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Research Article

Iterative Approximations for a Class of Generalized Nonexpansive Operators in Banach Spaces

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In this article, we prove some weak and strong convergence theorems for mappings satisfying condition (E) using the AK iterative scheme in the setting of Banach spaces. We offer a new example of mapping with condition (E) in support of our main result. Our results extend and improve many well-known corresponding results of the current literature.

1. Introduction

Let $\mathcal{X}$ be a Banach space, $\varnothing \neq \mathcal{C} \subseteq \mathcal{X}$ and $J\colon \mathcal{C} \longrightarrow \mathcal{C}$. An element $p \in \mathcal{C}$ is called a fixed point for J if $Jp = p$. We denote by $\mathrm{Fix}(J)$ the set of all fixed points of the map J. Throughout the work, we will denote by $\mathbb{N}$ the set of all natural numbers. When J is nonexpansive, that is, for all $z, z' \in \mathcal{C}$, one has $\|Jz - Jz'\| \leq \|z - z'\|$, then $\mathrm{Fix}(J)$ is nonempty provided that $\mathcal{X}$ is uniformly convex and $\mathcal{C}$ is convex closed bounded (see [1–3] and others). In 2008, Suzuki [4] introduced a new class of nonlinear mappings, which is the generalization of the class of nonexpansive mappings. A mapping $J\colon \mathcal{C} \longrightarrow \mathcal{C}$ is said to satisfy condition (C) (or Suzuki mapping) if for all $z, z' \in \mathcal{C}$,

$$\frac{1}{2}\|z - Jz\| \leq \left\|z - z'\right\| \text{ one has } \left\|Jz - Jz'\right\| \leq \left\|z - z'\right\|. \tag{1}$$

In 2011, Garcia-Falset et al. [5] extended condition (C) to the general formulations as follows. A mapping $J\colon \mathcal{C} \longrightarrow \mathcal{C}$ is said to satisfy condition (E_μ) if there exists some $\mu \geq 1$ such that

$$\left\|z - Jz'\right\| \leq \mu\|z - Jz\| + \left\|z - z'\right\| \text{ for all } z, z' \in \mathcal{C}. \tag{2}$$

A mapping J is said to satisfy condition (E) (or Garcia-Falset mapping) whenever J satisfies condition (E_μ) for some $\mu \geq 1$. Garcia-Falset et al. [5] proved that every Suzuki mapping satisfies condition (E) with $\mu = 3$. Notice also that the class of Garcia-Falset mappings also includes many other classes of generalized nonexpansive (see [6] for details). In this paper, we study in deep this general class of mapping.

The iterative approximation of fixed points for nonlinear operators is an active research area nowadays (see, e.g., [7–11] and others). The Banach contraction principle suggests a Picard iterative scheme for finding the unique fixed point of a given contraction mapping. However, the Picard iterative scheme does not always converge to a fixed point of a nonexpansive mapping. Thus, to overcome such difficulties and to obtain a better speed of convergence, many iterative schemes are available in the literature. Let $\mathcal{C}$ be a nonempty convex subset of a Banach space $\mathcal{X}$ and $J\colon \mathcal{C} \longrightarrow \mathcal{C}$. Assume that $a_n, b_n, c_n \in (0, 1)$ for all $n \in \mathbb{N}$. Then, the well-known Picard, Mann [12], Ishikawa [13], Noor [14], Agarwal [15], Abbas and Nazir [16], Thakur [17], M* [18], and AK [19] iterative schemes are, respectively, read as follows:

$$\begin{cases} z_1 = z \in \mathscr{C}, \\ z_{n+1} = Jz_n, \end{cases} \tag{3}$$

$$\begin{cases} z_1 = z \in \mathscr{C}, \\ z_{n+1} = (1 - a_n)z_n + a_n Jz_n, \end{cases} \tag{4}$$

$$\begin{cases} z_1 \in \mathscr{C}, \\ w_n = (1 - b_n)z_n + b_n Jz_n, \\ z_{n+1} = (1 - a_n)z_n + a_n Jw_n, \end{cases} \tag{5}$$

