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Research Article

Long Term Behavior for a Class of Stochastic Delay Lattice Systems in X_ρ Space

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In this paper, we focus on the asymptotic behavior of solutions to stochastic delay lattice equations with additive noise and deterministic forcing. We first show the existence of a continuous random dynamical system for the equations. Then we investigate the pullback asymptotical compactness of solutions as well as the existence and uniqueness of tempered random attractor in X_ρ space. Finally, ergodicity of the systems is achieved.

1. Introduction

We explore the asymptotic behavior of a class of stochastic lattice systems with time delay driven by additive white noise:

$$\frac{du_i}{dt} = \nu(u_{i-1} - 2u_i + u_{i+1}) - \lambda u_i + F_i(u_i) + f_i(u_i(t-\rho)) + g_i + \varepsilon a_i \frac{dw_i}{dt}, \quad t > \tau, \tag{1}$$

with initial data

$$u_i(\tau + s) = u_{\tau,i}(s), \quad s \in [-\rho, 0], \tag{2}$$

where $i \in \mathbb{Z}$, $\mathbb{Z}$ denotes the integer set, $\tau \in \mathbb{R}^+$. $u = (u_i(s))_{i\in\mathbb{Z}}$ is a sequence in X_ρ space (defined later), ν, λ, and ε are positive constants, ε is the intensity of noise, $F(u) = (F_i(u_i))_{i\in\mathbb{Z}}$ is a superlinear source term, $f(u(t-\rho)) = (f_i(u_i(t-\rho)))_{i\in\mathbb{Z}}$ is a nonlinear function satisfying certain structural conditions and capturing the time delay $\rho \geq 0$, $g = (g_i)_{i\in\mathbb{Z}} \in X$ (defined later), $a = (a_i)_{i\in\mathbb{Z}} \in X$, $w = (w_i)_{i\in\mathbb{Z}}$ is a two-side real valued Wiener process on a probability space.

Lattice differential equations are widely adopted in physics, biology, and engineering such as pattern formation, propagation of nerve pulses, electric circuits, and so on, see, e.g., ([1–7]). Stochastic lattice dynamical systems (SLDS) arise naturally, while random influences or uncertainties (called noises) are taken into account. These noises may play an important role as intrinsic phenomena rather than just compensation of defects in deterministic models [8].

The theory of attractors is a powerful tool to depict the asymptotic dynamics of an infinite-dimensional system. Random attractor is an important concept to describe asymptotic behavior for a random dynamical system and to capture the essential dynamics with possibly extremely wide fluctuations. Until now, random attractors have been investigated by many researchers, e.g., in [9–13] for *autonomous stochastic* equations, and in [14–21] for *nonautonomous stochastic* ones.

Lots of work have been done regarding the existence of global random attractors for SLDS with white noises of infinite sequences, see e.g., [22–29] and the references therein. Note that the stochastic equations considered in these papers do not contain nonlinearity with time delay. The differential equations with delays arise, for instance, from population dynamics where a time lag or after-effect is involved.

As far as we are aware, it seems that there are very few works in the literature dealing with random attractors of *stochastic lattice equations containing nonlinearity with time delay* except [30–33].

In the present paper, we consider the stochastic delay lattice equations with *superlinear* nonlinearity (delay terms). More precisely, we first prove the existence and uniqueness of tempered random attractors of equation (1). Then we show the ergodicity of the systems.

Denote

$$\ell^p = \left\{ u = (u_i)_{i\in\mathbb{Z}} : \sum_{i\in\mathbb{Z}} |u_i|^p < \infty \quad u_i \in \mathbb{R}, p \geq 1 \right\}. \tag{3}$$

Let $X := \ell^2(\mathbb{Z})(\ell^p : p = 2)$. It is known that X is a Hilbert space with the inner product $(u, v) = \sum_{i\in\mathbb{Z}} u_i v_i$, and the norm $\|u\|_X^2 = \sum_{i\in\mathbb{Z}} |u_i|^2$, where $u = (u_i)_{i\in\mathbb{Z}}, v = (v_i)_{i\in\mathbb{Z}} \in X$. Then we define a Banach space by $X_\rho := \mathscr{C}([-\rho, 0], X)$ with the norm

$$\|u\|_{X_\rho} = \sup_{s\in[-\rho,0]} \|u(s)\|_X, \quad \forall u = (u(s))_{s\in[-\rho,0]} \in X_\rho. \tag{4}$$

In the sequel, we use $\|\cdot\|$ and $(\cdot,\cdot)$ to denote the norm and inner product of X, respectively. The norm of ℓ^p is written as $\|\cdot\|_{\ell^p} (p \neq 2)$.

The letters C and $C_i (i = 1, 2, \cdots)$ are general positive constants, taking different values from line to line. Since their values are not significant, we do not care about their values and relationship between one and another.

This paper is organized as follows. In Section 2, we recall some basic concepts and already known results related to random dynamical systems and random attractors. In Section 3, we show that the stochastic delay lattice differential equation (1) generates an infinite dimensional random dynamical system. The existence of the global random attractor is given in Section 4. Finally, the proof of ergodicity of the systems is finished in Section 5.

2. Preliminaries

In the following, we recall some basic concepts on random dynamical systems and pullback attractors which are mentioned in [8, 22].

Let χ be a Banach space and $(\Omega, \mathscr{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ be a metric dynamical system, (χ, d) a complete separable metric space with Borel σ-algebra $\mathscr{B}(\chi)$. Suppose $\mathscr{D}$ is a collection of some families of nonempty subsets of χ.

Definition 1. A mapping $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times \chi \to \chi$ is called a continuous random dynamical system (RDS) on χ over $(\Omega, \mathscr{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$ if for all $\omega \in \Omega$ and $t, s \in \mathbb{R}^+$,

(i) $\Phi(\cdot,\cdot,\cdot) : \mathbb{R}^+ \times \Omega \times \chi \to \chi$ is $(\mathscr{B}(\mathbb{R}^+) \times \mathscr{F} \times \mathscr{B}(\chi), \mathscr{B}(\chi))$-measurable;
(ii) $\Phi(0, \omega, \cdot)$ is the identity on χ;
(iii) $\Phi(t + s, \omega, \cdot) = \Phi(t, \theta_s\omega, \cdot) \circ \Phi(s, \omega, \cdot)$;
(iv) $\Phi(t, \omega, \cdot) : \chi \to \chi$ is continuous.

Definition 2. A family $K = \{K(\omega) : \omega \in \Omega\} \in \mathscr{D}$ is called a $\mathscr{D}$-pullback absorbing set for Φ if for all $\omega \in \Omega$ and for every $D \in \mathscr{D}$, there exists $T = T(D, \omega) > 0$ such that

$$\Phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)) \subseteq K(\omega) \quad \text{for all } t \geq T. \tag{5}$$

If, in addition, for all $\omega \in \Omega$, $K(\omega)$ is a closed nonempty subset of χ and K is measurable in ω with respect to $\mathscr{F}$, then we say K is a closed measurable $\mathscr{D}$-pullback absorbing set for Φ.

Definition 3. The random dynamical system Φ is said to be $\mathscr{D}$-pullback asymptotically compact in χ if for $\omega \in \Omega$, the sequence

$$\{\Phi(t_n, \theta_{-t_n}\omega, x_n)\}_{n=1}^{\infty} \text{ has a convergent subsequence in } \chi. \tag{6}$$

whenever $t_n \to \infty$, and $x_n \in B(\theta_{-t_n}\omega)$ with $\{B(\omega) : \omega \in \Omega\} \in \mathscr{D}$.

Definition 4. A family $\mathscr{A} = \{\mathscr{A}(\omega) : \omega \in \Omega\} \in \mathscr{D}$ is called a $\mathscr{D}$-pullback attractor for Φ if for every $\omega \in \Omega$,

(i) $\mathscr{A}$ is measurable in ω with respect to $\mathscr{F}$ and $\mathscr{A}(\omega)$ is compact in χ;
(ii) $\mathscr{A}$ is invariant: $\Phi(t, \omega, \mathscr{A}(\omega)) = \mathscr{A}(t, \theta_t\omega)$, $\forall t \geq 0$;
(iii) $\mathscr{A}$ attracts every member of $\mathscr{D}$: for every $D \in \mathscr{D}$,

$$\lim_{t\to\infty} d_\chi(\Phi(t, \theta_{-t}\omega, D(\theta_{-t}\omega)), \mathscr{A}(\omega)) = 0, \tag{7}$$

where d_χ is the Hausdorff semi-distance in χ.

We borrow the following result for random dynamical systems from [28, 34] and omit its proof.

