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Research Article

Monotonicity Analysis of Fractional Proportional Differences

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In this work, the nabla discrete new Riemann–Liouville and Caputo fractional proportional differences of order $0 < \varepsilon < 1$ on the time scale $\mathbb{Z}$ are formulated. The differences and summations of discrete fractional proportional are detected on $\mathbb{Z}$, and the fractional proportional sums associated to $({}_c^R\nabla^{\varepsilon,\rho}\chi)(z)$ with order $0 < \varepsilon < 1$ are defined. The relation between nabla Riemann–Liouville and Caputo fractional proportional differences is derived. The monotonicity results for the nabla Caputo fractional proportional difference are proved; specifically, if $({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi)(z) > 0$ then $\chi(z)$ is $\varepsilon\rho-$increasing, and if $\chi(z)$ is strictly increasing on $\mathbb{N}_c$ and $\chi(c) > 0$, then $({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi)(z) > 0$. As an application of our findings, a new version of the fractional proportional difference of the mean value theorem (MVT) on $\mathbb{Z}$ is proved.

1. Introduction

Many problems in science, engineering, and media can be formulated using continuous and discrete fractional calculus [1–14]. The fractional sums and differences and their monotonicity properties are deeply studied in [15–25]. In [26], Atangana and Baleanu solved the fractional heat transfer model using new fractional derivatives with exponential kernels, and they presented many applications of the new notations of fractional derivatives. Applications of discrete fractional calculus are successfully discussed by many researchers in the last decade, for example, in [27–29]. Recently, studying the monotonicity for fractional difference operators with nonsingular discrete kernels is under focus [30, 31]. Monotonicity results for fractional difference operators with discrete exponential kernels were studied in [32] when the time step $h = 1$. In [3], deep monotonicity analysis is done for nabla $h-$discrete fractional differences with a discrete Mittag–Leffler kernel in the time scale $h\mathbb{Z}$ with $0 < \varepsilon < 1$ and $0 < h \leq 1$. The results of the research generalized those obtained in [22] where $0 < \varepsilon < 0.5$ and $h = 1$. After that, monotonicity analysis of fractional proportional differences is studied and then the results are prettified by formulating a new version of mean value theorem as an application. In [33], the nabla fractional sums and differences of order $0 < \varepsilon < 1$ on the time scale $h\mathbb{Z}$ where $0 < h \leq 1$ are formulated, and the monotonicity results for the nabla $h-$Caputo fractional difference operator were concluded. In this paper, the authors formulated the nabla discrete new Riemann–Liouville (RL) and Caputo fractional proportional differences of order $0 < \varepsilon < 1$ on the time scale $\mathbb{Z}$. They also proved a new version of the fractional proportional difference of the mean value theorem (MVT) on $\mathbb{Z}$.

The article is organized as follows: Section 2 presents the main definitions and needed preliminaries. In Section 3, the monotonicity results for fractional proportional differences are classified. In Section 4, we formulate a new version of the mean value theorem as an application. Finally, we provide the conclusions in Section 5.

2. Definitions and Preliminary Results

Definition 1. The discrete proportional difference of order $0 < \rho \leq 1$ for the function χ is defined by

$$\nabla^{\rho}\chi(z) = (1-\rho)\chi(z) + \rho\nabla\chi(z),$$
$$z \in \mathbb{N}_{c+1} = \{c+1, c+2, c+3, \ldots\}, \tag{1}$$

and $c \geq 0$ is an integer.

Definition 2. Let $z \in \mathbb{N}_c$, $0 < \rho \leq 1$, and $p = (\rho - 1/\rho)$, then $\widehat{e}_p(z, c) = \rho^{z-c}$.

Definition 3. For any real number α, the α rising function is $z^{\overline{\alpha}} = (\Gamma(z+\alpha)/\Gamma(z))$, such that $z \in \mathbb{C}\backslash\{\ldots, -2, -1, 0\}$, $0^{\overline{\alpha}} = 0$, where $\Gamma(z)$ is the gamma function.

Definition 4 (nabla fractional proportional sums).

For a function $\chi: \mathbb{N}_c \longrightarrow \mathbb{R}$, $\rho > 0$, and $\varepsilon \in \mathbb{C}$, $0 < \mathrm{Re}(\varepsilon) < 1$, the nabla left fractional proportional sum of χ starting at c is defined by

$$\begin{aligned}({}_c\nabla^{-\varepsilon,\rho}\chi)(z) &= \frac{1}{\Gamma(\varepsilon)}\int_c^z \widehat{e}_p(z-s,0)(z-\varsigma(s))^{\overline{\varepsilon-1}}\chi(s)\nabla s \\ &= \frac{1}{\Gamma(\varepsilon)}\sum_{\iota=c+1}^{z}\widehat{e}_p(z-\iota,0)(z-\varsigma(\iota))^{\overline{\varepsilon-1}}\chi(\iota), \quad z \in \mathbb{N}_c.\end{aligned} \tag{2}$$

For the function $\chi: {}_d\mathbb{N} = \{d, d-1, d-2, \ldots\} \longrightarrow \mathbb{R}$, the nabla right fractional proportional sum ending at d is defined by

$$\begin{aligned}\left(\nabla_d^{-\varepsilon,\rho}\chi\right)(z) &= \frac{1}{\Gamma(\varepsilon)}\int_z^d \widehat{e}_p(s-z,0)(s-\varsigma(z))^{\overline{\varepsilon-1}}\chi(s)\Delta s \\ &= \frac{1}{\Gamma(\varepsilon)}\sum_{\iota=z}^{d-1}\widehat{e}_p(\iota-z,0)(\iota-\varsigma(z))^{\overline{\varepsilon-1}}\chi(\iota), \quad z \in {}_d\mathbb{N}.\end{aligned} \tag{3}$$

We notice that by setting $\rho = 1$, the given definitions of the fractional sums are generalizations of the Riemann fractional sums.

