Research Article

Nontrivial Solutions for a Class of p -Kirchhoff Dirichlet Problem

Xian Hu and Yong-Yi Lan

School of Science, Jimei University, Xiamen 361021, China

Correspondence should be addressed to Yong-Yi Lan; lanyongyi@jmu.edu.cn

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This paper is devoted to the following p -Kirchhoff type of problems $\begin{cases} -(a+b\int_{\Omega}|\nabla u|^p dx)\Delta_p u = f(x,u), & x \in \Omega \\ u = 0, & x \in \partial\Omega \end{cases}$ with the

Dirichlet boundary value. We show that the p -Kirchhoff type of problems has at least a nontrivial weak solution. The main tools are variational method, critical point theory, and mountain-pass theorem.

1. Introduction and Main Results

Consider the following p -Kirchhoff type of problems with the Dirichlet boundary value:

$$\begin{cases} -(a+b\int_{\Omega}|\nabla u|^p dx)\Delta_p u = f(x,u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases} \quad (1)$$

where Ω is a smooth bounded domain in $\mathbb{R}^N$, $a > 0$ and $b \geq 0$ are real constants, Δ_p denotes the p -Laplacian operator by $\Delta_p u = \operatorname{div}(|\nabla u|^{p-2}\nabla u)$ ($1 < p < N$), and $f(x,u)$ is continuous on $\overline{\Omega} \times \mathbb{R}$.

We look for the weak solutions of (1) which are the same as the critical points of the functional $I: W_0^{1,p}(\Omega) \rightarrow \mathbb{R}$ defined by

$$I(u) = \frac{a}{p} \int_{\Omega} |\nabla u|^p dx + \frac{b}{2p} \left(\int_{\Omega} |\nabla u|^p dx \right)^2 - \int_{\Omega} F(x,u) dx, \quad (2)$$

where $F(x,u) = \int_0^u f(x,t) dt$ and $W_0^{1,p}(\Omega)$ is the Sobolev space endowed with the norm

$$\|u\| = \|\nabla u\|_p = \left(\int_{\Omega} |\nabla u|^p dx \right)^{1/p}. \quad (3)$$

I is of $W^1(\Omega)$ and $u, v \in W_0^{1,p}(\Omega)$ with derivatives given by

$$\begin{aligned} \langle I'(u), v \rangle &= \left(a + b \int_{\Omega} |\nabla u|^p dx \right) \int_{\Omega} |\nabla u|^{p-2} \nabla u \cdot \nabla v dx \\ &\quad - \int_{\Omega} f(x,u) v dx. \end{aligned} \quad (4)$$

Problem (1) began to attract the attention of researchers after the work of Lions [1], where a functional analysis approach was proposed to attack it. Since then, much attention has been paid to the existence of nontrivial solutions, sign-changing solutions, ground state solutions, multiplicity of solutions, and concentration of solutions.

Many researchers studied the existence of weak solution of (1) in N -dimensional whole spaces (see [2–7] and references therein). For example, Wu [2] showed that problem has a nontrivial solution and a sequence of high-energy solutions by using the mountain-pass theorem and the symmetric mountain-pass theorem. Similar consideration can be found in Nie and Wu [3], where radial potentials were considered. Jia and Li [6] studied multiplicity and concentration behaviour of positive solutions for Schrödinger-Kirchhoff type equations involving the p -Laplacian. Recently, Li and Niu [7] have obtained the existence of nontrivial solutions for the p -Kirchhoff type equations with critical exponents by the Ekeland variational principle and mountain-pass lemma.

Kirchhoff-type problem setting on a bounded domain also attracts a lot of attention, see [8–22] and the references therein. For example, we refer to some recent works [8, 9] in

which some interesting results on the problems with Dirichlet or Neumann boundary conditions have been obtained. Chung [10] studied Kirchhoff type problems with Robin boundary conditions and indefinite weights. Chen et al. [11] treated problem (1) when $f(x, t) = \lambda a(x)|u|^{q-2}u + b(x)|u|^{r-2}u$ ($1 < q < p = 2 < r < 2^*$), using the Nehari manifold and fibering maps, and they established the existence of multiple positive solutions for (1). In [14–19], the authors studied the existence of solutions of the p -Kirchhoff problem in the following form $-[M(\|u\|^p)]p - 1\Delta_p u = f(x, u)$, $x \in \Omega$ or p -Kirchhoff problem such as (1). In [16, 17], the authors obtained the existence and multiplicity of solutions for the p -Kirchhoff equation by using the genus theory. In [18, 19], using variational methods, the researchers studied fractional p -Kirchhoff equations and received the multiplicity and centration of solutions. In [20, 21], the authors used the fountain theorem and concentration-compactness principle to consider multiplicity of solutions for p -Kirchhoff equations. By applying a variant of the mountain-pass theorem and the Ekeland variational principle, Cheng et al. [22] obtained the existence of multiple nontrivial solutions for a class of Kirchhoff type problems with concave nonlinearity. In [23, 24], the authors studied the equations with nonlinearities which are superlinear in one direction and linear in the other. They use the Ambrosetti–Rabinowitz (AR) condition to express the superlinear growth at $+\infty$. The AR condition has been used in a technical but crucial way not only in establishing the mountain-pass geometry of the functional but also in obtaining the boundedness of Palais–Smale (PS) sequences. Since the publication of [25], the AR condition has been used extensively throughout the literature (see [26, 27]). However, this condition is very restrictive, eliminating many nonlinearities. There are always many functions that satisfy the AR condition. For example,

