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Research Article

General Forms of Solutions for Linear Impulsive Fuzzy Dynamic Equations on Time Scales

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A class of linear impulsive fuzzy dynamic equations on time scales is considered by using the generalized differentiability concept on time scales. Some novel criteria and general forms of solutions are established for such models whose significance lies in proposing the possibility to get unifying forms of solutions for discrete and continuous dynamical systems under uncertainty and to unify corresponding problems in the framework of fuzzy dynamic equations on time scales. Finally, some examples show the applicability of our results.

1. Introduction

In the real world, some processes vary continuously, while others vary discretely. These processes can be modeled by differential and difference equations, respectively. There are also some processes that vary both continuously and discretely. Usage of fuzzy differential and difference equations is a natural way to model dynamical systems under possibilistic uncertainty [1, 2]. First-order linear fuzzy differential (difference) equations are one of the simplest fuzzy equations which are very basic, important, and may appear in many applications. Thus, it is reasonable to seek conditions under which the resulting fuzzy systems would have a solution with a general form. Much progress has been seen in the fuzzy differential (difference) equation direction, and many criteria are established based on different approaches (for instance, fuzzy differential equations [1, 3–17] and fuzzy difference equations [18–23]). Careful investigation reveals that it is similar to explore the existence of solutions for fuzzy differential equations and their discrete analogue in the approaches, methods, and the main results. For example, extensive research shows that many results concerning the existence of fuzzy differential equations can be carried over to their discrete analogues [24–26]. However, other results seem to be completely different [3]. It is natural to ask whether we can explore such an existence problem in a unified way and offer more general conclusions.

For the certainty system, the theory of time scale calculus and dynamic equations on time scales provides us with a powerful tool for attacking such mixed processes [27]. The calculus on time scales (see [28–31]) was initiated by Hilger in [28] in order to unify continuous and discrete analysis under the certainty system, and it has a tremendous potential for applications and has recently received much attention.

The H-derivative of a fuzzy-number-valued function was introduced in [32], and it has its starting point in the Hukuhara derivative of set-valued functions. The first approach to modeling the uncertainty of dynamical systems uses the H-derivative or its generalized, and mainly the existence and uniqueness of the solution of a fuzzy differential equation are studied under this setting (see for example [11, 14, 33–35]). Fuzzy differential equations have been studied under other approaches (see [12, 36]). Furthermore, there are several works that have dealt with fuzzy-number-valued functions on time scales and focused on a class of new derivative of such fuzzy functions, see [37–40], as well as the Hukuhara derivative of set-valued functions has been extended onto the time scales by Hong in [41–43] and fuzzy or set dynamic equations have afterward been discussed in cited above references and [44, 45]. The aim of this paper is to establish a general form of solutions for linear fuzzy impulsive dynamic equations whose significance lies in proposing the possibility to get unifying forms of solutions for discrete and continuous dynamical systems under uncertainty and to build a unifying framework for the study of corresponding problems. As mentioned above, the notion of the H-derivative plays a fundamental role in the theory of fuzzy differential equations and the calculus on time scales has the features of unification and extension. In order to achieve our purpose, a derivative of fuzzy-number-valued functions on time scales, which is similar to the one in [37] and called the Δ_H-derivative in this paper, will be developed to suit our study of fuzzy dynamic equations. The proposed approach forms the appropriate environment within which the study of fuzzy dynamic systems on time scales can be developed.

This paper contains four sections. In Section 2, we recall several basic definitions and properties of time scales and generalized differentiability of fuzzy-number-valued functions on time scales proposed by [38] which is the extension of that on the real axis $\mathbb{R}$ introduced in [24]. Moreover, it contains the Δ_H-derivative introduced in [41]. In addition, some corresponding properties of the Δ_H-derivative are explored which provide the necessary background for our further consideration. Subsequently, in Section 3, we consider first order linear fuzzy dynamic equations on account of Δ_H-differentiability. The idea of the present section originates from the study of an analogous problem examined by Khastan et al. [3] for a variation of constant formula for the first-order linear fuzzy differential equations in $\mathbb{R}$. As distinct from [3], we consider the impulsive problem on an infinite time scale interval instead of the initial value problem on a finite real-number interval and present the solutions with general expressions in this setting. This study reveals that, when we deal with the existence of solutions with general expressions for linear fuzzy differential equations and the difference counterparts, it is unnecessary to prove results for fuzzy differential equations and separately again for their discrete analogues. In other words, one can get a unifying expression of solutions for such continuous and discrete uncertainty systems. Finally, several examples are given to illustrate the applicability of our results in Section 4.

2. Preliminaries

In this section, we first recall a notion of the time scale built by Hilger and Bernd Aulbach. For more details, we refer the reader to [28, 29].

A closed nonempty subset $\mathbb{T}$ of real axis $\mathbb{R}$ is called a time scale or measure chain. For $t \in \mathbb{T}$, we define the forward jump operator $\sigma\colon \mathbb{T} \longrightarrow \mathbb{T}$ by $\sigma(t) = \inf\{\tau \in \mathbb{T}\colon \tau > t\}$, while the backward jump operator $\rho\colon \mathbb{T} \longrightarrow \mathbb{T}$ is defined by $\rho(t) = \sup\{\tau \in \mathbb{T}\colon \tau < t\}$. The function $\mu\colon \mathbb{T} \longrightarrow [0,\infty)$ called the graininess function is defined by $\mu(t) = \sigma(t) - t$ for $t \in \mathbb{T}$. In this definition, we put $\inf \varnothing = \sup \mathbb{T}$(i.e. $\sigma(t) = t$ if $\mathbb{T}$ has a maximum t) and $\sup \varnothing = \inf \mathbb{T}$(i.e., $\rho(t) = t$ if $\mathbb{T}$ has a minimum t), where $\varnothing$ denotes the empty set. t is said to be right (left) scattered if $\sigma(t) > t\,(\rho(t) < t)$, and t is said to be right (left) dense if $\sigma(t) = t\,(\rho(t) = t)$. A point is said to be isolated (dense) if it is right-scattered (right-dense) and left-scattered (left-sense) at the same time. In this paper, we stipulate that the time scale $\mathbb{T}$ is $\mathbb{T} - \{M\}$ if $\mathbb{T}$ has a left-scattered maximum M.

A function f is right-dense continuous (rd-continuous, for short) if f is continuous at each right-dense point in $\mathbb{T}$ and its left-sided limits exist at each left-dense points in $\mathbb{T}$. For a function $f\colon \mathbb{T} \longrightarrow \mathbb{R}$ and $t \in \mathbb{T}$, S. Hilger defined the Δ-derivative of f at t, $f^{\Delta}(t)$, to be the number (when it exists), with the property that, for each $\varepsilon > 0$, there exists a neighborhood $U_{\mathbb{T}}$ of t (i.e. $U_{\mathbb{T}} = (t-\delta, t+\delta)\cap\mathbb{T}$) such that

$$\left| f(\sigma(t)) - f(s) - f^{\Delta}(t)(\sigma(t) - s)\right| < \varepsilon|\sigma(t) - s|, \tag{1}$$

for all $t \in U_{\mathbb{T}}$. A function f is said to be Δ-differentiable at t if its Δ-derivative exists at t and Δ-differentiable at $\mathbb{T}$ if its Δ-derivative exists at each $t \in \mathbb{T}$.

We also recall the concept of the matrix-valued functions introduced by [29]. An $m\times n$-matrix-valued function $A\colon \mathbb{T} \longrightarrow \mathbb{R}^{mn}$(a collection of all $m\times n$-real matrixes) is said to be Δ-differentiable on $\mathbb{T}$ provided each entry of A is Δ-differentiable on $\mathbb{T}$. In this case, we put

$$A^{\Delta} = \left(a_{ij}^{\Delta}\right)_{1\le i\le m, 1\le j\le n}, \quad \text{where } A = \left(a_{ij}\right). \tag{2}$$

An $n\times n$-matrix-valued function A on $\mathbb{T}$ is called regressive provided

$$I + \mu(t)A(t) \quad \text{is invertible for all } t \in \mathbb{T}. \tag{3}$$

Here, I stands for an $n\times n$-identity matrix (and so is it in what follows). Let

$$\begin{aligned}\mathscr{R} &= \{A \mid A\colon \mathbb{T} \longrightarrow \mathbb{R}^{nn} \text{ is a regressive and } rd\text{-continuous } n \\ &\quad \times n \text{ matrix} - \text{valued function}\},\\ \mathscr{R}_1^{+} &= \{p \mid p\colon \mathbb{T} \longrightarrow \mathbb{R} \text{ is a } rd\text{-continuous function and } 1 \\ &\quad + \mu(t)p(t) > 0 \text{ for } t \in \mathbb{T}\}.\end{aligned} \tag{4}$$

From now on, unless otherwise mentioned, the matrix-valued functions under consideration are always assumed to belong to $\mathscr{R}$.

For $A, B \in \mathscr{R}$, the "circle plus" and "circle minus" of matrix-valued functions are referred to as, respectively,

$$\begin{aligned}&(A\oplus B)(t) = A(t) + B(t) + \mu(t)A(t)B(t),\\ &(A\ominus B)(t) = (A\oplus(\ominus B))(t),\\ &\text{with } (\ominus A)(t) = -[I+\mu(t)A(t)]^{-1}A(t) = -A(t)[I+\mu(t)A(t)]^{-1}.\end{aligned} \tag{5}$$

A matrix exponential function $e_A(t,t_0)$ is defined as a unique matrix-valued solution of the following initial value problem:

$$\begin{aligned}Y^{\Delta} &= A(t)Y,\\ Y(t_0) &= I,\end{aligned} \tag{6}$$

where $A \in \mathscr{R}$ and $t_0 \in \mathbb{T}$. In [29], matrix exponential functions have been proved to possess the following properties:

(a1) $e_0(t,s) \equiv I, e_A(t,t) \equiv I, e_A(\sigma(t),s) = (I + \mu(t)A(t)) e_A(t,s)$

(a2) $e_A(s,t) = e_A^{-1}(t,s) = e^*_{\ominus A^*}(t,s)$, where A^* stands for the conjugate transpose of the matrix A

(a3) $e_A(t,s)e_A(s,r) = e_A(t,r)$

For an interval $J \subset \mathbb{T}$, if a function $g: J \longrightarrow \mathbb{R}$ is Δ-differentiable and $g^\Delta(t) = f(t)$; then, in [29], the authors defined the Cauchy integral by

$$\int_a^t f(s)\Delta s = g(t) - g(a). \tag{7}$$

In this case, f is said to be Δ-integrable on J. In particular, by $\int_a^\infty f(t)\Delta t := \lim_{b\longrightarrow\infty} \int_a^b f(t)\Delta t$, we mean that f is Δ-integrable on $J = [a,\infty) \cap \mathbb{T}$ provided this limit exists.

