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Research Article

Analysis of a Deterministic and a Stochastic SIS Epidemic Model with Double Epidemic Hypothesis and Specific Functional Response

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The purpose of this paper is to investigate the stability of a deterministic and stochastic SIS epidemic model with double epidemic hypothesis and specific nonlinear incidence rate. We prove the local asymptotic stability of the equilibria of the deterministic model. Moreover, by constructing a suitable Lyapunov function, we obtain a sufficient condition for the global stability of the disease-free equilibrium. For the stochastic model, we establish global existence and positivity of the solution. Thereafter, stochastic stability of the disease-free equilibrium in almost sure exponential and pth moment exponential is investigated. Finally, numerical examples are presented.

1. Introduction

Epidemiology is the study of the spread of infectious diseases with the objective to trace factors that are responsible for or contribute to their occurrence. Mathematical modeling has become an important tool in analyzing the epidemiological characteristics of infectious diseases and can provide useful control measures (see, for example, [1–5]).

In classical epidemic models, the susceptible individuals can be infected with only a disease. In the real world, the susceptible individuals can be infected by two or more kinds of diseases at the same time such as HBV coinfection with HCV and HDV and HIV coinfection with HBV, HCV, and TB. Recently, the authors of [6–9] investigated the epidemic model SIS (where infection with the disease does not confer permanent immunity against reinfection so that those who survived the infection revert to the class of wholly susceptible individuals [10]) with double epidemic hypothesis which has two epidemic diseases caused by two different viruses. In this paper, we consider a deterministic SIS model with double epidemic hypothesis described by the following differential system:

$$\begin{cases} \dot{S}(t) = A - \mu S(t) - \dfrac{\beta_1 S(t) I_1(t)}{1 + \alpha_1 S(t) + \gamma_1 I_1(t) + \mu_1 S(t) I_1(t)} - \dfrac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 S(t) + \gamma_2 I_2(t) + \mu_2 S(t) I_2(t)} + r_1 I_1(t) + r_2 I_2(t), \\ \dot{I}_1(t) = \dfrac{\beta_1 S(t) I_1(t)}{1 + \alpha_1 S(t) + \gamma_1 I_1(t) + \mu_1 S(t) I_1(t)} - (\mu + a_1 + r_1) I_1(t), \\ \dot{I}_2(t) = \dfrac{\beta_2 S(t) I_2(t)}{1 + \alpha_2 S(t) + \gamma_2 I_2(t) + \mu_2 S(t) I_2(t)} - (\mu + a_2 + r_2) I_2(t), \end{cases} \tag{1}$$

where $S(t)$ represents the number of susceptible at time t, $I_1(t)$ and $I_2(t)$ are the total population of the infected with virus V_1 and V_2 at time t, respectively, A represents the recruitment rate of the population, μ is the natural death rate of the population, r_i is the treatment cure rate of the disease caused by virus V_i, a_i is the disease-related death rate, and β_i is the infection coefficient, $i = 1, 2$. The incidence rate of disease is modeled by the specific functional response $\beta_i S I_i / (1 + \alpha_i S + \gamma_i I_i + \mu_i S I_i)$, where $\alpha_i, \gamma_i, \mu_i$ are saturation factors measuring the psychological or inhibitory effect. This specific functional response was introduced by Hattaf et al. [11], and here, it becomes to be a bilinear incidence rate if $\alpha_i = \gamma_i = \mu_i = 0$, a saturated incidence rate if $\alpha_i = \mu_i = 0$ or $\gamma_i = \mu_i = 0$, a Beddington–DeAngelis functional response [12, 13] if $\mu_i = 0$, and a Crowley–Martin functional response [14] if $\alpha_i \gamma_i = \mu_i$, $i = 1, 2$.

In the reality, epidemic systems are inevitably affected by environmental white noise. Therefore, it is necessary to study how the noise influences the epidemic models. Consequently, many authors have studied stochastic epidemic models, see, e.g., [15–17]. For this, we consider the case in which the rates β_i $(i = 1, 2)$ are subject to random fluctuations, namely, $\beta_i dt$ is replaced by $\beta_i dt + \sigma_i dB_i(t)$, where $B_i(t)$ $(i = 1, 2)$ are independent standard Brownian motions, and $\sigma_i > 0$ represents the intensity of $B_i(t)$ for $i = 1, 2$. Therefore, the corresponding stochastic system to (1) can be described by the following Itô equations:

$$\begin{cases} dS(t) = \left[A - \mu S(t) - \dfrac{\beta_1 S(t) I_1(t)}{f_1(S(t), I_1(t))} - \dfrac{\beta_2 S(t) I_2(t)}{f_2(S(t), I_2(t))} + r_1 I_1(t) + r_2 I_2(t)\right] dt - \dfrac{\sigma_1 S(t) I_1(t)}{f_1(S(t), I_1(t))} dB_1(t) - \dfrac{\sigma_2 S(t) I_2(t)}{f_2(S(t), I_2(t))} dB_2(t), \\ dI_1(t) = \left[\dfrac{\beta_1 S(t) I_1(t)}{f_1(S(t), I_1(t))} - (\mu + a_1 + r_1) I_1(t)\right] dt + \dfrac{\sigma_1 S(t) I_1(t)}{f_1(S(t), I_1(t))} dB_1(t), \\ dI_2(t) = \left[\dfrac{\beta_2 S(t) I_2(t)}{f_2(S(t), I_2(t))} - (\mu + a_2 + r_2) I_2(t)\right] dt + \dfrac{\sigma_2 S(t) I_2(t)}{f_2(S(t), I_2(t))} dB_2(t), \end{cases} \tag{2}$$

with $f_i(S, I_i) = 1 + \alpha_i S + \gamma_i I_i + \mu_i S I_i$, $i = 1, 2$.

The rest of this paper is organized in the following manner. In Section 2, we present a local stability analysis of the equilibria and a global stability analysis of the disease-free equilibrium for the deterministic model (1). In Section 3, we prove that the stochastic model (2) has a unique global positive solution, and we give sufficient conditions for the almost sure exponential stability and the pth moment exponential stability of the disease-free equilibrium. Numerical examples will be presented in Section 4. Finally, we close the paper with a brief conclusion.

