

Research Article

Bifurcation Behaviors of Steady-State Solution to a Discrete General Brusselator Model

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We study the local and global bifurcation of nonnegative nonconstant solutions of a discrete general Brusselator model. We generalize the linear u in the standard Brusselator model to the nonlinear $f(u)$. Assume that $f \in C([0, \infty), [0, \infty))$ is a strictly increasing function, and $f'(f^{-1}(a)) \in (0, \infty)$. Taking b as the bifurcation parameter, we obtain that the solution set of the problem constitutes a constant solution curve and a nonconstant solution curve in a small neighborhood of the bifurcation point $(b_0^1, f^{-1}(a), ab/[f^{-1}(a)]^2)$. Moreover, via the Rabinowitz bifurcation theorem, we obtain the global structure of the set of nonconstant solutions under the condition that $f(s)/s^2$ is nonincreasing in $(0, \infty)$. In this process, we also make a priori estimation for the nonnegative nonconstant solutions of the problem.

1. Introduction

In 1968, Prigogine and Lefever [1] introduced first the Brusselator model for a chemical reaction-diffusion of self-catalysis as follows:

$$\begin{cases} \frac{\partial u}{\partial t} - d_1 \Delta u = a - (b+1)u + u^2 v, & x \in \Omega, t > 0, \\ \frac{\partial v}{\partial t} - d_2 \Delta v = bu - u^2 v, & x \in \Omega, t > 0, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega, t > 0, \end{cases} \quad (1)$$

where $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) is a smooth and bounded domain, n denotes the outward unit normal vector on $\partial\Omega$, u and v represent the concentration of two intermediary reactants having the diffusion rates $d_1, d_2 \in (0, \infty)$ with $d_2 > d_1$, and $a, b > 0$ are the fixed concentrations. This chemical reaction plays an important role due to its similarities with neuronal and biological networks. Therefore, (1) has been extensively investigated in the last decades from both analytical and

numerical point of view (see [2–12]). Most of them are interested in finding spatially nonconstant solutions of the equilibrium problem

$$\begin{cases} -d_1 \Delta u = a - (b+1)u + u^2 v, & x \in \Omega, \\ -d_2 \Delta v = bu - u^2 v, & x \in \Omega, \\ \frac{\partial u}{\partial n} = \frac{\partial v}{\partial n} = 0, & x \in \partial\Omega. \end{cases} \quad (2)$$

From the definition of Strogatz [13], chaos sensitivity depends on initial conditions, which shows that nearby trajectories diverge exponentially. Continuous systems in a 2-dimensional phase space cannot experience such divergence; hence, chaotic behaviors can only be observed in deterministic continuous systems with a phase space of dimension 3, at least. On the contrary, in a discrete map, it is well known that chaos occurs also in one dimension. Therefore, discrete chaotic systems exhibit chaos whatever their dimension is.

It is worth to note that discrete models governed by difference equations are more appropriate than the

continuous one due to their efficient computational results and rich dynamical behavior (see [14, 15]). Therefore, the discrete Brusselator model has been studied by several authors, and they got some results ([16–18] and the references therein). In particular, Din [16] applied forward Euler's method to one-dimensional model (1) as follows:

$$\begin{cases} x_{n+1} = x_n + h(\alpha - (1 + \beta)x_n + x_n^2 y_n), \\ y_{n+1} = y_n + h(\beta x_n - x_n^2 y_n), \end{cases} \quad (3)$$

where $0 < h < 1$ represents the step size for Euler's method. The local dynamical behaviors are obtained for (3).

Note that [16–18] only studied the dynamical behaviors of the discrete-time Brusselator model. The reason is that the partial difference equation is very difficult for us. Indeed, the discrete-space Brusselator model is also worth studying due to the discontinuity of the space.

Therefore, we will consider the discrete space, more general form of (2) with $N = 1$:

$$\begin{cases} -d_1 \Delta^2 u(x-1) = a - (b+1)f(u(x)) + u^2(x)v(x), & x \in \mathbb{T}, \\ -d_2 \Delta^2 v(x-1) = bf(u(x)) - u^2(x)v(x), & x \in \mathbb{T}, \\ \Delta u(0) = \Delta u(T) = \Delta v(0) = \Delta v(T) = 0, \end{cases} \quad (4)$$

where $\Delta u(x) = u(x+1) - u(x)$, $\mathbb{T} := [1, T]_{\mathbb{Z}} = \{1, 2, \dots, T\}$, $T \geq 2$ is an integer, $d_1, d_2, a > 0$ are fixed parameters and

$d_2 > d_1$, and $b > 0$ is a bifurcation parameter. Clearly, $f(u) = (f(u)/u) \cdot u$; then, (4) is seen to be equivalent to

$$\begin{cases} -d_1 \Delta^2 u(x-1) = a - (b+1) \frac{f(u(x))}{u(x)} \cdot u(x) + u^2(x)v(x), & x \in \mathbb{T}, \\ -d_2 \Delta^2 v(x-1) = b \frac{f(u(x))}{u(x)} \cdot u(x) - u^2(x)v(x), & x \in \mathbb{T}, \\ \Delta u(0) = \Delta u(T) = \Delta v(0) = \Delta v(T) = 0, \end{cases} \quad (5)$$

where $f(u)/u$ can be regarded as a variable coefficient. It is well known that the linear terms $(b+1)u$ and bu in (2) cannot withstand any small perturbation.