In the following, we introduce the necessary definitions and notation for fuzzy numbers on time scales which are the extension of the corresponding concepts in $\mathbb{R}$ (see, for example, [46]). Let us denote by $\mathbb{T}_f$ the class of fuzzy subsets of $\mathbb{T}$ satisfying the following properties, that is, $u \in \mathbb{T}_f$, i.e. $u: \mathbb{T} \longrightarrow [0,1]$ and

(f1) u is normal, i.e., there exists $s_0 \in \mathbb{T}$ such that $u(s_0) = 1$

(f2) u is fuzzy convex on $\mathbb{T}$, i.e., $u(ta + (1-t)b) \geq \min\{u(a), u(b)\}$ for all $t \in [0,1]$ with $ta + (1-t)b \in \mathbb{T}$, where $a, b \in \mathbb{T}$

(f3) u is upper semicontinuous on $\mathbb{T}$

(f4) $[u]^0 = \overline{\{s \in \mathbb{T}: u(s) > 0\}} \cap \mathbb{T}$ is compact, where $\overline{A}$ denotes the closure of A in $\mathbb{R}$

Then, $\mathbb{T}_f$ is called the space of fuzzy numbers. Obviously, $\mathbb{T} \subset \mathbb{T}_f$. Here, $\mathbb{T} \subset \mathbb{T}_f$ is understood as $\mathbb{T} = \{\chi_{\{x\}} \mid x \text{ is an element in the time scale}\}$. For $0 < \alpha \leq 1$, set $[u]^\alpha = \{s \in \mathbb{T} \mid u(s) \geq \alpha\}$ and $[u]^0 = \overline{\{s \in \mathbb{T} \mid u(s) > 0\}} \cap \mathbb{T}$. From (f1)–(f4), it follows that the α-level set $[u]^\alpha$ is a nonempty compact interval of $\mathbb{T}$ for all $0 \leq \alpha \leq 1$ if u belongs to $\mathbb{T}_f$ (i.e., $[u]^\alpha$ is an intersection of a nonempty compact interval of $\mathbb{R}$ and $\mathbb{T}$). The notation

$$[u]^\alpha = [\underline{u}^\alpha, \overline{u}^\alpha] \cap \mathbb{T} =: [\underline{u}^\alpha, \overline{u}^\alpha], \quad \text{with } \underline{u}^\alpha, \overline{u}^\alpha \in \mathbb{T}, \tag{8}$$

denotes explicitly the α-level set of u on $\mathbb{T}$. We refer to $\underline{u}$ and $\overline{u}$ as the lower and upper branches of u, respectively. For $u \in \mathbb{T}_f$, we define the length of u as

$$\operatorname{diam}(u) = \overline{u}^\alpha - \underline{u}^\alpha. \tag{9}$$

For $u, v \in \mathbb{T}_f$ and $\lambda \in \mathbb{R}$, the sum $u + v$ and the product λu are defined by $[u+v]^\alpha = [u]^\alpha + [v]^\alpha$, $[\lambda u]^\alpha = \lambda[u]^\alpha$ for any $\alpha \in [0,1]$, where $[u]^\alpha + [v]^\alpha$ is defined as the same as the usual addition of two intervals (subsets) of $\mathbb{T}$ and $\lambda[u]^\alpha$ means the usual product between a scalar and a subset of $\mathbb{T}$. The metric structure is given by the Hausdorff distance $D: \mathbb{T}_f \times \mathbb{T}_f \longrightarrow \mathbb{R}_+ = [0,\infty) \subset \mathbb{R}$,

$$D(u,v) = \sup_{\alpha\in[0,1]} \max\left\{\left|\underline{u}^\alpha - \underline{u}^\alpha\right|, \left|\overline{u}^\alpha - \overline{v}^\alpha\right|\right\}. \tag{10}$$

$(\mathbb{T}_f, D)$ is a complete metric space [42, 46], and the following properties are well known:

$$\begin{aligned} D(u+w, v+w) &= D(u,v), \quad \forall u,v,w \in \mathbb{T}_f, \\ D(ku, kv) &= |k| D(u,v), \quad \forall k \in \mathbb{R}, u,v \in \mathbb{T}_f, \\ D(u+v, z+w) &\leq D(u,z) + D(v,w), \quad \forall u,v,z,w \in \mathbb{T}_f. \end{aligned} \tag{11}$$

Definition 1. Let $x, y \in \mathbb{T}_f$. If there exists $z \in \mathbb{T}_f$ such that $x = y + z$, then z is called the H-difference of x, y and it is denoted by $x -_H y$.

Let us remark that, in general, $x -_H y \neq x + (-1)y$. Usually, we denote $x + (-1)y$ by $x - y$, while $x -_H y$ stands for the H-difference. Similar to the analysis in [24, 37, 38], we have the following remark:

(1) We denote $\widehat{0} \in \mathbb{T}_f$ as a neutral element with respect to + if $u + \widehat{0} = \widehat{0} + u = u$ for all $u \in \mathbb{T}_f$. For instance, if $0 \in \mathbb{T}$, then $\widehat{0} = \chi_{\{0\}}$

(2) $(\lambda + \mu)u = (\lambda u) + (\mu u)$, with $\lambda\mu > 0$

(3) $\lambda(u + v) = (\lambda u) + (\lambda v)$

The following lemma appears in references [37, 38, 47].

Lemma 1. *If $u -_H v$ exists, it is unique and one has*

(i) $u -_H u = \widehat{0}$, $u -_H \widehat{0} = u$, $\widehat{0} -_H u = -u$ and $(u + v) -_H v = u$

(ii) $v -_H u = -(u -_H v) = (-u) -_H (-v)$ provided $v -_H u$ exists

In the sequel, we fix $\mathbb{T}_+ = \mathbb{T} \cap \mathbb{R}_+$. The strongly generalized differentiability on the real axis $\mathbb{R}$ was introduced in [24] and studied in [1, 26]. Motivated by these works, we introduce generalized differentiability on a time scale $\mathbb{T}$ which appears to [38] and can be regarded as a generalization of Δ_H-differentiability introduced in [41].

Definition 2. Let $F: \mathbb{T} \longrightarrow \mathbb{T}_f$, $t \in \mathbb{T}$ and a neighborhood $U_\mathbb{T}$ of t be defined by $U_\mathbb{T} = (t - \delta, t + \delta) \cap \mathbb{T}$ for some $\delta > 0$. Then,

(1) F is said to be Δ-right differentiable at t if there exists an element $\Delta_+ F(t)$ of $\mathbb{T}_f$ and, for any given $\varepsilon > 0$, there exists a neighborhood $U_\mathbb{T}$ of t such that either the H-differences $F(t+h) -_H F(\sigma(t))$ exists and

$$\begin{aligned} &(c_1) D\left[F(t+h) -_H F(\sigma(t)), \Delta_+ F(t)(h - \mu(t))\right] \\ &\leq \varepsilon(h - \mu(t)) \\ &\text{or } F(\sigma(t)) -_H F(t+h) \text{ exists and} \\ &(c_2) D\left[F(\sigma(t)) -_H F(t+h), \Delta_+ F(t)(\mu(t) - h)\right] \\ &\leq \varepsilon(h - \mu(t)), \end{aligned} \tag{12}$$

for all $t + h \in U_\mathbb{T}$ with $0 \leq h < \delta$. Moreover, $\Delta_+ F(t)$ is called the Δ-right derivative of F at t.

(2) F is said to be Δ-left differentiable at t if there exists an element $\Delta\_F(t)$ of $\mathbb{T}_f$ and, for any given $\varepsilon > 0$, there exists a neighborhood $U_{\mathbb{T}}$ of t such that either the H-differences $F(t-h) -_H F(\sigma(t))$ exists and

$$\begin{aligned}&(c_1)\, D\left[F(t-h) -_H F(\sigma(t)), \Delta\_F(t)(-h-\mu(t))\right]\\&\leq \varepsilon(h+\mu(t))\\&\text{or } F(\sigma(t)) -_H F(t-h) \text{ exists and}\\&(c_2)\, D\left[F(\sigma(t)) -_H F(t-h), \Delta\_F(t)(h+\mu(t))\right]\\&\leq \varepsilon(\mu(t)+h),\end{aligned} \tag{13}$$

for all $t-h \in U_{\mathbb{T}}$ with $0 \leq h < \delta$. Moreover, $\Delta\_F(t)$ is called the Δ-left derivative of F at t.

(3) F is said to be Δ-differentiable at t if F is both Δ-left and Δ-right differentiable at t and $\Delta\_F(t) = \Delta_+ F(t)$. The element $\Delta_+ F(t)$ or $\Delta\_F(t)$ is said to be the Δ_H-derivative of F at t and is denoted by $\Delta_H F(t)$. We say that F is Δ_H-differentiable at t if its Δ_H-derivative exists at t. Moreover, we say F is Δ_H-differentiable on $\mathbb{T}$ if its Δ_H-derivative exists at each $t \in \mathbb{T}$. The fuzzy set-valued function $\Delta_H F\colon \mathbb{T} \longrightarrow \mathbb{T}_f$ is then called the Δ_H-derivative of F on $\mathbb{T}$.

The principal properties of the Δ_H-derivative in the sense of Definition 2 have been proposed in [37–39, 41]. Next, we shall write some properties whose a majority of proofs are similar to the above mentioned references.

Proposition 1 (see [38]). *Let $F\colon \mathbb{T} \longrightarrow \mathbb{T}_f$. For $t \in \mathbb{T}$, we have the following results:*

(I) If F is Δ_H-differentiable at t, then F is continuous at t.

(II) If F is continuous at t and t is right scattered, then F is Δ_H-differentiable at t with

$$\begin{aligned}\Delta_H F(t) &= \frac{F(\sigma(t)) -_H F(t)}{\mu(t)}\\ \text{or } F(\sigma(t)) &= F(t) + \mu(t)\Delta_H F(t).\end{aligned} \tag{14}$$

(III) If t is right-dense, then F is Δ_H-differentiable at t if and only if

$$\begin{aligned}&\lim_{h\longrightarrow 0^+} \frac{F(t+h) -_H F(t)}{h}\\&\text{or } \lim_{h\longrightarrow 0^+} \frac{F(t) -_H F(t+h)}{-h},\\&\lim_{h\longrightarrow 0^+} \frac{F(t) -_H F(t-h)}{h}\\&\text{or } \lim_{h\longrightarrow 0^+} \frac{F(t-h) -_H F(t)}{-h},\end{aligned} \tag{15}$$

exist as a finite number and satisfy any one of the following equations:

$$\begin{aligned}\lim_{h\longrightarrow 0^+} \frac{F(t+h) -_H F(t)}{h} &= \lim_{h\longrightarrow 0^+} \frac{F(t) -_H F(t-h)}{h}\\&= \Delta_H F(t),\end{aligned} \tag{16}$$

$$\begin{aligned}\lim_{h\longrightarrow 0^+} \frac{F(t+h) -_H F(t)}{h} &= \lim_{h\longrightarrow 0^+} \frac{F(t-h) -_H F(t)}{-h}\\&= \Delta_H F(t),\end{aligned} \tag{17}$$

$$\begin{aligned}\lim_{h\longrightarrow 0^+} \frac{F(t) -_H F(t+h)}{-h} &= \lim_{h\longrightarrow 0^+} \frac{F(t) -_H F(t-h)}{h}\\&= \Delta_H F(t),\end{aligned} \tag{18}$$

$$\begin{aligned}\lim_{h\longrightarrow 0^+} \frac{F(t) -_H F(t+h)}{-h} &= \lim_{h\longrightarrow 0^+} \frac{F(t-h) -_H F(t)}{-h}\\&= \Delta_H F(t).\end{aligned} \tag{19}$$

Remark 1. Proposition 1 implies that, under the hypothesis that $\mathbb{T}$ is a discrete system, the fuzzy number-valued function $F\colon \mathbb{T} \longrightarrow \mathbb{T}_f$ is Δ_H-differentiable if and only if F is continuous and the corresponding H-differences exist. However, if $\mathbb{T}$ is a continuous system, Δ_H-derivative of F does not always exist even if the corresponding H-differences exist.

Remark 2. If $\mathbb{T} = \mathbb{R}$, equations (16)–(19) are identical to the relevant provisions in Definition 5 in [24] in which four cases for derivatives were considered in $\mathbb{R}$. In addition, equation (16) is identical to (16)-differentiability and equation (19) to (17)-differentiability in [3].

If $\mathbb{T} = \mathbb{Z}$, then the previous definition expresses some generalized difference operators, for example, corresponding to the difference operator $\Delta F_n = F_{n+1} -_H F_n$ given in [21].

For functions $F, G\colon \mathbb{T} \longrightarrow \mathbb{T}_f$, we define the sum $F+G$ by $(F+G)(t) = F(t) + G(t)$ and the H-difference $F -_H G$ by $(F -_H G)(t) = F(t) -_H G(t)$ for each $t \in \mathbb{T}$. We have the following.

Proposition 2. *Assume that $F, G\colon \mathbb{T} \longrightarrow \mathbb{T}_f$ are Δ_H-differentiable at $t \in \mathbb{T}$ and λ is any constant. Then,*

(i) The sum is Δ_H-differentiable at $t \in \mathbb{T}$. Moreover,

$$\Delta_H (F+G)(t) = \Delta_H F(t) + \Delta_H G(t); \tag{20}$$

(ii) λF is Δ_H-differentiable at t and $\Delta_H(\lambda F)(t) = \lambda \Delta_H F(t)$.