$$\begin{cases} z_1 = z \in \mathscr{C}, \\ v_n = (1 - c_n)z_n + c_n Jz_n, \\ w_n = (1 - b_n)z_n + b_n Jv_n, \\ z_{n+1} = (1 - a_n)z_n + a_n Jw_n, \end{cases} \tag{6}$$

$$\begin{cases} z_1 = z \in \mathscr{C}, \\ w_n = (1 - b_n)z_n + b_n Jz_n, \\ z_{n+1} = (1 - a_n)Jz_n + a_n Jw_n, \end{cases} \tag{7}$$

$$\begin{cases} z_1 = z \in \mathscr{C}, \\ v_n = (1 - c_n)z_n + c_n Jz_n, \\ w_n = (1 - b_n)Jz_n + b_n Jv_n, \\ z_{n+1} = (1 - a_n)Jw_n + a_n Jv_n, \end{cases} \tag{8}$$

$$\begin{cases} z_1 = z \in \mathscr{C}, \\ v_n = (1 - b_n)z_n + b_n Jz_n, \\ w_n = J((1 - a_n)z_n + a_n v_n), \\ z_{n+1} = Jw_n, \end{cases} \tag{9}$$

$$\begin{cases} z_1 = z \in \mathscr{C}, \\ v_n = (1 - b_n)z_n + b_n Jz_n, \\ w_n = J((1 - a_n)z_n + a_n Jv_n), \\ z_{n+1} = Jw_n, \end{cases} \tag{10}$$

$$\begin{cases} z_1 = z \in \mathscr{C}, \\ v_n = J((1 - b_n)z_n + b_n Jz_n), \\ w_n = J((1 - a_n)v_n + a_n Jv_n), \\ z_{n+1} = Jw_n. \end{cases} \tag{11}$$

In [15], Agarwal et al. proved that the iterative scheme (7) converges faster than the iterative schemes (3)–(5) for contraction mappings. In [17], Thakur et al. with the help of a numerical example proved that the iterative scheme (9) converges faster than all of the iterative schemes (3)–(8) for the general setting of Suzuki mappings. In [18], Ullah and Arshad with the help of a numerical example proved that the iterative scheme (10) is better than the leading iterative scheme (9) for Suzuki mappings in Banach spaces. In [19], the authors proved that the AK iterative scheme (11) is stable and converges faster than many well-known iterative schemes for contraction mappings. The purpose of this research is to study the iterative scheme (11) for the generalized class of Garcia-Falset mappings. We also give a new example of the Garcia-Falset mapping and show that its AK iteration process is more efficient than all of the above schemes.

2. Preliminaries

Let $\mathscr{C}$ be any nonempty subset of a Banach space $\mathscr{X}$ and let $\{z_n\}$ be a bounded sequence in $\mathscr{X}$. For $z \in \mathscr{X}$, we set

$$r(z, \{z_n\}) = \limsup_{n \longrightarrow \infty} \|z - z_n\|. \tag{12}$$

The asymptotic radius of $\{z_n\}$ relative to $\mathscr{C}$ is given by

$$r(\mathscr{C}, \{z_n\}) = \inf\{r(z, \{z_n\}) \colon z \in \mathscr{C}\}. \tag{13}$$

The asymptotic center of $\{z_n\}$ relative to $\mathscr{C}$ is the set

$$A(\mathscr{C}, \{z_n\}) = \{z \in \mathscr{C} \colon r(z, \{z_n\}) = r(\mathscr{C}, \{z_n\})\}. \tag{14}$$

When the space $\mathscr{X}$ is uniformly convex [20], then the set $A(\mathscr{C}, \{z_n\})$ is singleton. Notice also that the set $A(\mathscr{C}, \{z_n\})$ is convex as well as nonempty provided that $\mathscr{C}$ is weakly compact convex (see, e.g., [21, 22]).