Lemma 1. *Let $\chi: \mathbb{N}_c \longrightarrow \mathbb{R}$ be a function, $p = (\rho - 1/\rho)$, $0 < \varepsilon < 1$, and $0 < \rho \leq 1$, then*

$$\begin{aligned}&({}_c\nabla^{-\varepsilon,\rho}\ \nabla\chi)(z) \\ &= (\nabla_c\nabla^{-\varepsilon,\rho}\chi)(z) - \frac{(z-c)^{\overline{\varepsilon-1}}}{\Gamma(\varepsilon)}\widehat{e}_p(z-1,c)\chi(c).\end{aligned} \tag{4}$$

Proof.

$$\begin{aligned}({}_c\nabla^{-\varepsilon,\rho}\nabla\chi)(z) &= \frac{1}{\Gamma(\varepsilon)}\sum_{\iota=c+1}^{z}\widehat{e}_p(z-\iota,0)(z-\varsigma(\iota))^{\overline{\varepsilon-1}}\nabla\chi(\iota) \\ &= \frac{1}{\Gamma(\varepsilon)}\sum_{\iota=c+1}^{z}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{\varepsilon-1}}(\chi(\iota)-\chi(\iota-1)) \\ &= \frac{1}{\Gamma(\varepsilon)}\sum_{\iota=c+1}^{z}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{\varepsilon-1}}\chi(\iota) \\ &\quad - \frac{1}{\Gamma(\varepsilon)}\sum_{\iota=c+1}^{z}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{\varepsilon-1}}\chi(\iota-1) \\ &= \frac{1}{\Gamma(\varepsilon)}\sum_{\iota=c+1}^{z}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{\varepsilon-1}}\chi(\iota) \\ &\quad - \frac{1}{\Gamma(\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota-1}(z-\varsigma(\iota+1))^{\overline{\varepsilon-1}}\chi(\iota) \\ &= \frac{1}{\Gamma(\varepsilon)}\left(\sum_{\iota=c+1}^{z}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{\varepsilon-1}}\chi(\iota) - \sum_{\iota=c}^{z-1}\rho^{(z-1)-\iota}((z-1)-\varsigma(\iota))^{\overline{\varepsilon-1}}\chi(\iota)\right) \\ &= \frac{1}{\Gamma(\varepsilon)}\left(\sum_{\iota=c+1}^{z}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{\varepsilon-1}}\chi(\iota) - \sum_{\iota=c+1}^{z-1}\rho^{(z-1)-\iota}((z-1)-\varsigma(\iota))^{\overline{\varepsilon-1}}\chi(\iota)\right) \\ &= \frac{1}{\Gamma(\varepsilon)}\rho^{(z-1)-c}((z-1)-\varsigma(c))^{\overline{\varepsilon-1}}\chi(c) \\ &= \frac{1}{\Gamma(\varepsilon)}\nabla\sum_{\iota=c+1}^{z}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{\varepsilon-1}}\chi(\iota) - \frac{1}{\Gamma(\varepsilon)}\rho^{(z-1)-c}(z-1-c+1)^{\overline{\varepsilon-1}}\chi(c) \\ &= (\nabla_c\nabla^{-\varepsilon,\rho}\chi)(z) - \frac{(z-c)^{\overline{\varepsilon-1}}}{\Gamma(\varepsilon)}\widehat{e}_p(z-1,c)\chi(c).\end{aligned} \tag{5}$$

□

Lemma 2. *Let* $\chi: \mathbb{N}_c \longrightarrow \mathbb{R}$, $p = (\rho - 1/\rho)$, $0 < \varepsilon < 1$, *and* $0 < \rho \leq 1$, *then*

$$({}_c\nabla^{-\varepsilon,\rho}\nabla^{\rho}\chi)(z) = (\nabla^{\rho}{}_c\nabla^{-\varepsilon,\rho}\chi)(z) - \frac{\rho}{\Gamma(\varepsilon)}(z-c)^{\overline{\varepsilon-1}}\widehat{e}_p(z-1,c)\chi(c). \tag{6}$$

Proof.

$$\nabla^{\rho}\chi(z) = (1-\rho)\chi(z) + \rho\nabla\chi(z), \tag{7}$$

hence,

$$\begin{aligned}({}_c\nabla^{-\varepsilon,\rho}\nabla^{\rho}\chi)(z) &= {}_c\nabla^{-\varepsilon,\rho}((1-\rho)\chi(z) + \rho\nabla\chi(z))\\ &= (1-\rho)({}_c\nabla^{-\varepsilon,\rho}\chi)(z) + \rho({}_c\nabla^{-\varepsilon,\rho}\nabla\chi(z)) \text{ using Lemma 1}\\ &= (1-\rho)({}_c\nabla^{-\varepsilon,\rho}\chi)(z) + \rho\left((\nabla^{\rho}{}_c\nabla^{-\varepsilon,\rho}\chi)(z) - \frac{(z-c)^{\overline{\varepsilon-1}}}{\Gamma(\varepsilon)}\widehat{e}_p(z-1,c)\chi(c)\right)\\ &= \left((1-\rho)({}_c\nabla^{-\varepsilon,\rho}\chi)(z) + \rho\nabla^{\rho}({}_c\nabla^{-\varepsilon,\rho}\chi)(z)\right) - \frac{\rho}{\Gamma(\varepsilon)}(z-c)^{\overline{\varepsilon-1}}\widehat{e}_p(z-1,c)\chi(c)\\ &= (\nabla^{\rho}{}_c\nabla^{-\varepsilon,\rho}\chi)(z) - \frac{\rho}{\Gamma(\varepsilon)}(z-c)^{\overline{\varepsilon-1}}\widehat{e}_p(z-1,c)\chi(c).\end{aligned} \tag{8}$$

