanlinna's 5 IM (4 CM) Theorem (see, e.g., [3, 12]) during the past, about 90 years, especially of the hot research studies on the complex differences and complex difference equations (see, e.g., [1, 4, 8–10, 13–15]) recently. They have given examples to show that all cases of Theorem A and Theorem B can happen, and the numbers of shared values cannot be reduced. Li and Chen [16] turned to consider the following question: What can we say about the unicity of finite order transcendental meromorphic solutions of the equation

$$R_1(z)f(z+1) + R_2(z)f(z) = R_3(z), \tag{8}$$

where $R_1(z) \equiv 0, R_2(z), R_3(z)$ are rational functions? And we proved some interesting results and also provided some examples for sharpness of them. Two of those results read are as follows.

Theorem 3 (see [16]). *Let* $f(z)$ *and* $g(z)$ *be two finite order transcendental meromorphic solutions of equation (8), where* $R_3(z) \equiv 0$. *Suppose that* $f(z)$ *and* $g(z)$ *share* $0, \infty$ *CM. Then, either* $f(z) \equiv g(z)$ *or*

$$f(z) = \frac{R_3(z)}{2R_2(z)}\left(e^{a_1 z + a_0} + 1\right), \quad g(z) = \frac{R_3(z)}{2R_2(z)}\left(e^{-a_1 z - a_0} + 1\right), \tag{9}$$

where a_1, a_0 *are constants such that* $e^{-a_1} = e^{a_1} = -1$, *and the coefficients of (8) satisfy* $R_1(z)R_3(z+1) \equiv R_3(z)R_2(z+1)$.

Theorem 4 (see [16]). *Let* $f(z)$ *and* $g(z)$ *be two finite order transcendental meromorphic solutions of equation (8), where*

$$R_1(z) + R_2(z) \equiv R_3(z),$$
$$R_1(z)\left[R_3(z+1) - R_1(z+1)\right] \equiv \left[R_3(z) - R_2(z)\right]R_2(z+1). \tag{10}$$

If $f(z)$ and $g(z)$ share $1, \infty$ CM, then $f(z) \equiv g(z)$.

Remark 1. Notice that $f(z)$ and $g(z)$ share 0 CM if and only if $1/f(z)$ and $1/g(z)$ share ∞ CM; $f(z)$ and $g(z)$ share ∞ CM if and only if $1/f(z)$ and $1/g(z)$ share 0 CM; and $f(z)$ and $g(z)$ share 1 CM if and only if $1/f(z)$ and $1/g(z)$ share 1 CM. As a result, for the unicity of finite order transcendental meromorphic solutions equation (2), we only need to consider the case that two CM shared values are $1, \infty$. Indeed, we prove the following Theorem 5, whose proof is different from that in [5, 6, 16].

Theorem 5. *Let* $x(z)$ *and* $y(z)$ *be two finite order transcendental meromorphic solutions of equation (2). If* $x(z)$ *and* $y(z)$ *share* $1, \infty$ *CM and one of the following cases holds:*

(i) $R(z) \equiv P(z) \equiv -Q(z)$

(ii) $R(z) \equiv P(z)$ *and* $x(z)$ *has infinitely many poles of multiplicity* ≥ 2

(iii) $R(z) \equiv P(z)$, $\rho(x)$ *is not an integer, and* $x(z)$ *has at most finitely many simple poles, then* $x(z) \equiv y(z)$

We give some examples for the sharpness of Theorem 5 as follows.

Example 1

(1) $x(z) = 2/(e^{\pi i z} + 1)$ and $y(z) = 2/\ (e^{-\pi i z} + 1)$ satisfy the equation

$$y(z+1) = \frac{y(z)}{-1 + y(z)}. \tag{11}$$

Here, $x(z)$ and $y(z)$ share $1, \infty$ CM such that they have infinitely many poles and $\rho(x) = \rho(y) = 1$ and $R(z) \equiv P(z) \equiv -Q(z) \equiv 1$. This example shows that Theorem 5 may not hold for the case $R(z) \equiv P(z) \equiv -Q(z)$.

