Research Article

Point-Symmetric Extension-Based Interval Shannon-Cosine Spectral Method for Fractional PDEs

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The approximation accuracy of the wavelet spectral method for the fractional PDEs is sensitive to the order of the fractional derivative and the boundary condition of the PDEs. In order to overcome the shortcoming, an interval Shannon-Cosine wavelet based on the point-symmetric extension is constructed, and the corresponding spectral method on the fractional PDEs is proposed. In the research, a power function of cosine function is introduced to modulate Shannon function, which takes full advantage of the waveform of the *Shannon* function to ensure that many excellent properties can be satisfied such as the partition of unity, smoothness, and compact support. And the interpolative property of Shannon wavelet is held at the same time. Then, based on the point-symmetric extension and the general variational theory, an interval Shannon-Cosine wavelet is constructed. It is proved that the first derivative of the approximated function with this interval wavelet function is continuous. At last, the wavelet spectral method for the fractional PDEs is given by means of the interval Shannon-Cosine wavelet. By means of it, the condition number of the discrete matrix can be suppressed effectively. Compared with Shannon and Shannon-Gabor wavelet quasi-spectral methods, the novel scheme has stronger applicability to the shockwave appeared in the solution besides the higher numerical accuracy and efficiency.

1. Introduction

In recent years, fractional calculus has been attracting more and more researchers in different fields of science and engineering and has been theoretically developed quickly over the last two decades [1–4]. It has been proved that the fractional-order differential equation models are more consistent with the biological phenomena [5] and hydrodynamics [6–8] than those of integer-orders. The Caputo and Riemann–Liouville fractional derivatives are the classical definition, and both of them have a kernel with singularity. To solve the problem of singular kernel, Caputo and Fabrizio proposed a derivative with fractional order based on the exponential function; their derivative in fact does not have singular kernel. In order to overcome the shortcoming of the nonlocal property of the exponential function, Ravichandran et al. [9–12] proposed two generalized fractional derivatives in Caputo and Riemann–Liouville sense [13, 14].

Despite a few special fraction PDEs having analytical solution [15], most of them should be solved by the

numerical method. The solution of the fractional PDEs is sensitive to the iterative step, and so it is disabled to be solved by the traditional numerical method directly. Al-Mdallal has made outstanding contributions in this field, who has proposed many effective algorithms for solving the fractional PDEs, such as the fractional-order Legendre-collocation method [16] and fractional-Legendre spectral Galerkin method [17]. This inspired this work to try to construct a numerical method by means of the Shannon wavelet theory.

Sinc is a famous sampling function, but Shannon's reconstruction formula is rarely used in practice because of the slow decay of the *Shannon* function [18]. Taking the window functions to modulate the sinc function is the common method to improve its decay rate. Many windows have been proposed such as rectangular window, Bartlett window, Hanning window, Hamming window, and Blackman window.

In recent years, wavelet analysis theory has been developed to be a powerful tool to solve the fractional partial

differential equations in recent years [19–21]. Shannon wavelets have been constructed based on the sinc function. A complex Shannon wavelet is defined by

$$\psi(x) = \sqrt{f_b} \sin c(f_b x) e^{2i\pi f_c x}, \quad (1)$$

where f_b is the bandwidth and f_c is the wavelet center frequency. Hoffman et al. [22] have presented the Shannon-Gabor wavelet as follows:

$$\phi_G(x) = \frac{\sin(\pi x)}{\pi x} \exp\left(-\frac{x^2}{2\sigma^2}\right), \quad \sigma > 0, \quad (2)$$

where σ is the width parameter (or called window size).

Both of the two Shannon-type wavelets are obtained by taking the Gaussian window to modulate the sinc function. The presence of the Gaussian window destroys the normative property possessed by the Shannon wavelet, that is,

$$\hat{\phi}_G(0) = \int_{-\infty}^{\infty} \phi_G(x) dx = \operatorname{erf}\left(\frac{\pi\sigma}{\sqrt{2}}\right), \quad (3)$$

where $\operatorname{erf}(z) = (2/\sqrt{\pi}) \int_0^z e^{-t^2} dt$ is the error function. Obviously, $\hat{\phi}_G(0)$ is always less than unity except at the limit of $\sigma \rightarrow \infty$. This is the reason why the windowed *Shannon* wavelet is not recommended. They fail to satisfy the partition of unity; this has the disturbing consequence that the reconstruction error will not vanish as the sampling step tends to zero.

