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Research Article

Positive Solutions for a Class of Discrete Mixed Boundary Value Problems with the (p,q)-Laplacian Operator

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In this paper, we consider the existence of solutions for the discrete mixed boundary value problems involving (p,q)-Laplacian operator. By using critical points theory, we obtain the existence of at least two positive solutions for the boundary value problem under appropriate assumptions on the nonlinearity.

1. Introduction

In recent years, with the development of mechanical engineering, control system, computer science, and economics, the existence of solutions of difference equations has attracted wide attention (see [1–6]). For example, applying Ricceri variational principle to obtain the existence of multiple solutions [7–9], taking the invariant sets of descending flow to prove the existence of sign-changing solutions [10], making the linking theorem to get the existence and multiplicity of periodic solutions [11], and using critical point theory to obtain the existence of homoclinic solutions [12–15] and heteroclinic solutions [16].

As we know, the fixed-point method and upper and lower solution techniques are important tools to solve the existence of solutions for boundary value problems (see [17, 18]). But recently, it is more common to use critical point theory to study Dirichlet boundary value problems (see [19–23]). More result on difference equations by using critical point theories can be referred to [24–27].

In [28], D'Aguì et al. established the existence of at least two positive solutions for the following discrete Dirichlet boundary value problem:

$$\begin{cases} -\Delta\left(\phi_p(\Delta u(k-1))\right) + q(k)\phi_p(u(k)) = \lambda f(k,u(k)), & k \in \mathbb{Z}(1,N), \\ u(0) = u(N+1) = 0, \end{cases} \tag{1}$$

where $q(k) \geq 0$ for all $k \in \{1,2,\ldots,N\}$.

Unlike this, D'Aguì et al. in [29] proved that there are at least two nonzero weak solutions for the following mixed boundary value problem:

$$\begin{cases} -\left(q(x)|u'(k)|^{p-2}u'(x)\right)' + s(x)|u(k)|^{p-2}u(x) = \lambda f(x,u(x)), & x \in [a,b], \\ u(a) = u'(b) = 0, \end{cases} \tag{2}$$

where $p > 1, q, s \in L^{\infty}([a,b])$ with $q_0 = \operatorname{ess\,inf}_{[a,b]} q > 0, s_0 = \operatorname{ess\,inf}_{[a,b]} s \geq 0$, $f\colon [a,b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is an L^1-Carathéodory function and λ is a real positive parameter.

As a discrete analogy of the abovementioned problem, we consider the existence of positive solutions for the following discrete mixed boundary value problem:

$$\begin{cases} -\Delta\left(\phi_p(\Delta u(k-1))\right) + s(k)\phi_q(u(k)) = \lambda f(k, u(k)), k \in \mathbb{Z}(1, N), \\ u(0) = \Delta u(N) = 0, \end{cases} \quad \left(D_\lambda^f\right) \tag{3}$$

where $\mathbb{Z}(a, b)$ denote the discrete interval $\{a, a+1, \ldots, b\}$ for any integers a and b with $a < b$, N be a positive integer, $f(k, u)$ is continuous in u for each $k \in \mathbb{Z}(1, N)$, $\Delta u(k) = u(k+1) - u(k)$ is the forward difference operator, $\phi_r\colon \mathbb{R} \longrightarrow \mathbb{R}$ is the r-Laplacian given by $\phi_r(u) = |u|^{r-2}u$ with $u \in \mathbb{R}$, $1 < q \le p < +\infty$, $s(k) \ge 0$ for all $k \in \mathbb{Z}(1, N)$, and λ is a positive parameter.

In this paper, under suitable assumptions on the nonlinearity *f*, we use the theory of two nonzero critical points (see [30]) to ensure that there are at least two nonzero solutions for problem (D_λ^f). The two nonzero critical points theorem is an appropriate combination of local minimum theorem (see [31]) and classical Ambrosetti–Rabinowitz theorem (see [32]). An important hypothesis of mountain pass theorem is Palais–Smale condition. It satisfies the application of infinite dimensional space by requiring the condition that the nonlinear term is stronger than p-superlinearity at infinity. In order to obtain the existence of two nonzero solutions, we can assume the classical Ambrosetti–Rabinowitz condition and nonlinear algebraic condition (see (40) in Theorem 2) hold, that is, more widespread than the p-sublinearity at zero. Moreover, when we require that $f(k, 0) \ge 0$ for all $k \in \mathbb{Z}(1, N)$, we can use strong maximum principle to obtain the existence of positive solutions, which has been proved in Lemma 2.

Let $s_* = \min\{s(k)\colon k \in \mathbb{Z}(1, N)\}$, a special case of our main result is stated as follows.