$$f(x, t) = 2t \ln(1 + |t|). \quad (5)$$

Under the motivation of Pei and Zhang [28], the aim of this paper is to consider existence of nontrivial solutions for problem (1) with $1 < p < N$. We shall assume that the nonlinearity term $f(x, t)$ does not satisfy the AR condition and it is asymmetric as $t \in \mathbb{R}$ approaches $+\infty$ and $-\infty$.

Nevertheless, to ensure the global compactness, one needs to impose the subcritical growth condition on the nonlinearity $f(x, t)$: there exists a position constant C_0 such that

$$|f(x, t)| \leq C_0(1 + |t|^{q-1}), \quad \forall t \in \mathbb{R}, x \in \Omega, \quad (6)$$

where $p < q < p^* = pN/(N - p)$. However, under the motivation of Lan and Tang [29], we consider a class of elliptic partial differential equations with a more general growth condition. We assume that the following condition holds:

$$(F): \lim_{t \rightarrow \infty} \frac{f(x, t)}{|t|^{p^*-1}} = 0 \text{ uniformly for almost every } x \in \Omega. \quad (7)$$

It is weaker than subcritical growth condition.

We next state the other hypotheses on $f(x, t)$. Suppose $f(x, t) \in C(\overline{\Omega} \times \mathbb{R})$, and this satisfies:

$$(A_1) \lim_{t \rightarrow 0} (f(x, t)/|t|^{2p-2}t) = b\mu \quad \text{uniformly a.e. } x \in \Omega, \text{ where } \mu \in [0, +\infty)$$

$$(A_2) \lim_{t \rightarrow 0} (f(x, t)/|t|^{2p-2}t) = bv \quad \text{uniformly a.e. } x \in \Omega, \text{ where } v \in [0, +\infty)$$

$$(A_3) \lim_{t \rightarrow +\infty} (f(x, t)/t^{2p-1}) = +\infty \quad \text{uniformly a.e. } x \in \Omega$$

$$(A_4) (f(x, t)/|t|^{2p-2}t) \text{ is increasing when } t \geq 0, \text{ for a.e. } x \in \Omega$$

Theorem 1. Assume $f(x, u)$ satisfies (F) and $(A_1) - (A_3)$. If $\mu < \lambda_1 < v < \infty$, then problem (1) has at least one nontrivial solution when v is not any of the eigenvalues of $-\Delta_p$ on $W_0^{1,p}(\Omega)$.

Theorem 2. Let $a \leq (b/2)$ and assume $f(x, u)$ satisfies (F) and $(A_1) - (A_3)$. If $\mu < \lambda_1 = v$ and $\lim_{t \rightarrow -\infty} [f(x, t)t - pF(x, t)] = -\infty$ uniformly for a.e. $x \in \Omega$, then problem (1) has at least one nontrivial solution.

Theorem 3. Assume $f(x, u)$ satisfies (F) and $(A_1) - (A_4)$. If $\mu < \lambda_1$ and $v = +\infty$, then problem (1) has at least one nontrivial solution.

Here, λ_1 is the first eigenvalue of $(-\Delta_p, W_0^{1,p}(\Omega))$ and there is a corresponding eigenfunction $\phi_1 > 0$ in Ω . For any $1 \leq s < +\infty$, $L^s(\Omega)$ is the usual Lebesgue space endowed with the norm

$$\|u\|_s^s := \int_{\Omega} |u|^s dx. \quad (8)$$

The paper is organized as follows. In Section 1, we introduced the main purpose of this paper and get some conclusions. The proofs of main results will be given in Section 2. We denote various positive constants as C or C_i ($i = 0, 1, 2, 3, \dots$) for convenience.

2. Proofs of the Main Results

Proof of Theorem 1.