Similar to Haar wavelet [23–25], B-Spline wavelet [26, 27], and Legendre wavelet [28, 29], *Shannon* wavelet [30, 31] possesses almost all the excellent numerical properties such as interpolative, relative sparse, and orthogonal properties besides the compact support property. Therefore, it is necessary to construct a novel window for sinc function, which can satisfy the partition of unity, so that it can be utilized to solve fractional PDEs efficiently [32]. In order to overcome the disadvantages in the Shannon-Gabor wavelet, a Shannon-Cosine wavelet is constructed by Mei et al. [32, 33], in which the waveform is used to meet the requirement of the partition of unity. Unfortunately, there are too many parameters in the Shannon-Cosine wavelet function to be identified, and this makes the expression to be very complex. In this paper, a simplified Shannon-Cosine wavelet function is proposed and the corresponding interval wavelet is constructed based on the point-symmetric extension [34, 35]. And then, the interval Shannon-Cosine wavelet is employed to construct a wavelet spectral method for the fractional PDEs.

2. Simplified Shannon-Cosine Scaling Function

As a basis function $\phi(x) = \sin c(x)$ defined in Hilbert space, it could form a basic approximation space V as

$$V(\phi) = \left\{ s(x) = \sum_{k \in \mathbb{Z}} c(k) \phi(x - k), c \in l_2 \right\}. \quad (4)$$

This means that any function $s(x) \in V(\phi)$ is characterized by a sequence of coefficients $c(k)$. The coefficients $c(k)$ are the

samples of the signal, and that $\phi(x)$ is a kind of weight function. Therefore, it should meet the requirements as follows:

- ① The coefficients sequence should be square-summable: $c \in l_2$.
- ② The family of functions $\{\phi_k = \phi(x - k)\}_{k \in \mathbb{Z}}$ should form a Riesz basis of $V(\phi)$. This ensures that the representation is stable and unambiguously defined. The Riesz basis requirement has an equivalent expression in the Fourier domain:

$$A \leq \sum_{k \in \mathbb{Z}} |\hat{\phi}(\omega + 2k\pi)|^2 \leq B, \quad (5)$$

where $\hat{\phi}(\omega) = \int \phi(x) e^{-j\omega x} dx$ is the Fourier transform of $\phi(x)$.

- ③ $\phi(x)$ should satisfy the partition of unity condition $\int_{-\infty}^{\infty} \phi(x) dx = 1$.

Sinc is a famous sampling function [20] based on approximating the Dirac delta function as a band-limited function and is given by

$$\phi(x) = \frac{\sin(\pi x)}{\pi x}. \quad (6)$$

In order to overcome the shortcomings of the Shannon-Gabor function, we introduce the parametric cosine function instead of the exponential function to modulate Shannon function as follows:

$$S_C(x) = \frac{\sin(\pi x)}{\pi x} \cos^{2m}\left(\frac{\pi x}{N}\right) \left[\chi\left(x + \frac{N}{2}\right) - \chi\left(x - \frac{N}{2}\right) \right], \quad (7)$$

where N is a constant related to the support domain. $\chi(x)$ is the Heaviside function defined as follows:

$$\chi(x) = \begin{cases} 0, & x < 0, \\ \text{undefined}, & x = 0, \\ 1, & x > 0. \end{cases} \quad (8)$$

This ensures that new modulate Shannon function (7) is a real compact support function, and the support domain is $[-N/2, N/2]$. The function $S_C(x)$ is named as Shannon-Cosine scaling function.

The comparison between the sinc and the Shannon-Cosine scaling function is shown in Figure 1.

It is easy to prove that $S_C(x)$ has the interpolative property; the coefficients $c(k)$ in equation (4) are the samples of the signal. Therefore, $c(k)$ obtained from any signal $s \in L_2$ satisfies the first requirement.

Based on the Parseval equation, it is easy to understand that any basis function with $\int_{-\infty}^{\infty} \phi(x) dx = 1$ meets the second requirement. This means that the third requirement puts the strongest constraint of the selection on an admissible generating function $\phi(x)$. It is well known that the sinc function satisfies the partition of unity condition, and this results in the Gaussian-windowed sinc function are not meeting this requirement. In the next section, we are going to prove that it can be satisfied by choosing the support domain parameter N .

FIGURE 1: Comparison between Shannon and Shannon-Cosine scaling functions. (a) Shannon function. (b) Shannon-Cosine scaling function.