2. Deterministic SIS Epidemic Model

For biological reasons, we assume that the initial conditions of system (1) satisfy

$$\begin{aligned} S(0) &\geq 0, \\ I_1(0) &\geq 0, \\ I_2(0) &\geq 0. \end{aligned} \tag{3}$$

Thus, system (1) is positive [18], that is, $S(t) \geq 0, I_1(t) \geq 0$, and $I_2(t) \geq 0$ for all $t \geq 0$. In fact, by Proposition 2.1 in [19], we have

$$\begin{cases} \dot{S} = A + r_1 I_1 + r_2 I_2 \geq 0, & \text{for } S = 0 \text{ and } I_1, I_2 \geq 0, \\ \dot{I}_1 = 0 \geq 0, & \text{for } I_1 = 0 \text{ and } S, I_2 \geq 0, \\ \dot{I}_2 = 0 \geq 0, & \text{for } I_2 = 0 \text{ and } S, I_1 \geq 0. \end{cases} \tag{4}$$

By summing all the equations of system (1), we find that the total population size $N(t) = S(t) + I_1(t) + I_2(t)$ satisfies the inequality

$$\dot{N}(t) = A - \mu N(t) - a_1 I_1(t) - a_2 I_2(t) \leq A - \mu N(t), \tag{5}$$

which ensures that $\dot{N}(t) < 0$ if $N(t) > A/\mu$. The standard comparison theorem [20] can be used to deduce that

$$N(t) \le \frac{A}{\mu} - \left(\frac{A}{\mu} - N(0)\right)e^{-\mu t}. \tag{6}$$

Thus, the feasible solution set of the system equation of model (1) enters and remains in the region

$$\Gamma = \left\{(S, I_1, I_2) \in \mathbb{R}_+^3 : S + I_1 + I_2 \le \frac{A}{\mu}\right\}. \tag{7}$$

Therefore, model (1) is well posed epidemiologically and mathematically [21]. Hence, it is sufficient to study the dynamics of model (1) in Γ.

It is easy to see that system (1) has a disease-free equilibrium state $E_0 = (A/\mu, 0, 0)$. Therefore, the basic reproduction number is

$$R_0 = \max\{R_{01}, R_{02}\}, \tag{8}$$

where

$$\begin{aligned} R_{01} &= \frac{\beta_1 A}{(\mu + \alpha_1 A)(\mu + a_1 + r_1)}, \\ R_{02} &= \frac{\beta_2 A}{(\mu + \alpha_2 A)(\mu + a_2 + r_2)}. \end{aligned} \tag{9}$$

We mention that the expressions of R_{01} and R_{02} can also be obtained by applying the next generation matrix method provided by van den Driessche and Watmough [22].

Now, we investigate the local stability of the disease-free equilibrium E_0. The Jacobian matrix of system (1) at the equilibrium E_0 is as follows:

$$J_{E_0} = \begin{pmatrix} -\mu & \dfrac{-\beta_1 A}{\mu + \alpha_1 A} + r_1 & \dfrac{-\beta_2 A}{\mu + \alpha_2 A} + r_2 \\ 0 & \dfrac{\beta_1 A}{\mu + \alpha_1 A} - (\mu + a_1 + r_1) & 0 \\ 0 & 0 & \dfrac{\beta_2 A}{\mu + \alpha_2 A} - (\mu + a_2 + r_2) \end{pmatrix}. \tag{10}$$

The three eigenvalues of J_{E_0} are $\lambda_1 = -\mu < 0$, $\lambda_2 = (\mu + a_1 + r_1)(R_{01} - 1)$, and $\lambda_3 = (\mu + a_2 + r_2)(R_{02} - 1)$. Hence, the equilibrium E_0 will be locally asymptotically stable if $R_0 < 1$ and unstable when $R_0 > 1$.

The following theorem discusses the global stability of the disease-free equilibrium E_0.

Theorem 1. *If $R_0 \le 1$, then the disease-free equilibrium E_0 of (1) is globally asymptotically stable in Γ.*

Proof. Let U be the Lyapunov function defined as

$$U(t) = I_1(t) + I_2(t). \tag{11}$$

Differentiating U with respect to t along the positive solutions of system (1), we get

$$\dot{U}(t) = \left[\frac{\beta_1 S}{1 + \alpha_1 S + \gamma_1 I_1 + \mu_1 S I_1} - (\mu + a_1 + r_1)\right] I_1 + \left[\frac{\beta_2 S}{1 + \alpha_2 S + \gamma_2 I_2 + \mu_2 S I_2} - (\mu + a_2 + r_2)\right] I_2. \tag{12}$$

We have

$$\frac{\beta_i S}{1 + \alpha_i S + \gamma_i I_i + \mu_i S I_i} \le \frac{\beta_i S}{1 + \alpha_i S}, \quad i = 1, 2. \tag{13}$$

Since $S \le A/\mu$ and the functions $f_i : x \in \mathbb{R}_+ \mapsto (\beta_i x/(1 + \alpha_i x))$ are increasing, then $f_i(S) \le f_i(A/\mu) = \beta_i A/(\mu + \alpha_i A)$. Thus,

$$\dot{U}(t) \le (\mu + a_1 + r_1)(R_{01} - 1) I_1 + (\mu + a_2 + r_2)(R_{02} - 1) I_2. \tag{14}$$

Therefore, $R_0 \le 1$ ensures that $\dot{U}(t) \le 0$. Suppose that (S, I_1, I_2) is a solution of (1) contained entirely in the set $\Delta = \{(S, I_1, I_2) \in \Gamma : \dot{U}(t) = 0\}$. Then, $\dot{I}_1 + \dot{I}_2 = 0$. We discuss four cases:

Case 1. If $R_{01} < 1$ and $R_{02} < 1$, then

$$X_i = \frac{\beta_i S}{1 + \alpha_i S + \gamma_i I_i + \mu_i S I_i} - \mu + a_i + r_i \le \frac{\beta_i A}{\mu + \alpha_i A} - \mu + a_i + r_i = (\mu + a_i + r_i)(R_{0i} - 1) < 0, \quad i = 1, 2. \tag{15}$$

From the second and third equations of (1), we have $X_1 I_1 + X_2 I_2 = 0$, which implies, according to (15), that $I_1 = I_2 = 0$. On the other hand, solutions of (1) contained in the plane $I_1 = I_2 = 0$ satisfy $\dot{S} = A - \mu S$, which implies that $S \longrightarrow A/\mu$ as $t \longrightarrow \infty$.

Case 2. If $R_{01} < 1$ and $R_{02} = 1$, then $X_1 < 0$ and

$$X_2 = \frac{\beta_2 S}{1 + \alpha_2 S + \gamma_2 I_2 + \mu_2 S I_2} - \frac{\beta_2 A}{\mu + \alpha_2 A} = \frac{\beta_2 (S\mu - A) - \beta_2 A I_2 (\gamma_2 + \mu_2 S)}{(1 + \alpha_2 S + \gamma_2 I_2 + \mu_2 S I_2)(\mu + \alpha_2 A)} \leq 0. \quad (16)$$

Then, $X_1 I_1 + X_2 I_2 = 0$ implies that $I_1 = 0$ and consequently $X_2 I_2 = 0$. Suppose that $I_2 > 0$; then, $X_2 = 0$. Hence, $S\mu - A = AI_2(\gamma_2 + \mu_2 S) > 0$; then, $S > A/\mu$ which is a contradiction. Then, $I_1 = I_2 = 0$.