Step 1 (the $(PS)_c$ condition). Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a $(PS)_c$ sequence, then we have

$$I(u_n) \rightarrow c, I'(u_n) \rightarrow 0. \quad (9)$$

First, we prove that $\{u_n\}$ is bounded. Assume for contradiction that, up to a sequence, $\|u_n\| \rightarrow +\infty$ as $n \rightarrow \infty$. Since $I'(u_n) \rightarrow 0$, for every $w \in W_0^{1,p}(\Omega)$, we can write $\langle I'(u_n), w \rangle \geq I'(u_n)w = o(1)\|w\|$, namely,

$$\left(a + b \int_{\Omega} |\nabla u_n|^p dx \right) \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla w dx - \int_{\Omega} f(x, u_n) w dx = o(1)\|w\|. \quad (10)$$

Define $z_n = u_n / \|u_n\|$. Obviously, $z_n \in W_0^{1,p}(\Omega)$ with $\|z_n\| = 1$. Then, there exists a subsequence (denoted also by z_n) such that

$$\begin{aligned} z_n &\rightharpoonup z_0 \text{ in } W_0^{1,p}(\Omega), \\ z_n &\longrightarrow z_0 \text{ in } L^r(\Omega), \quad \text{where } p \leq r < p^*, \\ z_n(x) &\longrightarrow z_0(x) \text{ a.e. } x \in \Omega, \\ |z_n(x)| &\leq |q(x)| \text{ a.e. } x \in \Omega, \end{aligned} \quad (11)$$

where $z_0 \in W_0^{1,p}(\Omega)$ and $q \in W_0^{1,p}(\Omega)$. Dividing both sides of (10) by $\|u_n\|^{2p-1}$, we obtain

$$\begin{aligned} &\left(\frac{a}{\|u_n\|^p} + b \right) \int_{\Omega} |\nabla z_n|^{p-2} \nabla z_n \cdot \nabla w \, dx \\ &- \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|^{2p-1}} w \, dx = o(1) \frac{\|w\|}{\|u_n\|^{2p-1}}, \text{ for all } w \in W_0^{1,p}(\Omega). \end{aligned} \quad (12)$$

Passing to the limit, we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|^{2p-1}} w \, dx = b \int_{\Omega} |\nabla z_0|^{p-2} \nabla z_0 \cdot \nabla w \, dx. \quad (13)$$

Let us now prove that $z_0(x) \leq 0$ for a.e. $x \in \Omega$. To verify this, we chose $w = z_0^+ = \max\{z_0, 0\}$ in (13), and we have

$$\lim_{n \rightarrow \infty} \int_{\Omega^+} \frac{f(x, u_n)}{\|u_n\|^{2p-1}} z_0 \, dx = b \int_{\Omega^+} |\nabla z_0|^p \, dx < +\infty, \quad (14)$$

where $\Omega^+ = \{x \in \Omega \mid z_0(x) > 0\}$. Moreover, using $\lim_{n \rightarrow \infty} u_n(x) = +\infty$ a.e. $x \in \Omega^+$ and by (A_3) , we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{f(x, u_n)}{\|u_n\|^{2p-1}} z_0 &= \lim_{n \rightarrow \infty} \frac{f(x, u_n)}{u_n^{2p-1}} z_n^{2p-1}(x) z_0(x) \\ &= +\infty, \text{ a.e. } x \in \Omega^+. \end{aligned} \quad (15)$$

Therefore, if $|\Omega^+| > 0$, by Fatou's lemma, we will obtain that

$$\lim_{n \rightarrow \infty} \int_{\Omega^+} \frac{f(x, u_n(x))}{\|u_n\|^{2p-1}} z_0(x) \, dx = +\infty, \quad (16)$$

which contradicts (10). Thus, $|\Omega^+| = 0$ and the claim ($z_0(x) \leq 0$) is proved. Clearly, $z_0(x) \equiv 0$. By contradiction $z_0 = 0$, we know that $z_n \rightarrow 0$ in $W_0^{1,p}(\Omega)$ and $z_n \rightarrow 0$ in $L^r(\Omega)$, where $p \leq r \leq p^*$. Therefore, we obtain $\|z_n\| \rightarrow 0$, which is a contradiction. When $z_0 < 0$, we have $u_n = z_n \|u_n\| \rightarrow -\infty$ as $n \rightarrow \infty$. If this is the case, by $\langle I'(u_n), z_n \rangle = I'(u_n) z_n = o(1)$, we first obtain

$$\begin{aligned} &\left(a + b \int_{\Omega} |\nabla u_n|^p \, dx \right) \int_{\Omega} |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla z_n \, dx \\ &- \int_{\Omega} f(x, u_n) z_n \, dx = o(1). \end{aligned} \quad (17)$$

Dividing both sides of (17) by $\|u_n\|^{2p-1}$, we obtain

$$\begin{aligned} &\left(\frac{a}{\|u_n\|^p} + b \int_{\Omega} |\nabla z_n|^p \, dx \right) \int_{\Omega} |\nabla z_n|^p \, dx \\ &- \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|^{2p-1}} z_n \, dx = o(1). \end{aligned} \quad (18)$$

Passing to the limit, we get

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|^{2p-1}} z_n \, dx = b \left(\int_{\Omega} |\nabla z_0|^p \, dx \right)^2, \quad (19)$$

Then,

$$\lim_{n \rightarrow \infty} \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|^{2p-1}} z_n \, dx = b \|z_0\|^{2p}. \quad (20)$$