3. Normalization and Choice of the Support Domain Parameter N

Shannon function has the waveform shown in Figure 2; its support domain is $(-\infty, \infty)$ and it meets the normalization condition $\int_{-\infty}^{\infty} \phi(x) dx = 1$. It is easy to notice that the integration value of Shannon function in domain I_1 is larger than 1 and smaller than 1 in domain I_2 . This reminds us that the reasonable choice of the support domain can ensure that the Shannon-Cosine scaling function satisfied the partition of unity condition. Theorem 1 reveals the relation between the support domain parameter N and the partition of unity condition.

Theorem 1. *The Shannon-Cosine scaling function ($m=3$) satisfies the normalization condition as follows:*

$$\int_{-\infty}^{\infty} S_C(x) dx = 1. \quad (9)$$

Proof. The Fourier transform of $S_C(x)$ is given by

$$\begin{aligned} \widehat{S}_C(\omega) &= \int_{-\infty}^{\infty} S_C(x) e^{-i\omega x} dx \\ &= \frac{1}{2\pi} \sum_{n=0}^3 a_n \left[\text{Si}\left(\frac{1}{2}N \cdot (\pi + \omega) + n\pi\right) \right. \\ &\quad + \text{Si}\left(\frac{1}{2}N \cdot (\pi + \omega) - n\pi\right) + \text{Si}\left(\frac{1}{2}N \cdot (\pi - \omega) + n\pi\right) \\ &\quad \left. + \text{Si}\left(\frac{1}{2}N \cdot (\pi - \omega) - n\pi\right) \right]. \end{aligned} \quad (10)$$

Let $m=3$. Substituting $\omega = 0$ and equation (6) into equation (10), we obtain

FIGURE 2: Shannon function.

$$\begin{aligned} \widehat{S}_C(0) &= \int_{-\infty}^{\infty} S_C(x) dx \widehat{S}_C(0) \\ &= \frac{1}{\pi} \left[\frac{5}{16} \left(2\text{Si}\left(\frac{1}{2}N\pi\right) \right) + \frac{15}{32} \left(\text{Si}\left(\frac{1}{2}N\pi + \pi\right) \right. \right. \\ &\quad \left. \left. + \text{Si}\left(\frac{1}{2}N\pi - \pi\right) \right) + \frac{3}{16} \left(\text{Si}\left(\frac{1}{2}N\pi + 2\pi\right) \right. \right. \\ &\quad \left. \left. + \text{Si}\left(\frac{1}{2}N\pi - 2\pi\right) \right) + \frac{1}{32} \left(\text{Si}\left(\frac{1}{2}N\pi + 3\pi\right) \right. \right. \\ &\quad \left. \left. + \text{Si}\left(\frac{1}{2}N\pi - 3\pi\right) \right) \right]. \end{aligned} \quad (11)$$

In fact, $\widehat{S}_C(0)$ can be viewed as a continuous function with respect to the real number “ N ,” which is a parameter related to the length of the support domain. Let

$$\begin{aligned}
P(N) &= \widehat{S}_C(0) \\
&= \frac{1}{\pi} \left[\frac{5}{16} \left(2Si\left(\frac{1}{2}N\pi\right) \right) + \frac{15}{32} \left(Si\left(\frac{1}{2}N\pi + \pi\right) \right. \right. \\
&\quad \left. \left. + Si\left(\frac{1}{2}N\pi - \pi\right) \right) + \frac{3}{16} \left(Si\left(\frac{1}{2}N\pi + 2\pi\right) \right) \right. \\
&\quad \left. + Si\left(\frac{1}{2}N\pi - 2\pi\right) \right) + \frac{1}{32} \left(Si\left(\frac{1}{2}N\pi + 3\pi\right) \right. \\
&\quad \left. \left. + Si\left(\frac{1}{2}N\pi - 3\pi\right) \right) \right]. \quad (12)
\end{aligned}$$

Based on the definition of $Si(x)$, we know that $P(N)$ is an oscillator function around the constant 1, that is,

$$\begin{aligned}
P(N) &\begin{cases} >1, & \frac{N}{2} = 2n, \quad n \geq 2, \\ <1, & \frac{N}{2} = 2n+1, \quad n \geq 2, \end{cases} \\
P'(N) &= \frac{dP(N)}{dN} \\
&= -\frac{1440 \sin(N\pi/2)}{\pi N(N^2 - 4)(N^2 - 16)(N^2 - 36)}. \quad (13)
\end{aligned}$$

Obviously,

$$\frac{dP(N)}{dN} = P'(N) \begin{cases} <0, & \frac{N}{2} \in (2n, 2n+1), \quad n > 3, \\ =0, & \frac{N}{2} = 2n, \quad n \in \mathbb{Z}, n \geq 3, \\ >0, & \frac{N}{2} \in (2n+1, 2(n+1)), \quad n > 3. \end{cases} \quad (14)$$

That is, $N = 4n, n \in \mathbb{Z}, n \geq 3$, is in correspondence with the extreme point of the function $P(N)$, which is a

TABLE 1: Solution of $P(N) = 1$.