Proposition 5. *Let $\mathscr{D}$ be an inclusion closed collection of some families of nonempty subsets of χ, and Φ be a continuous RDS on χ over $\mathbb{R}$ and $(\Omega, \mathscr{F}, P, \{\theta_t\}_{t\in\mathbb{R}})$. Then Φ has a $\mathscr{D}$-pullback attractor $\mathscr{A}$ in $\mathscr{D}$ if Φ is $\mathscr{D}$-pullback asymptotically compact in χ and Φ has a closed measurable $\mathscr{D}$-pullback absorbing set K in $\mathscr{D}$. The $\mathscr{D}$-pullback attractor $\mathscr{A}$ is unique and is given by, for each $\omega \in \Omega$,*

$$\mathscr{A}(\omega) = \bigcap_{\kappa\geq t_K(\omega)} \overline{\bigcup_{t\geq\kappa} \Phi(t, \theta_{-t}\omega, K(\theta_{-t}\omega))}. \tag{8}$$

3. RDSs for Stochastic Delay Lattice Systems

In this section, we first state some assumptions that will be used throughout this paper. Then we illustrate the existence of RDSs for stochastic delay lattice systems.

From now on, the functions F, f are assumed to satisfy the following conditions with positive constants $p, \alpha_1, \alpha_2, C_f, L_f, L_F$, and $p \geq 2$.

(A_1) For all $s, t, \beta_{1,i}, \beta_{2,i} \in \mathbb{R}$,

$$F_i(s)s \leq -\alpha_1 |s|^p + \beta_{1,i}, \tag{9}$$

$$|F_i(s)| \leq \alpha_2 |s|^{p-1} + \beta_{2,i}, \tag{10}$$

and $\sum_{i\in\mathbb{Z}} \beta_{1,i} < \infty$, $\sum_{i\in\mathbb{Z}} \beta_{2,i} < \infty$. F also possesses the local Lipschitz condition, i.e., for any bounded interval $I \subset \mathbb{R}$, there exists a positive constant L_F, such that for every $s, t \in I$,

$$\left|F_i(t) - F_i(s)\right| \le L_F |t-s|, \tag{11}$$

(A_2) $f : \mathbb{R} \to \mathbb{R}$ is continuous and for all $(u_i)_{i\in\mathbb{Z}}, (v_i)_{i\in\mathbb{Z}} \in \ell^2$,

$$\left|f_i(u_i) - f_i(v_i)\right| \le L_f |u_i - v_i|, \tag{12}$$

$$\left|f_i(u_i)\right|^2 \le C_f^2 |u_i|^2 + |\eta_i|^2, \tag{13}$$

where $(\eta_i)_{i\in\mathbb{Z}} \in X$, and C_η is a constant satisfying $\sum_{i\in\mathbb{Z}} |\eta_i|^2 = C_\eta$.

(A_3) We also need this assumption:

$$\lambda \ge 2\sqrt{2}C_f. \tag{14}$$

(A_4) For sufficiently large $\lambda > 0$, there is a positive constant ϵ small enough, such that

$$\lambda - \epsilon - \frac{4}{\lambda} C_f^2 e^{\epsilon\rho} \ge 0. \tag{15}$$

(A_5) We can choose a positive constant ϑ such that

$$\frac{3\lambda}{2} - \vartheta - \frac{8}{\lambda} C_f^2 e^{\vartheta\rho} \ge 0. \tag{16}$$

For convenience, we now formulate system (1) as stochastic differential equations in X_ρ. Denote by B, B^*, and A the linear operators from X_ρ into X_ρ in the following way: for any $u = (u_i)_{i\in\mathbb{Z}} \in X_\rho$,

$$(Bu)_i = u_{i+1} - u_i, \quad (B^*u)_i = u_{i-1} - u_i, \tag{17}$$

and

$$(Au)_i = -u_{i-1} + 2u_i - u_{i+1}. \tag{18}$$

Then we have $A = BB^* = B^*B$, $(B^*u, v) = (u, Bv)$, and $(Au, u) \ge 0$ for all $u, v \in X_\rho$.

In the sequel, we consider the probability space $(\Omega, \mathcal{F}, P)$ where

$$\Omega = \{\omega \in \mathcal{C}(\mathbb{R}, X) : \omega(0) = 0\}, \tag{19}$$

$\mathcal{F}$ is the Borel σ-algebra induced by the compact-open topology of Ω, and $\mathcal{F}_0$ is the Borel σ-algebra on Ω. P is the corresponding Wiener measure on $(\Omega, \mathcal{F})$.

Let us recall a filtration over the parametric space $\left(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}}\right)$

$$\mathcal{F}_{t_0}^t = \sigma\{W(t_2) - W(t_1) : t_0 \le t_1 \le t_2 \le t\}, \tag{20}$$

which is the smallest σ-algebra generated by random variable $W(t_2) - W(t_1)$ for every $t_0 \le t_1 \le t_2 \le t$. This σ-algebra has the property: $\theta_h^{-1}\mathcal{F}_{t_0}^t = \mathcal{F}_{t_0+h}^{t+h}$. So $W(t)$ is adapted to $\mathcal{F}_{t_0}^t$.

There is a classical group $\{\theta_t\}_{t\in\mathbb{R}}$ acting on $(\Omega, \mathcal{F}, P)$, which is defined by

$$\theta_t\omega(\cdot) = \omega(\cdot + t) - \omega(t), \quad \omega \in \Omega, t \in \mathbb{R}. \tag{21}$$

Then $\left(\Omega, \mathcal{F}, P, \{\theta_t\}_{t\in\mathbb{R}}\right)$ is a parametric dynamical system (see [9] for more details).

Let $e^i \in X, i \in \mathbb{Z}$ denote the element having 1 at position i and all the other components 0. We have

$$\sum_{i\in\mathbb{Z}} a_i e^i w_i(t) = W(t) \quad \text{with} \quad (a_i)_{i\in\mathbb{Z}} \in X, \tag{22}$$

is the white noise taking valus in X defined on the probability space $(\Omega, \mathcal{F}, P)$.

Then problem (1) and (2) can be written as the following abstract form:

$$\frac{du}{dt} + \nu Au + \lambda u = F(u) + f(u(t-\rho)) + g + \varepsilon\frac{dW}{dt}, \quad t < \tau, \tag{23}$$

with the initial conditions

$$u(\tau + s) = u_\tau(s), \quad s \in [-\rho, 0]. \tag{24}$$

Next, we define a continuous RDS for lattice system (1) and (2) in X_ρ. This can be achieved by transferring the stochastic lattice system into a deterministic one with random parameters in a standard manner. Let $z(\theta_t\omega)$ satisfies the one-dimensional stochastic differential equation

$$dz(\theta_t\omega) + \alpha z(\theta_t\omega)dt = dw(t). \tag{25}$$

This equation has a random fixed point in the sense of random dynamical systems generating a stationary solution known as the stationary Ornstein–Uhlenbeck process (see [9] for more details)

$$z(\omega) = -\alpha\int_{-\infty}^0 e^{\alpha s}\omega(s)ds. \tag{26}$$

In fact, we have that there exists a $(\theta_t)_{t\in\mathbb{R}}$-invariant subset $\{\tilde{\Omega}\} \subseteq \Omega$ of full measure such that $z(\theta_t\omega)$ is continuous in t for every $\omega \in \tilde{\Omega}$, and the random variable $|z(\omega)|$ is tempered. Let $\tilde{\mathcal{F}}$ and $\tilde{P}$ be the restrictions of $\mathcal{F}$ and P, respectively. We will define a continuous RDS for lattice system (1) and (2) in X_ρ over $\mathbb{R}$ and $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}, \{\theta_t\}_{t\in\mathbb{R}}\right)$. For convenience, from now on, we will abuse the notation slightly and write the space $\left(\tilde{\Omega}, \tilde{\mathcal{F}}, \tilde{P}\right)$ as $(\Omega, \mathcal{F}, P)$.

Given a bounded nonempty subset D of X_ρ, the Hausdorff semidistance between D and the origin in X_ρ is denoted by $\|D\| = \sup_{\psi\in D}\|\psi\|$. Let $D = \{D(\omega) : \omega \in \Omega\}$ be a family of nonempty subsets of X_ρ. Such a D is said to be tempered in X_ρ if for every $\gamma > 0$,

$$\lim_{t\to-\infty} e^{\gamma t}\|D(\theta_t\omega)\| = 0. \tag{27}$$

Throughout the rest of this paper, we always use $\mathcal{D}$ to denote the collection of all families of tempered nonempty subsets of X_ρ.