Using Lebesgue's dominated convergence theorem and (A_2) for the left hand side of (20), we have

$$\begin{aligned} \lim_{n \rightarrow \infty} \int_{\Omega} \frac{f(x, u_n)}{\|u_n\|^{2p-1}} z_n \, dx &= \int_{\Omega} \lim_{n \rightarrow \infty} \frac{f(x, u_n)}{\|u_n\|^{2p-1}} z_n \, dx \\ &= \int_{\Omega} \lim_{n \rightarrow \infty} \frac{f(x, u_n)}{u_n^{2p-1}} z_n^{2p} \, dx \\ &= b v \int_{\Omega} z_0^{2p} \, dx. \end{aligned} \quad (21)$$

Then, (20) can be written as follows:

$$b v \int_{\Omega} z_0^{2p} \, dx = b \|z_0\|^{2p}. \quad (22)$$

This means that v is an eigenvalue, which contradicts our assumption. So, $\{u_n\}$ is bounded.

Now, we prove that $\{u_n\}$ has a convergent subsequence. By the continuity of embedding, we have $\|u_n\|_{p^*}^{p^*} \leq C_1 < \infty$ for all n . Going if necessary to a subsequence, we have

$$\begin{aligned} u_n &\rightharpoonup u \text{ in } W_0^{1,p}(\Omega), \\ u_n &\longrightarrow u \text{ in } L^q(\Omega), \end{aligned} \quad (23)$$

where $p \leq q < p^*$.

By (F), for every $\varepsilon > 0$, we can find a constant $C(\varepsilon) > 0$ such that

$$f(x, t) \leq C(\varepsilon) + \frac{1}{2C_1} \varepsilon |t|^{p^*}, \quad \text{for } t \in \mathbb{R}, \text{ a.e. } x \in \Omega. \quad (24)$$

Let $\delta = \varepsilon / (2C(\varepsilon))$, $E \subseteq \Omega$ means $E < \delta$, we have

$$\begin{aligned} \int_E f(x, u_n) u_n dx &\leq \int_E |f(x, u_n)| dx \\ &\leq \int_E C(\varepsilon) dx + \frac{1}{2C_1} \varepsilon \int_E |u_n|^{p^*} dx \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \quad (25)$$

Hence, $\{\int_\Omega (x, u_n) u_n dx, n \in N\}$ is equiabsolutely continuous. Moreover, $\{f(x, u_n) u_n, n \in N\}$ is equi-integrable and bounded in $L^1(\Omega)$, and $f(x, u_n) u_n \rightarrow f(x, u) u$ in measure. From the Vitali convergence theorem, it follows that

$$\int_\Omega f(x, u_n) u_n dx \rightarrow \int_\Omega f(x, u) u dx. \quad (26)$$

By (F), for every $\varepsilon > 0$, we can find a constant $C(\varepsilon) > 0$ such that

$$f(x, t) \leq C(\varepsilon) + \frac{1}{2c_1 c_2} \varepsilon |t|^{p^*-1}, \quad \text{for } t \in \mathbb{R}, \text{ a.e. } x \in \Omega, \quad (27)$$

where $c_1 \geq (\int_\Omega |u_n|^{p^*} dx)^{p^*-1/p^*}$ for all n and $c_2 = (\int_\Omega |u|^{p^*} dx)^{1/p^*}$ are positive constants with the assumption of $u \equiv 0$. Form Hölder's inequality, for every $E \subseteq \Omega$, we have

$$\begin{aligned} \int_E C(\varepsilon) |u| dx &\leq C(\varepsilon) (\text{meas } E)^{(p^*-1)/p^*} \left(\int_E |u|^{p^*} dx \right)^{1/p^*} \\ &= C(\varepsilon) (\text{meas } E)^{p^*-1/p^*} c_2, \\ \int_E |u|^{p^*-1} dx |u| dx &\leq \left(\int_E |u|^{p^*} dx \right)^{(p^*-1)/p^*} \left(\int_E |u|^{p^*} dx \right)^{1/p^*} \\ &\leq c_1 c_2. \end{aligned} \quad (28)$$

Let $\delta = (\varepsilon / 2c_2 C(\varepsilon))^{p^*/(p^*-1)} > 0$, $E \subseteq \Omega$ means $E < \delta$, we have

$$\begin{aligned} \left| \int_E f(x, u_n) u_n dx \right| &\leq \int_E |f(x, u_n) u_n| dx \\ &\leq \int_E C(\varepsilon) |u| dx + \frac{1}{2c_1 c_2} \varepsilon \int_E |u_n|^{p^*-1} |u| dx \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon. \end{aligned} \quad (29)$$

Hence, $\{\int_\Omega f(x, u_n) u_n dx, n \in N\}$ is equiabsolutely continuous. Moreover, $\{f(x, u_n) u_n, n \in N\}$ is equi-

integrable and bounded in $L^1(\Omega)$, and $f(x, u_n) u_n \rightarrow f(x, u) u$ in measure. From the Vitali convergence theorem, it follows that

$$\int_\Omega f(x, u_n) u_n dx \rightarrow \int_\Omega f(x, u) u dx. \quad (30)$$

Since

$$\begin{aligned} \langle I'(u_n), u_n \rangle &= (a + b \|u_n\|^p) \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u_n dx \\ &\quad - \int_\Omega f(x, u_n) u_n dx \rightarrow 0, \end{aligned} \quad (31)$$

it follows that

$$\begin{aligned} \langle I'(u_n), u \rangle &= (a + b \|u_n\|^p) \int_\Omega |\nabla u_n|^{p-2} \nabla u_n \cdot \nabla u dx \\ &\quad - \int_\Omega f(x, u_n) u dx \rightarrow 0. \end{aligned} \quad (32)$$

Thus, we have $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$, which means that I satisfies the (PS) condition.