Case 3. The case $R_{01} = 1$ and $R_{02} < 1$ is analogue to the previous case.

Case 4. If $R_{01} = 1$ and $R_{02} = 1$, then $X_1 I_1 + X_2 I_2 = 0$ such that $X_i = (\beta_i(S\mu - A) - \beta_i A I_i(\gamma_i + \mu_i S))/((1 + \alpha_i S + \gamma_i I_i + \mu_i S I_i)(\mu + \alpha_i A)) \leq 0$, $i = 1, 2$. Hence, $X_1 I_1 = X_2 I_2 = 0$, and by the same analysis in Case 2, we obtain that $I_1 = I_2 = 0$.

Hence, by LaSalle's invariance principle [23], every solution to equations of system (1), with initial conditions in Γ, approaches E_0 as $t \longrightarrow \infty$. Thus, E_0 is globally asymptotically stable.

Now, if $R_{01} > 1$, then system (1) has the disease-free equilibrium for I_2, $E_1^* = (S_1^*, \overline{I}_1^*, 0)$, where

$$\begin{aligned} S_1^* &= \frac{A - (\mu + a_1)\overline{I}_1^*}{\mu}, \\ \overline{I}_1^* &= \frac{2\varpi_1(\mu + \alpha_1 A)(R_{01} - 1)}{(\mu + a_1)(\beta_1 - \alpha_1\varpi_1) + \varpi_1(\gamma_1\mu + \mu_1 A) + \sqrt{\Delta_1}}, \end{aligned} \quad (17)$$

with $\varpi_1 = \mu + a_1 + r_1$ and

$$\begin{aligned} \Delta_1 &= \left[(\mu + a_1)(\beta_1 - \alpha_1\varpi_1) + \varpi_1(\gamma_1\mu + \mu_1 A)\right]^2 - 4\mu_1(\mu + a_1)\varpi_1\left[\beta_1 A - (\mu + \alpha_1 A)\varpi_1\right] \\ &= \left[(\mu + a_1)(\beta_1 - \alpha_1\varpi_1) + \varpi_1(\gamma_1\mu - \mu_1 A)\right]^2 + 4\mu_1\mu\varpi_1^2(\mu + a_1 + \gamma_1 A). \end{aligned} \quad (18)$$

Theorem 2. *If $R_{01} > 1$ and $R_{02} < 1$, then the equilibrium E_1^* is locally asymptotically stable.*

Proof. The Jacobian matrix of system (1) at the equilibrium E_1^* is determined by

$$J_{E_1^*} = \begin{pmatrix} -m_1 & -m_2 + r_1 & -m_3 + r_2 \\ m_4 & m_2 - m_5 & 0 \\ 0 & 0 & m_3 - m_6 \end{pmatrix}, \quad (19)$$

where

$$\begin{aligned} m_1 &= \mu + \frac{\beta_1\overline{I}_1^*(1 + \gamma_1\overline{I}_1^*)}{\left(1 + \alpha_1 S_1^* + \gamma_1\overline{I}_1^* + \mu_1 S_1^*\overline{I}_1^*\right)^2}, \\ m_2 &= \frac{\beta_1 S_1^*(1 + \alpha_1 S_1^*)}{\left(1 + \alpha_1 S_1^* + \gamma_1\overline{I}_1^* + \mu_1 S_1^*\overline{I}_1^*\right)^2}, \\ m_3 &= \frac{\beta_2 S_1^*}{1 + \alpha_2 S_1^*}, \\ m_4 &= \frac{\beta_1\overline{I}_1^*(1 + \gamma_1\overline{I}_1^*)}{\left(1 + \alpha_1 S_1^* + \gamma_1\overline{I}_1^* + \mu_1 S_1^*\overline{I}_1^*\right)^2}, \\ m_5 &= \mu + a_1 + r_1, \\ m_6 &= \mu + a_2 + r_2. \end{aligned} \quad (20)$$

Clearly, $\lambda_1 = \beta_2 S_1^*/(1 + \alpha_2 S_1^*) - (\mu + a_2 + r_2)$ is an eigenvalue of $J_{E_1^*}$. Since $S_1^* < A/\mu$ because $A - \mu S_1^* = (\mu + a_1)\overline{I}_1^* > 0$ and the function f_2: $x \in \mathbb{R}_+ \mapsto \beta_2 x/(1 + \alpha_2 x)$ is increasing, then $\lambda_1 < f_2(A/\mu) - (\mu + a_2 + r_2) = (\beta_2 A/(\mu + \alpha_2 A)) - (\mu + a_2 + r_2) = (\mu + a_2 + r_2)(R_{02} - 1)$. Hence, $\lambda_1 < 0$ if $R_{02} < 1$. The other two eigenvalues of $J_{E_1^*}$ are determined by the following equation:

$$\lambda^2 + a_1\lambda + a_0 = 0, \quad (21)$$

where

$$\begin{aligned} a_1 &= m_1 + m_5 - m_2, \\ a_0 &= (\mu + a_1)m_4 + \mu(m_5 - m_2). \end{aligned} \quad (22)$$

Since $m_5 - m_2 = ((\beta_1 S_1^*\overline{I}_1^* \quad (\gamma_1 + \mu_1 S_1^*))/((1 + \alpha_1 S_1^* + \gamma_1\overline{I}_1^* + \mu_1 S_1^*\overline{I}_1^*)^2)) > 0$, then $a_1 > 0$ and $a_0 > 0$. Thus, by the Routh–Hurwitz criterion, the eigenvalues $\lambda_j\,(j = 2, 3)$ of $J_{E_1^*}$ have negative real part. Therefore, the equilibrium E_1^* of system (1) is asymptotically stable if $R_{01} > 1$ and $R_{02} < 1$.

Furthermore, if $R_{02} > 1$, then system (1) has the disease-free equilibrium for I_1, $E_2^* = (S_2^*, 0, \overline{I}_2^*)$, where

$$\begin{aligned} S_2^* &= \frac{A - (\mu + a_2)\overline{I}_2^*}{\mu}, \\ \overline{I}_2^* &= \frac{2\varpi_2(\mu + \alpha_2 A)(R_{02} - 1)}{(\mu + a_2)(\beta_2 - \alpha_2\varpi_2) + \varpi_2(\gamma_2\mu + \mu_2 A) + \sqrt{\Delta_2}}, \end{aligned} \quad (23)$$

with $\varpi_2 = \mu + a_2 + r_2$ and

$$\begin{aligned}\Delta_2 &= [(\mu + a_2)(\beta_2 - \alpha_2\varpi_2) + \varpi_2(\gamma_2\mu + \mu_2 A)]^2 - 4\mu_2(\mu + a_2)\varpi_2[\beta_2 A - (\mu + \alpha_2 A)\varpi_2] \\ &= [(\mu + a_2)(\beta_2 - \alpha_2\varpi_2) + \varpi_2(\gamma_2\mu - \mu_2 A)]^2 + 4\mu_2\mu\varpi_2^2(\mu + a_2 + \gamma_2 A).\end{aligned} \tag{24}$$

Theorem 3. *If $R_{01} < 1$ and $R_{02} > 1$, then the equilibrium E_2^* is locally asymptotically stable.*

Proof. It is analogue to the previous proof.