The system (23) may be rewritten as an integral equation in X_ρ,

$$u(t) = u_\tau(0) + \int_\tau^t \left(-\nu Au(r) - \lambda u(r) + F(u(r)) + f(u(r-\rho)) + g\right)dr + \varepsilon W(t) - \varepsilon W(\tau), \quad t \ge \tau, w \in \Omega. \tag{28}$$

Theorem 6. *Let $T > 0$, then the following three properties hold:*

(1) *Equation (28) possesses a unique solution $u(t, u_\tau) \in \mathcal{L}^2(\Omega, \mathcal{C}([0,T], X))$;*

(2) *We have the following estimate, for every $\omega \in \Omega$:*

$$\sup_{t\in[\tau,T]}\|u(t)\|^2 \le C\Big(\|u_\tau(0)-\varepsilon W(\tau)\|^2+\varepsilon^2\sup_{t\in[\tau,T]}\|W(t)\|^2 + \int_\tau^T\big(\|W(r)\|^2+\|W(r)\|_{\ell^p}^p+\|W(r)\|_{\ell^1}+\|g\|^2\big)dr\Big); \tag{29}$$

(3) *The solution of (28) depends continuously on the initial data u_τ, that is to say, for all $\omega\in\Omega$, the mapping $u_\tau\in X\mapsto u(\cdot,\omega,u_\tau)\in\mathscr{C}([0,T],X)$ is continuous.*

Proof.

(1) Set $z(t)=u(t)-\varepsilon W(t)$.For every $\omega\in\Omega$, equation (28) has a solution $u\in\mathscr{L}^2(\Omega,\mathscr{C}([\tau,T],X))$ if and only if

$$z(t)=u_\tau(0)+\int_\tau^t\big(-\nu Az(r)-\lambda z(r)+F(z(r)+\varepsilon W(r))+f(z(r-\rho)+\varepsilon W(r-\rho))+g-\nu\varepsilon AW(r)-\lambda\varepsilon W(r)\big)dr-\varepsilon W(\tau), \tag{30}$$

has a solution $z\in\mathscr{L}^2(\Omega,\mathscr{C}([\tau,T],X))$for every $t\in[0,T]$ and every $\omega\in\Omega$. For each fixed $\omega\in\Omega$, (30) becomes a deterministic equation. As we know, it has a local solution $z\in\mathscr{C}([\tau,T_{max}],X)$, where $[\tau,T_{max}]$ is the maximal interval of the existence of the solution. Next we prove this local solution is a global one.

Suppose $\omega\in\Omega$, from (30) we have

$$\|z(t)\|^2=\|u_\tau(0)-\varepsilon W(\tau)\|^2+2\int_\tau^t\big[-\nu(Az(r),z(r))-\lambda\|z(r)\|^2 +(F(z(r)+\varepsilon W(r)),z(r))+(f(z(r-\rho)+\varepsilon W(r-\rho)),z(r)) +(g,z(r))-\nu\varepsilon(AW(r),z(r))-\lambda\varepsilon(W(r),z(r))\big]dr. \tag{31}$$

By the assumption (A_1) and Young's inequality, we obtain

$$\begin{aligned}&(F(z(r)+\varepsilon W(r)),z(r))\\ &=(F(z(r)+\varepsilon W(r)),z(r)+\varepsilon W(r))-(F(z(r)+\varepsilon W(r)),\varepsilon W(r))\\ &\le\sum_{i\in\mathbb{Z}}\big(-\alpha_1|z_i+\varepsilon a_ie^iw_i(r)|^p+\beta_{1,i}\big)\\ &\quad+\varepsilon\sum_{i\in\mathbb{Z}}\big(\big(\alpha_2|z_i+\varepsilon a_ie^iw_i(r)|^{p-1}+\beta_{2,i}\big)|a_ie^iw_i(r)|\big)\\ &\le\sum_{i\in\mathbb{Z}}\beta_{1,i}+C_1\sum_{i\in\mathbb{Z}}|a_ie^iw_i(r)|^p\varepsilon\sum_{i\in\mathbb{Z}}\big(\beta_{2,i}|a_ie^iw_i(r)|\big)\\ &\le C_2\big(1+\|W(r)\|_{\ell^p}^p+\|W(r)\|_{\ell^1}\big),\end{aligned} \tag{32}$$

It follows from assumption (A_2) and Young's inequality that

$$(f(z(r-\rho)+\varepsilon W(r-\rho)),z(r))\le\frac{\lambda}{8}\|z(r)\|^2+\frac{4}{\lambda}C_f^2\sum_{i\in\mathbb{Z}}|z_i(r-\rho)|^2 +\frac{4}{\lambda}(C_f\varepsilon)^2\sum_{i\in\mathbb{Z}}|a_ie^iw_i(r-\rho)|^2+\frac{2}{\lambda}C_\eta. \tag{33}$$

Utilizing Young's inequality, we gain the following three inequalities:

$$(g,z(r))\le\frac{\lambda}{8}\|z(r)\|^2+\frac{2}{\lambda}\|g\|^2, \tag{34}$$

$$-\nu\varepsilon(AW(r),z(r))\le\frac{\lambda}{8}\|z(r)\|^2+\frac{2}{\lambda}(\nu\varepsilon)^2\|A\|^2\|W(r)\|^2, \tag{35}$$

$$-\lambda\varepsilon(W(r),z(r))\le\frac{\lambda}{8}\|z(r)\|^2+\frac{2}{\lambda}(\lambda\varepsilon)^2\|W(r)\|^2. \tag{36}$$

By (31)–(36), we get

$$\begin{aligned}\|z(t)\|^2&\le\|u_\tau(0)-\varepsilon W(\tau)\|^2-\lambda\int_\tau^t\|z(r)\|^2dr\\ &+\frac{8}{\lambda}C_f^2\int_{\tau-\rho}^{t-\rho}\|z(r)\|^2dr+\frac{8}{\lambda}(C_f\varepsilon)^2\int_{\tau-\rho}^{t-\rho}\|W(r)\|^2dr\\ &+\frac{4}{\lambda}C_\eta(t-\tau)+\frac{4}{\lambda}\int_\tau^t\|g\|^2dr+\frac{4}{\lambda}(\nu\varepsilon)^2\int_\tau^t\|A\|^2\|W(r)\|^2dr\\ &+\frac{4}{\lambda}(\lambda\varepsilon)^2\int_\tau^t\|W(r)\|^2dr+2C_2\int_\tau^t\big(1+\|W(r)\|_{\ell^p}^p+\|W(r)\|_{\ell^1}\big)dr,\end{aligned} \tag{37}$$

where

$$\begin{aligned}\frac{8}{\lambda}C_f^2\int_{\tau-\rho}^{t-\rho}\|z(r)\|^2dr&\le\frac{8}{\lambda}C_f^2\int_{\tau-\rho}^{t}\|z(r)\|^2dr\\ &=\frac{8}{\lambda}C_f^2\Big(\int_{\tau-\rho}^{t}\|z(r)\|^2dr+\int_\tau^t\|z(r)\|^2dr\Big)\\ &\le\frac{8}{\lambda}C_f^2\Big(\rho\sup_{r\in[\tau-\rho,\tau]}\|z(r)\|+\int_\tau^t\|z(r)\|^2dr\Big).\end{aligned} \tag{38}$$

By the similar way, we receive

$$\frac{8}{\lambda}(C_f\varepsilon)^2\int_{\tau-\rho}^{t-\rho}\|W(r)\|^2dr\le\frac{8}{\lambda}(C_f\varepsilon)^2\Big(\rho\sup_{r\in[\tau-\rho,\tau]}\|W(r)\|+\int_\tau^t\|W(r)\|^2dr\Big). \tag{39}$$

It follows from (37)–(39) and assumption (A_3) that

$$\|z(t)\|^2\le C_2\|u_\tau(0)-\varepsilon W(\tau)\|^2 +C_1\int_\tau^t\big(\|W(r)\|^2+\|W(r)\|_{\ell^p}^p+\|W(r)\|_{\ell^1}+\|g\|^2+1\big)dr, \tag{40}$$

where C_1, C_2 are positive constants depending on ν, λ, ε, ρ,$_{Cf}$,$_{C\eta}$, $\sup_{r\in[\tau-\rho,\tau]}\|z(r)\|$, $\sup_{r\in[\tau-\rho,\tau]}\|W(r)\|$. This tells us that $\|z(t)\|$ is bounded by a continuous function, hence there exists a global solution on any $[\tau,T]$. For every $\omega\in\Omega$,

$$\sup_{t\in[\tau,T]}\|z(t)\|^2\le C\Big(\|u_\tau(0)-\varepsilon W(\tau)\|^2+\int_\tau^T\Big(\|W(r)\|^2 +\|W(r)\|_{\ell^p}^p+\|W(r)\|_{\ell^1}+\|g\|^2\Big)dr\Big). \tag{41}$$

Taking expectation on both sides of the above inequality, we known that $z(t)\in\mathscr{L}^2(\Omega,\mathscr{C}([\tau,T],X))$. It implies that (28) has a global solution $u(t)\in\mathscr{L}^2(\Omega,\mathscr{C}([\tau,T],X))$.