In fact, (5) has an important application value in biology and chemistry. Xu et al. [19] said that model (1) includes a basic assumption: the cells always live in a continuous patch environment. However, this may not be the case in reality, and the motion of individuals of given cells is random and isotropic, i.e., without any preferred direction, the cells are also absolute individuals. The cells or units are also absolute individuals in microscopic sense, and each isolated cell exchanges materials by diffusion with its neighbors. Thus, it is reasonable to consider a 1D or 2D spatially discrete reaction-diffusion system in order to explain the chemical system.

Kang [20] discussed the dynamics of the local map of a discrete version of the Brusselator model. To discretize system (1), he employed the following discretizations.

For the derivative in time, he used

$$\frac{\partial u(x, t)}{\partial t} \longrightarrow \frac{u(x, t + \Delta t) - u(x, t)}{\Delta t}. \quad (6)$$

For the space derivative, he used

$$\frac{\partial u(x, t)}{\partial x} \longrightarrow \frac{u(x + \Delta x, t) - u(x, t)}{\Delta x}, \quad (7)$$

$$\frac{\partial^2 u(x, t)}{\partial x^2} \longrightarrow \frac{u(x + \Delta x, t) - 2u(x, t) + u(x - \Delta x, t)}{(\Delta x)^2}. \quad (8)$$

It is important to note that Δ in (6)–(8) is different from Δ in this paper. Our discretization is consistent with Kang's, and we chose the step size to be 1. When $f(u) = u$, (5) is the steady-state form of the problem studied in [19, 20].

On the contrary, the Brusselator system has been investigated from the numerical point of view (see [21] and references therein). Most modern texts on numerical analysis give an introduction to numerical solutions of partial differential equations using the finite-difference approach. Twizell et al. [22] had given a second-order finite-difference scheme for the Brusselator reaction-diffusion system. It is well known that (2) is an important mathematical dynamics model in biology and chemistry. In some ways, (5) is even more practical than (2).

We will study the local and global bifurcation of nonnegative nonconstant solutions of (4) under the following assumptions:

- (H1) $f \in C([0, \infty), [0, \infty))$ is a strictly increasing function
- (H2) $f'(f^{-1}(a)) \in (0, \infty)$
- (H3) $f(s)/s^2$ is nonincreasing in $(0, \infty)$

Remark 1. If $f(u) = u$, then (4) is the discrete version of (2) with $N = 1$. Obviously, discrete Brusselator model (4) is a second-order difference boundary value problem.

The rest of the paper is organized as follows: in Section 2, we give a priori estimate and some preliminary results. Section 3 is devoted to studying the local bifurcation of nonnegative nonconstant solutions of (4) under conditions (H1) and (H2). Finally, in Section 4, we add condition (H3)

to obtain the global bifurcation of nonnegative nonconstant solutions of (4).

2. Preliminary Results

At first, let us look for the constant solution of (4). To get it, it suffices to look for the constant solution of the following problem:

$$\begin{cases} a - (b + 1)f(u(x)) + u^2(x)v(x) = 0, & x \in \mathbb{T}, \\ bf(u(x)) - u^2(x)v(x) = 0, & x \in \mathbb{T}. \end{cases} \quad (9)$$

By (H1), problem (4) has a unique constant solution $(f^{-1}(a), ab/[f^{-1}(a)]^2)$.

We can easily obtain the following a priori estimate of the nonnegative nonconstant solutions of (4).