Note that if $\rho = 1$, we get

$$({}_c\nabla^{-\varepsilon}\nabla\chi)(z) = (\nabla_c\nabla^{-\varepsilon}\chi)(z) - \frac{1}{\Gamma(\varepsilon)}(z-c)^{\overline{\varepsilon-1}}\chi(c). \tag{9}$$

□

Definition 5 (Riemann–Liouville (RL) fractional proportional differences)

For $0 < \rho \leq 1$, $\varepsilon \in \mathbb{C}$, $0 < \mathrm{Re}(\varepsilon) < 1$, and χ be a function defined on $\mathbb{N}_c$ or on ${}_d\mathbb{N}$, then the left Riemann–Liouville fractional proportional difference starting at c is defined by

$$\begin{aligned}({}_c^R\nabla^{\varepsilon,\rho}\chi)(z) &= \nabla^{\rho}{}_c\nabla^{-(1-\varepsilon),\rho}\chi(z)\\ &= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)}\int_c^z \widehat{e}_p(z-s,0)(z-\varsigma(s))^{\overline{-\varepsilon}}\chi(s)\nabla s\\ &= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)}\sum_{\iota=c+1}^{z}\widehat{e}_p(z-\iota,0)(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota)\\ &= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)}\sum_{\iota=c+1}^{z}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota),\end{aligned} \tag{10}$$

and the right Riemann–Liouville fractional proportional difference ending at d is defined by

$$\begin{aligned}({}^R\nabla_d^{\varepsilon,\rho}\chi)(z) &= -\Delta^{\rho}\nabla_d^{-(1-\varepsilon),\rho}\chi(z)\\ &= \frac{-\Delta^{\rho}}{\Gamma(1-\varepsilon)}\sum_{\iota=z}^{d-1}\rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}\chi(\iota).\end{aligned} \tag{11}$$

We notice that by setting $\rho = 1$, the given definitions of the fractional differences are generalizations of the Riemann fractional differences.

Definition 6 (Caputo fractional proportional differences)

For $0 < \rho \leq 1$, $\varepsilon \in \mathbb{C}$, $0 < \mathrm{Re}(\varepsilon) < 1$, and χ be a function defined on $\mathbb{N}_c$ or on ${}_d\mathbb{N}$, then the left Caputo fractional proportional difference starting at c is defined by

$$\begin{aligned}({}_c^C\nabla^{\varepsilon,\rho}\chi)(z) &= {}_c\nabla^{-(1-\varepsilon),\rho}\nabla^{\rho}\chi(z)\\ &= \frac{1}{\Gamma(1-\varepsilon)}\int_c^z \widehat{e}_p(z-s,0)(z-\varsigma(s))^{\overline{-\varepsilon}}(\nabla^{\rho}\chi(s))\nabla s\\ &= \frac{1}{\Gamma(1-\varepsilon)}\sum_{\iota=c+1}^{z}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\nabla^{\rho}\chi(\iota),\end{aligned} \tag{12}$$

and the right Caputo fractional proportional difference ending at d is defined by

$$\begin{aligned}({}^C\nabla_d^{\varepsilon,\rho}\chi)(z) &= \nabla_d^{-(1-\varepsilon),\rho}(-\Delta^{\rho}\chi(z))\\ &= \frac{1}{\Gamma(1-\varepsilon)}\sum_{\iota=z}^{d-1}\rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}(-\Delta^{\rho}\chi(\iota)).\end{aligned} \tag{13}$$

We notice that by setting $\rho = 1$, the given definitions of the fractional differences are generalizations of the Caputo fractional differences.

Proposition 1 (the relation between nabla RL and Caputo fractional proportional differences)

For any $\varepsilon \in \mathbb{C}$, $0 < Re(\varepsilon) < 1$, *and* $0 < \rho \leq 1$, *the relation between nabla RL and Caputo fractional proportional differences is given as follows:*

(i) $\left({}_{c}^{C}\nabla^{\varepsilon,\rho}\chi\right)(z) = \left({}_{c}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z) - (z-c)^{\overline{-\varepsilon}}/\Gamma(1-\varepsilon)\widehat{e}_{p}(z, c)\chi(c)$.

(ii) $\left({}_{c}^{C}\nabla^{\varepsilon,\rho}\chi\right)(z) = \left({}_{c}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z) - (d-z)^{\overline{-\varepsilon}}/\Gamma(1-\varepsilon)\widehat{e}_{p}(d, z)\chi(d)$.