Theorem 1. *Let $f\colon \mathbb{R} \longrightarrow \mathbb{R}$ be a continuous function such that*

$$\lim_{t \longrightarrow 0^+} \frac{f(t)}{t^{p-1}} \tag{4(a)}$$

$$\lim_{t \longrightarrow +\infty} \frac{f(t)}{t^{p-1}} \tag{4(b)}$$

then, for each $\lambda \in (0, (1 + s_* N^{p-1}/pN^p) min\ \{sup_{c>0}(c^q/max_{|\xi| \le c} \int_0^\xi f(t)dt), sup_{c>0}(c^p/max_{|\xi| \le c} \int_0^\xi f(t)dt)\})$, the problem

$$\begin{cases} -\Delta\left(\phi_p(\Delta u(k-1))\right) + s(k)\phi_q(u(k)) = \lambda f(u(k)), & k \in \mathbb{Z}(1, N), \\ u(0) = \Delta u(N) = 0, \end{cases} \tag{5}$$

admits at least two positive solutions.

The structure of the article is as follows. In Section 2, some basic definitions and properties are given. In Section 3, we give the main results. Under suitable hypothesis, Lemma 1 is used to obtain that the problem (D_λ^f) possesses at least two positive solutions. Finally, some examples are given to illustrate our main results.

2. Preliminaries

In this section, we recall some definitions, notations, and properties. Consider the N-dimensional Banach space:

$$S = \{u : \mathbb{Z}(0, N+1) \longrightarrow \mathbb{R} : u(0) = \Delta u(N) = 0\}, \tag{6}$$

and define the norm

$$\|u\| = \left(\sum_{k=1}^{N+1} |\Delta u(k-1)|^p \right)^{1/p}, \tag{7}$$

and $\|u\|_\infty = \max\{|u(k)| : k \in \mathbb{Z}(1, N)\}$ is another norm in S.

Proposition 1. *The following inequality holds:*

$$\|u\|_\infty \le \max\left\{ \left(\frac{pN^{p-1}}{1 + s_* N^{p-1}} \right)^{1/q} \left(\frac{\|u\|^p}{p} + \frac{\sum_{k=1}^N s(k)|u(k)|^q}{q} \right)^{1/q}, \left(\frac{pN^{p-1}}{1 + s_* N^{p-1}} \right)^{1/p} \left(\frac{\|u\|^p}{p} + \frac{\sum_{k=1}^N s(k)|u(k)|^q}{q} \right)^{1/p} \right\}. \tag{8}$$

Proof. Let $u \in S$, then there exist $k^* \in \mathbb{Z}(1, N)$ such that $|u(k^*)| = \max\{|u(k)| : k \in \mathbb{Z}(1, N)\}$.

Since

$$\left|u(k^*)\right| = \left| \sum_{k=1}^{k^*} \Delta u(k-1) \right| \le \left(\sum_{k=1}^{k^*} 1 \right)^{1-1/p} \left(\sum_{k=1}^{k^*} |\Delta u(k-1)|^p \right)^{1/p} \le N^{1-1/p} \|u\|, \tag{9}$$

then

$$\|u\|_\infty^p \le N^{p-1}\|u\|^p. \quad (10)$$

If $\|u\|_\infty > 1$, then

$$\begin{aligned}\frac{(1+s_* N^{p-1})\|u\|_\infty^q}{p} &\le \frac{\|u\|_\infty^p + N^{p-1}\sum_{k=1}^N s(k)|u(k)|^q}{p}\\ &\le \frac{N^{p-1}}{p}\left(\|u\|^p + \sum_{k=1}^N s(k)|u(k)|^q\right) \le \frac{N^{p-1}}{p}\|u\|^p + \frac{N^{p-1}}{q}\sum_{k=1}^N s(k)|u(k)|^q,\end{aligned} \quad (11)$$

that is,

$$\|u\|_\infty \le \left(\frac{pN^{p-1}}{1+s_* N^{p-1}}\right)^{1/q}\left(\frac{\|u\|^p}{p} + \frac{\sum_{k=1}^N s(k)|u(k)|^q}{q}\right)^{1/q}. \quad (12)$$

If $\|u\|_\infty \le 1$, then

$$\begin{aligned}\frac{(1+s_* N^{p-1})u_\infty^p}{p} &\le \frac{\|u\|_\infty^p + N^{p-1}\sum_{k=1}^N s(k)|u(k)|^q}{p}\\ &\le \frac{N^{p-1}}{p}\|u\|^p + \frac{N^{p-1}}{q}\sum_{k=1}^N s(k)|u(k)|^q,\end{aligned} \quad (13)$$

that is,

$$\|u\|_\infty \le \left(\frac{pN^{p-1}}{1+s_* N^{p-1}}\right)^{1/p}\left(\frac{\|u\|^p}{p} + \frac{\sum_{k=1}^N s(k)|u(k)|^q}{q}\right)^{1/p}. \quad (14)$$

In summary, we have

$$\|u\|_\infty \le \max\left\{\left(\frac{pN^{p-1}}{1+s_* N^{p-1}}\right)^{1/q}\left(\frac{\|u\|^p}{p} + \frac{\sum_{k=1}^N s(k)|u(k)|^q}{q}\right)^{1/q}, \left(\frac{pN^{p-1}}{1+s_* N^{p-1}}\right)^{1/p}\left(\frac{\|u\|^p}{p} + \frac{\sum_{k=1}^N s(k)|u(k)|^q}{q}\right)^{1/p}\right\}. \quad (15)$$