(2) By (41) and $z(t)=u(t)-\varepsilon W(t)$, for every $\omega\in\Omega$, we can conclude (29).

(3) Let $u_{\tau,Y_1}(s),u_{\tau,Y}(s)\in X_\rho$, $\|u_{\tau,Y_1}(s)\|,\|u_{\tau,Y}(s)\|<r_0$ for some $r_0>0$, and $Y_1(t)=:u(t,u_{\tau,Y_1}(s))$, $Y(t)=:u(t,u_{\tau,Y}(s))$ be the corresponding solutions of (28). Then it follows from (28) that

$$\|Y_1(t)-Y(t)\|^2 = \|u_{\tau,Y_1}(s)-u_{\tau,Y}(s)\|^2 + 2\int_\tau^t [-\nu(A(Y_1(r)-Y(r)),Y_1(r)-Y(r)) - \lambda\|Y_1(r)-Y(r)\|^2 + (F(Y_1(r))-F(Y(r)),Y_1(r)-Y(r)) + (f(Y_1(r-\rho))-f(Y(r-\rho)),Y_1(r)-Y(r))]dr. \quad (42)$$

Set

$$R = C\left(r_0^2 + \varepsilon^2 \sup_{t\in[\tau,T]} \|W(t)\|^2 + \int_\tau^T \left(\|W(r)\|^2 + \|W(r)\|_{\ell^p}^p + \|W(r)\|_{\ell^1} + \|g\|^2\right)dr\right). \quad (43)$$

By the assumption of local Lipschitz condition of F, we know there exists a constant L_R such that on the ball $B(0,R)$,

$$\|F(Y_1(r)) - F(Y(r))\| \le L_R \|Y_1(r)-Y(r)\|. \quad (44)$$

So

$$(F(Y_1(r)) - F(Y(r)), Y_1(r)-Y(r)) \le L_R \|Y_1(r)-Y(r)\|^2. \quad (45)$$

By Schwarz inequality, Young's inequality and the assumption (A_2), we get

$$\begin{aligned}\int_\tau^t &(f(Y_1(r-\rho)) - f(Y(r-\rho)), Y_1(r)-Y(r))dr \\ &\le \frac{1}{2}\int_\tau^t \left(\|f(Y_1(r-\rho)) - f(Y(r-\rho))\|^2 + \|Y_1(r)-Y(r)\|^2\right)dr \\ &\le \frac{1}{2}\int_\tau^t \left(L_f^2\|Y_1(r-\rho)-Y(r-\rho)\|^2 + \|Y_1(r)-Y(r)\|^2\right)dr \\ &\le \frac{1}{2}\int_\tau^t \left(L_f^2+1\right)\|Y_1(r)-Y(r)\|^2 dr + C_1,\end{aligned} \quad (46)$$

where $C_1 = (1/2)\int_{\tau-\rho}^{\tau} L_f^2 \|Y_1(r)-Y(r)\|^2 dr$. Therefore, we infer

$$\begin{aligned}\|Y_1(t)-Y(t)\|^2 &\le \left(\rho L_f^2+1\right)\left\|u_{\tau,Y_1}(s)-u_{\tau,Y}(s)\right\|^2 \\ &+ \left(2L_R + L_f^2 - 2\lambda + 1\right)\int_\tau^t \|Y_1(r)-Y(r)\|^2 dr.\end{aligned} \quad (47)$$

By Gronwall's inequality, we find

$$\|Y_1(t)-Y(t)\|^2 \le \left\|u_{\tau,Y_1}(s)-u_{\tau,Y}(s)\right\|^2 e^{|2L_R+L_f^2-2\lambda+1|(t-\tau)}, \quad t\in[\tau,T]. \quad (48)$$

Hence

$$\sup_{t\in[\tau,T]} \|Y_1(t)-Y(t)\|^2 \le \left\|u_{\tau,Y_1}(s)-u_{\tau,Y}(s)\right\|^2 e^{|2L_R+L_f^2-2\lambda+1|(T-\tau)}. \quad (49)$$

This inequality implies the uniqueness and continuous dependence on the initial data of the solution of (30). This proof is completed.

Similar to the proof of the Theorem 7 in [8] with minor modifications, we can prove. □

Theorem 7. *System (27) generates a continuous RDS $(\Phi(t,s))_{t\ge\tau}$ over $(\Omega,\mathcal{F},P,\{\theta_t\}_{t\in\mathbb{R}})$ and $\mathbb{R}$, where*

$$\Phi(t,s,\omega,u_\tau(s)) = u(t,s,\omega,u_\tau(s)) \quad s\in[-\rho,0], \quad (50)$$

for each $t\ge\tau$, $\omega\in\Omega$.

4. Existence of Pullback Attractors

This section is devoted to the proof of existence of tempered pullback attractors for the systems (1) and (2) in X_ρ. We first show the existence of the absorbing set for the system (23). Then we make uniform estimate on the tails of solutions of systems (1) and (2). Finally we derive the theorem for the existence of the tempered pullback attractors.

Theorem 8. *There exists a θ_t-invariant set $\Omega' \subset \Omega$ with full $\mathbb{P}$ measure and an absorbing set $K(\omega)$, $\omega\in\Omega'$, for $\Phi(t,s,\omega,u_\tau(s))$, that is to say, there is an absorbing time $T=T(D,\omega)>0$, for each $D\in\mathcal{D}$, $\omega\in\Omega'$ and $t\ge T$ such that*

$$\Phi(t,\theta_{-t}\omega, D(\theta_{-t}\omega)) \subset K(\omega), \quad (51)$$

what is more, $K\in\mathcal{D}$.

Proof. We apply an Ornstein–Uhlenbeck process on metric dynamical system $(\Omega,\mathcal{F},P,\{\theta_t\})$ in X. Suppose

$$z(\theta_t\omega) =: -\lambda\int_{-\infty}^0 e^{\lambda s}\theta_t\omega(s)ds \quad t>\tau, \lambda>0, \quad (52)$$

$$dz + \lambda z dt = dw(t) \quad t>\tau. \quad (53)$$

Moreover, there is a θ_t-invariant set $\Omega'\subset\Omega$ with full $\mathbb{P}$ measure such that, for all $\omega\in\Omega'$, (1) the mapping $t\to y(\theta_t\omega)$ is continuous; (2) the random variable $\|y(\omega)\|$ is tempered.

Set $v(t) = u(t) - \varepsilon y(\theta_t\omega)$, where $u(t)$ is a solution of (23). We have

$$\begin{aligned}\frac{dv}{dt} &= -\nu A v - \lambda v + F(v(t)+\varepsilon y(\theta_t\omega)) + f(t, v(t-\rho) \\ &+\varepsilon y(\theta_{t-\rho}\omega)) + g - \nu\varepsilon A y(\theta_t\omega), \quad t>\tau,\end{aligned} \quad (54)$$

with the initial conditions

$$v(\tau+s) = v_\tau(s), \quad s\in[-\rho,0]. \quad (55)$$

Taking the inner product of (54) with $v(t)$, we can obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\|v\|^2 + \nu(Av,v) + \lambda\|v\|^2 &= (F(v+\varepsilon y),v) \\ &+ \left(f\left(t, v(t-\rho)+\varepsilon y(\theta_{t-\rho}\omega)\right), v\right) \\ &+ (g,v) - \nu\varepsilon(Ay,v).\end{aligned} \quad (56)$$

By using the assumption (A_1) and Young's inequality, we arrive at

$$\begin{aligned}(F(v+\varepsilon y),v) &= (F(v+\varepsilon y), v+\varepsilon y) - (F(v+\varepsilon y),\varepsilon y) \\ &\le \sum_{i\in\mathbb{Z}}\left(-\alpha_1|v_i+\varepsilon y_i|^p + \beta_{1,i}\right) \\ &+ \sum_{i\in\mathbb{Z}}\left(\left(\alpha_2|v_i+\varepsilon y_i|^{p-1} + \beta_{2,i}\right)\varepsilon|y_i|\right) \\ &\le C\left(1 + \|y\|_{\ell^p}^p + \|y\|_{\ell^1}\right),\end{aligned} \quad (57)$$

where C is a positive constant depending only on $\varepsilon, p, \alpha_1, \alpha_2, \sum_{i\in\mathbb{Z}}\beta_{1,i}, \sum_{i\in\mathbb{Z}}\beta_{2,i}$.