We say that a Banach space $\mathscr{X}$ has Opial's property [23] if and only if for all $\{z_n\}$ in $\mathscr{C}$ which weakly converges to $z \in \mathscr{X}$ and for every $w \in \mathscr{X}$-z, one has

$$\limsup_{n \longrightarrow \infty} \|z_n - z\| < \limsup_{n \longrightarrow \infty} \|z_n - w\|. \tag{15}$$

The following lemma gives many examples of Garcia-Falset mappings.

Lemma 1 (see [5]). *Let J be a mapping on a subset $\mathscr{C}$ of a Banach space. If J satisfies condition (C), then J also satisfies condition (E) with $\mu = 3$.*

Lemma 2 (see [5]). *Let J be a mapping on a subset $\mathscr{C}$ of a Banach space. If J satisfies condition (E), then for all $p \in Fix(J)$ and $z \in \mathscr{C}$, we have $\|Jz - p\| \leq \|z - p\|$.*

Lemma 3 (see [5]). *Let J be a mapping on a subset $\mathscr{C}$ of a Banach space $\mathscr{X}$ having the Opial property. Assume that J satisfies the condition (E). If $\{z_n\}$ converges weakly to q and $\lim_{n \longrightarrow \infty} \|z_n - Jz_n\| = 0$, then $q \in Fix(J)$.*

In 1991, Schu [24] proved the following useful fact.

Lemma 4. *Let $\mathscr{X}$ be a uniformly convex Banach space and $0 < u \leq \theta_n \leq v < 1$ for all $n \in \mathbb{N}$. If $\{z_n\}$ and $\{w_n\}$ are two sequences in $\mathscr{X}$ such that $\limsup_{n \longrightarrow \infty} \|z_n\| \leq \lambda$, $\limsup_{n \longrightarrow \infty} \|w_n\| \leq \lambda$, and $\lim_{n \longrightarrow \infty} \|\theta_n z_n + (1 - \theta_n)w_n\| = \lambda$ for some $\lambda \geq 0$, then $\lim_{n \longrightarrow \infty} \|z_n - w_n\| = 0$.*

3. Convergence Theorems in Uniformly Convex Banach Spaces

In this section, we shall state and prove our main results. First, we give the following key lemma.

Lemma 5. *Let $\mathscr{C}$ be a nonempty closed convex subset of a Banach space $\mathscr{X}$ and let $J \colon \mathscr{C} \longrightarrow \mathscr{C}$ be a mapping satisfying* *condition (E) with* $Fix(J) \neq \emptyset$*. Let the sequence* $\{z_n\}$ *be defined by (11), then* $\lim_{n\longrightarrow\infty}\|z_n - p\|$ *exists for all* $p \in Fix(J)$.

Proof. Suppose $p \in \mathrm{Fix}(J)$. By Lemma 2, we have

$$\begin{aligned}\|v_n - p\| &= \|J((1-b_n)z_n + b_nJz_n) - p\| \le \|(1-b_n)z_n + b_nJz_n - p\| \\ &\le (1-b_n)\|z_n - p\| + b_n\|Jz_n - p\| \le (1-b_n)\|z_n \\ &\quad - p\| + b_n\|z_n - p\| \le \|z_n - p\|,\end{aligned} \tag{16}$$

which implies that

$$\begin{aligned}\|z_{n+1} - p\| &= \|Jw_n - p\| \le \|w_n - p\| \\ &= \|J((1-a_n)v_n + a_nJv_n) - p\| \\ &\le \|(1-a_n)v_n + a_nJv_n - p\| \\ &\le (1-a_n)\|v_n - p\| + a_n\|Jv_n - p\| \\ &\le (1-a_n)\|v_n - p\| + a_n\|v_n - p\| \\ &= \|v_n - p\| \le \|z_n - p\|.\end{aligned} \tag{17}$$

Thus, $\{\|z_n - p\|\}$ is bounded and nonincreasing, which implies that $\lim_{n\longrightarrow\infty}\|z_n - p\|$ exists for each $p \in \mathrm{Fix}(J)$.