Proof.

$$
\begin{aligned}
\left({}_{c}^{C}\nabla^{\varepsilon,\rho}\chi\right)(z) &= \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\nabla^{\rho}\chi(\iota) \\
&= \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}((1-\rho)\chi(\iota)+\rho\nabla\chi(\iota)) \\
&= \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}(1-\rho)\chi(\iota) \\
&\quad + \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}(\rho\nabla\chi(\iota)) \\
&= \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}(1-\rho)\chi(\iota) \\
&\quad + \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\rho(\chi(\iota)-\chi(\iota-1)) \\
&= \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}(1-\rho)\chi(\iota) \\
&\quad + \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\rho\chi(\iota) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z-1} \rho^{z-\iota-1}(z-\varsigma(\iota+1))^{\overline{-\varepsilon}}\rho\chi(k) \\
&= \frac{1-\rho}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota) \\
&\quad + \frac{\rho}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota) \\
&\quad - \frac{\rho}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z-1} \rho^{z-1-\iota}(z-1-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota) \\
&\quad - \frac{\rho}{\Gamma(1-\varepsilon)}\rho^{z-1-c}(z-1-\varsigma(c))^{\overline{-\varepsilon}}\chi(c) \\
&= \frac{1-\rho}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota) + \frac{\rho\nabla}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota) \\
&\quad - \frac{\rho}{\Gamma(1-\varepsilon)}\rho^{z-1-c}(z-1-c+1)^{\overline{-\varepsilon}}\chi(c)
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)} \sum_{\iota=c+1}^{z} \widehat{e}_{p}(z-\iota,0)(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota) \\
&\quad - \frac{(z-c)^{\overline{-\varepsilon}}}{\Gamma(1-\varepsilon)}\rho^{z-c}\chi(c) = ({}_{c}^{C}\nabla^{\varepsilon,\rho}\chi)(z) - \frac{(z-c)^{\overline{-\varepsilon}}}{\Gamma(1-\varepsilon)}\widehat{e}_{p}(z,c)\chi(c).
\end{aligned} \tag{14}
$$

$$
\begin{aligned}
({}^{C}\nabla_{d}^{\varepsilon,\rho}\chi)(z) &= \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}(-\Delta^{\rho})\chi(\iota) \\
&= \frac{-1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}((1-\rho)\chi(\iota)+\rho\Delta\chi(\iota)) \\
&= \frac{-1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}(1-\rho)\chi(\iota) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}\rho(\chi(\iota+1)-\chi(\iota)) \\
&= \frac{-1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}(1-\rho)\chi(\iota) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}\rho\chi(\iota+1) + \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}\rho\chi(\iota) \\
&= \frac{-1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}(1-\rho)\chi(\iota) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z+1}^{d} \rho^{\iota-1-z}(\iota-1-\varsigma(z))^{\overline{-\varepsilon}}\rho\chi(\iota) \\
&\quad + \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}\rho\chi(\iota) \\
&= \frac{-1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}(1-\rho)\chi(\iota) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)}\rho^{d-1-z}(d-1-\varsigma(z))^{\overline{-\varepsilon}}\rho\chi(d) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z+1}^{d-1} \rho^{\iota-(z+1)}(\iota-\varsigma(z+1))^{\overline{-\varepsilon}}\rho\chi(\iota) \\
&\quad + \frac{1}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}\rho\chi(\iota) \\
&= \frac{-(1-\rho)}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}\chi(\iota) - \frac{\rho\Delta}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}\rho\chi(\iota) \\
&\quad - \frac{1}{\Gamma(1-\varepsilon)}\rho^{d-z}(d-1-z+1)^{\overline{-\varepsilon}}\chi(d) \\
&= \frac{-\Delta^{\rho}}{\Gamma(1-\varepsilon)} \sum_{\iota=z}^{d-1} \rho^{\iota-z}(\iota-\varsigma(z))^{\overline{-\varepsilon}}\chi(\iota) - \frac{1}{\Gamma(1-\varepsilon)}\rho^{d-z}(d-z)^{\overline{-\varepsilon}}\chi(d) \\
&= ({}^{R}\nabla_{d}^{\varepsilon,\rho}\chi)(z) - \frac{(d-z)^{\overline{-\varepsilon}}}{\Gamma(1-\varepsilon)}\widehat{e}_{p}(d,z)\chi(d).
\end{aligned} \tag{15}
$$

Numerical calculations have been done in order to verify the first equation in Proposition 1. The values used are $c = 2.5$, $\rho = 0.7$, and $\varepsilon = 0.3$. The results are illustrated in Figure 1.

In addition to that, the data are presented in Table 1. □

Lemma 3. *Let* $0 < \varepsilon < 1$, $z, \iota \in \mathbb{N}_c$, *and* $\iota < z$, *then* $\nabla (z - \varsigma(\iota))^{\overline{-\varepsilon}} = -\varepsilon (z - \varsigma(\iota))^{\overline{-\varepsilon-1}}$.

Proof.

$$\begin{aligned}
\nabla (z - \varsigma(\iota))^{\overline{-\varepsilon}} &= (z - \varsigma(\iota))^{\overline{-\varepsilon}} - (z - 1 - \varsigma(\iota))^{\overline{-\varepsilon}} \\
&= (z - \iota + 1)^{\overline{-\varepsilon}} - (z - \iota)^{\overline{-\varepsilon}} \\
&= \frac{\Gamma(z - \iota - \varepsilon + 1)}{\Gamma(z - \iota + 1)} - \frac{\Gamma(z - \iota - \varepsilon)}{\Gamma(z - \iota)} \\
&= \frac{\Gamma(z - \iota - \varepsilon)}{\Gamma(z - \iota)} \left(\frac{z - \iota - \varepsilon}{z - \iota} - 1 \right) \\
&= \frac{\Gamma(z - \iota + 1 - \varepsilon - 1)}{\Gamma(z - \iota)} \left(\frac{-\varepsilon}{z - \iota} \right) \\
&= -\varepsilon \frac{\Gamma(z - \iota + 1 - \varepsilon - 1)}{\Gamma(z - \iota + 1)} \\
&= -\varepsilon (z - \iota + 1)^{\overline{-\varepsilon-1}} = -\varepsilon (z - \varsigma(\iota))^{\overline{-\varepsilon-1}}.
\end{aligned} \tag{16}$$

□

3. Monotonicity Results

The following two monotonicity definitions are given in [18].