(2) $x(z) = 1/(e^{\pi i z} + 1)$ and $y(z) = 1/(e^{-\pi i z} + 1)$ satisfy the equation

$$y(z+1) = \frac{y(z)}{-1 + 2y(z)}. \tag{12}$$

Here, $x(z)$ and $y(z)$ share $1, \infty$ CM such that they have infinitely many simple poles and $\rho(x) = \rho(y) = 1$ and $P(z) \equiv 2 \equiv R(z) \equiv -Q(z) \equiv 1$. This example shows that Theorem 5 may not hold for the case $R(z) \equiv P(z)$ if most (except finitely many) poles of $x(z)$ are simple or $\rho(x)$ is an integer.

Remark 2. It is interesting to ask a question: whether the shared condition "CM" is replaced by "IM" in Theorem 5. We have tried hard but failed to find some negative examples for this question. We conjecture that the conclusions in Theorem 5 still hold when the shared condition "CM" is replaced by "IM."

2. Proof of Theorem 5

To prove Theorem 5, we need the following lemma of Clunie (see, e.g., [1, 2]).

Lemma 1 (see [1, 2]). *Let $f(z)$ be a transcendental meromorphic solution of the equation*

$$f^n P(z,f) = Q(z,f), \tag{13}$$

where $P(z,f)$ and $Q(z,f)$ are polynomials in f and its derivatives with meromorphic coefficients, say $\{a_\lambda \mid \lambda \in I\}$, such that $m(r,a_\lambda) = S(r,f)$ for all $\lambda \in I$. If the total degree of $Q(z,f)$ as a polynomial in f and its derivatives is $\le n$, then

$$m(r, P(z,f)) = S(r,f). \tag{14}$$

Proof of Theorem 5. Since $x(z)$ and $y(z)$ are finite order transcendental solutions of equation (2) and share $1,\infty$ CM, without loss of generality, assume that $\rho(x) \ge \rho(y)$, and we get

$$x(z+1) = \frac{R(z)x(z)}{Q(z)+P(z)x(z)} := \frac{x(z)}{A(z)+B(z)x(z)}, \tag{15}$$

$$y(z+1) = \frac{R(z)y(z)}{Q(z)+P(z)y(z)} := \frac{y(z)}{A(z)+B(z)y(z)}, \tag{16}$$

$$\frac{y(z)-1}{x(z)-1} = e^{h(z)}, \tag{17}$$

where $h(z)$ is a polynomial such that $\deg h(z) \le \rho(x)$, and $A(z) = Q(z)/R(z), B(z) = P(z)/R(z)$ are rational functions.

If $e^h \equiv 1$, then our conclusion holds.

If $e^h \equiv 1$, then $e^{\bar{h}} \equiv 1$, and from (17), we have

$$\begin{aligned} y &= e^h x + 1 - e^h, \\ \bar{y} &= e^{\bar{h}}\bar{x} + 1 - e^{\bar{h}}. \end{aligned} \tag{18}$$

Here and in the following, we use the notations

$$\begin{aligned} \bar{f} &= f(z+1), \\ \bar{\bar{f}} &= f(z+2), \end{aligned} \tag{19}$$

for any given meromorphic function $f(z)$ for convenience.

Submitting (18) into (16), we have

$$e^{\bar{h}}\bar{x} = \frac{\left(Be^{\bar{h}} - B + 1\right)e^h x + C}{Be^h x + A + B\left(1-e^h\right)}, \tag{20}$$

where

$$C = A\left(e^{\bar{h}} - 1\right) + 1 - e^h + B\left(1-e^h\right)\left(e^{\bar{h}} - 1\right). \tag{21}$$

From (15) and (20), we obtain

$$\frac{\left(Be^{\bar{h}} - B + 1\right)e^h x + C}{Be^h x + A + B\left(1-e^h\right)} = e^{\bar{h}}\bar{x} = \frac{e^{\bar{h}}x}{A+Bx}, \tag{22}$$

or equally,

$$\begin{aligned} &B(B-1)\left(e^{\bar{h}}-1\right)e^h x^2 + \Big\{BC + A\left(Be^{\bar{h}} - B + 1\right)e^h \\ &\quad - \left[A + B\left(1-e^h\right)\right]e^{\bar{h}}\Big\}x + AC = 0. \end{aligned} \tag{23}$$