The following theorem will be used in the upcoming results. □

Theorem 1. *Let* $\mathscr{C}$ *be a nonempty closed convex subset of a uniformly convex Banach space* $\mathscr{X}$ *and let* $J: \mathscr{C} \longrightarrow \mathscr{C}$ *be a mapping satisfying condition (E). Let* $\{z_n\}$ *be the sequence defined by (11). Then,* $Fix(J) \neq \emptyset$ *if and only if* $\{z_n\}$ *is bounded and* $\lim_{n\longrightarrow\infty}\|Jz_n - z_n\| = 0$.

Proof. Let $\{z_n\}$ be bounded and $\lim_{n\longrightarrow\infty}\|Jz_n - z_n\| = 0$. Let $p \in A(\mathscr{C}, \{z_n\})$. We shall prove that $Jp = p$. Since J satisfies condition (E), we have

$$\begin{aligned}r(Jp, \{z_n\}) &= \limsup_{n\longrightarrow\infty}\|z_n - Jp\| \le \mu \limsup_{n\longrightarrow\infty}\|Jz_n - z_n\| \\ &\quad + \limsup_{n\longrightarrow\infty}\|z_n - p\| \\ &= \limsup_{n\longrightarrow\infty}\|z_n - p\| \\ &= r(p, \{z_n\}).\end{aligned} \tag{18}$$

It follows that $Jp \in A(\mathscr{C}, \{z_n\})$. Since $A(\mathscr{C}, \{z_n\})$ is a singleton set, we have $Jp = p$. Hence, $\mathrm{Fix}(J) \neq \emptyset$.

Conversely, we assume that $\mathrm{Fix}(J) \neq \emptyset$ and $p \in \mathrm{Fix}(J)$. We shall prove that $\{z_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\|z_n - Jz_n\| = 0$. By Lemma 5, $\lim_{n\longrightarrow\infty}\|z_n - p\|$ exists and $\{z_n\}$ is bounded. Put

$$\lim_{n\longrightarrow\infty}\|z_n - p\| = \lambda. \tag{19}$$

By (16), we have

$$\begin{aligned}&\|v_n - p\| \le \|z_n - p\| \\ &\Longrightarrow \limsup_{n\longrightarrow\infty}\|v_n - p\| \le \limsup_{n\longrightarrow\infty}\|z_n - p\| = \lambda.\end{aligned} \tag{20}$$

By Lemma 2, we have

$$\|Jz_n - p\| \le \|z_n - p\|, \tag{21}$$

$$\Longrightarrow \limsup_{n\longrightarrow\infty}\|Jz_n - p\| \le \limsup_{n\longrightarrow\infty}\|z_n - p\| = \lambda. \tag{22}$$

By (17), we have

$$\|z_{n+1} - p\| \le \|v_n - p\|. \tag{23}$$

So, we can get

$$\Longrightarrow \lambda \le \liminf_{n\longrightarrow\infty}\|v_n - p\|. \tag{24}$$

From (20) and (24), we get

$$\lambda = \lim_{n\longrightarrow\infty}\|v_n - p\|. \tag{25}$$

Using (19) and (25), we have

$$\begin{aligned}\lambda &= \lim_{n\longrightarrow\infty}\|v_n - p\| = \lim_{n\longrightarrow\infty}\|J((1-b_n)z_n + b_nJz_n) - p\| \\ &\le \lim_{n\longrightarrow\infty}\|(1-b_n)z_n + b_nJz_n - p\| \\ &= \lim_{n\longrightarrow\infty}\|(1-b_n)(z_n - p) + b_n(Jz_n - p)\| \\ &\le (1-b_n)\lim_{n\longrightarrow\infty}\|z_n - p\| + b_n\lim_{n\longrightarrow\infty}\|Jz_n - p\| \\ &\le (1-b_n)\lim_{n\longrightarrow\infty}\|z_n - p\| + b_n\lim_{n\longrightarrow\infty}\|z_n - p\| = \lambda.\end{aligned} \tag{26}$$

Hence,

$$\lambda = \lim_{n\longrightarrow\infty}\|(1-b_n)(z_n - p) + b_n(Jz_n - p)\|. \tag{27}$$