Put

$$F(k,t) := \int_0^t f(k,\xi)\,d\xi, \quad \forall (k,t) \in \mathbb{Z}(1,N)\times\mathbb{R}, \quad (16)$$

and consider the function $J_\lambda: S \longrightarrow \mathbb{R}$ for all $\lambda > 0$ by

$$J_\lambda = \Phi - \lambda\Psi, \quad (17)$$

where

$$\begin{aligned}\Phi &:= \Phi_1 + \Phi_2, \Phi_1(u) = \frac{\|u\|^p}{p},\\ \Phi_2(u) &= \frac{\sum_{k=1}^N s(k)|u(k)|^q}{q},\\ \Psi(u) &:= \sum_{k=1}^N F(k,u(k)).\end{aligned} \quad (18)$$

It is clear that $\Phi_1, \Phi_2, \Psi \in C^1(S,\mathbb{R})$ and their Gâteaux derivatives at the point $u \in S$ are given by

$$\begin{aligned}\Phi_1'(u)(v) &= \sum_{k=1}^{N+1}\phi_p(\Delta u(k-1))v(k)\\ &= -\sum_{k=1}^{N+1}\phi_p(\Delta u(k-1))v(k-1)\\ &= \sum_{k=1}^{N}\phi_p(\Delta u(k-1))v(k) - \sum_{k=0}^{N}\phi_p(\Delta u(k))v(k)\\ &= -\sum_{k=1}^{N}\Delta\phi_p(\Delta u(k-1))v(k),\\ \Phi_2'(u)(v) &= \sum_{k=1}^{N} s(k)\phi_q(u(k))v(k),\\ \Psi'(u)(v) &= \sum_{k=1}^{N} f(k,u(k))v(k),\end{aligned} \quad (19)$$

for all $u, v \in S$. So, we have

$$J_\lambda'(u)(v) = \sum_{k=1}^{N}\left[-\Delta\left(\phi_p(\Delta u(k-1))\right) + s(k)\phi_q(u(k)) - \lambda f(k, u(k))\right]v(k). \quad (20)$$

Hence, a critical point u of J_λ is a solution of problem (D_λ^f).

Now, we recall a definition and a two nonzero critical points theorem for the reader's convenience. □

Definition 1. Let X be a real Banach space; we say that a Gâteaux differentiable function $J_\lambda: X \longrightarrow \mathbb{R}$ satisfies the (PS)-condition, if any sequence $\{u_n\}_{n\in N} \subseteq X$ such that

(i) $J_\lambda(u_n) \longrightarrow c \in \mathbb{R}$, as $n \longrightarrow +\infty$

(ii) $J_\lambda'(u_n) \longrightarrow 0$, as $n \longrightarrow +\infty$, has a convergent subsequence

Lemma 1. *Let X be a real Banach space and $\Phi, \Psi \in C^1(S, \mathbb{R})$ such that $\inf_X(\Phi) = \Phi(0) = \Psi(0) = 0$. Assume that there are $r \in \mathbb{R}$ and $\omega \in X$, with $0 < \Phi(\omega) < r$, such that*

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u)}{r} < \frac{\Psi(\omega)}{\Phi(\omega)}, \quad (21)$$

and for each

$$\lambda \in \Lambda = \left(\frac{\Phi(\omega)}{\Psi(\omega)}, \frac{r}{\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u)}\right), \quad (22)$$

the functional $J_\lambda = \Phi - \lambda\Psi$ satisfies the (PS)-condition and it is unbounded from below.

Then, for each $\lambda \in \Lambda$, the functional J_λ admits at least two nonzero critical points $u_{\lambda,1}, u_{\lambda,2}$ such that $J_\lambda(u_{\lambda,1}) < 0 < J_\lambda(u_{\lambda,2})$.

In order to obtain the positive solution of problem (D_λ^f), we establish the following strong maximum principle.

Lemma 2. *Fix $u \in S$ such that either*

$$u(k) > 0 \text{ or } -\Delta\left(\phi_p(\Delta u(k-1))\right) + s(k)\phi_q(u(k)) \geq 0, \quad (23)$$

for all $k \in \mathbb{Z}(1, N)$. Then, either $u > 0$ in $\mathbb{Z}(1, N)$ or $u \equiv 0$.

Proof. Let $j \in \mathbb{Z}(1, N)$ such that

$$u(j) = \min\{u(k): k \in \mathbb{Z}(1, N)\}. \quad (24)$$

If $u(j) > 0$, then it is easy know that $u > 0$ in $\mathbb{Z}(1, N)$. If $u(j) \leq 0$, then by (23), we have

$$-\Delta\left(\phi_p(\Delta u(j-1))\right) \geq -s(j)\phi_q(u(j)) \geq 0, \quad (25)$$

that is,

$$\phi_p(\Delta u(j)) \leq \phi_p(\Delta u(j-1)). \quad (26)$$

Since $\phi_p(u)$ is increasing in u, we have

$$\Delta u(j) \leq \Delta u(j-1). \quad (27)$$

By the definition of $u(j)$, we know that

$$\begin{aligned} \Delta u(j) &\geq 0, \\ \Delta u(j-1) &\leq 0. \end{aligned} \quad (28)$$

By combining (27) with (28), we get $u(j+1) = u(j) = u(j-1)$. If $j-1 = 0$, we have $u(j) = u(j-1) = 0$. Otherwise, $j-1 \in \mathbb{Z}(1, N)$, replacing $j-1$ by j, we know $u(j-2) = u(j-1)$. Continuing in this way, we have $u(j) = u(j-1) = \cdots = u(0) = 0$. Similarly, we have $u(j) = u(j+1) = \cdots = u(N+1)$. Thus, $u(k) = u(0) = 0$ and $\forall k \in \mathbb{Z}(1, N)$.