Young's inequality and the assumption (A_2) also yield

$$\begin{aligned}&\left(f\left(v(t-\rho)+\varepsilon y\left(\theta_{t-\rho}\omega\right)\right),v\right)\\&\leq\frac{1}{\lambda}\left\|f\left(v(t-\rho)+\varepsilon y\left(\theta_{t-\rho}\omega\right)\right)\right\|^2+\frac{\lambda}{4}\|v\|^2\\&\leq\frac{\lambda}{4}\|v\|^2+\frac{1}{\lambda}\sum_{i\in\mathbb{Z}}\left(C_f^2\left|v_i(t-\rho)\;+\varepsilon y_i\left(\theta_{t-\rho}\omega_i\right)\right|^2+\left|\eta_i\right|^2\right)\\&\leq\frac{\lambda}{4}\|v\|^2+\frac{2}{\lambda}C_f^2\|v(t-\rho)\|^2+\frac{2}{\lambda}\left(C_f\varepsilon\right)^2\sum_{i\in\mathbb{Z}}\left|y_i\left(\theta_{t-\rho}\omega_i\right)\right|^2+\frac{1}{\lambda}C_\eta,\end{aligned}\tag{58}$$

$$(g,v)\leq\frac{\lambda}{8}\|v\|^2+\frac{2}{\lambda}\|g\|^2,\tag{59}$$

$$-\nu\varepsilon(Ay,v)\leq\frac{\lambda}{8}\|v\|^2+\frac{2}{\lambda}(\nu\varepsilon)^2\|A\|^2\|y\|^2.\tag{60}$$

Using (56)–(60), we have

$$\begin{aligned}&\frac{d}{dt}\left(e^{\epsilon t}\|v\|^2\right)+(\lambda-\epsilon)e^{\epsilon t}\|v\|^2\\&\leq e^{\epsilon t}\Big[2C\left(1+\|y\|_{\ell^p}^p+\|y\|_{\ell^1}\right)+\frac{4}{\lambda}C_f^2\|v(t-\rho)\|^2\\&\quad+\frac{4}{\lambda}\left(C_f\varepsilon\right)^2\left\|y\left(\theta_{t-\rho}\omega\right)\right\|^2+\frac{2}{\lambda}C_\eta+\frac{4}{\lambda}\|g\|^2+\frac{4}{\lambda}(\nu\varepsilon)^2\|A\|^2\|y\|^2\Big].\end{aligned}\tag{61}$$

Integrating (61) from τ to t and estimating the following terms, we find that

$$\begin{aligned}&\frac{4}{\lambda}C_f^2\int_\tau^t e^{\epsilon r}\|v(r-\rho)\|^2dr\leq\frac{4}{\lambda}C_f^2\left(\int_{\tau-\rho}^\tau e^{\epsilon(r+\rho)}\|v(r)\|^2dr\right.\\&\left.+\int_\tau^t e^{\epsilon(r+\rho)}\|v(r)\|^2dr\right)\leq\frac{4}{\lambda}C_f^2\rho\sup_{r\in[\tau-\rho,\tau]}e^{\epsilon(r+\rho)}\|v(r)\|^2\\&+\frac{4}{\lambda}C_f^2\int_\tau^t e^{\epsilon(r+\rho)}\|v(r)\|^2dr,\end{aligned}\tag{62}$$

and similarly,

$$\frac{4}{\lambda}\left(C_f\varepsilon\right)^2\int_\tau^t e^{\epsilon r}\left\|y\left(\theta_{r-\rho}\omega\right)\right\|^2dr\leq C_1+\frac{4}{\lambda}\left(C_f\varepsilon\right)^2\int_\tau^t e^{\epsilon(r+\rho)}\|y(\theta_r\omega)\|^2dr.\tag{63}$$

It follows from (61)–(63) and the assumption (A_4) that

$$\begin{aligned}e^{\epsilon t}\|v(t)\|^2\leq e^{\epsilon\tau}\|v(\tau)\|^2&+C\int_\tau^t e^{\epsilon r}\left(1+\|y\|_{\ell^p}^p+\|y\|_{\ell^1}+\|y\|^2\right)dr+\\&+\frac{2}{\lambda\epsilon}\left(C_\eta+2\|g\|^2\right)e^{\epsilon t}.\end{aligned}\tag{64}$$

Therefore,

$$\begin{aligned}\|v(t,\omega,v_\tau(s,\omega))\|^2&\leq\|v_\tau(s,\omega)\|^2e^{\epsilon(\tau-t)}+C\int_\tau^t e^{\epsilon(r-t)}\left(1+\|y(\theta_r\omega)\|_{\ell^p}^p\right.\\&\left.+\|y(\theta_r\omega)\|_{\ell^1}+\|y(\theta_r\omega)\|^2\right)dr+\frac{2}{\lambda\epsilon}\left(C_\eta+2\|g\|^2\right).\end{aligned}\tag{65}$$

Since $y(\theta_t\omega)$ is continuous and $\|y(\omega)\|$ is tempered, $\left(1+\|y\|_{\ell^p}^p+\|y\|_{\ell^1}+\|y\|^2\right)$is tempered. From proposition 4.3.3 [13] P187, there is a tempered function $r(\omega)>0$ such that

$$1+\|y(\theta_t\omega)\|_{\ell^p}^p+\|y(\theta_t\omega)\|_{\ell^1}+\|y(\theta_t\omega)\|^2\leq r(\theta_t\omega)\leq r(\omega)e^{(\epsilon/2)|t|}.\tag{66}$$

In (65), with ω replaced by $\theta_{-t}\omega$, from (66), we have

$$\begin{aligned}&\|v(t,\theta_{-t}\omega,v_\tau(s,\theta_{-t}\omega))\|^2\leq\|v_\tau(s,\theta_{-t}\omega)\|^2e^{\epsilon(\tau-t)}\\&+C\int_\tau^t e^{\epsilon(r-t)}\left(1+\|y(\theta_{r-t}\omega)\|_{\ell^p}^p+\|y(\theta_{r-t}\omega)\|_{\ell^1}+\|y(\theta_{r-t}\omega)\|^2\right)d\\&+\frac{2}{\lambda\epsilon}\left(C_\eta+2\|g\|^2\right)\leq\|v_\tau(s,\theta_{-t}\omega)\|^2e^{-\epsilon(t-\tau)}\\&+C\int_{\tau-t}^0 e^{\epsilon\varsigma}\left(1+\|y(\theta_\varsigma\omega)\|_{\ell^p}^p+\|y(\theta_\varsigma\omega)\|_{\ell^1}+\|y(\theta_\varsigma\omega)\|^2\right)d\varsigma\\&+\frac{2}{\lambda\epsilon}\left(C_\eta+2\|g\|^2\right)\leq\|v_\tau(s,\theta_{-t}\omega)\|^2e^{-\epsilon(t-\tau)}+\frac{2Cr(\omega)}{\epsilon}+\frac{2}{\lambda\epsilon}\left(C_\eta+2\|g\|^2\right).\end{aligned}\tag{67}$$

Choose

$$R(\omega)=\frac{4Cr(\omega)}{\epsilon}+\frac{4}{\lambda\epsilon}\left(C_\eta+2\|g\|^2\right).\tag{68}$$

Then, $R(\omega)$ is tempered and $\tilde{K}(\omega)=\left\{v\epsilon X_\rho:\|v\|^2\leq R(\omega)\right\}$ is an absorbing set for $v(t,\omega,v_\tau(s,\omega))$. i.e., there is an absorbing time $T=T(D,\omega)>0$, for each $D\in\mathscr{D}$, $\omega\in\Omega'$, and $t\geq T$ such that $v(t,\theta_{-t}\omega,D(\theta_{-t}\omega))\subset\tilde{K}(\omega)$. Furthermore, $\tilde{K}\in\mathscr{D}$.

Let

$$K(\omega)=\left\{u\in X_\rho:\ \|u\|^2\leq 2R(\omega)+2\left(\varepsilon\|y(\omega)\|\right)^2\right\}.\tag{69}$$

Then $K(\omega)$ is an absorbing set for $\Phi(t,\omega,u_\tau(s))$ because $\Phi(t,\omega,u_\tau(s))=v(t,\omega,u_\tau(s)-\varepsilon y(\theta_\tau\omega))+\varepsilon y(\theta_t\omega)$and $K\in\mathscr{D}$. This completes the proof of the theorem. □

In order to prove that the random dynamical system $\Phi(t,\omega,u_\tau(s,\omega))$ is asymptotic compact, we require the following lemma.