Step 2 (mountain-pass geometric structure). I has mountain-pass geometry, i.e., there exist $e \in W_0^{1,p}(\Omega)$ and $\rho, \alpha > 0$ such that

$$I(u) \geq \alpha > 0, \quad \text{when } \|u\| = \rho, \quad (33)$$

$$I(e) < 0, \quad \text{when } \|e\| > \rho. \quad (34)$$

Let $\phi_1 > 0$ be a λ_1 -eigenfunction with $\|\phi_1\| = 1$, and assume $(A_1) - (A_3)$ and (F) hold. If $v \in (\lambda_1, +\infty)$, for any $\varepsilon > 0$, there exist C_2 and C_3 so small such that for all $(x, t) \in \Omega \times \mathbb{R}$, and I has mountain-pass geometry. By (A_1) and (F), we have

$$F(x, s) \leq \frac{b}{2p} (\mu + \varepsilon) |s|^{2p} + C_2 |s|^{p^*}. \quad (35)$$

Choose $\varepsilon > 0$ such that $(\mu + \varepsilon) < \lambda_1$. By (35), using Poincaré inequality and Sobolev inequality, we get

$$\begin{aligned} I(u) &= \frac{a}{p} \int_\Omega |\nabla u|^p dx + \frac{b}{2p} \left(\int_\Omega |\nabla u|^p dx \right)^2 - \int_\Omega F(x, u) dx \\ &= \frac{a}{p} \|u\|^p + \frac{b}{2p} \|u\|^{2p} - \int_\Omega F(x, u) dx \\ &\geq \frac{a}{p} \|u\|^p + \frac{b}{2p} \|u\|^{2p} - \frac{b(\mu + \varepsilon)}{2p} \|u\|_{2p}^{2p} - C_2 \|u\|_{p^*}^{p^*} \\ &\geq \frac{a}{p} \|u\|^p + \frac{b}{2p} \left(1 - \frac{\mu + \varepsilon}{\lambda_1} \right) \|u\|^{2p} - C_2 K \|u\|^{p^*}. \end{aligned} \quad (36)$$

Then, (33) is proved if we choose $\|u\| = \rho > 0$ small enough.

On the other hand, by (A_2) , we have

$$F(x, s) \geq \frac{b}{2p} (v - \varepsilon) |s|^{2p} - C_3. \quad (37)$$

Take $\varepsilon > 0$ such that $v - \varepsilon > \lambda_1$, and let $e = t\phi_1$ such that $\|e\| = \|t\phi_1\| > \rho$. Using (37), we have

$$\begin{aligned} I(e) &= I(t\phi_1) = \frac{a|t|^p}{p} \int_{\Omega} |\nabla \phi_1|^p dx + \frac{bt^{2p}}{2p} \left(\int_{\Omega} |\nabla \phi_1|^p dx \right)^2 - \int_{\Omega} F(x, t\phi_1) dx \\ &\leq \frac{a|t|^p}{p} + \frac{bt^{2p}}{2p} - \frac{b(v - \varepsilon)}{2p} t^{2p} |\phi_1|_{2p}^{2p} + C_3 |\Omega| \\ &\leq \frac{a|t|^p}{p} + \frac{b}{2p} \left(1 - \frac{v - \varepsilon}{\lambda_1} \right) t^{2p} + C_3 |\Omega| \longrightarrow -\infty \text{ as } t \longrightarrow +\infty. \end{aligned} \quad (38)$$

Thus, it shows $\|e\| > \rho$ and $I(e) < 0$. Therefore, I has mountain-pass geometry.

Step 3 (a critical value of I). For e in Step 2, we define

$$\begin{aligned} \tau &= \left\{ \gamma: C[0, 1] \longrightarrow W_0^{1,p}(\Omega) \mid \gamma(0) = 0, \gamma(1) = e \right\}, \\ c &= \inf_{\gamma \in \tau} \max_{0 \leq t \leq 1} I(\gamma(t)). \end{aligned} \quad (39)$$

It turns out that the mountain-pass theorem holds. Then, c is a critical value of I . $\square$

Proof of Theorem 2.