Next, we investigate the local stability of system (1) at both-endemic equilibrium $E^* = (S^*, I_1^*, I_2^*)$. To obtain conditions for the existence of the equilibrium E^*, system (1) is rearranged to get I_1^* and I_2^* which gives

$$\begin{aligned}I_1^* &= \frac{(\beta_1 - \alpha_1\varpi_1)S^* - \varpi_1}{\varpi_1(\gamma_1 + \mu_1 S^*)}, \\ I_2^* &= \frac{(\beta_2 - \alpha_2\varpi_2)S^* - \varpi_2}{\varpi_2(\gamma_2 + \mu_2 S^*)}.\end{aligned} \tag{25}$$

We have $I_i^* > 0$ if $\beta_i - \alpha_i\varpi_i > 0$ for $i = 1, 2$, and $S^* > \max_{i=1,2}\{\varpi_i/(\beta_i - \alpha_i\varpi_i)\}$. In addition, S^* is given by the following cubic equation:

$$C_0 S^{*3} + C_1 S^{*2} + C_2 S^* - C_3 = 0, \tag{26}$$

where

$$\begin{aligned}C_0 &= \mu\mu_1\mu_2\varpi_1\varpi_2 > 0, \\ C_1 &= \varpi_1\varpi_2[\mu(\gamma_1\mu_2 + \gamma_2\mu_1) - A\mu_1\mu_2] \\ &\quad + \mu_1\varpi_1(\mu + a_2)(\beta_2 - \alpha_2\varpi_2) \\ &\quad + \mu_2\varpi_2(\mu + a_1)(\beta_1 - \alpha_1\varpi_1), \\ C_2 &= \varpi_1\varpi_2[\mu\gamma_1\gamma_2 - A(\gamma_1\mu_2 + \gamma_2\mu_1)] \\ &\quad + \varpi_1(\mu + a_2)[\gamma_1(\beta_2 - \alpha_2\varpi_2) - \mu_1\varpi_2] \\ &\quad + \varpi_2(\mu + a_1)[\gamma_2(\beta_1 - \alpha_1\varpi_1) - \mu_2\varpi_1], \\ C_3 &= \varpi_1\varpi_2[A\gamma_1\gamma_2 + \gamma_1(\mu + a_2) + \gamma_2(\mu + a_1)] > 0.\end{aligned} \tag{27}$$

With the help of Descartes' rule of signs [24], equation (26) has a unique positive real root S^* if any one of the following holds:

(i) $C_1 > 0$ and $C_2 > 0$
(ii) $C_1 > 0$ and $C_2 < 0$
(iii) $C_1 < 0$ and $C_2 < 0$

Hence, system (1) has a unique positive equilibrium E^* if $\beta_i - \alpha_i\varpi_i > 0$ for $i = 1, 2$, one of the conditions (i), (ii), and (iii) hold true, and $S^* > \max_{i=1,2}\{\varpi_i/(\beta_i - \alpha_i\varpi_i)\}$.

The Jacobian matrix of system (1) at the equilibrium E^* is determined by

$$J_{E^*} = \begin{pmatrix} -p_1 & -p_2 + r_1 & -p_3 + r_2 \\ p_4 & p_2 - p_5 & 0 \\ p_6 & 0 & p_3 - p_7 \end{pmatrix}, \tag{28}$$

where

$$\begin{aligned}p_1 &= \mu + \frac{\beta_1 I_1^*(1 + \gamma_1 I_1^*)}{(1 + \alpha_1 S^* + \gamma_1 I_1^* + \mu_1 S^* I_1^*)^2} \\ &\quad + \frac{\beta_2 I_2^*(1 + \gamma_2 I_2^*)}{(1 + \alpha_2 S^* + \gamma_2 I_2^* + \mu_2 S^* I_2^*)^2}, \\ p_2 &= \frac{\beta_1 S^*(1 + \alpha_1 S^*)}{(1 + \alpha_1 S^* + \gamma_1 I_1^* + \mu_1 S^* I_1^*)^2}, \\ p_3 &= \frac{\beta_2 S^*(1 + \alpha_2 S^*)}{(1 + \alpha_2 S^* + \gamma_2 I_2^* + \mu_2 S^* I_2^*)^2}, \\ p_4 &= \frac{\beta_1 I_1^*(1 + \gamma_1 I_1^*)}{(1 + \alpha_1 S^* + \gamma_1 I_1^* + \mu_1 S^* I_1^*)^2}, \\ p_5 &= \mu + a_1 + r_1, \\ p_6 &= \frac{\beta_2 I_2^*(1 + \gamma_2 I_2^*)}{(1 + \alpha_2 S^* + \gamma_2 I_2^* + \mu_2 S^* I_2^*)^2}, \\ p_7 &= \mu + a_2 + r_2.\end{aligned} \tag{29}$$

Theorem 4. *The endemic equilibrium E^* is locally asymptotically stable if it exists.*

Proof. The characteristic equation of Jacobian matrix J_{E^*} can be written as

$$\lambda^3 + Q_2\lambda^2 + Q_1\lambda + Q_0 = 0, \tag{30}$$

where

$$\begin{aligned}Q_2 &= p_1 + (p_5 - p_2) + (p_7 - p_3), \\ Q_1 &= (\mu + p_4)(p_7 - p_3) + (\mu + p_6)(p_5 - p_2) \\ &\quad + (p_5 - p_2)(p_7 - p_3) + (\mu + a_1)p_4 + (\mu + a_2)p_6, \\ Q_0 &= \mu(p_5 - p_2)(p_7 - p_3) + (\mu + a_1)(p_7 - p_3)p_4 \\ &\quad + (\mu + a_2)(p_5 - p_2)p_6.\end{aligned} \tag{31}$$

Note that

$$\begin{aligned}p_5 - p_2 &= \frac{\beta_1 S^* I_1^*(\gamma_1 + \mu_1 S^*)}{(1 + \alpha_1 S^* + \gamma_1 I_1^* + \mu_1 S^* I_1^*)^2} > 0, \\ p_7 - p_3 &= \frac{\beta_2 S^* I_2^*(\gamma_2 + \mu_2 S^*)}{(1 + \alpha_2 S^* + \gamma_2 I_2^* + \mu_2 S^* I_2^*)^2} > 0.\end{aligned} \tag{32}$$

Then, it is easy to show that $Q_2 > 0$, $Q_1 > 0$, $Q_0 > 0$, and $Q_2 Q_1 > Q_0$. Thus, by the Routh–Hurwitz criterion, all roots λ_i $(i = 1, 2, 3)$ of (30) have negative real part. Therefore, the equilibrium E^* of system (1) is asymptotically stable. □

3. Stochastic SIS Epidemic Model

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). We consider the following stochastic differential system:

$$\mathrm{d}x(t) = f(x(t),t)\mathrm{d}t + g(x(t),t)\mathrm{d}B(t), \quad t \geq 0, \tag{33}$$

where $x(t) \in \mathbb{R}^n$, $x(0) = x_0$ represents the initial value, and $f\colon \mathbb{R}^n \times [0,+\infty) \mapsto \mathbb{R}^n$ and $g\colon \mathbb{R}^n \times [0,+\infty) \mapsto \mathbb{R}^{n\times m}$ are locally Lipschitz functions in x. $\{B(t)\}_{t\geq 0}$ is an m-dimensional standard Wiener process defined on the above probability space.