Now, put

$$F^+(k, t) = \int_0^t f(k, \xi^+)\mathrm{d}\xi, \quad \forall (k, t) \in \mathbb{Z}(1, N) \times \mathbb{R}, \quad (29)$$

where $\xi^+ = \max\{\xi, 0\}$.

Define $J_\lambda^+ = \Phi_1 + \Phi_2 - \lambda\Psi^+$ and $\Psi^+(u): = \sum_{k=1}^N F^+(k, u(k))$. Standard arguments show that $J_\lambda^+ \in C^1(S, R)$ and the critical points of J_λ^+ are precisely the solutions of the following problem:

$$\begin{cases} -\Delta\left(\phi_p(\Delta u(k-1))\right) + s(k)\phi_q(u(k)) = \lambda f(k, u^+(k)), & k \in \mathbb{Z}(1, N), \\ u(0) = \Delta u(N) = 0. \end{cases} \quad \left(D_\lambda^{f^+}\right) \quad (30)$$

□

Lemma 3. *If $f(k, 0) \geq 0$ for each $k \in \mathbb{Z}(1, N)$, any nonzero critical point of the functional J_λ^+ is a positive solution of problem (D_λ^f).*

Proof. Since a critical point of J_λ^+ is a solution of problem $D_\lambda^{f^+}$, the conclusion follows by the discrete maximum principle ([33], Proposition 1).

Next, we suppose that $f(k,0) \geq 0$ and $f(k,x) = f(k,0)$ for all $x \leq 0$ and for all $k \in \mathbb{Z}(1,N)$. Put

$$L_\infty := \min_{k\in\mathbb{Z}(1,N)} \liminf_{t\longrightarrow+\infty} \frac{F(k,t)}{t^p},$$
$$\tilde{s} = \sum_{k=1}^{N} s(k), \tag{31}$$

we have the following result. □

Lemma 4. *If $L_\infty > 0$, then J_λ satisfies (PS)-condition and it is unbounded from below for all $\lambda \in (2^p N + \tilde{s} - 2^{p-1}/qL_\infty, +\infty)$.*

Proof. Let $\lambda > 2^p N + \tilde{s} - 2^{p-1}/qL_\infty$. We consider a sequence $\{u_n\}_{n\in N} \subseteq S$ such that $J_\lambda(u_n) \longrightarrow c \in \mathbb{R}$ and $J'_\lambda(u_n) \longrightarrow 0$, as $n \longrightarrow +\infty$. Let $u_n^+ = \max\{u_n, 0\}$ and $u_n^- = \max\{-u_n, 0\}$ for all $n \in N$. We first prove that $\{u_n^-\}$ is bounded. On one hand, we have

$$\left|\Delta u_n^-(k-1)\right|^p \leq -\phi_p(\Delta u_n(k-1))\Delta u_n^-(k-1),$$
$$s(k)|u_n^-(k)|^q = -s(k)|u_n(k)|^{q-2}u_n(k)u_n^-(k), \tag{32}$$

for all $k \in \mathbb{Z}(1,N)$. So,

$$\begin{aligned}\|u_n^-\|^p &= \sum_{k=1}^{N+1}\left|\Delta u_n^-(k-1)\right|^p \\ &\leq -\sum_{k=1}^{N+1}\phi_p(\Delta u_n(k-1))u_n^-(k) + \sum_{k=1}^{N=1}\phi_p(\Delta u_n(k-1))u_n^-(k-1) \\ &= -\sum_{k=1}^{N}\phi_p(\Delta u_n(k-1))u_n^-(k) + \sum_{k=1}^{N}\phi_p(\Delta u_n(k))u_n^-(k) \\ &= \sum_{k=1}^{N}\Delta\phi_p(\Delta u_n(k-1))u_n^-(k) = -\Phi_1'(u_n)(u_n^-),\end{aligned} \tag{33}$$
$$\sum_{k=1}^{N} s(k)\left|u_n^-(k)\right|^q = -\sum_{k=1}^{N} s(k)\phi_q(u_n(k))u_n^-(k) = -\Phi_1'(u_n)(u_n^-).$$