Definition 7. Let y: $\mathbb{N}_a \longrightarrow \mathbb{R}$ be a function satisfying $y(a) \geq 0$, $0 < \alpha < 1$. Then, $y(t)$ is called an α–increasing function on $\mathbb{N}_a$ if $y(t + 1) \geq \alpha y(t) \quad \forall t \in \mathbb{N}_a$.

Definition 8. Let y: $\mathbb{N}_a \longrightarrow \mathbb{R}$ be a function satisfying $y(a) \leq 0$, $0 < \alpha < 1$. Then, $y(t)$ is called an α–decreasing function on $\mathbb{N}_a$ if $y(t + 1) \leq \alpha y(t) \, \forall t \in \mathbb{N}_a$.

In the following, we report the new proportional monotonicity main results.

Theorem 1. *Let* χ: $\mathbb{N}_c \longrightarrow \mathbb{R}$ *be a function, and suppose that* $({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi)(z) \geq 0$ *for* $0 < \varepsilon < 1$ *and* $0 < \rho \leq 1$, $z \in \mathbb{N}_{c-1}$. *Then,* $\chi(z)$ *is* $\varepsilon\rho$ – *increasing.*

Proof.

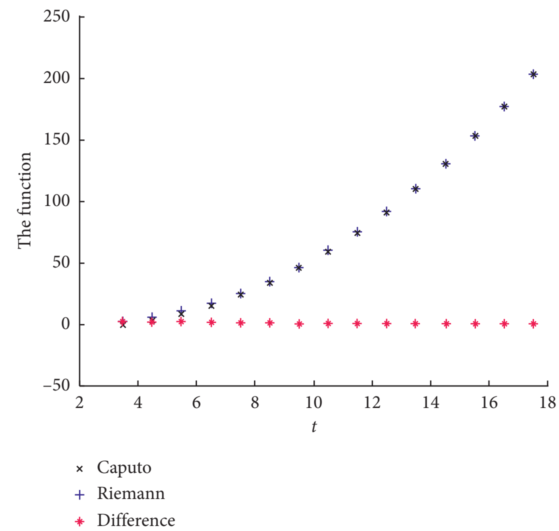

FIGURE 1: The relation between nabla Riemann and Caputo fractional proportional differences.

TABLE 1: The relation between nabla Riemann and Caputo fractional proportional differences.

z	Caputo	Riemann	Difference
3.5	2.181077	−0.415443	2.596520
4.5	5.786530	3.190010	2.596520
5.5	10.787558	8.649248	2.138310
6.5	17.274639	15.611509	1.663130
7.5	25.339691	24.081105	1.258585
8.5	35.058768	34.121524	0.937244
9.5	46.490945	45.800344	0.690601
10.5	59.680789	59.175722	0.505066
11.5	74.661373	74.294052	0.367321
12.5	91.456990	91.190999	0.265991
13.5	110.085354	109.893402	0.191952
14.5	130.559312	130.421178	0.138134
15.5	152.888147	152.788974	0.099173
16.5	177.078543	177.007482	0.071061
17.5	203.135301	203.084469	0.050832

$$\begin{aligned}
({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi)(z) &= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z} \widehat{e}_{p}(z-\iota,0)(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota) \\
&= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)} \sum_{\iota=c}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota).
\end{aligned} \tag{17}$$

Let

$$S(z) = \sum_{\iota=c}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota). \tag{18}$$

Then,

$$({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi)(z) = \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)} S(z). \tag{19}$$

Hence, from the assumption, we have $\nabla^{\rho} S(z) \geq 0$. That is,

$$\begin{aligned}\nabla^{\rho} S(z) &= (1-\rho)S(z) + \rho\nabla S(z)\\
&= (1-\rho)S(z) + \rho(S(z) - S(z-1))\\
&= S(z) - \rho S(z) + \rho S(z) - \rho S(z-1)\\
&= S(z) - \rho S(z-1)\\
&= \sum_{\iota=c}^{z} \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota)\\
&\quad - \rho\sum_{\iota=c}^{z-1}\rho^{z-1-\iota}(z-1-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota)\\
&= (z-\varsigma(z))^{\overline{-\varepsilon}}\chi(z) + \sum_{\iota=c}^{z-1}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota)\\
&\quad - \sum_{\iota=c}^{z-1}\rho^{z-1-\iota}(z-1-\varsigma(\iota))^{\overline{-\varepsilon}}\chi(\iota)\\
&= (z-z+1)^{\overline{-\varepsilon}}\chi(z) + \sum_{\iota=c}^{z-1}\rho^{z-\iota}\chi(\iota)\\
&\quad \cdot\left((z-\varsigma(\iota))^{\overline{-\varepsilon}} - (z-1-\varsigma(\iota))^{\overline{-\varepsilon}}\right)\\
&= (1)^{\overline{-\varepsilon}}\chi(z) + \sum_{\iota=c}^{z-1}\rho^{z-\iota}\chi(\iota)\nabla(z-\varsigma(\iota))^{\overline{-\varepsilon}}\\
&= \frac{\Gamma(1-\varepsilon)}{\Gamma(1)}\chi(z) + \sum_{\iota=c}^{z-1}\rho^{z-\iota}\chi(\iota)\left(-\varepsilon(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\right)\\
&= \Gamma(1-\varepsilon)\chi(z) - \varepsilon\sum_{\iota=c}^{z-1}\rho^{z-\iota}\chi(\iota)(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\\
&\geq 0.
\end{aligned} \tag{20}$$