Let us suppose that $f(0,t) = g(0,t) = 0$ for all $t \geq 0$ so that zero of $\mathbb{R}^n$ is an equilibrium point of system (33).

Definition 1 (see [25]). The trivial solution $x = 0$ of system (33) is said to be almost surely exponentially stable if for all $x_0 \in \mathbb{R}^n$, we have

$$\limsup_{t\longrightarrow\infty} \frac{1}{t}\ln|x(t)| < 0 \quad \text{(a.s.)}. \tag{34}$$

Denote by $\mathcal{C}^{2,1}(\mathbb{R}^n \times [0,+\infty); \mathbb{R}_+)$ the family of all nonnegative functions $V(x,t)$ defined on $\mathbb{R}^n \times [0,+\infty)$ such that they are continuously twice differentiable in x and once in t. Denote by $\mathbb{E}(X)$ the mathematical expectation of a random variable X. If $\mathcal{L}$ acts on a function $V \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [0,+\infty); \mathbb{R}_+)$, then

$$\begin{aligned}\mathcal{L}V(x,t) &= V_t(x,t) + V_x(x,t)f(x,t) \\ &\quad + \frac{1}{2}\operatorname{trace}\left[g(x,t)^T V_{xx}(x,t)g(x,t)\right],\end{aligned} \tag{35}$$

where $V_t(x,t) = \partial V/\partial t$, $V_x(x,t) = (\partial V/\partial x_1, \ldots, \partial V/\partial x_n)$, and $V_{xx}(x,t) = (\partial^2 V/\partial x_i \partial x_j)$.

By Itô's formula, we have

$$\mathrm{d}V(x,t) = \mathcal{L}V(x,t)\mathrm{d}t + V_x(x,t)g(x,t)\mathrm{d}B(t). \tag{36}$$

Lemma 1 *(see [26]). Suppose there exists a function $V \in \mathcal{C}^{2,1}(\mathbb{R}^n \times [0,+\infty); \mathbb{R}_+)$ satisfying the inequalities*

$$\begin{aligned}K_1|x|^p &\leq V(x,t) \leq K_2|x|^p, \\ \mathcal{L}V(x,t) &\leq -K_3|x|^p,\end{aligned} \tag{37}$$

where $p > 0$ and K_i $(i = 1,2,3)$ are positive constants. Then, the equilibrium of system (33) is pth moment exponentially stable. When $p = 2$, it is usually said to be exponentially stable in mean square, and the equilibrium $x = 0$ is globally asymptotically stable.

3.1. Existence and Uniqueness of the Global Positive Solution. The following theorem shows that the solution of our system (2) is global and positive.

Theorem 5. *For any initial value $(S(0), I_1(0), I_2(0)) \in \Gamma$, there is a unique solution $(S(t), I_1(t), I_2(t))$ to (2) on $t \geq 0$, and this solution remains in Γ with probability one.*

Proof. Let $(S(0), I_1(0), I_2(0)) \in \Gamma$. The total population in system (2) verifies the equation

$$\mathrm{d}N(t) = \left(A - \mu N(t) - a_1 I_1(t) - a_2 I_2(t)\right)\mathrm{d}t. \tag{38}$$

If $(S(s), I_1(s), I_2(s)) \in \mathbb{R}_+^3$ for all $s \in [0,t]$ (a.s.), then we get

$$\mathrm{d}N(s) \leq (A - \mu N(s))\mathrm{d}s \quad \text{(a.s.)}. \tag{39}$$

Hence, by integration, we have

$$N(s) \leq \frac{A}{\mu} - \left(\frac{A}{\mu} - N(0)\right)e^{-\mu s} \quad \text{(a.s.)}. \tag{40}$$

Then, $N(s) \leq A/\mu$ (a.s.), so

$$S(s), I_1(s), I_2(s) \in \left(0, \frac{A}{\mu}\right) \quad \text{for all } s \in [0,t] \quad \text{(a.s.)}. \tag{41}$$

Since the coefficients of system (2) are locally Lipschitz continuous, then by the work of Mao [25] for any initial value $(S(0), I_1(0), I_2(0)) \in \Gamma$, there is a unique local positive solution $(S(t), I_1(t), I_2(t))$ on $[0, \tau_e)$, where τ_e is the explosion time. To show that this solution is global, we only need to prove $\tau_e = \infty$ (a.s.).

Let $\varepsilon_0 > 0$ such that $S(0), I_1(0), I_2(0) > \varepsilon_0$. For $\varepsilon \leq \varepsilon_0$, we define the stopping time:

$$\tau_\varepsilon = \inf\{t \in [0, \tau_e)\colon S(t) \leq \varepsilon \text{ or } I_1(t) \leq \varepsilon \text{ or } I_2(t) \leq \varepsilon\}. \tag{42}$$

Then,

$$\tau = \lim_{\varepsilon\longrightarrow 0} \tau_\varepsilon = \inf\{t \in [0, \tau_e)\colon S(t) \leq 0 \text{ or } I_1(t) \leq 0 \text{ or } I_2(t) \leq 0\}. \tag{43}$$

Consider the function U defined for $(S, I_1, I_2) \in \mathbb{R}_+^3$ by

$$U(S, I_1, I_2) = -\ln\left(\frac{\mu}{A}S\right) - \ln\left(\frac{\mu}{A}I_1\right) - \ln\left(\frac{\mu}{A}I_2\right). \tag{44}$$