On the other hand, we assume that

$$f(k,u) = \begin{cases} f(k,u), & \text{if } u > 0, \\ f(k,0), & \text{if } u \leq 0, \end{cases} \tag{34}$$

for each $k \in \mathbb{Z}(1,N)$, then

$$\Psi'(u_n)(u_n^-) = \sum_{k=1}^{N} f(k,u_n(k))u_n^-(k) \geq 0. \tag{35}$$

Therefore,

$$\begin{aligned}\|u_n^-\|^p &\leq \|u_n^-\|^p + \sum_{k=1}^{N} s(k)|u(k)|^q \\ &\leq -\Phi_1'(u_n)(u_n^-) - \Phi_2'(u_n)(u_n^-) + \lambda\Psi'(u_n)(u_n^-) \\ &= -J_\lambda'(u_n)(u_n^-),\end{aligned} \tag{36}$$

for all $n \in N$, which leads to $\|u_n^-\|^{p-1} \longrightarrow 0$ as $n \longrightarrow +\infty$. So, we have $\|u_n^-\| \longrightarrow 0$ as $n \longrightarrow +\infty$. It means that there exists an $M > 0$ such that $u_n^- \leq M$. From (10) we know that $\|u_n^-\|_\infty \leq N^{1-1/p}M = \gamma$, for all $k \in \mathbb{Z}(1,N)$.

Next, we suppose that the sequence $\{u_n\}$ is unbounded, that is, $\{u_n^+\}$ is unbounded.

As $L_\infty > 0$, we know that there exists an $l \in R$ such that $L_\infty > l > 2^p N + \tilde{s} - 2^{p-1}/\lambda q$. From the definition of L_∞, there is $\delta_k > 0$ such that $F(k,t) > l|t|^p$ for all $t > \delta_k$. Furthermore, since $F(k,t)$ is a continuous function, there exists a constant $C(k) \geq 0$ such that $F(k,t) \geq l|t|^p - C(k)$ with $t \in [-\gamma, \delta_k]$. Thus, $F(k,t) \geq l|t|^p - C(k)$ for all $s \geq -\gamma$ and $k \in \mathbb{Z}(1,N)$. We can obtain that

$$\sum_{k=1}^{N} F(k,u_n(k)) \geq \sum_{k=1}^{N} l|u_n(k)|^p - C \geq l\|u_n\|_\infty^p - C, \tag{37}$$

for all $k \in \mathbb{Z}(1,N)$, where $C = \sum_{k=1}^{N} C(k)$, that is,

$$\Psi(u_n) \geq l\|u_n\|_\infty^p - C. \tag{38}$$

Hence, for all u_n such that $\|u_n\|_\infty \geq 1$, we conclude that

$$
\begin{aligned}
J_\lambda(u_n) &= \frac{\sum_{k=1}^{N+1}\left|\Delta u_n(k-1)\right|^p}{p} + \frac{\sum_{k=1}^{N} s(k)\left|u_n(k)\right|^q}{q} - \lambda\Psi(u_n) \\
&\le \frac{2^{p-1}}{p}\left(\sum_{k=1}^{N}\left|u_n(k)\right|^p + \sum_{k=1}^{N}\left|u_n(k-1)\right|^p\right) \\
&\quad + \frac{1}{q}\sum_{k=1}^{N} s(k)\left|u_n(k)\right|^q - \lambda\Psi(u_n) \\
&\le \frac{2^{p-1}(2N-1)}{p}\left\|u_n\right\|_\infty^p + \frac{\tilde{s}}{q}\left\|u_n\right\|_\infty^q - \lambda\Psi(u_n) \\
&\le \left(\frac{2^pN + \tilde{s} - 2^{p-1}}{q} - \lambda l\right)\left\|u_n\right\|_\infty^p + \lambda C.
\end{aligned}
\tag{39}
$$

Since $2^pN + \tilde{s} - 2^{p-1}/q - \lambda l < 0$, we can get $\lim_{n\longrightarrow+\infty} J_\lambda(u_n) = -\infty$ and this is absurd. Hence, J_λ satisfies (PS)-condition.

Let $\{u_n\}$ be such that $\{u_n^-\}$ is bounded and $\{u_n^+\}$ is unbounded. From the proof above we can see that J_λ is unbounded from below. □

3. Main Results

The main results of this paper are as follows.

Theorem 2. *Let $f\colon \mathbb{Z}(1,N)\times\mathbb{R}\longrightarrow\mathbb{R}$ be a continuous function satisfying $f(k,0)\ge 0$ for all $k\in\mathbb{Z}(1,N)$. If there are two constants c and d with $d<c$ such that*

$$
\begin{aligned}
&\frac{pN^{p-1}}{1+s_*N^{p-1}}\sum_{k=1}^{N}\max_{|\xi|\le c}F(k,\xi)\max\left\{\frac{1}{c^q},\frac{1}{c^p}\right\} \\
&\quad < \min\left\{\frac{\sum_{k=1}^{N}F(k,d)}{d^pp^{-1}+d^qq^{-1}\tilde{s}},\frac{qL_\infty}{2^pN+\tilde{s}-2^{p-1}}\right\}.
\end{aligned}
\tag{40}
$$

Then, for each $\lambda\in\Lambda_1$ with

$$
\Lambda_1 := \left(\max\left\{\frac{d^pp^{-1}+d^qq^{-1}\tilde{s}}{\sum_{k=1}^{N}F(k,d)},\frac{2^pN+\tilde{s}-2^{p-1}}{qL_\infty}\right\},\frac{\left(1+s_*N^{p-1}/pN^{p-1}\right)\min\{c^q,c^p\}}{\sum_{k=1}^{N}\max_{|\xi|\le c}F(k,\xi)}\right),
\tag{41}
$$

the problem (D_λ^f) admits at least two positive solutions.