Therefore,

$$\begin{aligned}\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z) &= \frac{\nabla^{\rho}}{\Gamma(1-\varepsilon)}S(z)\\
&= \frac{1}{\Gamma(1-\varepsilon)}\left(\Gamma(1-\varepsilon)\chi(z) - \varepsilon\sum_{\iota=c}^{z-1}\rho^{z-\iota}\chi(\iota)(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\right)\\
&= \chi(z) - \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\chi(\iota)(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\\
&= 0.
\end{aligned} \tag{21}$$

Hence,

$$\chi(z) \geq \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\chi(\iota)(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}. \tag{22}$$

Clearly, $\chi(c-1) = 0$. So, we can start the induction from the next step. When $z = c$, we get $\chi(c) \geq 0$; also, when $z = c+1$, we have

$$\begin{aligned}\chi(c+1) &\geq \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{c}\rho^{c+1-\iota}\chi(\iota)(c+1-\varsigma(\iota))^{\overline{-\varepsilon-1}}\\
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)}\rho^{c+1-c}\chi(c)(c+1-\varsigma(c))^{\overline{-\varepsilon-1}}\\
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)}\rho\chi(c)(c+1-c+1)^{\overline{-\varepsilon-1}}\\
&= \frac{\varepsilon\rho}{\Gamma(1-\varepsilon)}\chi(c)\frac{\Gamma(1-\varepsilon)}{\Gamma(2)}\\
&= \varepsilon\rho\chi(c).
\end{aligned} \tag{23}$$

Now for $z+1$, replace z by $z+1$, then we get

$$\begin{aligned}\chi(z+1) &\geq \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z}\rho^{z+1-\iota}\chi(\iota)(z+1-\varsigma(\iota))^{\overline{-\varepsilon-1}}\\
&\geq \frac{\varepsilon}{\Gamma(1-\varepsilon)}\rho^{z+1-z}\chi(z)(z+1-\varsigma(z))^{\overline{-\varepsilon-1}}\\
&= \frac{\varepsilon}{\Gamma(1-\varepsilon)}\rho\chi(z)(z+1-z+1)^{\overline{-\varepsilon-1}}\\
&= \frac{\varepsilon\rho}{\Gamma(1-\varepsilon)}\chi(z)2^{\overline{-\varepsilon-1}}\\
&= \varepsilon\rho\chi(z).
\end{aligned} \tag{24}$$

Hence, $\chi(z)$ is $\varepsilon\rho$–increasing which completes the proof.

Using Theorem 1 and Proposition 1 we can state the following Caputo fractional proportional difference monotonicity result.

□

Corollary 1. *Let $\chi\colon \mathbb{N}_{c-1} \longrightarrow \mathbb{R}$ be a function, and suppose that for $0 < \varepsilon < 1$ and $0 < \rho \leq 1$. Suppose that*

$$\left({}_{c-1}^{C}\nabla^{\varepsilon,\rho}\chi\right)(z) \geq \frac{-\widehat{e}_p(z,c-1)}{\Gamma(1-\varepsilon)}(z-c+1)^{\overline{-\varepsilon}}\chi(c-1), \quad z \in \mathbb{N}_{c-1}, \tag{25}$$

then $\chi(z)$ is $\varepsilon\rho$ – increasing.

Proof.

$$\begin{aligned}\left({}_{c-1}^{C}\nabla^{\varepsilon,\rho}\chi\right)(z) &= \left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z) - \frac{\widehat{e}_p(z,c-1)}{\Gamma(1-\varepsilon)}\\
&\quad\cdot(z-c+1)^{\overline{-\varepsilon}}\chi(c-1), \quad \forall z \in \mathbb{N}_{c-1},
\end{aligned} \tag{26}$$

now, from the assumption we have

$$\begin{aligned}\left({}_{c-1}^{C}\nabla^{\varepsilon,\rho}\chi\right)(z) &\geq \frac{-\widehat{e}_p(z,c-1)}{\Gamma(1-\varepsilon)}(z-c+1)^{\overline{-\varepsilon}}\chi(c-1),\\
&\qquad z \in \mathbb{N}_{c-1}, z \in \mathbb{N}_{c-1},
\end{aligned} \tag{27}$$

hence,

$$\left({}_{c-1}^{C}\nabla^{\varepsilon,\rho}\chi\right)(z)+\frac{\widehat{e}_{p}(z,c-1)}{\Gamma(1-\varepsilon)}(z-c+1)^{\overline{-\varepsilon}}\chi(c-1)\geq 0,$$
$$z\in\mathbb{N}_{c-1},z\in\mathbb{N}_{c-1}, \tag{28}$$

which means that $\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z)\geq 0$.

Now, from Theorem 1, we conclude that $\chi(z)$ is $\varepsilon\rho$ – increasing.

□

Theorem 2. *Let χ: $\mathbb{N}_{c-1}\longrightarrow\mathbb{R}$ be a function which satisfies $\chi(c)\geq 0$, and suppose that for $0<\varepsilon<1$ and $0<\rho\leq 1$. If $\chi(z)$ is increasing on $\mathbb{N}_c$, then we have*

$$\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z)\geq 0,\quad\forall z\in\mathbb{N}_{c-1}. \tag{29}$$

Proof. Since

$$\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z)=\chi(z)-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(\iota),$$
$$z\in\mathbb{N}_{c-1}, \tag{30}$$

when $z=c$, we have from the assumption $\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(c)=\chi(c)\geq 0$.