Calculating the differential of U along the solution trajectories of system (2) and using Itô's formula, for all $t \geq 0$ and $s \in [0, t\wedge\tau_\varepsilon]$, we get

$$\begin{aligned}\mathrm{d}U(S(s), I_1(s), I_2(s)) &= \mathcal{L}U\mathrm{d}s + \sigma_1 \frac{I_1 - S}{f_1(S, I_1)}\mathrm{d}B_1(s) \\ &\quad + \sigma_2 \frac{I_2 - S}{f_2(S, I_2)}\mathrm{d}B_2(s),\end{aligned} \tag{45}$$

where

$$\begin{aligned}\mathscr{L}U &= -\frac{A+r_1I_1+r_2I_2}{S}+3\mu+a_1+r_1+a_2+r_2+\beta_1\frac{I_1-S}{f_1(S,I_1)}+\beta_2\frac{I_2-S}{f_2(S,I_1)}\\ &\quad+\frac{\sigma_1^2}{2}\frac{I_1^2+S^2}{(f_1(S,I_1))^2}+\frac{\sigma_2^2}{2}\frac{I_2^2+S^2}{(f_2(S,I_2))^2}\\ &\le 3\mu+a_1+r_1+a_2+r_2+\frac{\beta_1I_1}{f_1(S,I_1)}+\frac{\beta_2I_2}{f_2(S,I_1)}+\frac{\sigma_1^2}{2}\frac{I_1^2+S^2}{(f_1(S,I_1))^2}+\frac{\sigma_2^2}{2}\frac{I_2^2+S^2}{(f_2(S,I_2))^2}.\end{aligned} \tag{46}$$

According to (41), we have $S(s), I_1(s), I_2(s)\in(0, A/\mu)$ for all $s\in[0, t\wedge\tau_\varepsilon]$ (a.s.). Hence,

$$\begin{aligned}&\frac{I_i(s)}{f_i(S(s),I_i(s))}\le\frac{A}{\mu},\\ &\frac{S(s)}{f_i(S(s),I_i(s))}\le\frac{A}{\mu},\\ &i=1,2.\end{aligned} \tag{47}$$

Therefore,

$$\mathrm{d}U\le\mathscr{M}\mathrm{d}s+\sigma_1\frac{I_1-S}{f_1(S,I_1)}\mathrm{d}B_1(s)+\sigma_2\frac{I_2-S}{f_2(S,I_2)}\mathrm{d}B_2(s), \tag{48}$$

where

$$\mathscr{M}=3\mu+a_1+r_1+a_2+r_2+\frac{\beta_1A}{\mu}+\frac{\beta_2A}{\mu}+\left(\frac{\sigma_1A}{\mu}\right)^2+\left(\frac{\sigma_2A}{\mu}\right)^2. \tag{49}$$

Integrating both sides of (48) from 0 to $t\wedge\tau_\varepsilon$ and after taking the expectation on both sides, we obtain that

$$\mathbb{E}[U(S(t\wedge\tau_\varepsilon),I_1(t\wedge\tau_\varepsilon),I_2(t\wedge\tau_\varepsilon))]\le U(S(0),I_1(0),I_2(0))+\mathscr{M}t. \tag{50}$$

Since $U(S(t\wedge\tau_\varepsilon),I_1(t\wedge\tau_\varepsilon),I_2(t\wedge\tau_\varepsilon))>0$, then

$$\begin{aligned}\mathbb{E}[U(S(t\wedge\tau_\varepsilon),I_1(t\wedge\tau_\varepsilon),I_2(t\wedge\tau_\varepsilon))]&\ge\mathbb{E}\left[U(S(t\wedge\tau_\varepsilon),I_1(t\wedge\tau_\varepsilon),I_2(t\wedge\tau_\varepsilon))\chi_{\{\tau_\varepsilon\le t\}}\right]\\ &\ge\mathbb{E}\left[U(S(\tau_\varepsilon),I_1(\tau_\varepsilon),I_2(\tau_\varepsilon))\chi_{\{\tau_\varepsilon\le t\}}\right],\end{aligned} \tag{51}$$

where $\chi_{\{\tau_k\le t\}}$ is the indicator function of $\{\tau_k\le t\}$. Note that there are some components of $(S(\tau_\varepsilon), I_1(\tau_\varepsilon), I_2(\tau_\varepsilon))$ equal to ε. Therefore,

$$U(S(\tau_\varepsilon),I_1(\tau_\varepsilon),I_2(\tau_\varepsilon))\ge-\ln\left(\frac{\mu}{A}\varepsilon\right)=\ln\left(\frac{A}{\mu\varepsilon}\right). \tag{52}$$

Thus,

$$\mathbb{E}[U(S(t\wedge\tau_\varepsilon),I_1(t\wedge\tau_\varepsilon),I_2(t\wedge\tau_\varepsilon))]\ge\ln\left(\frac{A}{\mu\varepsilon}\right)\mathbb{P}(\tau_\varepsilon\le t). \tag{53}$$

By combining (50) and (53), we get that, for all $t>0$,

$$\mathbb{P}(\tau_\varepsilon\le t)\le\frac{U(S(0),I_1(0),I_2(0))+\mathscr{M}t}{\ln(A/\mu\varepsilon)}. \tag{54}$$

Extending ε to 0, we obtain for all $t>0$, $\mathbb{P}(\tau\le t)=0$. Hence, $\mathbb{P}(\tau=\infty)=1$. As $\tau_e\ge\tau$, then $\tau=\tau_e=\infty$ (a.s.) which completes the proof. □

3.2. Almost Sure Exponential Stability. The goal of this section is to establish a sufficient condition for the almost sure exponential stability of the disease-free equilibrium E_0 in Γ. For this, we consider

$$\begin{aligned}\Psi(t)&=\left(\frac{A}{\mu}-S(t)\right)+I_1(t)+I_2(t),\\ V(S(t),I_1(t),I_2(t))&=\ln\Psi(t).\end{aligned} \tag{55}$$

Proposition 1. *$\Psi(t)$ almost surely converges exponentially to 0 if*

$$\limsup_{t\longrightarrow\infty}\frac{1}{t}\int_0^t\mathscr{L}V(S(s),I_1(s),I_2(s))\mathrm{d}s<0\quad\text{(a.s.)}. \tag{56}$$

Proof. By Itô's formula, we have

$$\begin{aligned}\mathrm{d}V(S(s),I_1(s),I_2(s))&=\mathscr{L}V(S(s),I_1(s),I_2(s))\mathrm{d}s\\ &\quad+\frac{2\sigma_1S(s)I_1(s)}{\Psi(s)f_1(S(s),I_1(s))}\mathrm{d}B_1(s)\\ &\quad+\frac{2\sigma_2S(s)I_2(s)}{\Psi(s)f_2(S(s),I_2(s))}\mathrm{d}B_2(s).\end{aligned} \tag{57}$$

Integrating both sides from 0 to t yields that

$$V(S(t), I_1(t), I_2(t)) = V(S(0), I_1(0), I_2(0)) + \int_0^t \mathscr{L}V(S(s), I_1(s), I_2(s))\mathrm{d}s + M_1(t) + M_2(t), \tag{58}$$

where $M_i(t) = \int_0^t (2\sigma_i S(s) I_i(s)/\Psi(s) f_i(S(s), I_i(s)))\mathrm{d}B_i(s)$, $i = 1, 2$, are continuous local martingales with $M_i(0) = 0$. Moreover, we have

$$\langle M_i, M_i\rangle_t = \int_0^t \left(\frac{2\sigma_i S(s) I_i(s)}{\Psi(s) f_i(S(s), I_i(s))}\right)^2 \mathrm{d}s \le C_i t, \tag{59}$$

where $C_i\ (i = 1, 2)$ are positive constants. Thus, the strong law of large numbers for local martingales [27] implies that

$$\lim_{t\longrightarrow\infty} \frac{M_i(t)}{t} = 0 \quad \text{(a.s.)}. \tag{60}$$

It follows that

$$\limsup_{t\longrightarrow\infty} \frac{V(S(t), I_1(t), I_2(t))}{t} = \limsup_{t\longrightarrow\infty} \frac{1}{t}\int_0^t \mathscr{L}V(S(s), I_1(s), I_2(s))\mathrm{d}s \quad \text{(a.s.)}. \tag{61}$$

The proposition is proved. □

Then, we obtain the following theorem.