In the sequel, we say that a fuzzy function is Δ_H-differentiable meaning that it is in two cases of (c^1) and (c_2)-differentiability (denoted by (i)-differentiable) or (c^2) and (c_1)-differentiability (denoted by (ii)-differentiable) on $\mathbb{T}$.

Proposition 3. *Let $F, G\colon \mathbb{T} \longrightarrow \mathbb{T}_f$ be Δ_H-differentiable such that F is (i)-differentiable and G is (ii)-differentiable or F is (ii)-differentiable and G is (i)-differentiable on $\mathbb{T}$. If the H-difference $F(t) -_H G(t)$ exists for $t \in \mathbb{T}$, then $F -_H G$ is Δ_H-differentiable and*

$$\Delta_H (F -_H G)(t) = \Delta_H F(t) + (-1) \cdot \Delta_H G(t). \tag{21}$$

Proof. For $t \in \mathbb{T}$, if t is a right-dense point, it is the same as the proof of Theorem 4 in [1]. If t is a right-scattered point, by means of Proposition 2-(II) and Lemma 1-(i) and (ii), we have

$$\begin{aligned}\Delta_H (F - G)(t) &= \frac{(F -_H G)(\sigma(t)) -_H (F -_H G)(t)}{\mu(t)} \\ &= \frac{F(\sigma(t)) -_H G(\sigma(t)) -_H (F(t) -_H G(t))}{\mu(t)} \\ &= \frac{F(\sigma(t)) -_H F(t)}{\mu(t)} + \frac{G(t) -_H G(\sigma(t))}{\mu(t)} \\ &= \Delta_H F(t) + (-1) \cdot \frac{G(\sigma(t)) -_H G(t)}{\mu(t)} = \Delta_H F(t) \\ &\quad + (-1) \cdot \Delta_H G(t).\end{aligned} \tag{22}$$

This proof is complete.

The following lemma roots in Theorem 5 of [1]. □

Lemma 2. *Let $a: \mathbb{T} \longrightarrow \mathbb{R}$ be Δ-differentiable and $G: \mathbb{T} \longrightarrow \mathbb{T}_f \Delta_H$-differentiable.*

(a) If $a(\sigma(t))a^\Delta(t) > 0$ and G is (i)-differentiable, then aG is (i)-differentiable and we have

$$\Delta_H (aG)(t) = a^\Delta(t)G(t) + a(\sigma(t))\Delta_H G(t). \tag{23}$$

(b) If $a(\sigma(t))a^\Delta(t) < 0$ and G is (ii)-differentiable, then aG is (ii)-differentiable and we have

$$\Delta_H (aG)(t) = a^\Delta(t)G(t) + a(\sigma(t))\Delta_H G(t). \tag{24}$$

(c) If $a(\sigma(t))a^\Delta(t) > 0$, G is (ii)-differentiable and the H-differences $(aG)(\sigma(t)) -_H (aG)(t+h)$ and $(aG)(t-h) -_H (aG)(\sigma(t))$ exist, then aG is (ii)-differentiable, and we have

$$\Delta_H (aG)(t) = a(\sigma(t))\Delta_H G(t) -_H \left(-a^\Delta(t)\right)G(t). \tag{25}$$

Proof. We get into details regarding the discussion of the cases (b) and (c), while the proof of (a) is similar to (b).

(b) For any $t \in \mathbb{T}$ and $\varepsilon > 0$, there exists a neighborhood $U_\mathbb{T}$ of t for some $\delta > 0$ such that

$$D\left[G(\sigma(t)) -_H G(t+h), \Delta_H G(t)(\mu(t) - h)\right] \le \varepsilon(h - \mu(t)), \tag{26}$$

$$\left|a(\sigma(t)) - a(t+h) - a^\Delta(t)(\mu(t) - h)\right| \le \varepsilon(h - \mu(t)), \tag{27}$$

for $0 \le h < \delta$ with $t + h \in U_\mathbb{T}$. On the contrary, we have

$$\begin{aligned}&D\left[a(\sigma(t))G(\sigma(t)) -_H a(t+h)G(t+h), \left(a^\Delta(t)G(t) + a(\sigma(t))\Delta_H G(t)\right)(\mu(t) - h)\right] \\ &\le D\left[a(\sigma(t))G(\sigma(t)) -_H a(\sigma(t))G(t+h), a(\sigma(t))\Delta_H \cdot G(t)(\mu(t) - h)\right] \\ &\quad + D\left[a(\sigma(t))G(t+h) -_H a(t+h)G(t+h), a^\Delta(t) \cdot G(t)(\mu(t) - h)\right] \\ &\le D\left[G(\sigma(t)) -_H G(t+h), \Delta_H G(t)(\mu(t) - h)\right] \cdot |a(\sigma(t))| \\ &\quad + D\left[a(\sigma(t))G(t+h) -_H a(t+h)G(t+h), a^\Delta(t) \cdot G(t)(\mu(t) - h)\right] \\ &\le D\left[G(\sigma(t)) -_H G(t+h), \Delta_H G(t)(\mu(t) - h)\right] \cdot |a(\sigma(t))| \\ &\quad + D\left[a(\sigma(t))G(t+h), a(t+h)G(t+h) + a^\Delta(t) \cdot G(t+h)(\mu(t) - h)\right] \\ &\quad + D\left[\{0\}, a^\Delta(t)G(t)(\mu(t) - h) -_H a^\Delta(t)G(t+h) \cdot (\mu(t) - h)\right].\end{aligned} \tag{28}$$

Note that $a(\sigma(t))$ has the same sign as $a(t+h)$ for sufficiently small $h > 0$. In addition, $a(\sigma(t))a^\Delta(t) < 0$ and $\mu(t) - h \le 0$ imply that $a^\Delta(t)(\mu(t) - h)$ has the same sign as $a(\sigma(t))$. Hence, $a(t+h)G(t+h) + a^\Delta(t)G(t+h)(\mu(t) - h) = [a(t+h) + a^\Delta(t)(\mu(t) - h)]G(t+h)$ and

$$\begin{aligned}&D\left[a(\sigma(t))G(t+h), a(t+h)G(t+h) + a^\Delta(t)G(t+h) \cdot (\mu(t) - h)\right] = \left|a(\sigma(t)) - \left(a(t+h) + a^\Delta(t)(\mu(t) - h)\right)\right| \\ &\cdot \|G(t+h)\|.\end{aligned} \tag{29}$$

It follows that

$$\begin{aligned}&D\left[a(\sigma(t))G(\sigma(t)) -_H a(t+h)G(t+h), \left(a^\Delta(t)G(t) + a(\sigma(t))\Delta_H G(t)\right)(\mu(t) - h)\right] \\ &\le D\left[G(\sigma(t)) -_H G(t+h), \Delta_H G(t)(\mu(t) - h)\right]|a(\sigma(t))| \\ &\quad + \left|a(\sigma(t)) - \left(a(t+h) + a^\Delta(t)(\mu(t) - h)\right)\right|\|G(t+h)\| \\ &\quad + D[G(t), G(t+h)]\left|a^\Delta(t)(\mu(t) - h)\right|.\end{aligned} \tag{30}$$

In view of this and the continuity of G, together with the inequalities (26) and (27), we see that aG satisfies the first inequality of Definition 2-(c^2). We can similarly check the second inequality of Definition 2-(c_1). Consequently, the desired conclusion arrives.

(c) As in case (b), the inequalities (26) and (27) are valid. Moreover, $-a^{\Delta}(t)(\mu(t)-h)$ has the same sign as $a(\sigma(t))$ and $a(t+h)$ under the hypothesis of (c). Therefore, for $0 \le h < \delta$ with $t+h \in U_{\mathbb{T}}$, we have

$$\begin{aligned}
&D\Big[a(\sigma(t))G(\sigma(t)) -_H a(t+h)G(t+h), \Big(a(\sigma(t))\Delta_H G(t) -_H \Big(-a^{\Delta}(t)\Big)G(t)\Big)(\mu(t)-h)\Big]\\
&\quad \le D\big[a(\sigma(t))G(\sigma(t)) -_H a(\sigma(t))G(t+h), a(\sigma(t))\Delta_H G(t)(\mu(t)-h)\big]\\
&\qquad + D\Big[a(\sigma(t))G(t+h) -_H a(t+h)G(t+h), \widehat{0} -_H \Big(-a^{\Delta}(t)G(t)(\mu(t)-h)\Big)\Big]\\
&\quad \le D\big[G(\sigma(t)) -_H G(t+h), \Delta_H G(t)(\mu(t)-h)\big]|a(\sigma(t))|\\
&\qquad + D\Big[a(\sigma(t))G(t+h), a(t+h)G(t+h) + \Big(-a^{\Delta}(t)\Big)G(t+h)(\mu(t)-h)\Big]\\
&\qquad + D\Big[\widehat{0}, -_H\Big(-a^{\Delta}(t)\Big)G(t+h)(\mu(t)-h) -_H \Big(-a^{\Delta}(t)G(t)(\mu(t)-h)\Big)\Big]\\
&\quad \le D\big[G(\sigma(t)) -_H G(t+h), \Delta_H G(t)(\mu(t)-h)\big]|a(\sigma(t))|\\
&\qquad + \Big|a(\sigma(t)) - \Big(a(t+h) + a^{\Delta}(t)(\mu(t)-h)\Big)\Big|\|G(t+h)\|\\
&\qquad + D[G(t), G(t+h)]\Big|a^{\Delta}(t)(\mu(t)-h)\Big|.
\end{aligned} \tag{31}$$

As in case (b), we arrive at the desired result. This proof is complete. □

Proposition 4. *Let $F\colon \mathbb{T} \longrightarrow \mathbb{T}_f$ and put $[F(t)]^{\alpha} = [\underline{F}^{\alpha}(t), \overline{F}^{\alpha}(t)]$ for each $\alpha \in [0,1]$.*

(i) If F is (i)-differentiable, then $\underline{F}^{\alpha}$ and $\overline{F}^{\alpha}$ are Δ-differentiable functions and

$$[\Delta_H F(t)]^{\alpha} = \left[\left(\overset{\alpha}{\underline{F}}\right)^{\Delta}(t), \left(\overline{F}^{\alpha}\right)^{\Delta}(t)\right]. \tag{32}$$

(ii) If F is (ii)-differentiable, then $\underline{F}^{\alpha}$ and $\overline{F}^{\alpha}$ are Δ-differentiable functions, and we have

$$[\Delta_H F(t)]^{\alpha} = \left[\left(\overline{F}^{\alpha}\right)^{\Delta}(t), \left(\overset{\alpha}{\underline{F}}\right)^{\Delta}(t)\right]. \tag{33}$$

Proposition 4 for its proof is similar to Theorem 5 in [11].

A fuzzy-number-valued function $F\colon J \subset \mathbb{T} \longrightarrow \mathbb{T}_f$ is called regulated provided its right-sided limit exists at any right-dense point in $\mathbb{T}$ and left-sided limit exists at any left-dense point in $\mathbb{T}$.

F is called right dense continuous, denoted rd-continuous, provided F is continuous at each right dense point in $\mathbb{T}$, its left-sided limits exist at each left dense points in $\mathbb{T}$. Similarly, we can define ld-continuity. The sets of all rd-continuous fuzzy-number-valued functions, and all such functions $F\colon J \longrightarrow \mathbb{T}_f$ whose rd-continuous Δ_H-derivative exist are denoted, respectively, by

$$\begin{aligned}
\mathscr{C}_{rd} &= \mathscr{C}_{rd}(J) = \mathscr{C}_{rd}\big(J, \mathbb{T}_f\big),\\
\mathscr{C}^1_{rd} &= \mathscr{C}^1_{rd}(J) = \mathscr{C}^1_{rd}\big(J, \mathbb{T}_f\big).
\end{aligned} \tag{34}$$

A function $f\colon J \subset \mathbb{T} \longrightarrow \mathbb{R}$ is called an integrable selector of the fuzzy-number-valued function $F\colon J \longrightarrow \mathbb{T}_f$ if f is Δ-integrable and $f(t) \in [F(t)]^{\alpha}$ for all $t \in J$ with $\alpha \in [0,1]$. F is called integrable on J if it has at least an integrable selector. The integral of F, denoted by $\int_J F(s)\Delta s$, is defined levelwise by

$$\left[\int_J F(s)\Delta s\right]^{\alpha} = \int_J [F(s)]^{\alpha}\Delta s = \left\{\int_J f(s)\Delta s\colon f \text{ is an integrable selector of } F \text{ on } J\right\}. \tag{35}$$

For the fuzzy version of the fundamental properties of calculus in the sense of (i)-differentiability, we refer to the analogue of set-valued functions in [41, 43], and in the sense of (ii)-differentiability we present the following results which are similar to those proposed in [3, 11]:

The integral $\mathscr{F}F(t) = \int_{t_0}^{t} F(s)\Delta s$ is (i)-differentiable and $\Delta_H F(t) = F(t)$. If $F \in \mathscr{C}_{rd}(\mathbb{T}, \mathbb{T}_f)$ and the function F is (ii)-differentiable, then $\Delta_H F(t) = F(t)$.