Lemma 1. *Let (H1), (H2), and (H3) hold. Then, any nonnegative nonconstant solution (u, v) of (4) satisfies*

$$f^{-1}\left(\frac{a}{b+1}\right) \leq u(x) \leq f^{-1}(a) + \frac{d_2}{d_1} \cdot \frac{ab}{(b+1)[f^{-1}(a/(b+1))]^2}, \quad x \in \mathbb{T}, \quad (10)$$

$$\frac{bf(f^{-1}(a) + (d_2/d_1) \cdot (ab/(b+1)[f^{-1}(a/(b+1))]^2))}{[f^{-1}(a) + (d_2/d_1) \cdot (ab/(b+1)[f^{-1}(a/(b+1))]^2)]^2} \leq v(x) \leq \frac{ab}{(b+1)[f^{-1}(a/(b+1))]^2}, \quad x \in \mathbb{T}.$$

Proof. Let $x_0 \in \mathbb{T}$ be the minimum point of u . We have

$$\begin{aligned} u(x_0 + 1) &\geq u(x_0), \\ u(x_0 - 1) &\geq u(x_0), \\ -d_1 \Delta^2 u(x_0 - 1) &= -d_1(u(x_0 + 1) - 2u(x_0) \\ &\quad + u(x_0 - 1)) \leq 0. \end{aligned} \quad (11)$$

Then,

$$\begin{aligned} a - (b + 1)f(u(x_0)) + u^2(x_0)v(x_0) &\leq 0, \\ a - (b + 1)f(u(x_0)) &\leq 0, \\ f(u(x_0)) &\geq \frac{a}{b+1}. \end{aligned} \quad (12)$$

Then, $u(x_0) \geq f^{-1}(a/(b+1))$ by (H1), and so

$$u(x) \geq u(x_0) \geq f^{-1}\left(\frac{a}{b+1}\right), \quad x \in \mathbb{T}. \quad (13)$$

Let $x_1 \in \mathbb{T}$ be the maximum point of v . Similarly, we can get that

$$\begin{aligned} v(x_1 + 1) &\leq v(x_1), v(x_1 - 1) \leq v(x_1), \\ -d_2 \Delta v(x_1 - 1) &= -d_2(v(x_1 + 1) - 2v(x_1) + v(x_1 - 1)) \geq 0, \end{aligned} \quad (14)$$

$$bf(u(x_1)) - u^2(x_1)v(x_1) \geq 0. \quad (15)$$

Then,

$$v(x) \leq v(x_1) \leq \frac{bf(u(x_1))}{u^2(x_1)}, \quad x \in \mathbb{T}. \quad (16)$$

Combining this with (13), from (H3), we show

$$v(x) \leq \frac{bf(u(x_1))}{u^2(x_1)} \leq \frac{ab}{(b+1)[f^{-1}(a/(b+1))]^2}, \quad x \in \mathbb{T}. \quad (17)$$

Let $w = d_1 u + d_2 v$. Then, it follows from (4) that

$$\begin{cases} -\Delta^2 w(x-1) = a - f(u(x)), & x \in \mathbb{T}, \\ \Delta w(0) = \Delta w(T) = 0. \end{cases} \quad (18)$$

Now, let $x_2 \in \mathbb{T}$ be the maximum point of w . Observe that

$$\begin{aligned} w(x_2 + 1) &\geq w(x_2), \\ w(x_2 - 1) &\geq w(x_2), \\ -\Delta^2 w(x_2 - 1) &= a - f(u(x_2)) \geq 0. \end{aligned} \quad (19)$$

Then, from (H1), it is easy to see $u(x_2) \leq f^{-1}(a)$. Combining this with (17), we know that, for any $x \in \mathbb{T}$,

$$\begin{aligned} d_1 u(x) \leq w(x) \leq w(x_2) &\leq d_1 f^{-1}(a) + d_2 \\ &\cdot \frac{ab}{(b+1)[f^{-1}(a/(b+1))]^2}. \end{aligned} \quad (20)$$

Then,

$$u(x) \leq f^{-1}(a) + \frac{d_2}{d_1} \cdot \frac{ab}{(b+1)[f^{-1}(a/(b+1))]^2}, \quad x \in \mathbb{T}. \quad (21)$$

If $x_3 \in \mathbb{T}$ is the minimum point of v , then

$$bf(u(x_3)) - u^2(x_3)v(x_3) \leq 0, \quad (22)$$

and so

$$v(x) \geq v(x_3) \geq \frac{bf(u(x_3))}{u^2(x_3)} \geq \frac{bf(f^{-1}(a) + (d_2/d_1) \cdot ab/(b+1)[f^{-1}(a/(b+1))]^2)}{[f^{-1}(a) + (d_2/d_1) \cdot ab/(b+1)[f^{-1}(a/(b+1))]^2]^2}, \quad x \in \mathbb{T}. \quad (23)$$

Consequently, the proof is completed. \square

Lemma 2 (see [23]). *Assume $T \geq 2$ is an integer. Then, the discrete second-order linear Neumann eigenvalue problem*

$$\begin{cases} -\Delta^2 \varphi(x-1) = \mu \varphi(x), & x \in \mathbb{T}, \\ \Delta \varphi(0) = \Delta \varphi(T) = 0, \end{cases} \quad (24)$$

has T real and simple eigenvalues, which can be ordered as follows:

$$0 = \mu_0 < \mu_1 < \mu_2 < \cdots < \mu_{T-1}. \quad (25)$$

Moreover, for $j \in \{1, \dots, T-1\}$, the eigenfunction φ_j corresponding to the eigenvalue μ_j has exactly $j-1$ simple generalized zeros.