Lemma 9. *Let $K(\omega)$ be the absorbing set and $u_\tau(s,\omega)\in K(\omega)$ the initial data. Then for each $\eta>0$, there exist an absorbing time $T(\eta,\omega)>0$ and $N(\eta,\omega)>0$ such that for every $t\geq T(\eta,\omega)$, the solution of (1) meets*

$$\sup_{-\rho\leq s\leq 0}\sum_{|i|\geq N}\left|u_i(s,\theta_{-t}\omega,u_\tau(s,\theta_{-t}\omega))\right|^2\leq\eta.\tag{70}$$

Proof. Let $\xi(s)$ be a smooth cut-off function with

$$\xi(s)=\begin{cases}0, & 0\leq s\leq 1,\\ \xi(s), & 1\leq s\leq 2,\\ 1, & s\geq 2,\end{cases}\tag{71}$$

where $0\leq\xi(s)\leq 1$, $s\in\mathbb{R}^+$, and with a constant C such that $|\xi'(s)|\leq C$.

With $k\in\mathbb{Z}^+$, taking the inner product of (54) with the sequence $(\xi(|i|/k)v_i)_{i\in\mathbb{Z}}$, we obtain

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|v_i|^2+\nu\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)(Av)_iv_i\\&+\lambda\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|v_i|^2=\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)F_i(v_i+\varepsilon y_i)v_i\\&+\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)f_i\left(v_i(t-\rho)+\varepsilon y_i\left(\theta_{t-\rho}\omega_i\right)\right)v_i\\&+\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)g_iv_i-\nu\varepsilon\sum_{i\in\mathbb{Z}}\theta\left(\frac{|i|}{k}\right)(Ay)_iv_i\end{aligned}\tag{72}$$

We now make the following estimate.

$$\begin{aligned}\nu\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)(A\nu)_i\nu_i &= \nu\sum_{i\in\mathbb{Z}}(\nu_{i+1}-\nu_i)\\&\quad\cdot\left[\left(\xi\left(\frac{|i+1|}{k}\right)-\xi\left(\frac{|i|}{k}\right)\right)\nu_{i+1}+\xi\left(\frac{|i|}{k}\right)(\nu_{i+1}-\nu_i)\right]\\&= \nu\sum_{i\in\mathbb{Z}}\left(\xi\left(\frac{|i+1|}{k}\right)-\xi\left(\frac{|i|}{k}\right)\right)(\nu_{i+1}-\nu_i)\nu_{i+1}\\&\quad+\nu\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)(\nu_{i+1}-\nu_i)^2\\&\geq \nu\sum_{i\in\mathbb{Z}}\left(\xi\left(\frac{|i+1|}{k}\right)-\xi\left(\frac{|i|}{k}\right)\right)(\nu_{i+1}-\nu_i)\nu_{i+1}.\end{aligned}\tag{73}$$

Using the property of the smooth cut-off function $\xi(s)$, we have

$$\begin{aligned}&\left|\sum_{i\in\mathbb{Z}}\left(\xi\left(\frac{|i+1|}{k}\right)-\xi\left(\frac{|i|}{k}\right)\right)(\nu_{i+1}-\nu_i)\nu_{i+1}\right|\\&\quad\leq\sum_{i\in\mathbb{Z}}\frac{|\xi'(\varsigma_i)|}{k}|\nu_{i+1}-\nu_i||\nu_{i+1}|\\&\quad\leq\frac{C}{k}\sum_{i\in\mathbb{Z}}\left(|\nu_{i+1}|^2+|\nu_i||\nu_{i+1}|\right)\leq\frac{2C}{k}\|\nu\|^2,\end{aligned}\tag{74}$$

which tells us that

$$\nu\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)(A\nu)_i\nu_i\geq\frac{-2C\nu}{k}\|\nu\|^2.\tag{75}$$

It follows from the assumption (A_1) and Young's inequality that

$$\begin{aligned}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)F_i(\nu_i+\varepsilon y_i)\nu_i &= \sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)F_i(\nu_i+\varepsilon y_i)(\nu_i+\varepsilon y_i)\\&\quad-\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)F_i(\nu_i+\varepsilon y_i)\varepsilon y_i\\&\leq\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left(-\alpha_1|\nu_i+\varepsilon y_i|^p+\beta_{1,i}\right)\\&\quad+\varepsilon\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left(\left(\alpha_2|\nu_i+\varepsilon y_i|^{p-1}+\beta_{2,i}\right)|y_i|\right)\\&\leq C\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left(1+|y_i|^p+|y_i|\right),\end{aligned}\tag{76}$$

where C is a positive constant depending on $\varepsilon, p, \alpha_1, \alpha_2, \sum_{i\in\mathbb{Z}}\beta_{1,i}, \sum_{i\in\mathbb{Z}}\beta_{2,i}, \xi(|i|/k)$ only.

By the assumption (A_2) and Young's inequality, we get

$$\begin{aligned}&\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)f_i\left(\nu_i(t-\rho)+\varepsilon y_i\left(\theta_{t-\rho}\omega_i\right)\right)\nu_i\\&\quad\leq\frac{\lambda}{8}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i|^2+\frac{4}{\lambda}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\\&\qquad\cdot C_f^2\left(|\nu_i(t-\rho)|^2+\left|\varepsilon y_i\left(\theta_{t-\rho}\omega_i\right)\right|^2\right)+\frac{2}{\lambda}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\eta_i|^2,\end{aligned}\tag{77}$$

$$\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)g_i\nu_i=\sum_{|i|\geq k}\xi\left(\frac{|i|}{k}\right)g_i\nu_i\leq\frac{\lambda}{8}\sum_{|i|\geq k}\xi\left(\frac{|i|}{k}\right)|\nu_i|^2+\frac{2}{\lambda}\sum_{|i|\geq k}|g_i|^2.\tag{78}$$

By (72), (73), (75)–(78), we arrive at

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}e^{\vartheta t}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i|^2+\left(\frac{3\lambda}{4}-\frac{\vartheta}{2}\right)e^{\vartheta t}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i|^2\\&\quad\leq e^{\vartheta t}\left[\frac{2C\nu}{k}\|\nu\|^2+C\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left(1+|y_i|^p+|y_i|\right)\right.\\&\qquad+\frac{4}{\lambda}C_f^2\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i(t-\rho)|^2+\frac{4}{\lambda}\left(C_f\varepsilon\right)^2\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left|y_i\left(\theta_{t-\rho}\omega_i\right)\right|^2\\&\qquad\left.+\frac{2}{\lambda}C_\eta+\frac{2}{\lambda}\sum_{|i|\geq k}|g_i|^2\right],\end{aligned}\tag{79}$$

where $\sum_{i\in\mathbb{Z}}\xi(|i|/k)|\eta_i|^2\leq\sum_{i\in\mathbb{Z}}|\eta_i|^2=C_\eta$.

Integrating the above inequality from T_k to t, we have

$$\begin{aligned}&e^{\vartheta t}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i|^2+\left(\frac{3\lambda}{2}-\vartheta\right)\int_{T_k}^t e^{\vartheta r}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i|^2dr\\&\quad\leq e^{\vartheta T_k}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i(T_k,\omega,\nu_\tau(s,\omega))|^2\\&\qquad+\frac{4C\nu}{k}\int_{T_k}^t e^{\vartheta r}\|\nu(r,\omega,\nu_\tau(s,\omega))\|^2dr+2C\int_{T_k}^t e^{\vartheta r}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\\&\qquad\cdot\left(1+|y_i|^p+|y_i|\right)dr+\frac{8}{\lambda}C_f^2\int_{T_k}^t e^{\vartheta r}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i(r-\rho)|^2dr\\&\qquad+\frac{8}{\lambda}\left(C_f\varepsilon\right)^2\int_{T_k}^t e^{\vartheta r}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left|y_i\left(\theta_{r-\rho}\omega_i\right)\right|^2dr\\&\qquad+\frac{4}{\lambda}\int_{T_k}^t e^{\vartheta r}\left(C_\eta+\sum_{|i|\geq k}|g_i|^2\right)dr.\end{aligned}\tag{80}$$

It's easy to know that

$$\begin{aligned}&\int_{T_k}^t e^{\vartheta r}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i(r-\rho)|^2dr\leq\int_{T_k-\rho}^t e^{\vartheta(r+\rho)}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i(r)|^2dr\\&\quad=\int_{T_k-\rho}^{T_k}e^{\vartheta(r+\rho)}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i(r)|^2dr+\int_{T_k}^t e^{\vartheta(r+\rho)}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i(r)|^2dr\\&\quad\leq C_1+\int_{T_k}^t e^{\vartheta(r+\rho)}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i(r)|^2dr,\end{aligned}\tag{81}$$

where $C_1=\rho\sum_{i\in\mathbb{Z}}\xi(|i|/k)\sup_{r\in[T_k-\rho,T_k]}e^{\vartheta(r+\rho)}|\nu_i(r)|^2$. And

$$\int_{T_k}^t e^{\vartheta r}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left|y_i\left(\theta_{r-\rho}\omega_i\right)\right|^2dr\leq C_2+\int_{T_k}^t e^{\vartheta(r+\rho)}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left|y_i\left(\theta_r\omega_i\right)\right|^2dr,\tag{82}$$

where $C_2=\rho\sum_{i\in\mathbb{Z}}\xi(|i|/k)\sup_{r\in[T_k-\rho,T_k]}e^{\vartheta(r+\rho)}|y_i(\theta_r\omega_i)|^2$.