Proof. Put Φ,Ψ as in (18). It is clear that $\inf_X(\Phi)=\Phi(0)=\Psi(0)=0$. According to Lemma 3, we know that a nonzero critical point in S of the functional J_λ^+ is precisely a positive solution of problem (D_λ^f). Next, we just need to prove condition (21) of Lemma 1.

We observe that $L_\infty>0$ from (40) and Λ_1 is nondegenerate. Fix $\lambda\in\Lambda_1$, Lemma 4 ensures that J_λ satisfies (PS)-condition for all $\lambda > 2^pN+\tilde{s}-2^{p-1}/qL_\infty$ and it is unbounded from below. We let $u\in\Phi^{-1}(-\infty,r]$, that is, $(\|u\|^p/p)+\left(\sum_{k=1}^{N}s(k)|u(k)|^q/q\right)\le r$. Put

$$
r = \frac{1+s_*N^{p-1}}{pN^{p-1}}\min\{c^q,c^p\}.
\tag{42}
$$

If $r=(1+s_*N^{p-1}/pN^{p-1})c^q$, it means that $c\ge 1$. According to (8), we obtain

$$
|u(k)|\le\|u\|_\infty\le\max\left\{\left(\frac{pN^{p-1}}{1+s_*N^{p-1}}\right)^{1/q}r^{1/q},\left(\frac{pN^{p-1}}{1+s_*N^{p-1}}\right)^{1/p}r^{1/p}\right\}=\max\left\{c,c^{q/p}\right\}=c.
\tag{43}
$$

If $r=(1+s_*N^{p-1}/pN^{p-1})c^p$, we know $0<c<1$, then

$$
|u(k)|\le\|u\|_\infty\le\max\left\{\left(\frac{pN^{p-1}}{1+s_*N^{p-1}}\right)^{1/q}r^{1/q},\left(\frac{pN^{p-1}}{1+s_*N^{p-1}}\right)^{1/p}r^{1/p}\right\}=\max\left\{c^{p/q},c\right\}=c.
\tag{44}
$$

To sum up, we know that $|u(k)|\le c$ for all $k\in\mathbb{Z}(1,N)$. Furthermore, we have

$$
\Psi(u)=\sum_{k=1}^{N}F(k,u(k))\le\sum_{k=1}^{N}\max_{|\xi|\le c}F(k,\xi),
\tag{45}
$$

for all $u\in S$ with $\Phi(u)\le r$. Hence,

$$
\frac{\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u)}{r}\le\frac{pN^{p-1}}{1+s_*N^{p-1}}\sum_{k=1}^{N}\max_{|\xi|\le c}F(k,\xi)\max\left\{\frac{1}{c^q},\frac{1}{c^p}\right\}.
\tag{46}
$$

Now, let $\omega(k)=d$ for all $k\in\mathbb{Z}(1,N)$ and $\omega(0)=\Delta\omega(N)=0$. Clearly, $\omega\in S$. It is easy to account that $\Phi(\omega)=d^p p^{-1}+d^q q^{-1}\tilde{s}$, then

$$\frac{\Psi(\omega)}{\Phi(\omega)}=\frac{\sum_{k=1}^{N}F(k,d)}{d^p p^{-1}+d^q q^{-1}\tilde{s}}. \tag{47}$$

Consequently, from (46), (47), and assumption (40), we can obtain

$$\frac{\sup_{u\in\Phi^{-1}(-\infty,r]}\Psi(u)}{r}<\frac{\Psi(\omega)}{\Phi(\omega)}. \tag{48}$$

Moreover, because $0<d<c$ and from (40), we obtain

$$0<d^p p^{-1}+d^q q^{-1}\tilde{s}<\frac{1+s_*N^{p-1}}{pN^{p-1}}\min\{c^q,c^p\}, \tag{49}$$

that is mean that $0<\Phi(\omega)<r$.