As the authors pointed out in [3], in general, the function $F(t)$ is not (ii)-differentiable. Indeed, suppose that is (ii)-differentiable, then the length of the support decreases in t, but the function $F(t)$, if f is fuzzy non-real-valued, has increasing length of the support. A (ii)-differentiable function needs to have decreasing length of support which is a contradiction.

Lemma 3 (see [42]). *Let $J = [t_0, T] \cap \mathbb{T}$ with $t_0, T \in \mathbb{T}$. Then, we have*

(i) Let $F \in \mathscr{C}_{rd}(J, \mathbb{T}_f)$ and define $U(t) = \gamma -_H \int_{t_0}^{t} -_H F(s)\Delta s$ for $t \in J$, where $\gamma \in \mathbb{T}_f$ is such that the previous H-difference exists for $t \in J$. Then, U is (ii)-differentiable and $\Delta_H U(t) = F(t)$;

(ii) Let F be (ii)-differentiable and $\Delta_H F$ be integrable on $\mathbb{T}_+$. Then, for each $t \in \mathbb{T}_+$ with $t \ge t_0 \in \mathbb{T}_+$, we have

$$F(t) = F(t_0) - {}_H\int_{t_0}^{t} -_H\Delta_H F(\tau)\Delta\tau; \tag{36}$$

(iii) $\int_{t_0}^{T} F(s)\Delta s = \int_{t_0}^{t} F(s)\Delta s + \int_{t}^{T} F(s)s\Delta s$. *Specially,* $\int_t^t F(s)\Delta s = \{0\}$ *for* $t \in J$;

(iv) $\int_t^{\sigma(t)} F(s)\Delta s = \mu(t)F(t)$, *for* $t \in J$;

(v) Let F *is (i)-differentiable on* $\mathbb{T}_+$. *Then,* $F(t) = \gamma + \int_{t_0}^{t} F(s)\Delta s$ *with* $\gamma \in \mathbb{T}_f$ *is (i)-differentiable and* $\Delta_H F(t) = F(t)$;

(vi) If $F, G \in C_{rd}(E)$, *then* $D[F(\cdot), G(\cdot)]\colon J \longrightarrow \mathbb{R}_+$ *is* Δ*-integrable and*

$$D\left[\int_{t_0}^{T} F(s)\Delta s, \int_{t_0}^{T} G(s)\Delta s\right] \le \int_{t_0}^{T} D[F(s), G(s)]\Delta s. \tag{37}$$

3. General Forms of Solutions for LIFDE

Let $\mathbb{T}_+ = \{t \in \mathbb{T} \mid t \ge 0\}$, $\mathbb{J} = \{t_k \in \mathbb{T}_+ \mid 0 \le t_0 < t_1 < t_2 < \cdots < t_k < \cdots, \lim_{k\longrightarrow\infty} t_k = \infty\}$, $J_- = [0, t_0] \cap \mathbb{T}_+$ and $J_k = (t_k, t_{k+1}] \cap \mathbb{T}_+$ for $k = 0, 1, \ldots$. Let $x_{t_k^+} = x(t_k^+)$ represent the right limit of $x(t)$ at t_k if t_k is right-dense and $x_{t_k^+} = x(\sigma(t_k))$ if t_k is right-scattered for $k = 1, 2, \ldots$. We emphasize the following notation:

$$\begin{aligned} PC &= \Big\{U\colon \mathbb{T}_+ \longrightarrow \mathbb{T}_f \,\Big|\, U \in \mathscr{C}_{rd}\big((t_{k-1}, t_k]\big) \text{ and } \lim_{t\longrightarrow t_k^+} U(t) \\ &\qquad = U(t_k^+) \text{ exists for } k = 1, 2, \ldots\Big\}. \\ BC &= \big\{U \in PC \mid \|U(t)\| = D(U(t), \widehat{0}) \text{ is bounded in } \mathbb{T}_+\big\}, \\ PC^1 &= \{U \in BC \mid U \text{ is } \Delta_H \\ &\qquad - \text{differentiable in each interval } (t_{k-1}, t_k)\}. \end{aligned} \tag{38}$$

It is clear that (BC, D_0) is a complete metric space if it is endowed with the distance $D_0(U, V) = \sup_{t\in\mathbb{T}_+} D(U(t), V(t))$.

Consider the first linear impulsive fuzzy dynamic equation (LIFDE):

$$\begin{cases} \Delta_H U(t) = r(t)U(t) + F(t), & t \in \mathbb{T}_+\mathbb{J}, \\ U_{t_k^+} = L_k U(t_k), & t_k \in \mathbb{J}(k = 0, 1, 2, \ldots), \\ U(0) = U_0 \in \mathbb{T}_f, & t_0 \in \mathbb{T}_+, \end{cases} \tag{39}$$

where $r\colon \mathbb{T}_+ \longrightarrow \mathbb{R}, F \in PC$ and $L_k\colon PC^1 \longrightarrow PC^1$ is a continuous linear operator, i.e., for any $v, w \in \mathbb{T}_f$ and $a, b \in \mathbb{R}$, one has $L_k(av \pm_g bw) = aL_k(v) \pm_g bL_k(w)$ whenever the H-difference exists. Let $U \in PC^1$ be a fuzzy-number-valued function such that $\Delta_H U$ exists at every point $t \in \mathbb{T}_+ \backslash \mathbb{J}$. By a (i)-solution of LIFDE (39), we mean U and $\Delta_H U$ exist in the case of (i)-differential and satisfy problem (39). The definition that U is a (ii)-solution of (39) is similar.

To explore the existence of solutions to LIFDE (39), we need the following essential preliminaries. In virtue of Theorem 5.24 in [29], the problem

$$\begin{cases} u^\Delta(t) = A(t)u(t) + f(t), & t \in \mathbb{T}, \\ u(0) = v_0, \end{cases} \tag{40}$$

has a unique solution $v\colon \mathbb{T} \longrightarrow \mathbb{R}^n$ given by

$$v(t) = e_A(t)v_0 + \int_0^t e_A(t, \sigma(s)) f(s)\Delta s, \tag{41}$$

where $A \in \mathscr{R}$, $e_A(t) = e_A(t, 0)$ and $w\colon \mathbb{T} \longrightarrow \mathbb{R}^n$ is rd-continuous.

Let $A \in \mathscr{R}$, $f\colon \mathbb{T}_+ \longrightarrow \mathbb{R}^n$ be an rd-continuous function and $\lim_{t\longrightarrow t_k^+} f(t) = f(t_k^+)$ exist for $k = 1, 2, \ldots$, and let L_k be a continuous linear operator acting in $\mathbb{R}^n$ for $k = 1, 2, \ldots$. By an analogue of the proof of the above result, we can prove that the linear impulsive dynamic equation:

$$\begin{cases} u^\Delta(t) = A(t)u(t) + f(t), & t \in J_k, \\ u_{t_k^+} = \Phi_k = L_k(u(t_k)), \\ u(t_k) = u_{k-1}(t_k), \end{cases} \tag{42}$$

for each $t_k \in \mathbb{J}$ has a unique solution

$$\begin{cases} u_k(t) = e_A(t, t_k^+)\Phi_k + \displaystyle\int_{t_k}^{t} e_A(t, \sigma(\tau)) f(\tau)\Delta\tau, & t \in J_k, \\ u_k(t_k) = u_{k-1}(t_k), \end{cases} \tag{43}$$

for $k = 0, 1, 2, \ldots$, where $u_{0-1} = v(t_0)$. Thus, we obtain that the linear impulsive dynamic equation:

$$\begin{cases} u^\Delta(t) = A(t)u(t) + f(t), & t \in \mathbb{T}_+ \backslash \mathbb{J}, \\ u_{t_k^+} = \Phi_k = L_k(u(t_k)), & t_k \in \mathbb{J}(k = 0, 1, 2, \ldots), \\ u(0) = v_0, \end{cases} \tag{44}$$

has a unique solution $w = w(v_0)$ on $\mathbb{T}_+$ which is left continuous on $\mathbb{T}_+$ and defined by

$$w(t) = \begin{cases} v(t), & t \in J_-, \\ u_0(t), & t \in J_0, \\ \cdots & \cdots \\ u_k(t), & t \in J_k; \\ \cdots & \cdots. \end{cases} \tag{45}$$

Similar to the formulation in [3], we study LIFDE (39) in three cases $r(t) < 0, r(t) > 0$ and $r(t) = 0$ for $t \in \mathbb{T}_+$, where r is a function given in LIPDE (39). We first observe that the hyperbolic functions proposed by Bohner and Peterson [31] can be extended to

$$\cosh_r(t,s) = \frac{e_r(t,s)+e_{-r}(t,s)}{2},$$
$$\sinh_r(t,s) = \frac{e_r(t,s)-e_{-r}(t,s)}{2}. \tag{46}$$

Let $E_r(t,s) = e_r(t,s)e_{-r}(t,s)$. Obviously, for all $t,s \in \mathbb{T}$, the hyperbolic functions possess the following properties:

(p1) $\cosh_r(s,s) = 1$, $\sinh_r(s,s) = 0$

(p2) $\cosh_r^{\Delta_t}(t,s) = r\sinh_r(t,s)$, $\sinh_r^{\Delta_t}(t,s) = r\cosh_r(t,s)$, where $\alpha^{\Delta_t}(t,s)$ means the Δ-derivative of α with respect to the variable t

(p3) $\cosh_r(s,t) = (1/E_r(t,s))\cosh_r(t,s)$, $\sinh_r(s,t) = -(1/E_r(t,s))\sinh_r(t,s)$

(p4) $\sinh_r(t,s) + \cosh_r(t,s) = e_r(t,s)$, $\cosh_r(t,s) - \sinh_r(t,s) = e_{-r}(t,s)$ and $\cosh_r^2(t,s) - \sinh_r^2(t,s) = E_r(t,s)$

We are now in a position to state and verify our main results.

Theorem 1. *If $r \in \mathscr{R}_1^+$ satisfies $r(t) < 0$ for all $t \in \mathbb{T}_+$, then*

(i) LIFDE (39) has a (i)-solution on $\mathbb{T}_+$ given by

$$U(t) = \begin{cases} V(t), & t \in J_-, \\ U_k(t), & t \in J_k\,(k = 0,1,2,\ldots), \end{cases} \tag{47}$$

where

$$\begin{aligned} V(t) &= \cosh_r(t)\left\{U_0 + \int_0^t \left[F(\tau)\frac{\cosh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)} -_H F(\tau)\frac{\sinh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)}\right]\Delta\tau\right\} \\ &\quad + \sinh_r(t)\left\{U_0 + \int_0^t \left[F(\tau)\frac{\cosh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)} -_H F(\tau)\frac{\sinh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)}\right]\Delta\tau\right\}, \\ U_k(t) &= \cosh_r(t,t_k^+)\left\{L_kU_k(t_k) + \int_{t_k}^t \left[F(\tau)\frac{\cosh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)} -_H F(\tau)\frac{\sinh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)}\right]\Delta\tau\right\} \\ &\quad + \sinh_r(t,t_k^+)\left\{L_kU_k(t_k) + \int_{t_k}^t \left[F(\tau)\frac{\cosh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)} -_H F(\tau)\frac{\sinh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)}\right]\Delta\tau\right\}, \end{aligned} \tag{48}$$

with $U_0(t_0) = V(t_0), U_k(t_k) = U_{k-1}(t_k)$ for $k = 0,1,2,\ldots$, and

(ii) The (ii)-solution of the LIFDE (39) on $\mathbb{T}_+$ is given by

$$U(t) = \begin{cases} V(t) = e_r(t)\left[U_0 -_H \int_0^t -_H F(\tau)e_{\ominus r}(\sigma(\tau))\Delta\tau\right], & t \in J_-, \\ U_k(t) = e_r(t,t_k^+)\left[L_kU_k(t_k) -_H \int_{t_k}^t F(\tau)e_{\ominus r}(\sigma(\tau),t_k)\Delta\tau\right], & t \in J_k, \end{cases} \tag{49}$$

provided the H-differences exist and $U_0(t_0) = V(t_0)$, $U_k(t_k) = U_{k-1}(t_k)$ for $k = 0,1,2,\ldots$.