Next, we discuss three cases. □

Case 1. $R(z) \equiv P(z) \equiv -Q(z)$. Then, $A(z) \equiv -1$, $B(z) \equiv 1$, and

$$C = A\left(e^{\bar{h}} - 1\right) + e^{\bar{h}}\left(1-e^h\right). \tag{24}$$

Thus, (23) is of the form

$$\left(1 - e^{\bar{h}+h}\right)x = A\left(e^{\bar{h}} - 1\right) + e^{\bar{h}}\left(1-e^h\right). \tag{25}$$

We claim that $e^{\bar{h}+h} \equiv 1$. Otherwise, $e^{\bar{h}+h} \equiv 1$, and then $h(z) \equiv c_1$ and (25) yields that $A(z) \equiv A = e^{c_1} := c_2 \notin \{0, \pm 1\}$.

If there is a point z_1 such that $x(z_1) = 1$, then $y(z_1) = 1$. We can easily deduce from (2), (16), and (18) that

$$\begin{aligned} \frac{1}{c_2+1} = \frac{y(z_1)}{c_2 + y(z_1)} &= y(z_1+1) = c_2 x(z_1+1) + 1 - c_2 \\ &= \frac{c_2}{c_2+1} + 1 - c_2, \end{aligned} \tag{26}$$

which gives $c_2 \in \{0, 1\}$, a contradiction to the fact that $c_2 \notin \{0, \pm 1\}$.

If 1 is a Picard exceptional value of $x(z)$, then 1 is also a Picard exceptional value of $y(z)$. What is more, from (2) and (16), we see that $1/(1+c_2)$ is a Picard exceptional value of $x(z)$ and $y(z)$. Since $1/(1+c_2) \ne 1$, $x(z)$ has no other Picard exceptional value. Choose a point z_2 such that $x(z_2) = c_2/(1+c_2)$, then

$$y(z_2) = c_2 x(z_2) + 1 - c_2 = \frac{1}{1+c_2}. \tag{27}$$

This indicates that $1/(1+c_2)$ is not a Picard exceptional value of $y(z)$, a contradiction.

Now, we have proved that $e^{\bar{h}+h} \equiv 1$. From (25), we get

$$x = \frac{e^{\bar{h}+h} - (A+1)e^{\bar{h}} + A}{e^{\bar{h}+h} - 1}. \tag{28}$$

Since $x(z)$ is transcendental, we see that $\deg h(z) = n \ge 1$. Set

$$h(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0, \tag{29}$$

where $a_0, a_1, \ldots, a_n$ are constants such that $a_n = r_1 e^{i\theta_1} \ne 0$.

Substituting (28) into (15), we get

$$\begin{aligned} \frac{e^{\bar{\bar{h}}+\bar{h}} - (\bar{A}+1)e^{\bar{\bar{h}}} + \bar{A}}{e^{\bar{\bar{h}}+\bar{h}} - 1} &= \bar{x} = \frac{x}{A+x} \\ &= \frac{e^{\bar{h}+h} - (A+1)e^{\bar{h}} + A}{(A+1)e^{\bar{h}+h} - (A+1)e^{\bar{h}}}, \end{aligned} \tag{30}$$

which gives $F(z) = 0$, where

$$F = Ae^{\bar{\bar{h}}+2\bar{h}+h} - (\bar{A}+1)(A+1)e^{\bar{\bar{h}}+\bar{h}+h} + [(\bar{A}+1)(A+1) - A]\, e^{\bar{\bar{h}}+\bar{h}} + [(\bar{A}+1)(A+1) - A]e^{\bar{h}+h} - (\bar{A}+1)(A+1)e^{\bar{h}} + A. \tag{31}$$