Hence, from (80)–(82) and the assumption (A_5), then multiplying both sides of it by $e^{-\vartheta t}$, we get, for $t\geq T_k=T_k(\omega)>\tau$,

$$\begin{aligned}&\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i(t,\omega,\nu_\tau(s,\omega))|^2\leq e^{\vartheta(T_k-t)}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|\nu_i(T_k,\omega,\nu_\tau(s,\omega))|^2\\&\quad+\frac{4C\nu}{k}e^{-\vartheta t}\int_{T_k}^t e^{\vartheta r}\|\nu(r,\omega,\nu_\tau(s,\omega))\|^2dr\\&\quad+2Ce^{-\vartheta t}\int_{T_k}^t e^{\vartheta r}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left(1+|y_i|^p+|y_i|\right)dr\\&\quad+\frac{8}{\lambda}\left(C_f\varepsilon\right)^2e^{-\vartheta t}\int_{T_k}^t e^{\vartheta(r+\rho)}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|y_i|^2dr\\&\quad+\frac{4}{\vartheta\lambda}\left(C_\eta+\sum_{|i|\geq k}|g_i|^2\right)\left(1-e^{\vartheta(T_k-t)}\right)+(C_3+C_4)e^{-\vartheta t},\end{aligned}\tag{83}$$

where $C_3 = (8/\lambda)C_f^2 C_1, C_4 = (8/\lambda)(C_f\varepsilon)^2 C_2$.

We replace ω by $\theta_{-t}\omega$, and estimate each term on the right-hand side of the above inequality. With t instead of T_k in (65), we have

$$\begin{aligned} &e^{\vartheta(T_k-t)}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left|v_i(T_k,\theta_{-t}\omega,v_\tau(s,\theta_{-t}\omega))\right|^2 \\ &\leq e^{\vartheta(T_k-t)}\left\|v(T_k,\tau-T_k,\theta_{-t}\omega,v_\tau(s,\theta_{-t}\omega))\right\|^2 \\ &\leq e^{\vartheta(T_k-t)}\left(\left\|v_\tau(s,\theta_{-t}\omega)\right\|^2 e^{-\epsilon(T_k-\tau)}+\frac{2Cr(\omega)}{\epsilon}+\frac{2}{\lambda\epsilon}\left(C_\eta+2\|g\|^2\right)\right). \end{aligned} \tag{84}$$

As we know, $v_\tau(s,\theta_{-t}\omega)\in K(\theta_{-t}\omega)$, namely, $\|v_\tau(s,\theta_{-t}\omega)\|\leq R(\theta_{-t}\omega)$ (tempered), so there exists a $T_1(\eta,\omega)>T_k(\omega)$, such that when $t>T_1$,

$$e^{\vartheta(T_k-t)}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left|v_i(T_k,\theta_{-t}\omega,v_\tau(s,\theta_{-t}\omega))\right|^2\leq\frac{1}{28}\eta. \tag{85}$$

Also by (65), we estimate

$$\begin{aligned} &\frac{4C\nu}{k}e^{-\vartheta t}\int_{T_k}^{t}e^{\vartheta r}\left\|v(r,\theta_{-t}\omega,v_\tau(s,\theta_{-t}\omega))\right\|^2dr \\ &\leq\frac{4C\nu}{k}e^{-\vartheta t}\left[\frac{e^{\tau\epsilon}}{\vartheta-\epsilon}\left\|v_\tau(s,\theta_{-t}\omega)\right\|^2\left(e^{(\vartheta-\epsilon)t}-e^{(\vartheta-\epsilon)T_k}\right)\right. \\ &\quad\left.+\left(\frac{2Cr(\omega)}{\vartheta\epsilon}+\frac{2}{\lambda\vartheta\epsilon}\left(C_\eta+2\|g\|^2\right)\right)\left(e^{\vartheta t}-e^{\vartheta T_k}\right)\right], \end{aligned} \tag{86}$$

where $\vartheta-\epsilon\neq 0$. Therefore there exist $T_2(\eta,\omega)>T_k(\omega)$ and $N_1(\eta,\omega)>0$, such that as $t>T_2, k>N_1$,

$$\frac{4C\nu}{k}e^{-\vartheta t}\int_{T_k}^{t}e^{\vartheta r}\left\|v(r,\theta_{-t}\omega,v_\tau(s,\theta_{-t}\omega))\right\|^2dr\leq\frac{1}{28}\eta. \tag{87}$$

Using (66), for $k>N_2(\eta,\omega)>0$ we deduce

$$\begin{aligned} &2Ce^{-\vartheta t}\int_{T_k}^{t}e^{\vartheta r}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left(1+|y_i(\theta_{-t}\omega)|^p+|y_i(\theta_{-t}\omega)|\right)dr \\ &\leq\frac{2Cr(\omega)}{\vartheta}e^{-(\epsilon/2)t}\left(1-e^{\vartheta(T_k-t)}\right), \end{aligned} \tag{88}$$

so there is a $T_3(\eta,\omega)>T_k(\omega)$, such that as $t>T_3$,

$$2Ce^{-\vartheta t}\int_{T_k}^{t}e^{\vartheta r}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left(1+|y_i(\theta_{-t}\omega)|^p+|y_i(\theta_{-t}\omega)|\right)dr\leq\frac{1}{28}\eta. \tag{89}$$

Applying (66), we also get

$$\begin{aligned} &\frac{8}{\lambda}(C_f\varepsilon)^2e^{-\vartheta t}\int_{T_k}^{t}e^{\vartheta(r+\rho)}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|y_i(\theta_{-t}\omega)|^2dr \\ &\leq\frac{8}{\lambda}(C_f\varepsilon)^2e^{-\vartheta t}\int_{T_k}^{t}e^{\vartheta(r+\rho)}r(\omega)e^{-(\epsilon/2)t}dr \\ &=\frac{8}{\vartheta\lambda}(C_f\varepsilon)^2r(\omega)e^{-\vartheta t-(\epsilon/2)t}\left(e^{\vartheta(t+\rho)}-e^{\vartheta(T_k+\rho)}\right). \end{aligned} \tag{90}$$

Thus there exists $T_4(\eta,\omega)>T_k(\omega)$, such that as $t>T_4$,

$$\frac{8}{\lambda}(C_f\varepsilon)^2e^{-\vartheta t}\int_{T_k}^{t}e^{\vartheta(r+\rho)}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)|y_i(\theta_{-t}\omega)|^2dr\leq\frac{1}{28}\eta. \tag{91}$$

Since $g\in X$, there is $N_3(\eta,\omega)>0$, we obtain, for $k>N_3(\eta,\omega)>0$,

$$\frac{4}{\vartheta\lambda}\left(C_\eta+\sum_{|i|\geq k}|g_i|^2\right)\leq\frac{1}{28}\eta. \tag{92}$$

By the same argument, we have $T_5(\eta,\omega)>T_k(\omega)$, when $t>T_5$, satisfying

$$\left|-\frac{4}{\vartheta\lambda}\left(C_\eta+\sum_{|i|\geq k}|g_i|^2\right)e^{\vartheta(T_k-t)}\right|\leq\frac{1}{28}\eta. \tag{93}$$

Obviously, there is a $T_6(\eta,\omega)>T_k(\omega)$, when $t>T_6$, fulfilling

$$(C_3+C_4)e^{-\vartheta t}\leq\frac{1}{28}\eta. \tag{94}$$

Take $T(\eta,\omega)=\max\{T_1,T_2,T_3,T_4,T_5,T_6\}<T_k(\omega), N(\eta,\omega)=\max\{N_1,N_2,N_3\}<0$. We collect (83)–(94) to yield, for every $t>T, k>N$,

$$\begin{aligned} &\sup_{-\rho\leq s\leq 0}\sum_{|i|\geq 2k}\left|u_i(s,\theta_{-t}\omega,u_\tau(s,\theta_{-t}\omega))\right|^2 \\ &\leq\sup_{-\rho\leq s\leq 0}\sum_{i\in\mathbb{Z}}\xi\left(\frac{|i|}{k}\right)\left|u_i(s,\theta_{-t}\omega,u_\tau(s,\theta_{-t}\omega))\right|^2 \\ &\leq 2\sup_{-\rho\leq s\leq 0}\sum_{|i|\geq N}\left(\left|v_i(s,\theta_{-t}\omega,v_\tau(s,\theta_{-t}\omega))\right|^2+\varepsilon^2|y_i(\theta_{-t}\omega)|^2\right)\leq\eta, \end{aligned} \tag{95}$$

which concludes the proof.