Proof. Proposition 4 shows us how to translate the LIFDE (39) into a system of ordinary dynamic equations (ODEs), that is, if $r(t) < 0$ with $t \in \mathbb{T}_+$ and U is (i)-differentiable, then $[\Delta_H U(t)]^\alpha = [(\underline{U}^\alpha)^\Delta(t), (\overline{U}^\alpha)^\Delta(t)]$ with $[U]^\alpha = [\underline{U}^\alpha, \overline{U}^\alpha]$ for all $\alpha \in [0,1]$, and (39) is translated into the following impulsive system of ODEs:

$$\begin{cases} (\underline{U}^\alpha)^\Delta(t) = r(t)\overline{U}^\alpha(t) + \underline{F}^\alpha(t), & t \in \mathbb{T}_+\backslash\mathbb{J}, \\ (\overline{U}^\alpha)^\Delta(t) = r(t)\underline{U}^\alpha(t) + \overline{F}^\alpha(t), & t \in \mathbb{T}_+\backslash\mathbb{J}, \\ (\underline{U}^\alpha)_{t_k^+} = L_k\underline{U}^\alpha(t_k), \\ (\overline{U}^\alpha)_{t_k^+} = L_k\overline{U}^\alpha(t_k), \\ t_k \in \mathbb{J}\,(k = 0,1,2,\ldots), \\ \underline{U}^\alpha(0) = (\underline{U}^\alpha)_0, \\ \overline{U}^\alpha(0) = (\overline{U}^\alpha)_0, \end{cases} \tag{50}$$

where $[F(t)]^\alpha = [\underline{F}^\alpha(t), \overline{F}^\alpha(t)]$ for all $\alpha \in [0,1]$. For solving system (50), we translate it into system (44) with

$$u(t) = \begin{pmatrix} \underline{U}^\alpha(t) \\ \overline{U}^\alpha(t) \end{pmatrix}, \quad u_{t_k^+} = \begin{pmatrix} (\underline{U}^\alpha)_{t_k^+} \\ (\overline{U}^\alpha)_{t_k^+} \end{pmatrix}, \quad f(t) = \begin{pmatrix} \underline{F}^\alpha(t) \\ \overline{F}^\alpha(t) \end{pmatrix}, \quad v_0 = \begin{pmatrix} (\underline{U}^\alpha)_0 \\ (\overline{U}^\alpha)_0 \end{pmatrix}, \quad (\alpha \in [0,1]), \quad A(t) = \begin{pmatrix} 0 & r(t) \\ r(t) & 0 \end{pmatrix}. \tag{51}$$

Similarly, if U is (ii)-differentiable then $[\Delta_H U(t)]^\alpha = [(\overline{U}^\alpha)^\Delta(t), (\underline{U}^\alpha)^\Delta(t)]$ and (39) is translated into (44) with $u, u_{t_k^+}, v_0$ given as in (51) and

$$A(t) = \begin{pmatrix} r(t) & 0 \\ 0 & r(t) \end{pmatrix}, \quad f(t) = \begin{pmatrix} \overline{F}^\alpha(t) \\ \underline{F}^\alpha(t) \end{pmatrix}. \tag{52}$$

(i) Under the case of the (i)-differential, we check that A given by (51) belongs to $\mathscr{R}$. From $r \in \mathscr{R}_1^+$, it follows that the matrix

$$I + \mu(t)A(t) = \begin{pmatrix} 1 & \mu(t)r(t) \\ \mu(t)r(t) & 1 \end{pmatrix}, \tag{53}$$

is invertible for each $t \in \mathbb{T}_+$, that is, $A \in \mathscr{R}$. Moreover, we easily check that

$$e_A(t,s) = \begin{pmatrix} \cosh_r(t,s) & \sinh_r(t,s) \\ \sinh_r(t,s) & \cosh_r(t,s) \end{pmatrix}, \tag{54}$$

with A given in (51).

Now, by substituting this matrix exponential function for $e_A(t, t_k^+)$ and $e_A(t, \sigma(\tau))$ of (43), we have

$$\begin{cases} u_k(t) = \begin{pmatrix} \cosh_r(t,t_k^+) & \sinh_r(t,t_k^+) \\ \sinh_r(t,t_k^+) & \cosh_r(t,t_k^+) \end{pmatrix} \Phi_k \\ \quad + \displaystyle\int_{t_k}^t \begin{pmatrix} \cosh_r(t,\sigma(\tau)) & \sinh_r(t,\sigma(\tau)) \\ \sinh_r(t,\sigma(\tau)) & \cosh_r(t,\sigma(\tau)) \end{pmatrix} f(\tau)\Delta\tau, \quad t \in J_k, \\ u_k(t_k) = u_{k-1}(t_k), \end{cases} \tag{55}$$

where $u_{0-1}(t_0) = v(t_0)$. Since $e_A(t,\sigma(\tau)) = e_A(t,t_k)\, e_A(t_k,\sigma(\tau))$, we have

$$u_k(t) = \begin{pmatrix} \cosh_r(t,t_k^+) & \sinh_r(t,t_k^+) \\ \sinh_r(t,t_k^+) & \cosh_r(t,t_k^+) \end{pmatrix} \left[\Phi_k + \int_{t_k}^t \begin{pmatrix} \cosh_r(t_k,\sigma(\tau)) & \sinh_r(t_k,\sigma(\tau)) \\ \sinh_r(t_k,\sigma(\tau)) & \cosh_r(t_k,\sigma(\tau)) \end{pmatrix} f(\tau)\Delta\tau\right]. \tag{56}$$

Therefore, for $t \in J_k$ with $k = 0, 1, 2, \ldots$, we have

$$\begin{aligned} \begin{pmatrix} \underline{U}^\alpha(t) \\ \overline{U}^\alpha(t) \end{pmatrix} = \begin{pmatrix} \underline{U_k}^\alpha(t) \\ \overline{U_k}^\alpha(t) \end{pmatrix} &= \begin{pmatrix} \cosh_r(t,t_k^+) & \sinh_r(t,t_k^+) \\ \sinh_r(t,t_k^+) & \cosh_r(t,t_k^+) \end{pmatrix} \\ &\quad \times \left[\begin{pmatrix} L_k\underline{U_k}^\alpha(t_k) \\ L_k\overline{U_k}^\alpha(t_k) \end{pmatrix} + \int_{t_k}^t \begin{pmatrix} \cosh_r(t_k,\sigma(\tau)) & \sinh_r(t_k,\sigma(\tau)) \\ \sinh_r(t_k,\sigma(\tau)) & \cosh_r(t_k,\sigma(\tau)) \end{pmatrix} \begin{pmatrix} \underline{F}^\alpha(\tau) \\ \overline{F}^\alpha(\tau) \end{pmatrix} \Delta\tau\right] \\ &= \begin{pmatrix} \cosh_r(t,t_k^+) & \sinh_r(t,t_k^+) \\ \sinh_r(t,t_k^+) & \cosh_r(t,t_k^+) \end{pmatrix} \\ &\quad \times \begin{pmatrix} L_k\underline{U_k}^\alpha(t_k) + \displaystyle\int_{t_k}^t \left[\underline{F}^\alpha(\tau)\cosh_r(t_k,\sigma(\tau)) + \overline{F}^\alpha(\tau)\sinh_r(t_k,\sigma(\tau))\right]\Delta\tau \\ L_k\overline{U_k}^\alpha(t_k) + \displaystyle\int_{t_k}^t \left[\underline{F}^\alpha(\tau)\sinh_r(t_k,\sigma(\tau)) + \overline{F}^\alpha(\tau)\cosh_r(t_k,\sigma(\tau))\right]\Delta\tau \end{pmatrix}. \end{aligned} \tag{57}$$

Then, by the property (p3), the solution of the linear dynamic equation system is

$$\begin{aligned}
\underline{u}^{\alpha}(t) &= \underline{U_k}^{\alpha}(t)\\
&= \cosh_r(t,t_k^+)\left\{L_k\underline{U_k}^{\alpha}(t_k) + \int_{t_k}^{t}\left[\underline{F}^{\alpha}(\tau)\frac{\cosh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)} - \overline{F}^{\alpha}(\tau)\frac{\sinh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)}\right]\Delta\tau\right\}\\
&\quad + \sinh_r(t,t_k^+)\left\{L_k\overline{U_k}^{\alpha}(t_k) + \int_{t_k}^{t}\left[-\underline{F}^{\alpha}(\tau)\frac{\sinh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)} + \overline{F}^{\alpha}(\tau)\frac{\cosh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)}\right]\Delta\tau\right\},\\
\overline{U}^{\alpha}(t) &= \overline{U_k}^{\alpha}(t)\\
&= \sinh_r(t,t_k^+)\left\{L_k\underline{U_k}^{\alpha}(t_k) + \int_{t_k}^{t}\left[\underline{F}^{\alpha}(\tau)\frac{\cosh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)} - \overline{F}^{\alpha}(\tau)\frac{\sinh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)}\right]\Delta\tau\right\}\\
&\quad + \cosh_r(t,t_k^+)\left\{L_k\overline{U_k}^{\alpha}(t_k) + \int_{t_k}^{t}\left[-\underline{F}^{\alpha}(\tau)\frac{\sinh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)} + \overline{F}^{\alpha}(\tau)\frac{\cosh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)}\right]\Delta\tau\right\}.
\end{aligned} \tag{58}$$

Thus, we obtain the (i)-solution of LIFDE (39) on J_k for $k = 0,1,2,\ldots$ when $r(t) < 0$ with $t \in \mathbb{T}_+$ as follows:

$$\begin{aligned}
U(t) &= U_k(t)\\
&= \cosh_r(t,t_k^+)\left\{L_kU_k(t_k) + \int_{t_k}^{t}\left[F(\tau)\frac{\cosh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)} -_H F(\tau)\frac{\sinh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)}\right]\Delta\tau\right\}\\
&\quad + \sinh_r(t,t_k^+)\left\{L_kU_k(t_k) + \int_{t_k}^{t}\left[F(\tau)\frac{\cosh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)} -_H F(\tau)\frac{\sinh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)}\right]\Delta\tau\right\}.
\end{aligned} \tag{59}$$

Similarly, if $t \in J_-$, we have

$$\begin{aligned}
U(t) = V(t) &= \cosh_r(t)\left\{U_0 + \int_{0}^{t}\left[F(\tau)\frac{\cosh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)} -_H F(\tau)\frac{\sinh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)}\right]\Delta\tau\right\}\\
&\quad + \sinh_r(t)\left\{U_0 + \int_{0}^{t}\left[F(\tau)\frac{\cosh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)} -_H F(\tau)\frac{\sinh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)}\right]\Delta\tau\right\}.
\end{aligned} \tag{60}$$

Let us remark that the H-difference

$$F(\tau)\frac{\cosh_r(\sigma(\tau),s)}{E_r(\sigma(\tau),s)} -_H F(\tau)\frac{\sinh_r(\sigma(\tau),s)}{E_r(\sigma(\tau),s)}, \tag{61}$$

always exists for $s \in [0,\sigma(\tau)]$ and $r(t) < 0$. As explicated in [3], the diameters of the α-level sets of $F(\tau)\cosh_r(\sigma(\tau),s)/E_r(\sigma(\tau),s)$ and $F(\tau)\sinh_r(\sigma(\tau),s)/E_r(\sigma(\tau),s)$ are, respectively,

$$\begin{aligned}
&\operatorname{diam}\left([F(\tau)]^{\alpha}\right)\frac{\cosh_r(\sigma(\tau),s)}{E_r(\sigma(\tau),s)},\\
&\operatorname{diam}\left([F(\tau)]^{\alpha}\right)\left[\frac{-\sinh_r(\sigma(\tau),s)}{E_r(\sigma(\tau),s)}\right].
\end{aligned} \tag{62}$$

While the former is greater than the latter since we have the inequality:

$$\frac{\sinh_r(\sigma(\tau), s)}{E_r(\sigma(\tau), s)} + \frac{\cosh_r(\sigma(\tau), s)}{E_r(\sigma(\tau), s)} = \frac{e_r(\sigma(\tau), s)}{E_r(\sigma(\tau), s)} = \frac{1}{e_{-r}(\sigma(\tau), s)} > 0, \tag{63}$$

from $r \in \mathscr{R}_1^+$ and $r(t) < 0$.