Hence, the problem (D_λ^f) admits at least two positive solutions by Lemma 1 and Lemma 3 for all $\lambda\in\Lambda_1$. □

Remark 1. If $f(k,t)$ is a nonnegative function and there are two positive constants c,d with $d<c$ such that

$$\begin{aligned}&\frac{pN^{p-1}}{1+s_*N^{p-1}}\max\left\{\frac{\sum_{k=1}^{N}F(k,c)}{c^q},\frac{\sum_{k=1}^{N}F(k,c)}{c^p}\right\}\\&<\min\left\{\frac{\sum_{k=1}^{N}F(k,d)}{d^p p^{-1}+d^q q^{-1}\tilde{s}},\frac{qL_\infty}{2^pN+\tilde{s}-2^{p-1}}\right\},\end{aligned} \tag{50}$$

then the result of Theorem 2 is also valid for each $\lambda\in\Lambda_2$ with

$$\Lambda_2:=\left(\max\left\{\frac{d^p p^{-1}+d^q q^{-1}\tilde{s}}{\sum_{k=1}^{N}F(k,d)},\frac{2^pN+\tilde{s}-2^{p-1}}{qL_\infty}\right\},\frac{1+s_*N^{p-1}}{pN^{p-1}}\min\left\{\frac{c^q}{\sum_{k=1}^{N}F(k,c)},\frac{c^p}{\sum_{k=1}^{N}F(k,c)}\right\}\right). \tag{51}$$

There are some consequences of Theorem 2 as follows.

Corollary 1. *Let $g:\mathbb{R}\longrightarrow[0,+\infty)$ be a continuous function such that $f(k,t)=\alpha(k)g(t)$, where $\alpha(k)>0$ for all $k\in\mathbb{Z}(1,N)$. Put $A=\sum_{k=1}^{N}\alpha(k)$, $G(t)=\int_0^t g(\xi)d\xi$ for all $t\in\mathbb{R}$ and $L_\infty^*:=\min_{k\in[1,N]}\alpha(k)\liminf_{t\longrightarrow+\infty}(G(t)/t^p)>0$.*

If there exists $c>d>0$ such that

$$\begin{aligned}&\frac{pN^{p-1}}{1+s_*N^{p-1}}AG(c)\max\left\{\frac{1}{c^q},\frac{1}{c^p}\right\}\\&<\min\left\{\frac{AG(d)}{d^p p^{-1}+d^q q^{-1}\tilde{s}},\frac{qL_\infty^*}{2^pN+\tilde{s}-2^{p-1}}\right\},\end{aligned} \tag{52}$$

then the problem D_λ^f has at least two positive solutions for each $\lambda\in\Lambda_3$ with

$$\Lambda_3:=\left(\max\left\{\frac{d^p p^{-1}+d^q q^{-1}\tilde{s}}{AG(d)},\frac{2^pN+\tilde{s}-2^{p-1}}{qL_\infty^*}\right\},\frac{1+s_*N^{p-1}}{pN^{p-1}}\frac{\min\{c^q,c^p\}}{AG(c)}\right). \tag{53}$$

Proof. Consider the function $f:\mathbb{Z}(1,N)\times\mathbb{R}\longrightarrow\mathbb{R}$ is given as

$$f(k,\xi)=\alpha(k)g(\xi),\quad \forall k\in\mathbb{Z}(1,N),\ \xi\in\mathbb{R}, \tag{54}$$

so that

$$\sum_{k=1}^{N}\max_{|\xi|\le c}F(k,\xi)=AG(c),\ \sum_{k=1}^{N}F(k,d)=AG(d). \tag{55}$$

Then, the conclusion can be obtained by Theorem 2. □

Corollary 2. *Assume f be a continuous function with $f(k,0)\ge 0$ and*

$$\limsup_{t\longrightarrow 0^+}\frac{F(k,t)}{t^p}=+\infty, \tag{56}$$

$$\lim_{t\longrightarrow+\infty}\frac{F(k,t)}{t^p}=+\infty, \tag{57}$$

for all $k\in\mathbb{Z}(0,N)$. Put $\lambda^*=(1+s_*N^{p-1}/pN^{p-1})min\{sup_{c>0}(c^q/\sum_{k=1}^{N}max_{|\xi|\le c}F(k,\xi)), sup_{c>0}(c^p/\sum_{k=1}^{N}max_{|\xi|\le c}F(k,\xi))\}$. Then, for each $\lambda\in(0,\lambda^*)$, the problem (D_λ^f) admits at least two positive solutions.

Proof. We know that $L_\infty=+\infty$ from (57). Fix $\lambda\in(0,\lambda^*)$, and then there exists $c>0$ such that

$$\begin{aligned}\lambda&<\frac{1+s_*N^{p-1}}{pN^{p-1}}\\&\cdot\min\left\{\sup_{c>0}\frac{c^q}{\sum_{k=1}^{N}\max_{|\xi|\le c}F(k,\xi)},\sup_{c>0}\frac{c^p}{\sum_{k=1}^{N}\max_{|\xi|\le c}F(k,\xi)}\right\}.\end{aligned} \tag{58}$$

From (56), we can also obtain

$$\limsup_{t\longrightarrow 0^+}\frac{\sum_{k=1}^{N}F(k,t)}{t^p}=+\infty, \tag{59}$$

and then there exists $d\in(0,c)$ such that $(\sum_{k=1}^{N}F(k,d)/d^p p^{-1}+d^q q^{-1}\tilde{s})>(1/\lambda)$. Therefore, Theorem 2 ensures the conclusion. □