For any fixed $T \geq 2$, it is well known that

$$\mu_j = \begin{cases} 0, & j = 0, \\ 2 - \cos\left(\frac{j\pi}{T}\right), & j \in (1, T-1]_{\mathbb{Z}}, \end{cases} \quad (26)$$

and the corresponding eigenfunctions are

$$\varphi_j(x) = \begin{cases} 1, & j = 0, \\ \cos\left(\frac{j\pi}{T}x\right), & j \in (1, T-1]_{\mathbb{Z}}. \end{cases} \quad (27)$$

Lemma 3 (see [18], Theorem 2.5). *Let a be a constant. Then, for $\Delta C(i) = 0$,*

$$\sum \cos ai = \frac{\sin a(i - (1/2))}{2 \sin(a/2)} + C(i), \quad (a \neq 2n\pi). \quad (28)$$

Lemma 4 (see [18], Theorem 2.7). *If z_n is an indefinite sum of y_n , then*

$$\sum_{k=m}^{n-1} y_k = [z_k]_m^n = z_n - z_m. \quad (29)$$

Let

$$\begin{aligned} X &:= \{(u, v) \mid u, v: \widehat{\mathbb{T}} \longrightarrow \mathbb{R}, \Delta u(0) = \Delta u(T) = \Delta v(0) = \Delta v(T) = 0\}, \\ X_1 &:= \{u \mid u: \mathbb{T} \longrightarrow \mathbb{R}\}, \quad Y := X_1 \times X_1, \\ (w_1, w_2)_Y &= \sum_{x=1}^T u_1(x)u_2(x) + \sum_{x=1}^T v_1(x)v_2(x), \end{aligned} \quad (30)$$

where $w_1 = (u_1, v_1), w_2 = (u_2, v_2) \in Y$ and $\widehat{\mathbb{T}} := [0, T+1]_{\mathbb{Z}}$.

3. Local Bifurcation

By the second part, $\bar{w} := (f^{-1}(a), ab/[f^{-1}(a)]^2)$ is the unique constant solution of (4).

Define the mapping $P: (0, \infty) \times X \longrightarrow Y$:

$$P(b, w) = \begin{pmatrix} d_1 \Delta^2 u(x-1) + a - (b+1)f(u(x)) + u^2(x)v(x) \\ d_2 \Delta^2 v(x-1) + bf(u(x)) - u^2(x)v(x) \end{pmatrix}. \quad (31)$$

For the fixed $b > 0$, $w = (u, v)$ is a solution of (4) if and only if (b, w) is a zero-point of P . Note that $P(b, \bar{w}) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ since \bar{w} is the constant solution of (4).

Let

$$\begin{cases} u = f^{-1}(a) + \sum_{k=1}^{\infty} \varepsilon^k u_k, \\ v = \frac{ab}{[f^{-1}(a)]^2} + \sum_{k=1}^{\infty} \varepsilon^k v_k, \end{cases} \quad (32)$$

$$b = b_0 + \sum_{k=1}^{\infty} \varepsilon^k b_k. \quad (33)$$

We also have to Taylor expand f at the point $f^{-1}(a)$. The purpose of the rest of this section is to solve b_0 and prove that (b_0, \bar{w}) is the bifurcation point of $P(b, w) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$.

First of all, we substitute (32) and (33) into (4) and let the higher-order term of ε be equal to 0. Then, we can get the problem

$$\begin{cases} -d_1\Delta^2 u_1(x-1) = \left(-b_0 f'(f^{-1}(a)) - f'(f^{-1}(a)) + \frac{2ab_0}{f^{-1}(a)}\right)u_1(x) + [f^{-1}(a)]^2 v_1(x), & x \in \mathbb{T}, \\ -d_2\Delta^2 v_1(x-1) = \left(b_0 f'(f^{-1}(a)) - \frac{2ab_0}{f^{-1}(a)}\right)u_1(x) - [f^{-1}(a)]^2 v_1(x), & x \in \mathbb{T}, \\ \Delta u_1(0) = \Delta u_1(T) = \Delta v_1(0) = \Delta v_1(T) = 0. \end{cases} \quad (34)$$

In (34), by using undetermined coefficient method, it follows that

$$b_0 = \frac{f^{-1}(a)\{d_1 d_2 \mu_j^2 + d_2 f'(f^{-1}(a))\mu_j + d_1 [f^{-1}(a)]^2 \mu_j + [f^{-1}(a)]^2 f'(f^{-1}(a))\}}{2ad_2 \mu_j - d_2 f^{-1}(a) f'(f^{-1}(a))\mu_j} := b_0^j, \quad j = 1, 2, \dots, T-1. \quad (35)$$