Finally, note that $\cosh_r(\sigma(t), s) \cdot \cosh_r^{\Delta_t}(t, s) = r\cosh_r(\sigma(t), s) \cdot \sinh_r(t, s) > 0$ and $\sinh_r(\sigma(t), s) \cdot \sinh_r^{\Delta_t}(t, s) = r(t)\sinh_r(\sigma(t), s) \cdot \cosh_r(t, s) > 0$ for $r(t) < 0$, we see that $U(t)$ is (i)-differentiable on $\mathbb{T}_+ \backslash \mathbb{J}$ in view of Lemma 2-(a). Consequently, LIFDE (39) has a (i)-solution and (47) holds.

(ii) For $t \in J_k$, under the hypothesis of (ii)-differentiability, LIFDE (39) is translated into the corresponding system (42) with A and f given as (52). Obviously, $A \in \mathscr{R}$ and

$$e_A(t, s) = e_r(t, s)I. \tag{64}$$

By means of (43), we have

$$\begin{cases} u_k(t) = e_r(t, t_k^+)\left[I\Phi_k + \int_{t_k}^t e_{\ominus r}(\sigma(\tau), t_k)\,\mathrm{If}(\tau)\Delta\tau\right], & t \in J_k, \\ u_k(t_k) = u_{k-1}(t_k). \end{cases} \tag{65}$$

Repeating the arguments of (i), we obtain that the solution of the corresponding ODEs system is

$$\underline{U_k}^\alpha(t) = e_r(t, t_k^+)\left[L_k\underline{U_k}^\alpha(t_k) + \int_{t_k}^t \overline{F}^\alpha(\tau)e_{\ominus r}(\sigma(\tau), t_k)\Delta\tau\right],$$

$$\overline{U_k}^\alpha(t) = e_r(t, t_k^+)\left[L_k\overline{U_k}^\alpha(t_k) + \int_{t_k}^t \underline{F}^\alpha(\tau)e_{\ominus r}(\sigma(\tau), t_k)\Delta\tau\right], \tag{66}$$

for all $\alpha \in [0, 1]$. We assert that the (ii)-solution of LIFDE (39) on J_k is

$$U(t) = U_k(t) = e_r(t, t_k^+)\left[L_kU_k(t_k) -_H \int_{t_k}^t -F(\tau)e_{\ominus r}(\sigma(\tau), t_k)\Delta\tau\right], \tag{67}$$

where $k = 0, 1, 2, \ldots$ and $r(t) < 0$. In fact, we observe that $e_r(\sigma(t), t_k)e_r^{\Delta_t}(t, t_k) = r(t)e_r(\sigma(t), t_k)e_r(t, t_k) < 0$. If we denote $G_k(t) = L_kU_k(t_k) -_H \int_{t_k}^t -F(\tau)e_{\ominus r}(\sigma(\tau), t_k)\Delta\tau$, then Lemma 3-(i) guarantees that G_k is (ii)-differentiable and

$$\Delta_H G_k(t) = F(t)e_{\ominus r}(\sigma(t), t_k). \tag{68}$$

Now, the conditions in Lemma 2-(b) are met, so

$$\Delta_H U_k(t) = r(t)e_r(t, t_k^+)G_k(t) + e_r(\sigma(t), t_k)\Delta_H G_k(t) = r(t)U_k(t) + F(t), \tag{69}$$

for $k = 0, 1, 2 \ldots$. Similarly, for $t \in J_-$, we have

$$U(t) = V(t) = e_r(t, 0)\left[U_0 -_g \int_0^t -_g F(\tau)e_{\ominus r}(\sigma(\tau), 0)\Delta\tau\right]. \tag{70}$$

We obtain that LIFDE (39) has a (ii)-solution satisfying (49). The proof is complete. □

Theorem 2. *If $-r \in \mathscr{R}_1^+$ and $r(t) > 0$ with $t \in \mathbb{T}_+$, then*

(i) LIFDE (39) has a (ii)-solution on $\mathbb{T}_+$ given as in (47), where

$$\begin{cases} V(t) = \cosh_r(t)\left\{U_0 -_H \int_0^t \left[F(\tau)\frac{\sinh_r(\sigma(\tau), 0)}{E_r(\sigma(\tau), 0)} -_H F(\tau)\frac{\cosh_r(\sigma(\tau), 0)}{E_r(\sigma(\tau), 0)}\right]\Delta\tau\right\} -_H [-\sinh_r(t)] \\ \quad \cdot \left\{U_0 -_H \int_0^t \left[F(\tau)\frac{\sinh_r(\sigma(\tau), 0)}{E_r(\sigma(\tau), 0)} -_H F(\tau)\frac{\cosh_r(\sigma(\tau), 0)}{E_r(\sigma(\tau), 0)}\right]\Delta\tau\right\}, \quad t \in J_-, \\ U_k(t) = \cosh_r(t, t_k^+)\left\{L_kU_k(t_k) -_H \int_{t_k}^t \left[F(\tau)\frac{\sinh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)} -_H F(\tau)\frac{\cosh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)}\right]\Delta\tau\right\} -_H [-\sinh_r(t, t_k^+)] \\ \quad \cdot \left\{L_kU_k(t_k) -_H \int_{t_k}^t \left[F(\tau)\frac{\sinh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)} -_H F(\tau)\frac{\cosh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)}\right]\Delta\tau\right\}, t \in J_k, \end{cases} \tag{71}$$

provided that the above H-differences exist and $U_0(t_0) = V(t_0), U_k(t_k) = U_{k-1}(t_k)$ *for* $k = 0, 1, 2, \ldots$.

(ii) The (i)-solution of LIFDE (39) on $\mathbb{T}_+$ *is given by*

$$U(t) = \begin{cases} V(t) = e_r(t)\left[U_0 + \int_0^t F(\tau)e_{\ominus r}(\sigma(\tau))\Delta\tau\right], & t \in J_-, \\ U_k(t) = e_r(t, t_k^+)\left[L_k U_k(t_k) + \int_{t_k}^t F(\tau)e_{\ominus r}(\sigma(\tau), t_k)\Delta\tau\right], & t \in J_k, \end{cases} \tag{72}$$

where $U_0(t_0) = V(t_0), U_k(t_k) = U_{k-1}(t_k)$ *for* $k = 0, 1, 2, \ldots$.

Proof. (i) LIFDE (39) with (ii)-differentiability is transformed into the linear impulsive dynamic equation system (44) with $u(t), u_{t_k^+}, A(t), v_0$ given in (51) and $f(t)$ as in (52). By (44), we obtain

$$\begin{cases} u_k(t) = A(t, t_k^+)\left[\Phi_k + \int_{t_k}^t A(t_k, \sigma(\tau)) f(\tau)\Delta\tau\right], & t \in J_k, \\ u_k(t_k) = u_{k-1}(t_k), \end{cases} \tag{73}$$

where $u_{0-1}(t_0) = v(t_0)$. Therefore, for each $\alpha \in [0, 1]$, we have

$$\begin{aligned} \underline{u}^\alpha(t) &= \underline{U_k}^\alpha(t) \\ &= \cosh_r(t, t_k^+)\left\{L_k\underline{U_k}^\alpha(t_k) + \int_{t_k}^t \left[\overline{F}^\alpha(\tau)\frac{\cosh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)} - \underline{F}^\alpha(\tau)\frac{\sinh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)}\right]\Delta\tau\right\} \\ &\quad + \sinh_r(t, t_k^+)\left\{L_k\overline{U_k}^\alpha(t_k) + \int_{t_k}^t \left[-\overline{F}^\alpha(\tau)\frac{\sinh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)} + \underline{F}^\alpha(\tau)\frac{\cosh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)}\right]\Delta\tau\right\}, \\ \overline{U}^\alpha(t) &= \overline{U_k}^\alpha(t) \\ &= \sinh_r(t, t_k^+)\left\{L_k\underline{U_k}^\alpha(t_k) + \int_{t_k}^t \left[\overline{F}^\alpha(\tau)\frac{\cosh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)} - \underline{F}^\alpha(\tau)\frac{\sinh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)}\right]\Delta\tau\right\} \\ &\quad + \cosh_r(t, t_k^+)\left\{L_k\overline{U_k}^\alpha(t_k) + \int_{t_k}^t \left[-\overline{F}^\alpha(\tau)\frac{\sinh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)} + \underline{F}^\alpha(\tau)\frac{\cosh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)}\right]\Delta\tau\right\}. \end{aligned} \tag{74}$$

Therefore, for $t \in J_k (k = 0, 1, 2, \ldots)$, by $r(t) > 0$, we check that

$$U_k(t) = \cosh_r(t, t_k^+)G_k(t) -_H [-\sinh_r(t, t_k^+)]G_k(t), \tag{75}$$

with

$$G_k(t) = L_k U_k(t_k) -_H \int_{t_k}^t \left[F(\tau)\frac{\sinh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)} -_H F(\tau)\frac{\cosh_r(\sigma(\tau), t_k)}{E_r(\sigma(\tau), t_k)}\right]\Delta\tau, \tag{76}$$

is (ii)-differentiable and U_k is a (ii)-solution of LIFDE (39) on J_k.

The following argument is due to the proof of Theorem 3.3 in [3]. First, by our hypothesis G_k is well defined. Second, Lemma 3-(i) guarantees that G_k is (ii)-differentiable and the diameter of G_k is nonincreasing in the variable t for fixed $\alpha \in [0, 1]$. Note that $\cosh_r(t, t_k^+) - \sinh_r(t, t_k^+) = e_{-r}(t, t_k^+)$ is nonnegative and decreasing in t. Thus, $\operatorname{diam}[U_k(t)]^\alpha$ is nonincreasing in t for fixed $\alpha \in [0, 1]$. Therefore, the H-differences $U_k(\sigma(t)) -_H U_k(t + h)$ and $U_k(t - h) -_H U_k(\sigma(t))$ exist. Third, we check that the (ii)-derivative of U_k is $r(t)U_k(t) + F(t)$ for $t \in E_k (k = 0, 1, 2, \ldots)$. If t is a right-dense point, in view of an analogous argument of Theorem 3.3 in [3], we can check that

$$\lim_{h\longrightarrow 0^+}\frac{U_k(t) -_H U_k(t+h)}{-h} = \lim_{h\longrightarrow 0^+}\frac{U_k(t-h) -_H U_k(t)}{-h} = r(t)U_k(t) + F(t). \tag{77}$$

In the light of Proposition 1-(III), we have $\Delta_H U_k(t) = r(t)U_k(t) + F(t)$. If t is a right-scattered point, we denote

$$\begin{aligned}\xi_k(t) &= \int_{t_k}^{t}\left[\overline{F}^{\alpha}(\tau)\frac{\cosh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)} - \underline{F}^{\alpha}(\tau)\frac{\sinh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)}\right]\Delta\tau,\\ \eta_k(t) &= \int_{t_k}^{t}\left[-\overline{F}^{\alpha}(\tau)\frac{\sinh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)} + \underline{F}^{\alpha}(\tau)\frac{\cosh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)}\right]\Delta\tau.\end{aligned} \tag{78}$$