7.437330346546332	9.323969185188616	11.261199765911442
13.219709116325248	15.189881678437814	17.167295939754695
19.149560460820794	21.135248098522425	23.123447719961405
25.113547757267952	27.105121575295925	29.097861737012863
31.091541469097137	33.085988700389862	35.081073641777039
37.076687097549438	39.072752714157104	41.069202184677124
43.065983057022095	45.063048839569092	47.060366630554199
49.057900428771973	51.055631637573242	53.053528785705566
55.051587104797363	57.049772262573242	59.048089981079102
61.046504974365234	63.045036315917969	65.043693542480469
67.042350769042969	69.041099548339844	

monotone function on the intervals $(2n, 2n+1)$ and $(2n+1, 2n+2)$. Based on the mean value theorem, there must exist unique points $N/2 \in (2n, 2n+1)$ and $N/2 \in (2n, 2n+2)$ so as to $P(N) = 1$, i.e.,

$$P(N) = \widehat{S}_C(0) = \int_{-\infty}^{\infty} S_C(x) dx = 1. \quad (15)$$

This completes the proof. $\square$

By means of the interval bisection method, it is easy to obtain the value of N which is in correspondence with $P(N) = 1$ as in Table 1.

It is easy to prove that the simplified Shannon-Cosine wavelet is equivalent to the Shannon-Cosine wavelet proposed in [36] using the parameter $m \leq 3$, but only the parameter N should be chosen in applications. So, it is convenient for solving the fractional derivative PDEs.

4. Point-Symmetric Interval Wavelet

In order to eliminate the boundary effect introduced by the wavelet transform, the point-symmetric extension and the general variational theory are employed to construct the interval wavelet. Compared to other extension methods such as the zero extension, symmetric extension, and periodic extension, the point-symmetric extension can ensure that the function is smooth at the endpoint.

Theorem 2. *If the continuous function $f(x)$ is symmetric on point $(x_0, f(x_0))$, then the first-order derivative of the function $f(x)$ is continuous at this point.*

Proof. As the continuous function $f(x)$ is symmetric on point $(x_0, f(x_0))$, let $\Delta x > 0$; we have

$$\begin{aligned}
f(x_0 - \Delta x) &= 2f(x_0) - f(x_0 + \Delta x), \\
\lim_{x_0 - \Delta x \rightarrow x_0} \frac{f(x_0 - \Delta x) - f(x_0)}{x_0 - \Delta x - x_0} &= \lim_{x_0 - \Delta x \rightarrow x_0} \frac{f(x_0 - \Delta x) - f(x_0)}{-\Delta x} \\
&= \lim_{x_0 - \Delta x \rightarrow x_0} \frac{2f(x_0) - f(x_0 + \Delta x) - f(x_0)}{-\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}, \\
\lim_{x_0 + \Delta x \rightarrow x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{x_0 + \Delta x - x_0} &= \lim_{x_0 + \Delta x \rightarrow x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x} \\
&= \lim_{\Delta x \rightarrow 0} \frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.
\end{aligned} \tag{16}$$

Therefore, we have

$$\lim_{x_0 - \Delta x \rightarrow x_0} \frac{f(x_0 - \Delta x) - f(x_0)}{x_0 - \Delta x - x_0} = \lim_{x_0 + \Delta x \rightarrow x_0} \frac{f(x_0 + \Delta x) - f(x_0)}{x_0 + \Delta x - x_0}. \tag{17}$$

This illustrates that the first derivative of the function $f(x)$ continues to reach the point $(x_0, f(x_0))$. This completes the proof.