Applying Lemma 4, we obtain

$$\lim_{n\longrightarrow\infty}\|Jz_n - z_n\| = 0. \tag{28}$$

First, we discuss the strong convergence of $\{z_n\}$ defined by (11) for mappings with condition (E). □

Theorem 2. *Let* $\mathscr{C}$ *be a nonempty convex compact subset of a uniformly convex Banach space* $\mathscr{X}$ *and let* J *and* $\{z_n\}$ *be as in Theorem 1 and* $Fix(J) \neq \emptyset$*. Then,* $\{z_n\}$ *converges strongly to a fixed point of* J.

Proof. By Theorem 1, $\lim_{n\longrightarrow\infty}\|Jz_n - z_n\| = 0$. By compactness of $\mathscr{C}$, we can find a subsequence $\{z_{nl}\}$ of $\{z_n\}$ such that $\{z_{nl}\}$ converges strongly to $q \in \mathscr{C}$ for some q. Since J satisfies condition (E), there exists some $\mu \ge 1$, such that

$$\|z_{n_l} - Jq\| \le \mu\|z_{n_l} - Jz_{n_l}\| + \|z_{n_l} - q\|. \tag{29}$$

Letting $l \longrightarrow \infty$, we get $Jq = q$. By Lemma 5, $\lim_{n\longrightarrow\infty}\|z_n - q\|$ exists. Hence, q is the strong limit of $\{z_n\}$.

Proof of the following result is elementary and hence omitted. □

Theorem 3. *Let* $\mathscr{C}$ *be a nonempty closed convex subset of a uniformly convex Banach space* $\mathscr{X}$ *and let* J *and* $\{z_n\}$ *be as in Theorem 1. If* $Fix(J) \neq \emptyset$ *and* $liminf_{n\longrightarrow\infty}$

dist $(z_n, Fix(J)) = 0$, *then* $\{z_n\}$ *converges strongly to a fixed point of* J.

Now, we establish a strong convergence result for Garcia-Falset mappings using the AK iteration process with the help of condition (I).

Definition 1 (see [25]). Let $\mathscr{C}$ be a nonempty subset of a Banach space $\mathscr{X}$. A mapping $J: \mathscr{C} \longrightarrow \mathscr{C}$ is said to satisfy condition (I) if there is a function $\alpha: [0,\infty) \longrightarrow [0,\infty)$ satisfying $\alpha(0) = 0$ and $\alpha(t) > 0$ for all $t \in (0,\infty)$ such that $\|z - Jz\| \geq \alpha(\text{dist}(z, \text{Fix}(J)))$ for all $z \in \mathscr{C}$.

Theorem 4. *Let* $\mathscr{C}$ *be a nonempty closed convex subset of a uniformly convex Banach space* $\mathscr{X}$ *and let* J *and* $\{z_n\}$ *be as in Theorem 1 and* $Fix(J) \neq \varnothing$. *If* J *satisfies condition (I), then* $\{z_n\}$ *converges strongly to a fixed point of* J.

Proof. From Theorem 1, it follows that

$$\liminf_{n\longrightarrow\infty}\|Jz_n - z_n\| = 0. \tag{30}$$

Since J satisfies condition (I), we have

$$\|z_n - Jz_n\| \geq \alpha(\text{dist}(z_n, \text{Fix}(J))). \tag{31}$$

From (30), we get

$$\liminf_{n\longrightarrow\infty} \alpha(\text{dist}(z_n, \text{Fix}(J))) = 0. \tag{32}$$

$\alpha: [0,\infty) \longrightarrow [0,\infty)$ is a nondecreasing function with $\alpha(0) = 0$ and $\alpha(t) > 0$ for each $t \in (0,\infty)$. Hence,

$$\liminf_{n\longrightarrow\infty} \text{dist}(z_n, \text{Fix}(J)) = 0. \tag{33}$$

The conclusion follows from Theorem 3.