Moreover, it is not difficult to prove (34) has a nontrivial solution (u_1, v_1) :

$$\begin{cases} u_1 = c_1(j)\cos\left(\frac{j\pi}{l}x\right) = c_1(j)\varphi_j(x), & c_1(j) = \frac{d_2 \mu_j}{d_1 \mu_j + f'(f^{-1}(a))}, \\ v_1 = \cos\left(\frac{j\pi}{l}x\right) = \varphi_j(x). \end{cases} \quad (36)$$

Next, we substitute (32) and (33) into (4) and let the higher-order term of ε^2 be equal to 0; then, (4) becomes the following system:

$$\begin{cases} d_1\Delta^2 u_2(x-1) + \left(-b_0 f'(f^{-1}(a)) - f'(f^{-1}(a)) + \frac{2ab_0}{f^{-1}(a)}\right)u_2(x) + [f^{-1}(a)]^2 v_2(x) = -F_1, & x \in \mathbb{T}, \\ d_2\Delta^2 v_2(x-1) + \left(b_0 f'(f^{-1}(a)) - \frac{2ab_0}{f^{-1}(a)}\right)u_2(x) - [f^{-1}(a)]^2 v_2(x) = F_1, & x \in \mathbb{T}, \\ \Delta u_2(0) = \Delta u_2(T) = \Delta v_2(0) = \Delta v_2(T) = 0, \end{cases} \quad (37)$$

where

$$F_1 = \left(\frac{2ab_1}{f^{-1}(a)} - b_1 f'(f^{-1}(a))\right)u_1 + 2f^{-1}(a)u_1 v_1 + \frac{ab_0}{[f^{-1}(a)]^2}u_1^2. \quad (38)$$

In order to solve b_1 from (37), let us consider the following adjoint system of the homogeneous system related to (37):

$$\begin{cases} d_1 \Delta^2 y_2(x-1) + \left(-b_0 f'(f^{-1}(a)) - f'(f^{-1}(a)) + \frac{2ab_0}{f^{-1}(a)} \right) y_2(x) + \left(b_0 f'(f^{-1}(a)) - \frac{2ab_0}{f^{-1}(a)} \right) z_2(x) = 0, & x \in \mathbb{T}, \\ d_2 \Delta^2 z_2(x-1) + [f^{-1}(a)]^2 y_2(x) - [f^{-1}(a)]^2 z_2(x) = 0, & x \in \mathbb{T}, \\ \Delta y_2(0) = \Delta y_2(T) = \Delta z_2(0) = \Delta z_2(T) = 0. \end{cases} \quad (39)$$

It is not difficult to verify that (39) has a solution (y_2, z_2) :

$$\begin{cases} y_2 = c_2(j) \cos\left(\frac{j\pi}{l}x\right) = c_2(j) \varphi_j(x), & c_2(j) = 1 + \frac{d_2 \mu_j}{[f^{-1}(a)]^2}, \\ z_2 = \cos\left(\frac{j\pi}{l}x\right) = \varphi_j(x). \end{cases} \quad (40)$$

By virtue of the solvability condition for (37), it is obvious that

$$\sum_{x=1}^T (z_2(x) - y_2(x)) F_1 = 0. \quad (41)$$

In fact,

$$\begin{aligned} & \sum_{x=1}^T (z_2(x) - y_2(x)) F_1 \\ &= \sum_{x=1}^T \left(-\frac{d_2 \mu_j}{[f^{-1}(a)]^2} \cdot \varphi_j(x) \right) \left[\left(\frac{2ab_1}{f^{-1}(a)} - b_1 f'(f^{-1}(a)) \right) \cdot \left(-\frac{d_2 \mu_j}{d_1 \mu_j + f'(f^{-1}(a))} \cdot \varphi_j(x) \right) \right. \\ & \quad \left. + 2f^{-1}(a) \left(-\frac{d_2 \mu_j}{d_1 \mu_j + f'(f^{-1}(a))} \right) \varphi_j^2(x) + \frac{ab_0}{[f^{-1}(a)]^2} \cdot \frac{d_2^2 \mu_j^2}{[d_1 \mu_j + f'(f^{-1}(a))]^2} \cdot \varphi_j^2(x) \right] = 0. \end{aligned} \quad (42)$$

We know that

$$\sum_{x=1}^T \varphi_j^3(x) = \sum_{x=1}^T \cos^3 \frac{j\pi}{T} x = \frac{1}{4} \sum_{x=1}^T \left(\cos \frac{3j\pi}{T} x + 3 \cos \frac{j\pi}{T} x \right). \quad (43)$$