Step 1 (the $(C)_c$ condition). Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a $(C)_c$ sequence, then we have

$$I(u_n) \longrightarrow c, \|I'(u_n)\| (1 + \|u_n\|) \longrightarrow 0. \quad (40)$$

Similar to the proof of Theorem 1, we have $z_0(x) \leq 0, x \in \Omega, z_0(x) \equiv 0$ and

$$bv \int_{\Omega} |z_0|^{2p} dx = b \|z_0\|^{2p}. \quad (41)$$

By maximum principle, z_0 is an eigenfunction of λ_1 . When $z_0 < 0$, we have $u_n = z_n \|u_n\| \longrightarrow -\infty$ as $n \longrightarrow \infty$. By our assumption,

$$\lim_{n \rightarrow \infty} (f(x, u_n(x))u_n(x) - pF(x, u_n(x))) = -\infty, \quad (42)$$

uniformly in $x \in \Omega$, which implies that

$$\int_{\Omega} (f(x, u_n(x))u_n(x) - pF(x, u_n(x))) dx = -\infty \text{ as } n \longrightarrow \infty. \quad (43)$$

On the other hand,

$$\begin{aligned} pI(u_n) - \langle I'(u_n), u_n \rangle &= a \int_{\Omega} |\nabla u_n|^p dx + \frac{b}{2} \left(\int_{\Omega} |\nabla u_n|^p dx \right)^2 - p \int_{\Omega} F(x, u_n(x)) dx \\ &\quad - a \int_{\Omega} |\nabla u_n|^p dx - b \left(\int_{\Omega} |\nabla u_n|^p dx \right)^2 + \int_{\Omega} f(x, u_n(x))u_n(x) dx \\ &= -\frac{b}{2} \|u_n\|^p + \int_{\Omega} (f(x, u_n(x))u_n(x) - pF(x, u_n(x))) dx \longrightarrow pc \text{ as } n \longrightarrow \infty, \end{aligned} \quad (44)$$

which implies that

$$\int_{\Omega} (f(x, u_n(x))u_n(x) - pF(x, u_n(x))) dx \longrightarrow pc + \frac{b}{2} \|u_n\|^p \longrightarrow +\infty \text{ as } n \longrightarrow \infty. \quad (45)$$

It contradicts (43). Hence, $\{u_n\}$ is bounded. According to Step 1, the proof of Theorem 1, we have $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$, which means that I satisfies (C_c) . Step 2 (mountain-pass geometric structure). Let $\phi_1 > 0$ be a λ_1 -eigenfunction with $\|\phi_1\| = 1$, and assume $(A_1) -$

(A_3) and (F) hold. If $v = \lambda_1$, similar to the proof of Theorem 1, (33) holds. We only need to show (34) holds. Let $e = -t\phi_1$ be such that $\|e\| = \|-t\phi_1\| > \rho$. By (A_2) , we have

$$\begin{aligned} I(e) &= I(-t\phi_1) = \frac{a|t|^p}{p} \int_{\Omega} |\nabla \phi_1|^p dx + \frac{bt^{2p}}{2p} \left(\int_{\Omega} |\nabla \phi_1|^p dx \right)^2 - \int_{\Omega} F(x, -t\phi_1) dx \\ &\leq \frac{a|t|^p}{p} + \frac{bt^{2p}}{2p} - \frac{1}{p} \int_{\Omega} f(x, -t\phi_1) (-t\phi_1) dx - \int_{\Omega} \left\{ F(x, -t\phi_1) - \frac{1}{p} f(x, -t\phi_1) (-t\phi_1) \right\} dx \\ &\leq \frac{a|t|^p}{p} + \frac{bt^{2p}}{2p} - \frac{b}{p} (\lambda_1 |-t\phi_1|^{2p}) - \int_{\Omega} \left\{ F(x, -t\phi_1) - \frac{1}{p} f(x, -t\phi_1) (-t\phi_1) \right\} dx \\ &\leq \frac{a|t|^p}{p} - \frac{bt^{2p}}{2p} - \int_{\Omega} \left\{ F(x, -t\phi_1) - \frac{1}{p} f(x, -t\phi_1) (-t\phi_1) \right\} dx. \end{aligned} \quad (46)$$

By (43), we get

$$\int_{\Omega} \left\{ F(x, -t\phi_1) - \frac{1}{p} f(x, -t\phi_1) (-t\phi_1) \right\} dx \rightarrow \infty. \quad (47)$$

By our assumption $a \leq (b/2)$, we have

$$I(e) = I(-t\phi_1) \rightarrow -\infty \text{ as } t \rightarrow +\infty. \quad (48)$$

Thus, (34) is proved. Therefore, I has mountain-pass geometry. $\square$

Proof of Theorem 3.