Finally, we establish a weak convergence of $\{z_n\}$ for mappings with condition (E). □

Theorem 5. *Let* $\mathscr{X}$ *be a uniformly Banach space with the Opial property,* $\mathscr{C}$ *a nonempty closed convex subset of* $\mathscr{X}$, *and let* J *and* $\{z_n\}$ *be as in Theorem 1 and* $Fix(J) \neq \varnothing$. *Then,* $\{z_n\}$ *converges weakly to a fixed point of* J.

Proof. By Theorem 1, $\{z_n\}$ is bounded and $\lim_{n\longrightarrow\infty}\|Jz_n - z_n\| = 0$. By the Milman–Pettis theorem, the space $\mathscr{X}$ is reflexive. Thus, by Eberlin's theorem, there exists a subsequence $\{z_{nk}\}$ of $\{z_n\}$ such that $\{z_{nk}\}$ converges weakly to some $q_1 \in \mathscr{C}$. By Lemma 3, we have $q_1 \in \text{Fix}(J)$. It is sufficient to show that $\{z_n\}$ converges weakly to q_1. In fact, if $\{z_n\}$ does not converge weakly to q_1. Then, there exists a subsequence $\{z_{nl}\}$ of $\{z_n\}$ and $q_2 \in \mathscr{C}$ such that $\{z_{nl}\}$ converges weakly to q_2 and $q_2 \neq q_1$. Again by Lemma 3, $q_2 \in \text{Fix}(J)$. By Lemma 5 together with the Opial property, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\|z_n - q_1\| &= \lim_{k\longrightarrow\infty}\|z_{n_k} - q_1\| < \lim_{k\longrightarrow\infty}\|z_{n_k} - q_2\| \\ &= \lim_{n\longrightarrow\infty}\|z_n - q_2\| = \lim_{l\longrightarrow\infty}\|z_{n_l} - q_2\| \\ &< \lim_{l\longrightarrow\infty}\|z_{n_l} - q_1\| = \lim_{n\longrightarrow\infty}\|z_n - q_1\|.\end{aligned} \tag{34}$$

This is a contradiction. Hence, the conclusions are reached. □

4. Numerical Example

In this section, we present a new example of the Garcia-Falset mapping which is not a Suzuki mapping. Using this example, we compare the rate of convergence of the AK iterative scheme with the other iterative schemes. This example also shows that the converse of Lemma 1 may not hold in general.

Example 1. Let $\mathscr{C} = [0,1]$. Set J on $\mathscr{C}$ as follows:

$$Jz = \begin{cases} 0, & \text{if } 0 \leq z < \dfrac{1}{150}, \\ \dfrac{2z}{3}, & \text{if } \dfrac{1}{150} \leq z \leq 1. \end{cases} \tag{35}$$

Let $\mu = 3$. We shall prove that $|z - Jz'| \leq 3|z - Jz| + |z - z'|$ for all $z, z' \in \mathscr{C}$.

Case (i): if $z, z' \in [0, (1/150))$, then $Jz = Jz' = 0$. Now,

$$|z - Jz'| = |z| \leq 3|z| = 3|z - Jz| \leq 3|z - Jz| + |z - z'|. \tag{36}$$

Case (ii): if $z, z' \in [(1/150), 1]$, then $Jz = 2z/3$ and $Jz' = 2z'/3$. Now,

$$\begin{aligned}|z - Jz'| &\leq |z - Jz| + |Jz - Jz'| = |z - Jz| + \left|\frac{2z}{3} - \frac{2z'}{3}\right| \\ &= |z - Jz| + \frac{2}{3}|z - z'| \leq |z - Jz| + |z - z'| \\ &\leq 3|z - Jz| + |z - z'|.\end{aligned} \tag{37}$$

Case (iii): if $z \in [(1/150), 1]$ and $z' \in [0, (1/150))$, then $Jz = 2z/3$ and $Jz' = 0$. Now,

$$|z - Jz'| = |z| = 3\left|\frac{z}{3}\right| = 3|z - Jz| \leq 3|z - Jz| + |z - z'|. \tag{38}$$