Theorem 6. *If $(\beta_1\sigma_2)^2 + (\beta_2\sigma_1)^2 < (\mu/2)(\sigma_1\sigma_2)^2$, then the disease-free equilibrium E_0 of stochastic system (2) is almost surely exponentially stable in Γ.*

Proof. It suffices to prove that $\Psi(t)$ converges to 0 exponentially (a.s.). Then, by Proposition 1, it suffices to prove that

$$\limsup_{t\longrightarrow\infty} \frac{1}{t}\int_0^t \mathscr{L}V(S(s), I_1(s), I_2(s))\mathrm{d}s < 0 \quad \text{(a.s.)}. \tag{62}$$

By Itô's formula, we have

$$\begin{aligned}
\mathscr{L}V(S(s), I_1(s), I_2(s)) &= \frac{1}{\Psi}\left[-A + \mu S + \frac{2\beta_1 S I_1}{f_1(S, I_1)} + \frac{2\beta_2 S I_2}{f_2(S, I_2)} - r_1 I_1 - r_2 I_2 - (\mu + a_1 + r_1)I_1 - (\mu + a_2 + r_2)I_2\right] \\
&\quad - 2\left(\frac{\sigma_1 S I_1}{\Psi f_1(S, I_1)}\right)^2 - 2\left(\frac{\sigma_2 S I_2}{\Psi f_2(S, I_2)}\right)^2 \\
&= \frac{1}{\Psi}\left[\frac{2\beta_1 S I_1}{f_1(S, I_1)} + \frac{2\beta_2 S I_2}{f_2(S, I_2)} - \mu\left(\frac{A}{\mu} - S\right) - (\mu + a_1 + 2r_1)I_1 - (\mu + a_2 + 2r_2)I_2\right] \\
&\quad - 2\left(\frac{\sigma_1 S I_1}{\Psi f_1(S, I_1)}\right)^2 - 2\left(\frac{\sigma_2 S I_2}{\Psi f_2(S, I_2)}\right)^2.
\end{aligned} \tag{63}$$

Since

$$\mu\left(\frac{A}{\mu} - S\right) + (\mu + a_1 + 2r_1)I_1 + (\mu + a_2 + 2r_2)I_2 \ge \mu\left(\frac{A}{\mu} - S\right) + \mu I_1 + \mu I_2 = \mu\Psi, \tag{64}$$

we have

$$\mathscr{L}V(S(s), I_1(s), I_2(s)) \le \frac{2\beta_1 S I_1}{\Psi f_1(S, I_1)} + \frac{2\beta_2 S I_2}{\Psi f_2(S, I_2)} - \mu - 2\left(\frac{\sigma_1 S I_1}{\Psi f_1(S, I_1)}\right)^2 - 2\left(\frac{\sigma_2 S I_2}{\Psi f_2(S, I_2)}\right)^2. \tag{65}$$

Set $X_1 = SI_1/\Psi f_1(S, I_1)$ and $X_2 = SI_2/\Psi f_2(S, I_2)$. Then,

$$\mathscr{L}V(S(s), I_1(s), I_2(s)) \leq 2\beta_1 X_1 - 2\sigma_1^2 X_1^2 + 2\beta_2 X_2 - 2\sigma_2^2 X_2^2 - \mu. \tag{66}$$

Since $2\beta_i X_i - 2\sigma_i^2 X_i^2 - (\mu/2) = -2\sigma_i^2 \quad ((\beta_i/\sigma_i^2) - X_i)^2 + (4\beta_i^2 - \mu\sigma_i^2/2\sigma_i^2)$, $i = 1, 2$, hence,

$$\begin{aligned}\mathscr{L}V(S(s), I_1(s), I_2(s)) &\leq \frac{4\beta_1^2 - \mu\sigma_1^2}{2\sigma_1^2} + \frac{4\beta_2^2 - \mu\sigma_2^2}{2\sigma_2^2} \\ &= \frac{2(\beta_1\sigma_2)^2 + 2(\beta_2\sigma_1)^2 - \mu(\sigma_1\sigma_2)^2}{(\sigma_1\sigma_2)^2}.\end{aligned} \tag{67}$$

Therefore,

$$\limsup_{t\longrightarrow\infty} \frac{1}{t}\int_0^t \mathscr{L}V(S(s), I_1(s), I_2(s))\mathrm{d}s \leq \frac{2(\beta_1\sigma_2)^2 + 2(\beta_2\sigma_1)^2 - \mu(\sigma_1\sigma_2)^2}{(\sigma_1\sigma_2)^2} < 0 \quad \text{(a.s.)}. \tag{68}$$

This completes the proof. □

3.3. Moment Exponential Stability. In this section, we investigate the pth moment exponential stability of the disease-free equilibrium E_0 in Γ of stochastic system (2).

We use Lemma 1 to prove the following theorem.

Theorem 7. *Let $p \geq 2$. If*

$$\max\left\{R_{01} + \frac{(p-1)\sigma_1^2}{2(\mu + a_1 + r_1)}\left(\frac{A}{\mu + \alpha_1 A}\right)^2, R_{02} + \frac{(p-1)\sigma_2^2}{2(\mu + a_2 + r_2)}\left(\frac{A}{\mu + \alpha_2 A}\right)^2\right\} < 1, \tag{69}$$

then the disease-free equilibrium E_0 of stochastic system (2) is pth moment exponentially stable in Γ.