Step 1 (the $(C)_c$ condition). Let $\{u_n\} \subset W_0^{1,p}(\Omega)$ be a $(C)_c$ sequence. We can prove that $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. Assume $\|u_n\| \rightarrow +\infty$ as $n \rightarrow \infty$. Similar to the proof of Theorem 1, we have $z_0 \leq 0$, and when $z_0 < 0$, $u_n = z_n \|u_n\| \rightarrow -\infty$ as $n \rightarrow \infty$. Let

$$s_n = \frac{\sqrt[p]{4pc/b}}{\|u_n\|}, w_n = s_n u_n = \frac{\sqrt[p]{4pc/b} u_n}{\|u_n\|}. \quad (49)$$

Since $\{w_n\}$ is bounded in $W_0^{1,p}(\Omega)$, we have a subsequence denoted also by $\{w_n\}$ such that $w_n \rightarrow w_0$ in $W_0^{1,p}(\Omega)$. From the proof of Theorem 1, we know $|\Omega^+| = 0$, so $\Omega^- = \{x \in \Omega: w_0(x) < 0\}$, and $v = +\infty$. By (A_3) , we have

$$\frac{f(x, u_n)}{|u_n|^{2p-2} u_n} \geq M \text{ uniformly for all } x \in \Omega^-, \quad (50)$$

where M is a large enough constant. Since $\langle I'(u_n, u_n) \rangle \rightarrow 0$, then we get

$$a\|u_n\|^p + b\|u_n\|^{2p} = \int_{\Omega} f(x, u_n(x)) u_n dx. \quad (51)$$

Therefore, we have

$$\begin{aligned} a\sqrt{\frac{4pc}{b}} + 4pc &= \lim_{n \rightarrow \infty} (a\|w_n\|^p + b\|w_n\|^{2p}) = \lim_{n \rightarrow \infty} \int_{\Omega} \frac{f(x, w_n)}{|w_n|^{2p-2} w_n} |w_n|^{2p} dx \\ &\geq \lim_{n \rightarrow \infty} \int_{\Omega^-} \frac{f(x, w_n)}{|w_n|^{2p-2} w_n} |w_n|^{2p} dx \\ &\geq M \int_{\Omega^-} |w_0|^{2p} dx. \end{aligned} \quad (52)$$

So, $w_0 \equiv 0$ for a.e. $x \in \Omega$. However, if $w_0 \equiv 0$, then $\int_{\Omega} F(x, w_n) dx \rightarrow 0$. Hence,

$$I(w_n) = \frac{a}{p} \|w_n\|^p + \frac{b}{2p} \|w_n\|^{2p} + o(1) \geq 2c + o(1). \quad (53)$$

On the other hand, by $\langle I'(u_n), u_n \rangle \rightarrow 0$ as $n \rightarrow \infty$, for a suitable subsequence, we may assume that

$$\begin{aligned} -\frac{1}{n} \langle I'(u_n), u_n \rangle &= (a + b \|u_n\|^p) \|u_n\|^p \\ &- \int_{\Omega} f(x, u_n(x)) u_n(x) dx < \frac{1}{n} \text{ for all } n. \end{aligned} \quad (54)$$

From (54), we have

$$b \|u_n\|^{2p} < -a \|u_n\|^p + \int_{\Omega} f(x, u_n(x)) u_n(x) dx + \frac{1}{n}. \quad (55)$$

Then, for any $t \geq 0$,

$$\begin{aligned} I(tu_n) &= \frac{at^p}{p} \|u_n\|^p + \frac{bt^{2p}}{2p} \|u_n\|^{2p} - \int_{\Omega} F(x, tu_n(x)) dx \\ &\leq \frac{at^p}{p} \|u_n\|^p + \frac{t^{2p}}{2p} \left(-a \|u_n\|^p + \frac{1}{n} + \int_{\Omega} f(x, tu_n(x)) u_n(x) dx \right) - \int_{\Omega} F(x, tu_n(x)) dx \\ &= \frac{t^{2p}}{2np} + \frac{at^p}{p} \left(1 - \frac{t^p}{2} \right) \|u_n\|^p + \int_{\Omega} \left(\frac{t^{2p}}{2p} f(x, tu_n(x)) u_n(x) - F(x, tu_n(x)) \right) dx. \end{aligned} \quad (56)$$

We set

$$h(t) = \frac{t^{2p}}{2p} f(x, u_n(x)) u_n(x) - F(x, tu_n(x)). \quad (57)$$

Then,

$$\begin{aligned} h'(t) &= t^{2p-1} f(x, u_n) u_n - f(x, tu_n) u_n \\ &= t^{2p-1} u_n \left(f(x, u_n) - \frac{f(x, tu_n)}{t^{2p-1}} \right) \\ &= t^{2p-1} u_n \left(\frac{f(x, 1u_n)}{1^{2p-1}} - \frac{f(x, tu_n)}{t^{2p-1}} \right). \end{aligned} \quad (58)$$

By (A_4) , hence

$$\begin{cases} h'(t) \geq 0, & 0 < t \leq 1, \\ h'(t) \leq 0, & t \geq 1. \end{cases} \quad (59)$$

So, $h(t) \leq h(1)$ for all $t \geq 0$.