From the above cases, we conclude that J satisfies condition (E). Now choose $z = 1/250$ and $z' = 1/150$. Then, $1/2|z - Jz| < |z - z'|$ but $|Jz - Jz'| > |z - z'|$. Hence, J is not a Suzuki mapping. For all $n \in \mathbb{N}$, let $a_n = 0.70$, $b_n = 0.65$, and $c_n = 0.90$. Table 1 and Figure 1 show that the AK iteration

TABLE 1: Computation table obtained from the AK, M*, Thakur, Abbas, Agarwal, Noor, Ishikawa, and Mann iteration process using Example 1.

n	AK	M*	Thakur	Abbas	Agarwal	Noor	Ishikawa	Mann
1	0.9	0.9	0.9	0.9	0.9	0.9	0.9	0.9
2	0.1601	0.2662	0.3393	0.3906	0.5090	0.5444	0.5999	0.6900
3	0.0284	0.0787	0.1279	0.1695	0.2878	0.3293	0.3986	0.5290
4	0.0050	0.0232	0.0482	0.0735	0.1628	0.1991	0.2653	0.4055
5	**0**	0.0068	0.0181	0.0319	0.0920	0.1204	0.1766	0.3109
6	0	**0**	0.0068	0.0138	0.0520	0.0728	0.1175	0.2383
7	0	0	**0**	0.0060	0.0294	0.0440	0.0782	0.1827
8	0	0	0	**0**	0.0166	0.0266	0.0520	0.1401
9	0	0	0	0	0.0094	0.0161	0.0346	0.1074
10	0	0	0	0	0.0053	0.0097	0.0230	0.0823
11	0	0	0	0	**0**	0.0029	0.0153	0.0631
12	0	0	0	0	0	0.0008	0.0102	0.0484
13	0	0	0	0	0	0.0002	0.0067	0.0371
14	0	0	0	0	0	**0**	0.0020	0.0284
15	0	0	0	0	0	0	0.0001	0.0218
16	0	0	0	0	0	0	**0**	0.0167
17	0	0	0	0	0	0	0	0.0128
18	0	0	0	0	0	0	0	0.0098

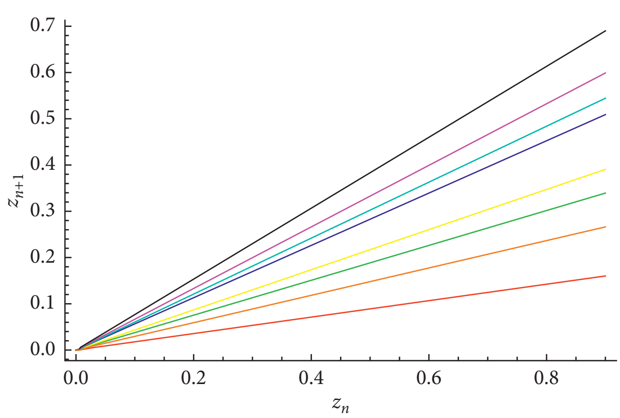

FIGURE 1: Convergence behavior of AK (red line), M* (orange line), Thakur (green line), Abbas (yellow line), Agarwal (blue line), Noor (cyan line), Ishikawa (magenta line), and Mann (black line) iterates for mapping J defined in Example 1 where $z_1 = 0.9$.

scheme converges faster to the fixed point $p = 0$ of the mapping J as compared with the other known iterative schemes.

5. Conclusions

We have proved several strong and weak convergence results for mappings with condition (E) (Garcia-Falset mappings) in the context of Banach spaces. In view of the above discussion, the results for an operator satisfying condition (C) or else nonexpansive are special cases of our new results. Hence, our results are more general than the results of Ullah and Arshad [18], Abbas and Nazir [16], Thakur et al. [17], Phuengrattana [26], and many others. Moreover, our results extend the idea of Ullah and Arshad [19] from the setting of contraction mappings to the general setting of Garcia-Falset mappings.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally and significantly in writing this article. All authors read and approved the final manuscript.

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