Proof. Let $p \geq 2$ and $(S(0), I_1(0), I_2(0)) \in \Gamma$. We define the Lyapunov function V as follows:

$$V = \omega\left(\frac{A}{\mu} - S\right)^p + \frac{1}{p}\sum_{i=1}^{2} I_i^p, \tag{70}$$

where ω is a positive constant which will be determined later. By Itô's formula, we have

$$\begin{aligned}\mathscr{L}V &= -\omega p\left(\frac{A}{\mu} - S\right)^{p-1}\left(A - \mu S - \sum_{i=1}^{2}\frac{\beta_i S I_i}{f_i(S, I_i)} + \sum_{i=1}^{2} r_i I_i\right) + \frac{\omega p(p-1)}{2}\sum_{i=1}^{2}\left(\frac{\sigma_i S I_i}{f_i(S, I_i)}\right)^2\left(\frac{A}{\mu} - S\right)^{p-2} \\ &\quad + \sum_{i=1}^{2}\left(\frac{\beta_i S}{f_i(S, I_i)} - (\mu + a_i + r_i)\right)I_i^p + \frac{p-1}{2}\sum_{i=1}^{2}\left(\frac{\sigma_i S}{f_i(S, I_i)}\right)^2 I_i^p \\ &\leq -\omega\mu p\left(\frac{A}{\mu} - S\right)^p + \omega p\sum_{i=1}^{2}\frac{\beta_i A}{\mu + \alpha_i A}\left(\frac{A}{\mu} - S\right)^{p-1} I_i + \frac{\omega p(p-1)}{2}\sum_{i=1}^{2}\sigma_i^2\left(\frac{A}{\mu + \alpha_i A}\right)^2\left(\frac{A}{\mu} - S\right)^{p-2} I_i^2 \\ &\quad + \sum_{i=1}^{2}\left(\frac{\beta_i A}{\mu + \alpha_i A} - (\mu + a_i + r_i)\right)I_i^p + \frac{p-1}{2}\sum_{i=1}^{2}\sigma_1^2\left(\frac{A}{\mu + \alpha_i A}\right)^2 I_i^p.\end{aligned} \tag{71}$$

From Young's inequality, for $\varepsilon > 0$, we have

$$\begin{aligned} \left(\frac{A}{\mu} - S\right)^{p-1} I_i &\leq \frac{p-1}{p}\varepsilon\left(\frac{A}{\mu} - S\right)^p + \frac{1}{p}\varepsilon^{1-p} I_i^p, \\ \left(\frac{A}{\mu} - S\right)^{p-2} I_i^2 &\leq \frac{p-2}{p}\varepsilon\left(\frac{A}{\mu} - S\right)^p + \frac{2}{p}\varepsilon^{(2-p)/p} I_i^p, \\ i &= 1, 2. \end{aligned} \tag{72}$$

Then,

$$\mathscr{L}V \leq -Q_0\left(\frac{A}{\mu} - S\right)^p - \sum_{i=1}^{2} Q_i I_i^p, \tag{73}$$

where

$$\begin{aligned} Q_0 &= \omega\left[\mu p - (p-1)\left(\sum_{i=1}^{2}\frac{\beta_i A}{\mu + \alpha_i A} + \frac{p-2}{2}\sum_{i=1}^{2}\sigma_i^2\left(\frac{A}{\mu + \alpha_i A}\right)^2\right)\varepsilon\right], \\ Q_i &= \mu + a_i + r_i - \frac{\beta_i A}{\mu + \alpha_i A} - \frac{p-1}{2}\sigma_i^2\left(\frac{A}{\mu + \alpha_i A}\right)^2 - \omega\left(\frac{\beta_i A}{\mu + \alpha_i A}\varepsilon^{1-p} + (p-1)\sigma_i^2\left(\frac{A}{\mu + \alpha_i A}\right)^2\varepsilon^{(2-p)/p}\right), \\ i &= 1, 2. \end{aligned} \tag{74}$$

Now, we choose ε sufficiently small such that $Q_0 > 0$. In view of condition (69), we have $\mu + a_i + r_i - (\beta_i A/(\mu + \alpha_i A)) - ((p-1)/2)\sigma_i^2 (A/(\mu + \alpha_i A))^2 > 0$ for $i = 1, 2$; hence, we can choose ω positive such that $Q_i > 0$ for $i = 1, 2$. According to Lemma 1, the proof is completed. □

Remark 1. From Lemma 1, Theorem 7, and the case $p = 2$, we get that if

$$\max\left\{R_{01} + \frac{\sigma_1^2}{2(\mu + a_1 + r_1)}\left(\frac{A}{\mu + \alpha_1 A}\right)^2, R_{02} + \frac{\sigma_2^2}{2(\mu + a_2 + r_2)}\left(\frac{A}{\mu + \alpha_2 A}\right)^2\right\} < 1, \tag{75}$$

then the disease-free equilibrium E_0 of stochastic system (2) is globally asymptotically stable in Γ.

4. Numerical Examples

In this section, we give some numerical examples in order to illustrate our theoretical results in Theorem 1 and Theorem 6.

Example 1. We consider the deterministic SIS system with parameters $A = 0.9$, $\mu = 0.3$, $\beta_1 = 0.25$, $\beta_2 = 0.2$, $a_1 = 0.2$, $a_2 = 0.3$, $r_1 = 0.3$, $r_2 = 0.2$, $\alpha_1 = 0.2$, $\gamma_1 = 0.1$, $\mu_1 = 0.06$, $\alpha_2 = 0.15$, $\gamma_2 = 0.2$, and $\mu_2 = 0.07$. By calculation, we have $R_0 = \max\{R_{01}, R_{02}\} = \max\{0.5859, 0.5172\} < 1$. Hence, according to Theorem 1, the disease-free equilibrium is globally asymptotically stable, which means that the disease dies out.

Example 2. In this example, we consider the stochastic SIS system with parameters the same as in Example 1 and $\sigma_1 = 0.9$, $\sigma_2 = 0.95$, Then, we have $0.08880625 = (\beta_1\sigma_2)^2 + (\beta_2\sigma_1)^2 < (\mu/2)(\sigma_1\sigma_2)^2 = 0.10965375$. Thus, from Theorem 6, we can conclude that the disease-free equilibrium is almost surely exponentially stable.

Now, we choose $\sigma_1 = 0.4$ and $\sigma_2 = 0.5$. Then, we have $0.022025 = (\beta_1\sigma_2)^2 + (\beta_2\sigma_1)^2 > (\mu/2)(\sigma_1\sigma_2)^2 = 0.006$. Therefore, the condition of Theorem 6 is not satisfied. Consequently, if the magnitude of the intensity of noise $\sigma_i\,(i = 1, 2)$ is large, then the disease in the stochastic model will go extinct.

5. Conclusion

In this paper, we have proposed and analyzed a new stochastic SIS epidemic model with double epidemic hypothesis and specific functional response by introducing random perturbations of white noise. Firstly, in the absence of noise, we have derived sufficient conditions for local asymptotic stability of the equilibria; also, we have proved the global stability for disease-free equilibrium. Next, we have established global existence and positivity of the solution for our stochastic model. In addition, we have given a sufficient condition for the almost sure exponential stability and pth moment exponential stability of the disease-free equilibrium of model (2). It is shown that the magnitude of the intensity of noise $\sigma_i\,(i = 1, 2)$ will have an effective impact on stochastic stability of E_0.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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