Similarly, we set

$$g(t) = \frac{at^p}{p} \left(1 - \frac{t^p}{2} \right). \quad (60)$$

Then,

$$g'(t) = a(t^{p-1} - t^{2p-1}). \quad (61)$$

Hence,

$$\begin{cases} g'(t) \geq 0, & 0 < t \leq 1, \\ g'(t) \leq 0, & t \geq 1. \end{cases} \quad (62)$$

So, $g(t) \leq g(1)$ for all $t \geq 0$.

Therefore,

$$\begin{aligned} I(tu_n) &= \frac{t^{2p}}{2np} + g(t) \|u_n\|^p + \int_{\Omega} h(t) dx \\ &\leq \frac{t^{2p}}{2np} + g(1) \|u_n\|^p + \int_{\Omega} h(1) dx \\ &= \frac{t^{2p}}{2np} + \frac{a}{2p} \|u_n\|^p + \int_{\Omega} \left(\frac{1}{2p} f(x, u_n(x)) u_n(x) - F(x, u_n(x)) \right) dx. \end{aligned} \quad (63)$$

On the other hand, from (54), we have

Then,

$$b\|u_n\|^{2p} > -a\|u_n\|^p - \frac{1}{n} + \int_{\Omega} f(x, u_n(x))u_n(x)dx. \quad (64)$$

$$\begin{aligned} I(u_n) &= \frac{a}{p}\|u_n\|^p + \frac{b}{2p}\|u_n\|^{2p} - \int_{\Omega} F(x, u_n(x))dx \\ &\geq \frac{a}{p}\|u_n\|^p + \frac{1}{2p}\left(-a\|u_n\|^p - \frac{1}{n} + \int_{\Omega} f(x, u_n(x))u_n(x)dx\right) - \int_{\Omega} F(x, u_n(x))dx \\ &= \frac{a}{2p}\|u_n\|^p - \frac{1}{2np} + \int_{\Omega} \left(\frac{1}{2p}f(x, u_n(x))u_n(x) - F(x, u_n(x))\right)dx. \end{aligned} \quad (65)$$

That is,

$$\frac{a}{2p}\|u_n\|^p + \int_{\Omega} \left(\frac{1}{2p}f(x, u_n(x))u_n(x) - F(x, u_n(x))\right)dx \leq \frac{1}{2pn} + I(u_n). \quad (66)$$

Thus, we have

$$I(tu_n) \leq \frac{t^{2p}}{2pn} + \frac{1}{2pn} + I(u_n) = \frac{1+t^{2p}}{2pn} + I(u_n), \quad (67)$$

for all $t \geq 0$. By $\|u_n\| \rightarrow \infty$ as $n \rightarrow \infty$, we have $s_n \rightarrow 0$ (defined by (49)) as $n \rightarrow \infty$. Thus, we get

$$I(w_n) = I(s_n u_n) \leq \frac{1+(s_n)^{2p}}{2np} + I(u_n) \rightarrow c + o(1) \text{ as } n \rightarrow \infty. \quad (68)$$

Obviously, it contradicts (53). So, $\{u_n\}$ is bounded in $W_0^{1,p}(\Omega)$. According to the proof of Theorem 1, we have $u_n \rightarrow u$ in $W_0^{1,p}(\Omega)$ which means that I satisfies (C_c) .

Step 2 (mountain-pass geometric structure). Let $\phi_1 > 0$ be a λ_1 -eigenfunction with $\|\phi_1\| = 1$, and assume $(A_1) - (A_4)$ and (F) hold. If $v = +\infty$, for any $\varepsilon > 0$, there exists C_4 so small such that for all $(x, t) \in \Omega \times \mathbb{R}$, and I has mountain-pass geometry. Similar to the proof of Theorem 1, (33) holds. We only need to show that (34) holds. By (A_2) , we have

$$F(x, s) \geq \frac{b}{2p}(\lambda_1 + \varepsilon)|s|^{2p} - C_4. \quad (69)$$

Let $e = t\phi_1$ such that $\|e\| = \|t\phi_1\| > \rho$. Using (69), we have

$$\begin{aligned} I(e) &= I(t\phi_1) = \frac{a|t|^p}{p} \int_{\Omega} |\nabla \phi_1|^p dx + \frac{bt^{2p}}{2p} \left(\int_{\Omega} |\nabla \phi_1|^p dx \right)^2 - \int_{\Omega} F(x, t\phi_1) dx \\ &\leq \frac{a|t|^p}{p} + \frac{bt^{2p}}{2p} - \frac{b(\lambda_1 + \varepsilon)}{2p} t^{2p} |\phi_1|_{2p}^{2p} + C_4 |\Omega| \\ &\leq \frac{a|t|^p}{p} + \frac{b}{2p} \left(1 - \frac{\lambda_1 + \varepsilon}{\lambda_1} \right) t^{2p} + C_4 |\Omega| \rightarrow -\infty \text{ as } t \rightarrow +\infty. \end{aligned} \quad (70)$$

Thus, it shows $\|e\| > \rho$ and $I(e) < 0$. Therefore, I has mountain-pass geometry. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors declare that the study was realized in collaboration with the same responsibility. All authors read and approved the final manuscript.

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