From Lemmas 3 and 4, for any $j \in \{1, 2, \dots, T-1\}$ and $j \neq (2nT/3), n \in \{1, 2, \dots\}$, we obtain

$$\begin{aligned}
 \sum_{x=1}^T \varphi_j^3(x) &= \frac{1}{4} \left[\frac{\sin(3j\pi/T)(x - (1/2))}{2 \sin(3j\pi/2T)} \right]_1^{T+1} + \frac{3}{4} \left[\frac{\sin(j\pi/T)(x - (1/2))}{2 \sin(j\pi/2T)} \right]_1^{T+1} \\
 &= \frac{1}{4} \cdot \frac{\sin((2T+1)3j\pi/2T) - \sin(3j\pi/2T)}{2 \sin(3j\pi/2T)} + \frac{3}{4} \cdot \frac{\sin((2T+1)j\pi/2T) - \sin(j\pi/2T)}{2 \sin(j\pi/2T)} \\
 &= \frac{1}{4} \cdot \frac{\cos((2T+2)3j\pi/2T) \cdot \sin(2T \cdot 3j\pi/2T)}{\sin(3j\pi/2T)} + \frac{3}{4} \cdot \frac{\cos((2T+2)j\pi/2T) \cdot \sin(2T \cdot j\pi/2T)}{\sin(j\pi/2T)} = 0.
 \end{aligned} \tag{44}$$

Then $b_1^j := b_1 = 0$, and so F_1 will reduce to

$$\begin{aligned}
 F_1 &= 2f^{-1}(a)c_1(j)\varphi_j^2(x) + \frac{ab_0}{[f^{-1}(a)]^2}c_1^2(j)\varphi_j^2(x) \\
 &= 2f^{-1}(a)c_1(j)\cos^2\left(\frac{j\pi}{l}x\right) + \frac{ab_0}{[f^{-1}(a)]^2}c_1^2(j)\cos^2\left(\frac{j\pi}{l}x\right) \\
 &= f^{-1}(a)c_1(j)\left(\cos\left(\frac{2j\pi}{l}x\right) + 1\right) + \frac{1}{2} \cdot \frac{ab_0}{[f^{-1}(a)]^2}c_1^2(j)\left(\cos\left(\frac{2j\pi}{l}x\right) + 1\right) \\
 &= \frac{1}{2} \left[2f^{-1}(a)c_1(j) + \frac{ab_0}{[f^{-1}(a)]^2}c_1^2(j) \right] + \frac{1}{2} \left[2f^{-1}(a)c_1(j) + \frac{ab_0}{[f^{-1}(a)]^2}c_1^2(j) \right] \varphi_{2j}(x).
 \end{aligned} \tag{45}$$

Therefore, a particular solution (u_2, v_2) of (37) can be obtained as follows: where

$$\begin{cases} u_2 = a_1(j) + a_2(j)\cos\left(\frac{2j\pi}{l}x\right) = a_1(j) + a_2(j)\varphi_{2j}(x), \\ v_2 = a_3(j) + a_4(j)\cos\left(\frac{2j\pi}{l}x\right) = a_3(j) + a_4(j)\varphi_{2j}(x), \end{cases} \tag{46}$$

$$\begin{aligned}
 a_2(j) &= \frac{[f^{-1}(a)]^2 c_1 d_2 \mu_{2j} + (ab_0 d_2 \mu_{2j} c_1^2 / 2 f^{-1}(a))}{d_1 d_2 \mu_{2j}^2 f^{-1}(a) + f'(f^{-1}(a)) f^{-1}(a) ([f^{-1}(a)]^2 + d_2 \mu_{2j} (1 + b_0)) - (2ab_0 d_2 - d_1 [f^{-1}(a)]^3) \mu_{2j}}, \\
 a_1(j) &= 0, \\
 a_3(j) &= -\frac{c_1}{[f^{-1}(a)]^2} \left(f^{-1}(a) + \frac{ab_0}{2[f^{-1}(a)]^2} c_1 \right), \\
 a_4(j) &= -\frac{d_1 \mu_{2j} + f'(f^{-1}(a))}{d_2 \mu_{2j}} a_2(j).
 \end{aligned} \tag{47}$$