With the same method like the proof of the Theorem 11 in [8], we can obtain the asymptotic compactness of the continuous RDS Φ.

Theorem 10. *Φ is asymptotically compact for all $\omega\in\Omega'$: each sequence $q_n\in\Phi(t_n,\theta_{-t_n}\omega,K(\tau,\theta_{-t_n}\omega))$ has a convergent subsequence in X_ρ as $t_n\to\infty$.*

We are now in a position to present our main result about the existence of $\mathscr{D}$-pullback attractor.

Theorem 11. *Suppose that assumptions (A_1)–(A_6) hold. Then the continuous RDS Φ associated with problems (1) and (2) has a unique $\mathscr{D}$-pullback attractor, which is characterized by, for each $\omega\in\Omega$,*

$$\mathcal{A}(\omega)=\bigcap_{\kappa\geq t_K(\omega)}\overline{\bigcup_{t\geq\kappa}\Phi(t,\theta_{-t}\omega,K(\theta_{-t}\omega))}. \tag{96}$$

Proof. Note that the existence, uniqueness and characterization of the $\mathscr{D}$-pullback attractor of Φ follows from Proposition 5 immediately, based on Theorem 8. Lemma 9. and Theorem 10. ☐

5. Ergodicity of the Systems

We discuss the ergodicity of the systems in this section.

There exists at least an invariant measure for the random dynamical system $\Phi(t,\omega,u_\tau(s,\omega))$ induced by (23) and (24) (see [10]). The Markov semigroup $\{P_t\}_{t\geq 0}$ associated with them is denoted by

$$P_t\varphi(x) = \mathbb{E}(\varphi(u(t,0,x))), \quad t \geq 0 \quad x \in X_\rho \quad \varphi \in \mathscr{C}_{X_\rho}, \quad (97)$$

where $\mathscr{C}_{X_\rho}$ is a space of bounded and uniformly continuous function on X_ρ.

The related transition probability $P_t(x,\cdot)$ is

$$P_t(x,\Gamma) = \mathscr{L}(u(t,0,x))(\Gamma), \quad t \geq 0 \quad x \in X_\rho \quad \Gamma \in \mathscr{B}_{X_\rho}, \quad (98)$$

where $\mathscr{L}(\cdot)$ is a Banach algebra of all linear bounded operators from X_ρ to X_ρ, and $\mathscr{B}_{X_\rho}$ is a σ-field of all Borel subsets of X_ρ.

Now we prove equation (23) is ergodic, i.e., the invariant measure is unique.

Assume $u(t,\varrho,u_\varrho)$, $\varrho \in \mathbb{R}$, $t > \varrho$ to be a mild solution of equation

$$\frac{du}{dt} + \nu Au + \lambda u = F(u) + f(u_t) + g + \varepsilon\frac{dW}{dt}, \quad (99)$$

with the initial conditions

$$u_\varrho = u(\varrho + s), \quad s \in [-\varrho, 0]. \quad (100)$$

Suppose $\{S(t)\}_{t\geq\varrho}$ is a semigroup formed by $-\nu A - \lambda$, from Theorem 6, we have

$$u(t,\varrho,u_\varrho) = S(t-\varrho)u_\varrho - \int_\varrho^t S(t-h)(F(u) + f(u_t))dh + \varepsilon\int_\varrho^t S(t-h)dW(h). \quad (101)$$

Put $\zeta(t) = \varepsilon\int_\varrho^t S(t-h)dW(h)$ and $z(t) = u(t,\varrho,u_\varrho) - \varepsilon\int_\varrho^t S(t-h)dW(h)$. Then $z(t)$ is a mild solution of

$$\frac{dz}{dt} = -(\nu A + \lambda)z(t) + F(z(t)) + f(z_t) + g. \quad (102)$$

Theorem 12. *The system (23) and (24) is ergodic.*

Proof. We need to prove that there is a unique invariant measure μ for Markov semigroup $\{P_t\}_{t\geq0}$. By the same discussion with a little change like (67), we get.

$$\|z(t)\| \leq \|z_\tau\|e^{(\epsilon/2)(\varrho-t)} + \sqrt{\frac{2Cr(\omega)}{\epsilon} + \frac{2}{\lambda\epsilon}\left(C_\eta + 2\|g\|^2\right)}. \quad (103)$$

We know $\sup_{t\geq\rho}\mathbb{E}\|\zeta(t)\| < +\infty$, then

$$\mathbb{E}\|u(t)\| \leq \mathbb{E}\|z(t)\| + \sup_{t\geq\varrho}\mathbb{E}\|\zeta(t)\| \leq \|z_\tau\|e^{(\epsilon/2)(\varrho-t)} + \sup_{t\geq\varrho}\mathbb{E}\|\zeta(t)\| + \sqrt{\frac{2Cr(\omega)}{\epsilon} + \frac{2}{\lambda\epsilon}\left(C_\eta + 2\|g\|^2\right)}. \quad (104)$$

Let $t \geq 2\varrho$, $s \in [-\varrho, 0]$, then $t + s \geq \varrho$ and

$$\mathbb{E}\|u_t\| \leq \|z_\tau\|e^{(\epsilon/2)(\varrho-t)} + \sup_{t\geq\varrho}\mathbb{E}\|\zeta(t)\| + \sqrt{\frac{2Cr(\omega)}{\epsilon} + \frac{2}{\lambda\epsilon}\left(C_\eta + 2\|g\|^2\right)}. \quad (105)$$

So, for arbitrary $u_{\varrho,1}$, $u_{\varrho,2} \in X_\rho$, $t + s \geq \varrho$, we induce

$$\left\|u(t+s,\varrho,u_{\varrho,1}) - u(t+s,\varrho,u_{\varrho,2})\right\| \leq \left\|u_{\varrho,1} - u_{\varrho,2}\right\|e^{(\epsilon/2)(\varrho-t)}. \quad (106)$$

For $\varrho_0 \in \mathbb{R}$ and $\varrho_0 \leq \varrho < 2\varrho \leq t$, we obtain

$$\mathbb{E}\left\|u(t+s,\varrho,u_\varrho) - u(t+s,\varrho_0,u_\varrho)\right\| \leq \mathbb{E}\left\|u_\varrho - u(\varrho,\varrho_0,u_\varrho)\right\|e^{(\epsilon/2)(\varrho-t)} \leq 2\left(\|z_\tau\| + \sup_{t\geq\varrho}\mathbb{E}\|\zeta(t)\| + \sqrt{\frac{2Cr(\omega)}{\epsilon} + \frac{2}{\lambda\epsilon}\left(C_\eta + 2\|g\|^2\right)}\right)e^{(\epsilon/2)(\varrho-t)}. \quad (107)$$

Hence, we know there is a random variable ϖ such that

$$\lim_{\varrho\to-\infty}\mathbb{E}\left\|u(0,\varrho,u_\varrho) - \varpi\right\| = 0. \quad (108)$$

For arbitrary $u_\varrho \in X_\rho$, we are led to

$$P_t(u_\varrho,\cdot) = \mathscr{L}(u(t,u_\varrho)) = \mathscr{L}(u(0,-t,u_\varrho)) \to \mu. \quad (109)$$

This tells us that, the law $\mu = \mathscr{L}(\varpi)$ is the unique invariant measure for Markov semigroup $\{P_t\}_{t\geq0}$. This accomplishes the proof. ☐

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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