Since $A(z) \equiv 0$ is a rational function, there exist some $d > 0$ and $r_2 > 1$ such that for all $z = re^{i\theta}$, $r > r_2$, and we have

$$|A| \geq r^{-d}. \tag{32}$$

Notice that

$$\begin{aligned} h\left(re^{-i\theta_1/n}\right) &= r_1 r^n (1 + o(1)), \\ \bar{h}\left(re^{-i\theta_1/n}\right) &= r_1 r^n (1 + o(1)), \\ \bar{\bar{h}}\left(re^{-i\theta_1/n}\right) &= r_1 r^n (1 + o(1)), \end{aligned} \tag{33}$$

as $r \longrightarrow +\infty$. From (31) and (32), we can deduce that

$$\lim_{r \longrightarrow +\infty} \left|F\left(re^{-i\theta_1/n}\right)\right| = \lim_{r \longrightarrow +\infty} e^{4r_1 r^n (1+o(1))}(1 + o(1)) = +\infty, \tag{34}$$

a contradiction to the fact that $F(z) = 0$.

Case 2. $R(z) \equiv P(z)$ and $x(z)$ has infinitely many poles of multiplicity ≥ 2. From (23), we have

$$x^2 = Dx + E, \tag{35}$$

where

$$\begin{aligned} D &= -\frac{BC + A\left(Be^{\bar{h}} - B + 1\right)e^h - \left[A + B\left(1 - e^h\right)\right]e^{\bar{h}}}{B(B-1)\left(e^{\bar{h}} - 1\right)e^h}, \\ E &= -\frac{AC}{B(B-1)\left(e^{\bar{h}} - 1\right)e^h}. \end{aligned} \tag{36}$$

Subcase 1. $h(z)$ is a constant. Then, $D(z)$ and $E(z)$ are rational functions and hence have at most finitely many poles. Choose a pole of $x(z)$ with multiplicity $k_1 \geq 1$, denoted by z_3, such that $D(z_3) \neq \infty$, $E(z_3) \neq \infty$. Then, z_3 is a pole of $x^2(z)$ with multiplicity $2k_1$ and a pole of $D(z)x(z) + E(z)$ with multiplicity k_1. However, from (35), we see that it is impossible.

Subcase 2. $h(z)$ is a nonconstant polynomial such that $\deg h(z) = n \geq 1$. Then, from

$$\left(e^{\bar{h}} - 1\right)' = \bar{h}' e^{\bar{h}}, \tag{37}$$

we see that $e^{\bar{h}} - 1$ has at most n zeros of multiplicity ≥ 2. Then, $D(z)$ and $E(z)$ are meromorphic functions which have at most finitely many poles of multiplicity ≥ 2. Choose a pole of $x(z)$ with multiplicity $k_2 \geq 2$ denoted by z_4 such that z_4 is not a pole of $D(z), E(z)$ of multiplicity ≥ 2. Then, z_4 is a pole of $x^2(z)$ with multiplicity $2k_2 \geq 4$ and a pole of $D(z)x(z) + E(z)$ with multiplicity at most $k_2 + 1$. However, from (35) and $k_2 + 1 < 2k_2$, we find that it is also impossible.

Case 3. $R(z) \equiv P(z)$, $\rho(x)$ is not an integer, and $x(z)$ has at most finitely many simple poles. Then, $\deg h(z) < \rho(x)$ since $\deg h(z) \leq \rho(x)$. From Case 2, we can suppose that $x(z)$ has at most finitely many poles of multiplicity ≥ 2 and use (35) directly. Now, $x(z)$ has at most finitely many poles.

On the one hand, we have

$$m(r, x) = T(r, x) - N(r, x) = T(r, x) + S(r, x). \tag{38}$$

On the other hand, since $\deg h(z) < \rho(x)$, it is easy to find that

$$m(r, D) \leq T(r, D) = S(r, x), m(r, E) \leq T(r, E) = S(r, x). \tag{39}$$

Applying Lemma 1 to (35), we get

$$m(r, x) = S(r, x), \tag{40}$$

which contradicts to (38). Our proof of Theorem 5 is thus completed.

3. Conclusion

Our result shows that the finite order transcendental meromorphic solution of equation (2) is mainly determined by its poles and 1-value points.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors have drafted the manuscript, read, and approved the final manuscript.

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