Since $b_1 = 0$, we have to solve b_2 . We substitute (32) and (33) into (4) and let the higher-order term of ε^3 be equal to 0; then, a problem similar to (37) is obtained:

$$\begin{cases} d_1 \Delta^2 u_3(x-1) + \left(-b_0 f'(f^{-1}(a)) - f'(f^{-1}(a)) + \frac{2ab_0}{f^{-1}(a)} \right) u_3(x) + [f^{-1}(a)]^2 v_3(x) = -F_2, & x \in \mathbb{T}, \\ d_2 \Delta^2 v_3(x-1) + \left(b_0 f'(f^{-1}(a)) - \frac{2ab_0}{f^{-1}(a)} \right) u_3(x) - [f^{-1}(a)]^2 v_3(x) = F_2, & x \in \mathbb{T}, \\ \Delta u_3(0) = \Delta u_3(T) = \Delta v_3(0) = \Delta v_3(T), \end{cases} \quad (48)$$

where

$$F_2 = \left(-b_2 f'(f^{-1}(a)) + \frac{2ab_2}{f^{-1}(a)} \right) u_1 + 2f^{-1}(a)u_1v_2 + 2f^{-1}(a)u_2v_1 + u_1^2v_1 + \frac{2ab_0}{[f^{-1}(a)]^2}u_1u_2. \quad (49)$$

Clearly, (39) is also the adjoint system of the homogeneous system related to (48); then,

$$\sum_{x=1}^T (z_2(x) - y_2(x))F_2 = 0. \quad (50)$$

According to values of u_1, u_2, v_1 , and v_2 , we have

$$\begin{aligned} & \sum_{x=1}^T (z_2(x) - y_2(x))F_2 \\ &= \sum_{x=1}^T \left(-\frac{d_2 \mu_j}{[f^{-1}(a)]^2} \varphi_j(x) \right) \left[\frac{2a - f'(f^{-1}(a))f^{-1}(a)}{f^{-1}(a)} \cdot b_2 c_1 \varphi_j(x) + 2f^{-1}(a)c_1 \varphi_j(x)(a_3 + a_4)\varphi_{2j}(x) \right. \\ & \quad \left. + 2f^{-1}(a)a_2 \varphi_{2j}(x)\varphi_j(x) + c_1^2 \varphi_j^3(x) + \frac{2ab_0 a_2 c_1}{[f^{-1}(a)]^2} \varphi_j(x)\varphi_{2j}(x) \right]. \end{aligned} \quad (51)$$

From Lemmas 3 and 4, for any $j \in \{1, 2, \dots, T-1\}$ and $j \neq (nT/2), n \in \{1, 2, \dots\}$, we know that

$$\begin{aligned} \sum_{x=1}^T \varphi_j^2(x)\varphi_{2j}(x) &= \sum_{x=1}^T \cos^2\left(\frac{j\pi}{T}x\right) \cdot \cos\left(\frac{2j\pi}{T}x\right) = \frac{1}{2} \sum_{x=1}^T \left(\frac{1}{2} \cos \frac{4j\pi}{T}x + \cos \frac{2j\pi}{T}x \right) + \frac{T}{4} \\ &= \frac{1}{4} \cdot \left[\frac{\sin(4j\pi(x-(1/2))/T)}{2 \sin(4j\pi/2T)} \right]_1^{T+1} + \frac{1}{2} \cdot \left[\frac{\sin(2j\pi(x-(1/2))/T)}{2 \sin(2j\pi/2T)} \right]_1^{T+1} + \frac{T}{4} \\ &= \frac{1}{4} \cdot \frac{2 \cos((2T+2)2j\pi/T) \sin((2j\pi \cdot 2T)/T)}{2 \sin(2j\pi/T)} + \frac{1}{2} \cdot \frac{\cos((2T+2)j\pi/T) \sin((j\pi \cdot 2T)/T)}{2 \sin(2j\pi/T)} + \frac{T}{4} = \frac{T}{4} \neq 0. \end{aligned} \quad (52)$$

Thus,

$$b_2^j := b_2 \neq 0, \quad j = 1, 2, \dots, T-1. \quad (53)$$

From the above analysis, we obtain the main result of this section.

Theorem 1. Assume that (H1) and (H2) hold. Then, for any positive integer $j < T, j \neq (nT/2)$ and $j \neq (2nT/3), n \in \{1, 2, \dots\}$, (b_0^j, \bar{w}) is a bifurcation point of $P(b, w) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Moreover, there is a nontrivial solution $\phi(\varepsilon) = (b(\varepsilon),$

$u(\varepsilon), v(\varepsilon)$ of (4) if ε is small enough, where $b, u,$ and v are continuous with respect to ε :

$$u(\varepsilon) = f^{-1}(a) + \varepsilon c_1(j)\varphi_j + \varepsilon^2(a_1(j) + a_2(j)\varphi_{2j}) + o(\varepsilon^2),$$

$$v(\varepsilon) = \frac{ab}{[f^{-1}(a)]^2} + \varepsilon\varphi_j + \varepsilon^2(a_3(j) + a_4(j)\varphi_{2j}) + o(\varepsilon^2),$$

$$b(\varepsilon) = b_0^j + \varepsilon^2 b_2^j + o(\varepsilon^2). \tag{54}$$

The set of zero-points of P constitutes two curves in a neighborhood of bifurcation point (b_0^j, \bar{w}) .