Remark 2. If $f(k,t)$ is a nonnegative function for all $(k,t) \in \mathbb{Z}(1,N) \times [0,+\infty)$ As long as condition (56) holds for at least one $k \in \mathbb{Z}(1,N)$, then Corollary 2 ensures that the solutions are obtained for each

$$\lambda \in \left(0, \frac{1+s_* N^{p-1}}{pN^{p-1}} \min\left\{\sup_{c>0} \frac{c^q}{\sum_{k=1}^N F(k,c)}, \sup_{c>0} \frac{c^p}{\sum_{k=1}^N F(k,c)}\right\}\right). \tag{60}$$

Remark 3. When $f(k,t) = f(t)$ for all $k \in \mathbb{Z}(1,N)$, Theorem 1 can be ensured by Corollary 2. Obviously, condition (4(a)) implies $f(0) \geq 0$. Specially, if f is nonnegative, we only need condition (4(a)) to get the corresponding solution for each

$$\lambda \in \left(0, \frac{1+s_* N^{p-1}}{pN^{p}} \min\left\{\sup_{c>0} \frac{c^q}{F(c)}, \sup_{c>0} \frac{c^p}{F(c)}\right\}\right). \tag{61}$$

Example 1. Let $p = 4, q = 2, N = 3, s(k) = 12$, and $f(k,t) = e^t$.

Put

$$x(c) = \frac{c^2}{e^c - 1}. \tag{62}$$

Then,

$$x'(c) = \frac{c((2-c)e^c - 2)}{(e^c - 1)^2}. \tag{63}$$

Let $z(c) = (2-c)e^c - 2$, then $z'(c) = (1-c)e^c$. So, $z(c)$ is increasing in $c \in (0,1)$ and decreasing in $c \in (1,+\infty)$. Since $z(0) = 0$ and $z(+\infty) = -\infty$, there exists an unique $c_1 \in (1,+\infty)$ such that $z(c_1) = 0$. Thus, $x(c)$ in increasing in $c \in (0,c_1)$ and decreasing in $c \in (c_1,+\infty)$. This means that $\sup_{c>0} x(c) = x(c_1)$. In fact, $c_1 \approx 1.5936$.

Similarly, put $y(c) = c^4/e^c - 1$, we can show that there exists a unique $c_2 \in (3,+\infty)$ such that $\sup_{c>0} y(c) = y(c_2)$. In fact, $c_2 \approx 3.9207$.

Since

$$y(c_2) = \frac{c_2^4}{e^{c_2} - 1} > \frac{c_1^4}{e^{c_1} - 1} = c_1^2 x(c_1) > x(c_1), \tag{64}$$

then

$$\begin{aligned} &\frac{1+s_* N^{p-1}}{pN^p} \min\left\{\sup_{c>0} x(c), \sup_{c>0} y(c)\right\} \\ &= \frac{1+s_* N^{p-1}}{pN^p} x(c_1) \approx 0.6496. \end{aligned} \tag{65}$$

Therefore, for each $\lambda \in (0, 0.6496)$, the problem

$$\begin{cases} -\Delta(\phi_4(\Delta u(k-1))) + 12\phi_2(u(k)) = \lambda e^{u(k)}, & k = 1,2,3, \\ u(0) = \Delta u(3) = 0, \end{cases} \tag{66}$$

admits at least two positive solutions.

Example 2. Let $N = 3$, $p = 3$, and $q = 2$ and f be a function as follows:

$$f(k,t) = \begin{cases} 0, & \text{if } t < 0, \\ \sqrt[3]{kt}, & \text{if } 0 \leq t \leq 1, \\ \sqrt[3]{kt} + 225t^2 - 225, & \text{if } t > 1. \end{cases} \tag{67}$$

From Remark 1, we can choose $c = 1$ and $d = 0.02$. Easy calculation shows that

$$\begin{aligned} \frac{pN^{p-1}}{1+s_* N^{p-1}} \sum_{k=1}^N F(k,c) \max\left\{\frac{1}{c^q}, \frac{1}{c^p}\right\} &= \frac{pN^{p-1}}{1+s_* N^{p-1}} \sum_{k=1}^3 \int_0^1 \sqrt[3]{kt}\,dt \\ &\approx 1.3693, \\ \frac{\sum_{k=1}^N F(k,d)}{d^p p^{-1} + d^q q^{-1} \tilde{s}} &= \frac{\sum_{k=1}^3 \int_0^{0.02} \sqrt[3]{kt}\,dt}{d^p p^{-1} + d^q q^{-1} \tilde{s}} \approx 3.1386, \\ \frac{qL_\infty}{2^p N + \tilde{s} - 2^{p-1}} &= \frac{1}{66} \lim_{t \to +\infty} \frac{\sqrt[3]{kt} + 225t^2 - 225}{t^2} \\ &\approx 3.4091, \end{aligned} \tag{68}$$

which satisfy condition (50). Thus, for each $\lambda \in (0.3186, 0.7303)$, the problem

$$\begin{cases} -\Delta(\phi_3(\Delta u(k-1))) + 8\phi_2(u(k)) = \lambda f(k,u(k)), & k = 1,2,3, \\ u(0) = \Delta u(3) = 0, \end{cases} \tag{69}$$

admits at least two positive solutions.

Data Availability

No data were used to support the study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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