For convenience, the values of $f(x)$ at $x_{-n}, x_{-n+1}, \dots, x_0, x_1, \dots, x_{2^j+1}$ are denoted as $f_{-n}, f_{-n+1}, \dots, f_0, f_1, \dots, f_{2^j+1}$. Based on the general variational theory, we can deduce the format of the interval wavelet function as follows:

① As $x = -n, -n+1, \dots, -1$,

$$f_{-1} = (2f_0 - f_1)\cos^2(\pi/2n)$$

$$f_{-2} = (2f_0 - f_2)\cos^2(2\pi/2n)$$

$\dots$

$$f_{-k} = (2f_0 - f_k)\cos^2(k\pi/2n)$$

$\dots$

$$f_{-N} = (2f_0 - f_N)\cos^2(\pi/2)$$

② As $x = 2^j + 1, 2^j + 2, \dots, 2^j + n$,

$$f_{2^j+1} = (2f_{2^j} - f_{2^j-1})\cos^2(\pi/2n)$$

$$f_{2^j+2} = (2f_{2^j} - f_{2^j-2})\cos^2(2\pi/2n)$$

$\dots$

$$f_{2^j+k} = (2f_{2^j} - f_{2^j-k})\cos^2(k\pi/2n)$$

$\dots$

$$f_{2^j+n} = (2f_{2^j} - f_{2^j-n})\cos^2(n\pi/2n)$$

And so, the function $f(x)$ can be expressed as

$$\begin{aligned}
f(x) &= \sum_{k=-n}^{2^j+n} \phi(2^j x - k) f_k^j \\
&= \sum_{k=-n}^{-1} \phi(2^j x - k) (2f_0 - f_{-k}^j) \cos^2\left(\frac{k\pi}{2n}\right) + \sum_{k=0}^{2^j} \phi(2^j x - k) f_k^j + \sum_{k=2^j+1}^{2^j+n} \phi(2^j x - k) (2f_{2^j} - f_{2^j-k}^j) \cos^2\left(\frac{(k-2^j)\pi}{2n}\right) \\
&= - \sum_{k=-n}^{-1} \phi(2^j x - k) \cos^2\left(\frac{k\pi}{2n}\right) f_{-k}^j + 2 \sum_{k=-n}^{-1} \phi(2^j x - k) \cos^2\left(\frac{k\pi}{2n}\right) f_0 + \sum_{k=0}^{2^j} \phi(2^j x - k) f_k^j \\
&\quad + 2 \sum_{k=2^j+1}^{2^j+n} \phi(2^j x - k) \cos^2\left(\frac{(k-2^j)\pi}{2n}\right) f_{2^j} - \sum_{k=2^j+1}^{2^j+n} \phi(2^j x - k) \cos^2\left(\frac{(k-2^j)\pi}{2n}\right) f_{2^j+1-k}^j \\
&= - \sum_{k=1}^n \phi(2^j x + k) \cos^2\left(\frac{k\pi}{2n}\right) f_k^j + 2 \sum_{k=1}^n \phi(2^j x + k) \cos^2\left(\frac{k\pi}{2n}\right) f_0 + \sum_{k=0}^{2^j} \phi(2^j x - k) f_k^j \\
&\quad + 2 \sum_{k=-2^j-n}^{-2^j-1} \phi(2^j x + k) \cos^2\left(\frac{(k+2^j)\pi}{2n}\right) f_{2^j} - \sum_{k=2^j+1}^{2^j+n} \phi(2^j x - k) \cos^2\left(\frac{(k-2^j)\pi}{2n}\right) f_{2^j+1-k}^j.
\end{aligned} \tag{18}$$

Let $m = 2^{j+1} - k$; we have

$$\begin{aligned} f(x) = & - \sum_{k=1}^n \phi(2^j x + k) \cos^2\left(\frac{k\pi}{2n}\right) f_k^j + 2 \sum_{k=-n}^{-1} \phi(2^j x - k) \cos^2\left(\frac{k\pi}{2n}\right) f_0^j + \sum_{k=0}^{2^j} \phi(2^j x - k) f_k^j \\ & + 2 \sum_{k=2^j+1}^{2^j+n} \phi(2^j x - k) \cos^2\left(\frac{(2^j - k)\pi}{2n}\right) f_{2^j}^j - \sum_{m=2^j-n}^{2^j-1} \phi(2^j x - 2^{j+1} + m) \cos^2\left(\frac{(2^j - m)\pi}{2n}\right) f_m^j, \end{aligned} \quad (19)$$

where

So, we have

$$\begin{aligned} \sum_{k=0}^{2^j} \phi(2^j x - k) f_k^j &= \phi(2^j x) f_0^j + \sum_{k=1}^n \phi(2^j x - k) f_k^j \\ &+ \sum_{k=n+1}^{2^j-n-1} \phi(2^j x - k) f_k^j + \sum_{k=2^j-n}^{2^j-1} \phi(2^j x - k) f_k^j + \phi(2^j x - 2^j) f_{2^j}^j. \end{aligned} \quad (20)$$