Then,

$$\begin{aligned}\xi_k^{\Delta}(t) &= \overline{F}^{\alpha}(t)\frac{\cosh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)} - \underline{F}^{\alpha}(t)\frac{\sinh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)},\\ \eta_k^{\Delta}(t) &= -\overline{F}^{\alpha}(t)\frac{\sinh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)} + \underline{F}^{\alpha}(t)\frac{\cosh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)},\\ \xi_k(\sigma(t)) - \xi_k(t) &= \int_{t}^{\sigma(t)}\left[\overline{F}^{\alpha}(\tau)\frac{\cosh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)} - \underline{F}^{\alpha}(\tau)\frac{\sinh_r(\sigma(\tau),t_k)}{E_r(\sigma(\tau),t_k)}\right]\Delta\tau\\ &= \mu(t)\overline{F}^{\alpha}(t)\frac{\cosh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)} - \mu(t)\underline{F}^{\alpha}(t)\frac{\sinh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)},\\ \eta_k(\sigma(t)) - \eta_k(t) &= -\mu(t)\overline{F}^{\alpha}(t)\frac{\sinh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)} + \mu(t)\underline{F}^{\alpha}(t)\frac{\cosh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)}.\end{aligned} \tag{79}$$

From Proposition 1-(II) and the property (p2) it follows

$$\begin{aligned}\sinh_r(\sigma(t),t_k) - \sinh_r(t+h,t_k) &= r(t)\cosh_r(t,t_k)\mu(t),\\ \cosh_r(\sigma(t),t_k) - \cosh_r(t+h,t_k) &= r(t)\sinh_r(t,t_k)\mu(t).\end{aligned} \tag{80}$$

Due to the above results, for each $\alpha \in [0,1]$, we uniformly have

$$\begin{aligned}&\underline{U_k}^{\alpha}(\sigma(t)) - \underline{U_k}^{\alpha}(t) - \left[r(t)\overline{U_k}^{\alpha}(t) + \overline{F}^{\alpha}(t)\right]\mu(t)\\ &= \cosh_r(\sigma(t),t_k^+)\left[L_k\underline{U_k}^{\alpha}(t_k) + \xi_k(\sigma(t))\right] + \sinh_r(\sigma(t),t_k^+)\left[L_k\overline{U_k}^{\alpha}(t_k) + \eta_k(\sigma(t))\right]\\ &\quad - \cosh_r(t,t_k^+)\left[L_k\underline{U_k}^{\alpha}(t_k) + \xi_k(t)\right] - \sinh_r(t,t_k^+)\left[L_k\overline{U_k}^{\alpha}(t_k) + \eta_k(t)\right]\\ &\quad - r(t)\mu(t)\left\{\sinh_r(t,t_k^+)\left[L_k\underline{U_k}^{\alpha}(t_k) + \xi_k(t)\right] + \cosh_r(t,t_k^+)\left[L_k\overline{U_k}^{\alpha}(t_k) + \eta_k(t)\right]\right\} - \overline{F}^{\alpha}(t)\mu(t)\\ &= \left[\cosh_r(\sigma(t),t_k) - \cosh_r(t,t_k) - r(t)\mu(t)\sinh_r(t,t_k)\right]\left[L_k\underline{U_k}^{\alpha}(t_k) + \xi_k(\sigma(t))\right]\\ &\quad + \left[\sinh_r(\sigma(t),t_k) - \sinh_r(t,t_k) - r(t)\mu(t)\cosh_r(t,t_k)\right]\left[L_k\overline{U_k}^{\alpha}(t_k) + \eta_k(\sigma(t))\right]\\ &\quad + \cosh_r(t,t_k)\left(\xi_k(\sigma(t)) - \xi_k(t)\right) + r(t)\mu(t)\sinh_r(t,t_k)\left[\xi_k(\sigma(t)) - \xi_k(t)\right]\\ &\quad + \sinh_r(t,t_k)\left(\eta_k(\sigma(t)) - \eta_k(t)\right) + r(t)\mu(t)\cosh_r(t,t_k)\left[\eta_k(\sigma(t)) - \eta_k(t)\right] - \overline{F}^{\alpha}(t)\mu(t)\end{aligned}$$

$$\begin{aligned}
&= \mu(t)\left[\cosh_r(t,t_k) + r(t)\sinh_r(t,t_k)\right]\left[\overline{F}^{\alpha}(t)\frac{\cosh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)} - \underline{F}^{\alpha}(t)\frac{\sinh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)}\right] \\
&\quad + \mu(t)\left[\sinh_r(t,t_k) + r(t)\cosh_r(t,t_k)\right]\left[-\overline{F}^{\alpha}(t)\frac{\sinh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)} + \underline{F}^{\alpha}(t)\frac{\cosh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)}\right] - \overline{F}^{\alpha}(t)\mu(t) \\
&= \mu(t)\cosh_r(\sigma(t),t_k)\left[\overline{F}^{\alpha}(t)\frac{\cosh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)} - \underline{F}^{\alpha}(t)\frac{\sinh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)}\right] \\
&\quad + \mu(t)\sinh_r(\sigma(t),t_k)\left[-\overline{F}^{\alpha}(t)\frac{\sinh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)} + \underline{F}^{\alpha}(t)\frac{\cosh_r(\sigma(t),t_k)}{E_r(\sigma(t),t_k)}\right] - \overline{F}^{\alpha}(t)\mu(t) = 0,
\end{aligned} \tag{81}$$

that is, $\underline{U_k^{\alpha}}(\sigma(t)) - \underline{U_k^{\alpha}}(t) = [r(t)\overline{U_k}^{\alpha}(t) + \overline{F}^{\alpha}(t)]\mu(t)$. Analogously, we can prove

$$\overline{U_k}^{\alpha}(\sigma(t)) - \overline{U_k}^{\alpha}(t) = \left[r(t)\underline{U_k^{\alpha}}(t) + \underline{F}^{\alpha}(t)\right]\mu(t), \tag{82}$$

for all $\alpha \in [0,1]$. Hence, Proposition 1-(II) guarantees that $\Delta_H U_k(t) = r(t)U_k(t) + F(t)$for all right-scattered point $t \in J_k$. In consequence, U_k is a (ii)-solution of LIFDE (39) on J_k for $k = 0,1,2\ldots$. Similarly, for $t \in J_-$, we have

$$\begin{aligned}
U(t) = V(t) &= \cosh_r(t)\left\{U_0 -_H \int_0^t \left[F(\tau)\frac{\sinh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)} -_H F(\tau)\frac{\cosh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)}\right]\Delta\tau\right\} -_H [-\sinh_r(t)] \\
&\quad \left\{U_0 -_H \int_0^t \left[F(\tau)\frac{\sinh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)} -_H F(\tau)\frac{\cosh_r(\sigma(\tau),0)}{E_r(\sigma(\tau),0)}\right]\Delta\tau\right\}.
\end{aligned} \tag{83}$$

Now, (71) holds, and (i) is proved.

(ii) For $t \in J_k$, under the hypothesis of (i)-differentiability, LIFDE (39) is translated into the corresponding system (42) with $A \in \mathscr{R}$ given in (52) and f given in (51). Load $e_A(t,s) = e_r(t,s)I$ in (43) and repeat the process of the proof of Theorem 1, we have, for all $\alpha \in [0,1]$,

$$\begin{aligned}
\underline{U_k}^{\alpha}(t) &= e_r(t,t_k^+)\left[L_k\underline{U_k}^{\alpha}(t_k) - \int_{t_k}^t -\overline{F}^{\alpha}(\tau)e_{\ominus r}(\sigma(\tau),t_k)\Delta\tau\right], \\
\overline{U_k}^{\alpha}(t) &= e_r(t,t_k^+)\left[L_k\overline{U_k}^{\alpha}(t_k) - \int_{t_k}^t -\underline{F}^{\alpha}(\tau)e_{\ominus r}(\sigma(\tau),t_k)\Delta\tau\right].
\end{aligned} \tag{84}$$

We now check that the (i)-solution of LIFDE (39) on J_k for $k = 0,1,2,\ldots$ and $r(t) > 0 (t \in \mathbb{T}_+)$ is

$$U_k(t) = e_r(t,t_k^+)\left[L_kU_k(t_k) + \int_{t_k}^t F(\tau)e_{\ominus r}(\sigma(\tau),t_k)\Delta\tau\right]. \tag{85}$$

Note that $e_r(\sigma(t),t_k)e_r^{\Delta}(t,t_k) = r(t)e_r(\sigma(t),t_k)e_r(t,t_k) > 0$ for $r(t) > 0$. From Lemma 3-(v), it follows that $L_kU_k(t_k) + \int_{t_k}^t F(\tau)e_{\ominus r}(\sigma(\tau),t_k)\Delta\tau$ is (i)-differentiable on $\mathbb{T}_+$, and from Lemma 2-(a) it follows that

$$\begin{aligned}
\Delta_H U_k(t) &= r(t)e_r(t,t_k^+)\left[L_kU_k(t_k) + \int_{t_k}^t F(\tau)e_{\ominus r}(\sigma(\tau),t_k)\Delta\tau\right] \\
&\quad + e_r(\sigma(t),t_k^+)F(t)e_{\ominus r}(\sigma(t),t_k) = r(t)U_k(t) + F(t),
\end{aligned} \tag{86}$$

for $k = 0,1,2,\ldots$. Similarly, we have for $t \in J_-$

$$U(t) = V(t) = e_r(t)\left[U_0 + \int_0^t F(\tau)e_{\ominus r}(\sigma(\tau))\Delta\tau\right]. \tag{87}$$

This shows that (71) holds, and LIFDE (39) has a (i)-solution on $\mathbb{T}_+$. This proof is complete. □

Theorem 3. *If $r \equiv 0$, then LIFDE (39) has a (i)-solution*

$$U(t) = \begin{cases} V(t) = U_0 + \int_0^t F(\tau)\Delta\tau, & t \in J_-, \\ U_k(t) = L_kU_k(t_k) + \int_{t_k}^t F(\tau)\Delta\tau, & t \in J_k (k = 0,1,2,\ldots), \end{cases} \tag{88}$$

and a (ii)-solution

$$U(t) = \begin{cases} V(t) = U_0 -_H \int_0^t (-F(\tau))\Delta\tau, & t \in J_-, \\ U_k(t) = L_kU_k(t_k) -_H \int_{t_k}^t (-F(\tau))\Delta\tau, & t \in J_k (k = 0,1,2,\ldots), \end{cases} \tag{89}$$

provided the above H-differences exist and $U_0(t_0) = V(t_0), U_k(t_k) = U_{k-1}(t_k)$ *for* $k = 0, 1, 2, \ldots$.

Remark 3. If $\mathbb{T} = \mathbb{R}$, then $E_r(t,s) \equiv 1$, $\sinh_r(t,s) = \sinh\left\{\int_s^t r(\tau)\mathrm{d}\tau\right\}$, and $\cosh_r(t,s) = \cosh\left\{\int_s^t r(\tau)\mathrm{d}\tau\right\}$. Thus, our results are the extension and improvement of the corresponding results in [3]. In particular, the present results are identical to those in [3] in case that we consider t restricted to J_- and under H-difference.

If $\mathbb{T} = \mathbb{Z}$, then $E_r(t,s) = \prod_{\tau=s}^{t-1}(1 - r^2(\tau))$, $\sinh_r(t,s) = 1/2\left(\prod_{\tau=s}^{t-1}(1 + r(\tau)) - \prod_{\tau=s}^{t-1}(1 - r(\tau))\right)$ and $\cosh_r(t,s) = 1/2\left(\prod_{\tau=s}^{t-1}(1 + r(\tau)) + \prod_{\tau=s}^{t-1}(1 - r(\tau))\right)$ provided that r is never ± 1 and $s < t$. Thus, we obtain the general form of solutions to linear impulsive fuzzy difference equation:

$$\begin{cases} U(n+1) = r(n)U(n) + U(n) + F(n), & n \in \mathbb{Z}_+ \backslash \mathbb{J}, \\ U_{t_k^+} = L_k U(t_k), & t_k \in \mathbb{J}(k = 0, 1, 2, \ldots), \\ U(0) = U_0 \in \mathbb{T}_f, & t_0 \in \mathbb{T}_+. \end{cases} \tag{90}$$

In addition, we also extend these "classical cases" to cases "in between", for instance, $\mathbb{T} = \left\{\sum_{k=1}^n (1/k) \colon n \in \mathbb{N}\right\}$. In this case,

$$\begin{aligned} E_r(t,s) &= \binom{n-s+r}{n-s} \cdot \binom{n-s-r}{n-s}, \\ \sinh_r(t,s) &= \frac{1}{2}\left[\binom{n-s+r}{n-s} - \binom{n-s-r}{n-s}\right], \\ \cosh_r(t,s) &= \frac{1}{2}\left[\binom{n-s+r}{n-s} + \binom{n-s-r}{n-s}\right], \end{aligned} \tag{91}$$

where $t = \sum_{k=1}^n (1/k)$. We can also consider the so-called q-difference problems.