Clearly, $\chi(c-1)=0$. So, we can start the induction from the next step.

Assume that $\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(i)\geq 0$, $\forall i<z$. We shall show that $\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z)\geq 0$.

Since, from assumption, $\chi(z)$ is increasing, it follows that $\chi(z)\geq\chi(z-1)\geq\chi(c)\geq 0$, $\forall z\in\mathbb{N}_c$:

$$\begin{aligned}
\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z)&=\chi(z)-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(\iota)\\
&=\chi(z)-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\rho^{z-z+1}(z-\varsigma(z-1))^{\overline{-\varepsilon-1}}\chi(z-1)\\
&\quad-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-2}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(\iota)\\
&=\chi(z)-\varepsilon\rho\chi(z-1)-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-2}\rho^{z-\iota}\\
&\quad\cdot(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(\iota)\\
&\quad+\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-2}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(z-1)\\
&\quad-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-2}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(z-1)\\
&=\chi(z)-\varepsilon\rho\chi(z-1)\\
&\quad+\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-2}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}(\chi(z-1)-\chi(\iota))\\
&\quad-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-2}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(z-1)\\
&\geq\chi(z)-\varepsilon\rho\chi(z-1)-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-2}\rho^{z-\iota}\\
&\quad\cdot(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(z-1)\\
&=\chi(z)-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\\
&\quad\cdot(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(z-1)\\
&=\chi(z)-\chi(z-1)+\chi(z-1)\\
&\quad-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-2}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(z-1)\\
&\geq\chi(z-1)-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\\
&\quad\cdot(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(z-1)\\
&=\chi(z-1)\left(1-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\right)\\
&\geq 0.
\end{aligned} \tag{31}$$

□

Theorem 3. *Let χ: $\mathbb{N}_{c-1}\longrightarrow\mathbb{R}$ be a function which satisfies $\chi(c)>0$ and be strictly increasing on $\mathbb{N}_c$, where $0<\varepsilon<1$ and $0<\rho\leq 1$. Then,*

$$\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)>0,\quad z\in\mathbb{N}_{c-1}. \tag{32}$$

Proof. Since

$$\begin{aligned}
\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z)&=\chi(z)-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}\\
&\quad\cdot(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(\iota),\quad z\in\mathbb{N}_{c-1},
\end{aligned} \tag{33}$$

when $z=c$, we have $\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(c)=\chi(c)>0$. Clearly, $\chi(c-1)=0$, and so we can start the induction from the next step.

Assume that $\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(i)>0$, $\forall i<z$. We shall show that $\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z)>0$.

Since, from assumption, $\chi(z)$ is increasing it follows that $\chi(z)>\chi(z-1)>\chi(c)>0,\forall z\in\mathbb{N}_c$:

$$\begin{aligned}({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi)(z) &= \chi(z) - \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(\iota)\\ &> \chi(z)-\chi(z-1)+\chi(z-1)\\ &\quad - \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(z-1)\\ &> \chi(z-1) - \frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\\ &\quad\cdot \rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\chi(z-1)\\ &= \chi(z-1)\left(1-\frac{\varepsilon}{\Gamma(1-\varepsilon)}\sum_{\iota=c}^{z-1}\rho^{z-\iota}(z-\varsigma(\iota))^{\overline{-\varepsilon-1}}\right)>0.\end{aligned} \tag{34}$$

□

Theorem 4. *Let* χ: $\mathbb{N}_{c-1}\longrightarrow\mathbb{R}$ *be a function, and suppose that* $({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi)$ $(z)\leq 0$ *for* $0<\varepsilon<1$ *and* $0<\rho\leq 1, z\in\mathbb{N}_{c-1}$. *Then,* $\chi(z)$ *is* $\varepsilon\rho$ − *decreasing.*

Proof. Let θ: $\mathbb{N}_{c-1}\longrightarrow\mathbb{R}$ be a function such that $\theta(z)=-\chi(z)$; hence,

$$({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\theta)(z) = ({}_{c-1}^{R}\nabla^{\varepsilon,\rho}(-\chi))(z) = -({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi)(z)\geq 0. \tag{35}$$

Now by Theorem 1, we conclude that $\theta(z)$ is $\varepsilon\rho$ − increasing.

Hence,

$$\theta(z+1)\geq\varepsilon\rho\theta(z), \tag{36}$$

which is

$$\begin{aligned}-\chi(z+1)&\geq\varepsilon\rho(-\chi(z)),\\ \chi(z+1)&\leq\varepsilon\rho\chi(z),\end{aligned} \tag{37}$$

that is to say, $\chi(z)$ is $\varepsilon\rho$ − decreasing.

□

Theorem 5. *Let a function* χ: $\mathbb{N}_{c-1}\longrightarrow\mathbb{R}$ *be decreasing on* $\mathbb{N}_c$ *such that* $\chi(c)\leq 0$. *Then, for* $0<\varepsilon<1$ *and* $0<\rho\leq 1$, *we have*

$$({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi)(z)\leq 0,\quad \forall z\in\mathbb{N}_{c-1}. \tag{38}$$

Proof. The proof follows by applying Theorem 2 to $\theta(z)=-\chi(z)$.