$$\begin{aligned} f(x) = & \left(\phi(2^j x) + 2 \sum_{k=-n}^{-1} \phi(2^j x - k) \cos^2\left(\frac{k\pi}{2n}\right) \right) f_0^j + \sum_{k=1}^n \left(\phi(2^j x - k) - \phi(2^j x + k) \cos^2\left(\frac{k\pi}{2n}\right) \right) f_k^j \\ & + \sum_{k=N+1}^{2^j-N-1} \phi(2^j x - k) f_k^j + \sum_{k=2^j-N}^{2^j-1} \left(\phi(2^j x - k) - \phi(2^j x - 2^{j+1} + k) \cos^2\left(\frac{(2^j - k)\pi}{2n}\right) \right) f_k^j \\ & + \left(\phi(2^j x - 2^j) + 2 \sum_{k=2^j+1}^{2^j+N} \phi(2^j x - k) \cos^2\left(\frac{(2^j - k)\pi}{2n}\right) \right) f_{2^j}^j. \end{aligned} \quad (21)$$

Therefore, the interval interpolative wavelet function can be expressed as

$$w(2^j \mathbf{x} - \mathbf{k}) = w_{j,k} = \begin{cases} \phi(2^j \mathbf{x} - \mathbf{k}) + 2 \sum_{k=1}^P \phi(2^j \mathbf{x} + \mathbf{k}) \cos^2\left(\frac{k\pi}{2P}\right), & \mathbf{k} = 0, \\ \phi(2^j \mathbf{x} - \mathbf{k}) - \phi(2^j \mathbf{x} + \mathbf{k}) \cos^2\left(\frac{k\pi}{2P}\right), & \mathbf{k} = 1, 2, \dots, P, \\ \phi(2^j \mathbf{x} - \mathbf{k}), & \mathbf{k} = N + 1, \dots, 2^j - P - 1, \\ \phi(2^j \mathbf{x} - \mathbf{k}) - \phi(2^j \mathbf{x} - 2^{j+1} + \mathbf{k}) \cos^2\left(\frac{(2^j - k)\pi}{2P}\right), & \mathbf{k} = 2^j - P, \dots, 2^j - 1, \\ \phi(2^j \mathbf{x} - 2^j) + 2 \sum_{k=2^j+1}^{2^j+N} \phi(2^j \mathbf{x} - \mathbf{k}) \cos^2\left(\frac{(2^j - k)\pi}{2P}\right), & \mathbf{k} = 2^j. \end{cases} \quad (22)$$

According to theory 2, it is easy to understand that equation (22) is able to reduce the boundary effect efficiently. In theory, the proposed interval wavelet can prevent the steep shock wave appearing near the boundary, which can introduce the large condition number of the discrete matrix and result in the decrease of the numerical precision.

5. Application in Solving Fractional PDEs

5.1. Interval Wavelet Spectral Method for Fractional Fokker–Planck Equation. The fractional Fokker–Planck equation is a typical fractional PDE, which is often used to describe a subdiffusive behavior of a particle under the combined influence of external nonlinear force field and a Boltzmann thermal heat bath. In the presence of an external force field $F(\mathbf{x}) = -v'(x)$, the evolution of a test particle is usually described in terms of the Fokker–Planck equation (FPE):

$$\frac{\partial u(x, t)}{\partial t} = {}_0D_t^{1-\alpha} \left[\frac{\partial}{\partial x} \frac{v'(x)}{m\eta_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2} \right] u(x, t), \quad (23)$$

$$a \leq x \leq b, 0 \leq t \leq T,$$

which defines the probability $u(x, t)$ of finding the particle at a certain position x at a given time t . m denotes the mass of the diffusing particle, $K_\alpha > 0$ denotes the generalized diffusion coefficient with dimension $[K_\alpha] = \text{cm}^2 \text{sec}^{-\alpha}$, and η_α is the generalized friction coefficient with dimension $[\eta_\alpha] = \text{sec}^{\alpha-2}$. The corresponding initial condition is

$$u(x, 0) = \varphi(x), \quad a \leq x \leq b, \quad (24)$$

and the boundary conditions are

$$\begin{aligned} u(a, t) &= p_1(t), \\ u(b, t) &= p_2(t), \end{aligned} \quad (25)$$

$$0 < t \leq T.$$

Equation (2) uses the Riemann–Liouville fractional derivative of order $1 - \alpha$, defined by

$${}_0D_t^{1-\alpha} u(x, t) = \frac{1}{\Gamma(\alpha)} \frac{\partial}{\partial t} \int_0^1 \frac{u(x, \eta)}{(t - \eta)^{1-\alpha}} d\eta, \quad 0 \leq \alpha < 1, \quad (26)$$

where $\Gamma(\alpha)$ is the gamma function.