Remark 4. Although problem (44) has a unique solution on $\mathbb{T}_+$, the solution of LIFDE (39) is not unique in general.

4. Examples

In this section, we present several examples to further illustrate the applicability of the results involved in the above sections.

Example 1. Consider the impulsive fuzzy dynamic equation:

$$\begin{cases} \Delta_H U(t) = aU(t) + t, & t \in \mathbb{T}_+ \backslash \mathbb{J}, \\ U(t_k^+) = L_k U(t_k), & k = 0, 1, 2, \ldots, \\ U(t_0) = U_0, \end{cases} \tag{92}$$

where $a < 0$, $a \in \mathscr{R}_+^1$, and L_k is linear bounded. It is not difficult to infer the existence of the H-differences $U_0 -_H \int_0^t \tau e_{\ominus a}(\sigma(\tau))\Delta\tau$ and $L_k U(t_k) -_H \int_{t_k}^t \tau e_{\ominus a}(\sigma(\tau), t_k)\Delta\tau$. Thus, by Theorem 1, (ii)-solution of LIFDE (92) on $\mathbb{T}_+$ is given by

$$U(t) = \begin{cases} V(t) = e_a(t)\left[U_0 -_H \int_0^t \tau e_{\ominus a}(\sigma(\tau))\Delta\tau\right], & t \in J_-, \\ U_k(t) = e_a(t, t_k^+)\left[L_k U_k(t_k) -_H \int_{t_k}^t \tau e_{\ominus a}(\sigma(\tau), t_k)\Delta\tau\right], & t \in J_k, \end{cases} \tag{93}$$

where $U_0(t_0) = V(t_0), U_k(t_k) = U_{k-1}(t_k)$ for $k = 0, 1, 2, \ldots$.

Remark 5. If $\mathbb{T} = \mathbb{R}$, then we get

$$\begin{aligned} V(t) &= e^{at}\left[U_0 -_H \int_0^t -\tau e^{-a\tau}\mathrm{d}\tau\right], & t \in J_-, \\ U_k(t) &= e^{a(t-t_k)}\left[L_k U_k(t_k) -_H \int_{t_k}^t \tau e^{-a(\tau - t_k)}\mathrm{d}\tau\right], & t \in J_k(k = 0, 1, 2 \ldots). \end{aligned} \tag{94}$$

We obtain by calculating

$$U_k(t) = \frac{1}{a}\left(t + \frac{1}{a}\right)\widetilde{1} + \left[L_k U_k(t_k) + \left(\frac{t}{a} + \frac{1}{a^2}\right)\right] e^{a(t-t_k)}, \tag{95}$$

with $\widetilde{1} = \chi_{\{1\}}, t \in J_k$. We observe that in this case $D[U(t), (1/a)(t + (1/a))\widetilde{1}] \le \|L_k U_k(t_k) + (t/a) + (t/a^2)\| e^{at}$, and this implies that $\lim_{t\to\infty} D[U(t), (1/a)(t + (1/a))\widetilde{1}] = 0$. An interpretation in the light of [1] is that the uncertainty asymptotically disappears on the fuzzy system.

If $\mathbb{T} = \mathbb{Z}$, then

$$\begin{aligned} V(t) &= (1+a)^t \left[U_0 -_H \sum_{\tau=0}^{t-1} (-\tau)(1+a)^{-(\tau+1)}\right], && t \in J_-, \\ U_k(t) &= (1+a)^{(t-t_k)} \left[L_k U_k(t_k) -_H \sum_{\tau=t_k}^{t-1} (-\tau)(1+a)^{-(\tau+1-t_k)}\right], && t \in J_k (k = 0, 1, 2, \ldots). \end{aligned} \tag{96}$$

For $t \in J_k$, we have

$$\begin{aligned} U(t) = U_k(t) &= L_k U_k(t_k)(1+a)^{t-t_k} -_H (1+a)^t \sum_{\tau=t_k}^{t-1} (-\tau)(1+a)^{-(\tau+1)} \\ &= \frac{(1+a)^{1-t_k}}{a}\widetilde{1} + \left[L_k U(t_k) - \frac{1+a}{a}\right](1+a)^{t-t_k}. \end{aligned} \tag{97}$$

Therefore, in the discrete case, the phenomenon that the uncertainty asymptotically disappears on the fuzzy system arises only $a > -1$.

Example 2. Let us consider the LIFDE

$$\begin{cases} \Delta_H U(t) = 2tU(t) + t\gamma, & t \in \mathbb{T}_+ \backslash \mathbb{J}, \\ U_{t_k^+} = 4U(t_k), & k = 0, 1, 2, \ldots, \\ U(0) = \gamma, \end{cases} \tag{98}$$

where $[\gamma]^\alpha = [\alpha - 1, 1 - \alpha]$ with $\alpha \in [0, 1]$.

5. Conclusion

Let $\mathbb{T} = \mathbb{R}$ and $t_k = \sqrt{(k+1)\ln 2}$ for $k = 0, 1, 2, \ldots$. As a result of Theorem 2, the (i)-solution of (98) is

$$U(t) = \begin{cases} V(t) = \frac{1}{2}\left(3e^{t^2} - 1\right)\gamma, & t \in J_- = [0, \sqrt{\ln 2}\,] \cap \mathbb{T}_+, \\ U_k(t) = 4U_k(\sqrt{(k+1)\ln 2}\,)e^{t^2-(k+1)\ln 2} + \frac{1}{2}\left(e^{t^2-(k+1)\ln 2} - 1\right)\gamma, & t \in J_k (k = 0, 1, 2, \ldots). \end{cases} \tag{99}$$

To seek the (ii)-solution, we note that $\cosh_r(t, t_k) = \cosh(t^2 - t_k^2)$ and $\sinh_r(t, t_k) = \sinh(t^2 - t_k^2)$ when $r(t) = 2t$. The solution of the ODEs system corresponding to (43) is

$$\begin{aligned} \underline{U_k}^\alpha(t) &= 4\underline{U_k}^\alpha(t_k)\cosh\left(t^2 - t_k^2\right) + 4\overline{U_k}^\alpha(t_k)\sinh\left(t^2 - t_k^2\right) \\ &\quad - \frac{\alpha - 1}{2}\left[\cosh\left(t^2 - t_k^2\right) + \sinh\left(t^2 - t_k^2\right)\right) - 1\right]\left(\cosh\left(t^2 - t_k^2\right) - \sinh\left(t^2 - t_k^2\right)\right), \\ \overline{U_k}^\alpha(t) &= 4\underline{U_k}^\alpha(t_k)\sinh\left(t^2 - t_k^2\right) + 4\overline{U_k}^\alpha(t_k)\cosh\left(t^2 - t_k^2\right) \\ &\quad - \frac{1 - \alpha}{2}\left[\cosh\left(t^2 - t_k^2\right) + \sinh\left(t^2 - t_k^2\right)\right) - 1\right]\left(\cosh\left(t^2 - t_k^2\right) - \sinh\left(t^2 - t_k^2\right)\right). \end{aligned} \tag{100}$$

It is easy to see that the H-difference

$$\gamma -_H \int_0^t \left[\tau\gamma \sinh\left(\tau^2\right)\tau\gamma\cosh\left(\tau^2\right)\right] d\tau, \tag{101}$$

exists on $[0, \sqrt{\ln 2}\,]$, and the (ii)-solution $V(t)$ of (98) on $[0, \sqrt{\ln 2}\,]$ can be written as

$$V(t) = \frac{1}{2}\left(3e^{-t^2} - 1\right)\gamma. \tag{102}$$

In particular, $V(\sqrt{\ln 2}) = (1/4)\gamma$. From $L_0 U_0(t_0) = 4 \times (1/4)\gamma = \gamma$, it follows that the H-difference

$$L_0 U_0(t_0) -_H \int_{t_0}^{t} \left[\tau\gamma \sinh\left(\tau^2 - t_0^2\right)\tau\gamma \cosh\left(\tau^2 - t_0^2\right)\right] d\tau,$$
$$= \gamma -_H \int_{\sqrt{\ln 2}}^{t} \left[\tau\gamma \sinh\left(\tau^2 - \ln 2\right)\tau\gamma \cosh\left(\tau^2 - \ln 2\right)\right] d\tau, \tag{103}$$

exists on $(\sqrt{\ln 2}, \sqrt{2\ln 2}]$, and the (ii)-solution $U_0(t)$ of (98) on $(\sqrt{\ln 2}, \sqrt{2\ln 2}]$ can be written as

$$\begin{aligned} U_0(t) &= \cosh\left(t^2 - \ln 2\right)\left[\gamma -_H \int_{\sqrt{\ln 2}}^{t} \left[\tau\gamma \sinh\left(\tau^2 - \ln 2\right) -_H \tau\gamma \cosh\left(\tau^2 - \ln 2\right)\right] d\tau\right] \\ &\quad -_H \left(-\sinh\left(t^2 - \ln 2\right)\left[\gamma -_H \int_{\sqrt{\ln 2}}^{t} \left[\tau\gamma \sinh\left(\tau^2 - \ln 2\right) - \tau\gamma \cosh\left(\tau^2 - \ln 2\right)\right] d\tau\right]\right. \\ &= \left(1 - \frac{1}{2}\left(\sinh\left(t^2 - \ln 2\right) + \cosh\left(t^2 - \ln 2\right) - 1\right)\right)\left(\cosh\left(t^2 - \ln 2\right) - \sinh\left(t^2 - \ln 2\right)\right)\gamma \\ &= \frac{1}{2}\left(3e^{-\left(t^2 - \ln 2\right)} - 1\right). \end{aligned} \tag{104}$$

In particular, $U_0(\sqrt{2\ln 2}) = (1/4)\gamma$. On the analogy of this process, we obtain that the (ii)-solution of (98) on $J_k = (\sqrt{(k+1)\ln 2}, \sqrt{(k+2)\ln 2}]$ can be expressed by

$$U_k(t) = \frac{1}{2}\left(3e^{-\left(t^2 - (k+1)\ln 2\right)} - 1\right), \quad \text{for } t \in J_k,$$
$$U_k(\sqrt{(k+2)\ln 2}) = \frac{1}{4}\gamma, \quad k = 1, 2, \ldots \tag{105}$$

Let $\mathbb{T} = \mathbb{Z}$ and $t_k = (2k+1)\,(k = 0, 1, 2, \ldots)$. $r(t) = 2t$ implies that $e_r(t, s) = \prod_{\tau=s}^{t-1}(1 + 2\tau)$. Therefore, the (i)-solution of (98) is

$$U(t) = \begin{cases} V(t) = \gamma, & t \in J_-, \\ U_k(t) = \prod_{\tau=2k+1}^{t-1}(1 + 2\tau) \cdot \left[4U_{k-1}(t_k) + \sum_{\tau=2k+1}^{t-1} \dfrac{\tau}{\prod_{s=2k+1}^{\tau}(1 + 2s)}\gamma\right], & t \in J_k\,(k = 0, 1, 2, \ldots). \end{cases} \tag{106}$$

Here, $J_- = [0, 1] \cap \mathbb{Z}_+$, $J_k = (2k+1, 2k+3] \cap \mathbb{Z}_+$, $U_{0-1}(t_0) = \gamma$, and $U_0(t_1) = 3 \cdot 5(4\gamma) + (1/5)\gamma + 2\gamma, \ldots$

Similarly, we can present the expression of the (ii)-solutions of (98) in the discrete case.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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