Using Theorem 4.3 in [4] we can state the following. □

Theorem 6 (see [4]). *For any* $0<\varepsilon<1$, $0<\rho\leq 1$, $p=(\rho-1/\rho)$, *and* χ: $\mathbb{N}_{c+1}\longrightarrow\mathbb{R}$, *the following equality holds:*

$$\left({}_{c}^{R}\nabla^{-\varepsilon,\rho}\ {}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z) = \chi(z) - \frac{\widehat{e}_p(z,c)}{\Gamma(\varepsilon)}(z-c+1)^{\overline{\varepsilon-1}}\chi(c). \tag{39}$$

4. Application: Mean Value Theorem (MVT)

First, for the sake of simplification, depending on Theorem 6, we shall write

$$\left({}_{c}^{R}\nabla^{-\varepsilon,\rho}\ {}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z) = \chi(z) - S(z,c)\chi(c), \tag{40}$$

where

$$S(z,c) = \frac{\widehat{e}_p(z,c)}{\Gamma(\varepsilon)}(z-c+1)^{\overline{\varepsilon-1}}. \tag{41}$$

Theorem 7 (the fractional proportional difference MVT)

Let Θ *and* θ *be functions defined on* $\mathbb{N}_c\cap{}_d\mathbb{N}=\{c,c+1,c+2,\ldots,d-2,d-1,d\}$, *where* $c\equiv d\,mod 1$. *Assume that* θ *is strictly increasing,* $\theta(c)>0$, *and* $0<\varepsilon<1$ *and* $0<\rho\leq 1$. *Then, there exist* $s_1,s_2\in\mathbb{N}_c\cap{}_d\mathbb{N}$ *such that*

$$\frac{({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\Theta)(s_1)}{({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\theta)(s_1)} \leq \frac{\Theta(d)-S(d,c)\Theta(c)}{\theta(d)-S(d,c)\theta(c)} \leq \frac{({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\Theta)(s_2)}{({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\theta)(s_2)}. \tag{42}$$

Proof. First we need to show that $\theta(d)-S(d,c)\theta(c)>0$. Since θ is strictly increasing, then by Theorem 3 we have

$$({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\theta)(z)>0,\quad \forall z,\in\mathbb{N}_c\cap{}_d\mathbb{N}. \tag{43}$$

Applying the fractional sum operator associated to $({}_{c}^{R}\nabla^{\varepsilon,\rho}\theta)(z)$ on both sides of the inequality, by means of (40), we get

$${}_{c}^{R}\nabla^{-\varepsilon,\rho}({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\theta)(z) > ({}_{c-1}^{R}\nabla^{-\varepsilon,\rho}(0))\quad \forall z,\in\mathbb{N}_c\cap{}_d\mathbb{N}, \tag{44}$$

or we have

$$\theta(z)-S(z,c)\theta(c)>0.\quad \forall z\in\mathbb{N}_c\cap{}_d\mathbb{N}. \tag{45}$$

For $z=d$, we get

$$\theta(d)-S(d,c)\theta(c)>0. \tag{46}$$

To prove the theorem, we use the proof by contradiction. Assume (42) is not true, then either

$$\frac{\Theta(d)-S(d,c)\Theta(c)}{\theta(d)-S(d,c)\theta(c)} < \frac{({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\Theta)(z)}{({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\theta)(z)},\quad \forall z\in\mathbb{N}_c\cap{}_d\mathbb{N}, \tag{47}$$

or

$$\frac{\Theta(d)-S(d,c)\Theta(c)}{\theta(d)-S(d,c)\theta(c)} > \frac{({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\Theta)(z)}{({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\theta)(z)},\quad \forall z\in\mathbb{N}_c\cap{}_d\mathbb{N}. \tag{48}$$

Again, since θ is strictly increasing, then by Theorem 3 we conclude that

$$({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\Theta)(z),\quad \forall z\in\mathbb{N}_c\cap{}_d\mathbb{N}. \tag{49}$$

Hence, (47) becomes

$$\frac{\Theta(d) - S(d,c)\Theta(c)}{\theta(d) - S(d,c)\theta(c)} \left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\theta\right)(z) < \left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\Theta\right)(z), \quad \forall z \in \mathbb{N}_c \cap {}_d\mathbb{N}. \tag{50}$$

Applying the fractional sum operator on both sides of the inequality at $z = d$ and by making use of (43), we see that

$$\frac{\Theta(d) - S(d,c)\Theta(c)}{\theta(d) - S(d,c)\theta(c)} \left(\theta(d) - S(d,c)\theta(c)\right) < \left(\Theta(d) - S(d,c)\Theta(c)\right), \tag{51}$$

and hence, $\Theta(d) < \Theta(d)$, which is a contradiction. In a similar way, (48) leads to contradiction.

□

5. Conclusions

The conclusions of this article are summarized as follows:

(1) The summation and difference of a discrete fractional proportional have been detected.

(2) The nabla discrete new Riemann–Liouville and Caputo fractional proportional differences of order $0 < \varepsilon < 1$ on the time scale $\mathbb{Z}$ have been formulated.

(3) The fractional proportional sums associated to $\left({}_{c}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z)$ with order $0 < \varepsilon < 1$ have been defined.

(4) The relation between nabla Riemann–Liouville and Caputo fractional proportional differences has been detected.

(5) The monotonicity results for the nabla Caputo fractional proportional difference which are if $\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z) > 0$, then $\chi(z)$ is $\varepsilon\rho$ –increasing; if $\chi(z)$ is strictly increasing on $\mathbb{N}_c$ and $\chi(c) > 0$, then $\left({}_{c-1}^{R}\nabla^{\varepsilon,\rho}\chi\right)(z) > 0$ has been proved as well.

(6) A new version of the fractional proportional difference of the mean value theorem on $\mathbb{Z}$ has been proved as an application.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors participated in obtaining the main results of this manuscript and drafted the manuscript. All authors read and approved the final manuscript.

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