According to the properties of the Riemann–Liouville fractional derivative, it is easy to know that, if $(x, t) \in C_{x,t}^{2,1}([a, b] \times [0, T])$, equation (2) can be rewritten as follows:

$$D_t^\alpha u(x, t) - \frac{u(x, 0)t^{-\alpha}}{\Gamma(1-\alpha)} = \left[\frac{\partial}{\partial x} \frac{v'(x)}{m\eta_\alpha} + K_\alpha \frac{\partial^2}{\partial x^2} \right] u(x, t), \quad (27)$$

$$a \leq x \leq b, 0 \leq t \leq T.$$

According to the wavelet spectral method, the fractional Fokker–Planck equation can be approximately represented as

$$\begin{aligned} & \tau^{-\alpha} \left[u_j(x_i, t_n) + \sum_{k=1}^{n-1} g_k u_j(x_i, t_{n-k}) - \sum_{k=0}^{n-1} g_k u_j(x_i, t_0) \right] \\ &= f'(x_i) u_j(x_i, t_n) + \sum_{m=0}^{2^j} u_j(x_m, t_n) [w'(x_i - x_m) \\ & \quad + K_\alpha w''(x_i - x_m)], \end{aligned} \quad (28)$$

where $i = 0, 1, 2, \dots, 2^j$. Let

$$\begin{aligned} V_j^n &= (u_j(x_0, t_n), u_j(x_1, t_n), \dots, u_j(x_{2^j}, t_n))^T, \\ F &= \text{diag}(f'(x_0), f'(x_1), \dots, f'(x_{2^j})), \\ W_1 &= \begin{bmatrix} w'(x_0 - x_0) & w'(x_0 - x_1) & \cdots & w'(x_0 - x_{2^j}) \\ w'(x_1 - x_0) & w'(x_1 - x_1) & \cdots & w'(x_1 - x_{2^j}) \\ \vdots & \vdots & \ddots & \vdots \\ w'(x_{2^j} - x_0) & w'(x_{2^j} - x_1) & \cdots & w'(x_{2^j} - x_{2^j}) \end{bmatrix}, \\ W_2 &= \begin{bmatrix} w''(x_0 - x_0) & w''(x_0 - x_1) & \cdots & w''(x_0 - x_{2^j}) \\ w''(x_1 - x_0) & w''(x_1 - x_1) & \cdots & w''(x_1 - x_{2^j}) \\ \vdots & \vdots & \ddots & \vdots \\ w''(x_{2^j} - x_0) & w''(x_{2^j} - x_1) & \cdots & w''(x_{2^j} - x_{2^j}) \end{bmatrix}. \end{aligned} \quad (29)$$

Then, the system of (28) can be expressed in the matrix format:

$$(W_1 + K_\alpha W_2 + F - \tau^{-\alpha} I) V_j^n = \sum_{k=1}^{n-1} g_k V_j^{n-k} - \sum_{k=0}^{n-1} g_k V_j^0. \quad (30)$$

5.2. Numerical Experiments. In this section, a simple Fokker–Planck equation with the exact analytical solution is taken as the example to illustrate the effectiveness. Consider the Fokker–Planck equation as follows:

$$\frac{\partial u(x, t)}{\partial t} = {}_0D_t^{1-\alpha} \left[\frac{\partial}{\partial x} (-1) + \frac{\partial^2}{\partial x^2} \right] u(x, t), \quad 0 \leq x \leq 1, t > 0. \quad (31)$$

With the initial condition

$$u(x, 0) = x(1 - x), \quad 0 \leq x \leq 1, \quad (32)$$

the boundary conditions are

$$\begin{aligned} u(0, t) &= -\frac{3t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad t > 0, \\ u(1, t) &= -\frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}, \quad t > 0. \end{aligned} \quad (33)$$

The exact analytic solution is

$$u(x, t) = x(1 - x) + (2x - 3) \frac{t^\alpha}{\Gamma(1+\alpha)} - \frac{2t^{2\alpha}}{\Gamma(1+2\alpha)}. \quad (34)$$

TABLE 2: Influence of α on the numerical precision ($t = 0.0001$, $T = 0.1$).

j	α	e_1			e_2	
		Point-symmetric interval WPM	Lagrange interval WPM ($L = 2$)	Dynamic interval WPM	Lagrange interval WPM ($L = 2$)	Dynamic interval WPM
6	0.8	0.0012	6.2776×10^{-6}	3.5310×10^{-4}	9.5851×10^{-4}	4.9985×10^{-6}
	0.6	0.0079	9.6280×10^{179}	0.0096	0.0066	0.0076
	0.4	0.0509	0	0.0588	0.0433	0.0499
	0.2	0.3488	0	0.3962	0.2909	0.3284
7	0.8	0.0012	1.9290×10^{155}	0.0014	9.9232×10^{-4}	$\inf^4$
	0.6	0.008	0	0.0092	0.0067	0.0077
	0.4	0.0511	0	0.0586	0.0437	0.0502
	0.2	0.3503	0	0.3932	0.2936	0.3282

FIGURE 3: Trend of the solution with t .

All the comparisons in this section are made qualitatively by comparing the calculation precision in the same time step and space mesh grid size. The first measure of error e_1 is given by

$$e_1 = \|V_j^n - V_{\text{exact}}^n\|_{\infty}. \quad (35)$$

It provides a measure of the accuracy of the solution near the boundary. The second measure of error e_2 is given by

$$e_2 = \sqrt{\frac{1}{2^j + 1} \sum_{i=0}^{2^j} (u(x_i) - u_{\text{exact}}(x_i))^2}. \quad (36)$$

It provides a general measure of the accuracy of the solution over the main body of the distribution and was often used to investigate the accuracy of the FEM.

In [24], the dynamic interval wavelet spectral method (WPM) is employed to solve the fractional Fokker–Planck equation, in which the Lagrange interpolation-based interval wavelet spectral method is taken to compare with their method. Compared to these two methods, as shown in Table 2, the proposed method is more robust and

insensitive to the parameters j and α . With the increase in the discrete point amount and decrease in the parameter α , the numerical precision of the proposed method is better than of the dynamic interval WPM.

It should be noticed that the numerical precision of the dynamic interval WPM is better than the proposed method as $j = 6$ and $\alpha = 0.8$. The amount of the discrete points is 2^j . This denotes that the numerical precision obtained by fewer discrete points is better than that obtained by more points. This is obviously unreasonable.

In fact, the dynamic interval wavelet function is constructed based on the Lagrange interpolation. The condition number of the discrete matrix is becoming very large with the increase in the Lagrange polynomial degree [36]. This is the primary reason why the numerical precision with fewer discrete points is better than more points by the dynamic interval WPM. Theorem 2 illustrates that the point-symmetric interval wavelet method can overcome this shortcoming of the dynamic interval wavelet as it does not have the steep wave close to the boundary point introduced by the Lagrange interpolation.

The trend of the solution with the parameter “ t ” obtained by the proposed method is shown in Figure 3. This illustrates

that the point-symmetric interval WPM is a robust method for the fractional PDEs.

6. Conclusions

By means of the waveform of the sinc function, a family of simplified Shannon-Cosine scaling functions is presented, which can be utilized to construct the wavelet spectral method for solving the fractional PDEs, combining with the point-symmetric extension, the solutions of which are permitted to have different smoothness. Compared with the sinc and Shannon-Gabor functions, the Shannon-Cosine scaling functions possess almost all the excellent numerical properties such as the compact support, interpolation, and derivability. The point-symmetric extension-based interval wavelet is a basis with robust properties, which can prevent the increase in the condition number introduced by the Lagrange interpolation. Besides, the proposed interval wavelet need not choose the smoothness of the approximation function near the boundary points, and this is helpful to improve the efficiency of the algorithm. Based on the choice scheme of the parameters appeared in the simplified Shannon-Cosine wavelet function, the adaptability of the spectral method to the smoothness of the solution can be improved greatly at even fewer collocation points. Compared with the Shannon-Cosine wavelets, there are fewer parameters in the simplified wavelet function, and this brings us a lot of conveniences in solving PDEs.

Data Availability

The MATLAB source code supporting the findings of this study has been deposited in the GitHub repository (<https://github.com/meishuli/meishuli/tree/Matlab-source-code>).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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