Let \mathbb{C} be the closure of the nonconstant solution set of $P(b, w) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and Γ_j be a connected component of $\mathbb{C} \cup \{(b_0^j, \bar{w})\}$ and $(b_0^j, \bar{w}) \in \Gamma_j$. In a small neighborhood of bifurcation point (b_0^j, \bar{w}) , the curve Γ_j is determined by the eigenfunction φ_j , where φ_j has exactly $j - 1$ simple generalized zeros.

4. Global Bifurcation

Theorem 2. *Let (H1), (H2), and (H3) hold. If $\mu_j \neq [f^{-1}(a)]^2/d_2, j = 1, 2, \dots, T$, then projection of continuum Γ_j is unbounded on the b -axis.*

Proof. (4) can be written as follows:

$$\begin{cases} -\Delta^2 u(x-1) = g(u(x), v(x)), & x \in \mathbb{T}, \\ -\Delta^2 v(x-1) = h(u(x), v(x)), & x \in \mathbb{T}, \\ \Delta u(0) = \Delta u(T) = \Delta v(0) = \Delta v(T) = 0, \end{cases} \tag{55}$$

where

$$g(u, v) = \frac{1}{d_1}(a - (b+1)f(u) + u^2v), \tag{56}$$

$$h(u, v) = \frac{1}{d_2}(bf(u) - u^2v).$$

Let $\tilde{u} = u - f^{-1}(a), \tilde{v} = v - ab/[f^{-1}(a)]^2$. Then, (55) is equivalent to the following problem:

$$\begin{cases} -\Delta^2 \tilde{u}(x-1) = g_0 \tilde{u}(x) + g_1 \tilde{v}(x) + \tilde{g}(\tilde{u}(x), \tilde{v}(x)), & x \in \mathbb{T}, \\ -\Delta^2 \tilde{v}(x-1) = h_0 \tilde{u}(x) + h_1 \tilde{v}(x) + \tilde{h}(\tilde{u}(x), \tilde{v}(x)), & x \in \mathbb{T}, \\ \Delta \tilde{u}(0) = \Delta \tilde{u}(T) = \Delta \tilde{v}(0) = \Delta \tilde{v}(T) = 0, \end{cases} \tag{57}$$

where \tilde{g} and \tilde{h} are higher-order terms of \tilde{u}, \tilde{v} and

$$\begin{aligned} g_0 &= g_u(u, v)|_{(f^{-1}(a), ab/[f^{-1}(a)]^2)} = \frac{1}{d_1} \left(-(b+1)f'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} \right), \\ g_1 &= g_v(u, v)|_{(f^{-1}(a), ab/[f^{-1}(a)]^2)} = \frac{[f^{-1}(a)]^2}{d_1}, \\ h_0 &= h_u(u, v)|_{(f^{-1}(a), ab/[f^{-1}(a)]^2)} = \frac{1}{d_2} \left(bf'(f^{-1}(a)) - \frac{2ab}{f^{-1}(a)} \right), \\ h_1 &= h_v(u, v)|_{(f^{-1}(a), ab/[f^{-1}(a)]^2)} = \frac{[f^{-1}(a)]^2}{d_2}. \end{aligned} \tag{58}$$

In this way, we convert the constant solution $\bar{w} = (f^{-1}(a), ab/[f^{-1}(a)]^2)$ of (4) to the trivial solution $\theta = (0, 0)$ of (57).

Let $H_1: Y \rightarrow X$ and $H_2: Y \rightarrow X$ be the inverse of operators $(f'(f^{-1}(a))/d_1)I - A$ and $([f^{-1}(a)]^2/d_2)I - A$

with Neumann boundary conditions, respectively, where $A: [0, \infty) \rightarrow (-\infty, 0]$ and $Au(x) = \Delta^2 u(x-1)$. Set $U = (\tilde{u}, \tilde{v})$:

$$\begin{aligned} K(b)U &= \left(\frac{1}{d_1} \left[-bf'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} \right] H_1(\tilde{u}) + \frac{[f^{-1}(a)]^2}{d_1} H_1(\tilde{v}), \frac{1}{d_2} \left[bf'(f^{-1}(a)) - \frac{2ab}{f^{-1}(a)} \right] H_2(\tilde{u}) \right), \\ W(U) &= (H_1(\tilde{g}(\tilde{u}, \tilde{v})), H_2(\tilde{h}(\tilde{u}, \tilde{v}))). \end{aligned} \tag{59}$$

It can be verified that (57) is equivalent to

$$U = K(b)U + W(U), \quad (60)$$

in X . For any fixed $b > 0$, $K(b)$ and $W(U)$ are linear compact operators in X and $W(U) = o(\|U\|)$. By the Rabinowitz global bifurcation theorem [24], we need to verify

- (i) 1 is an eigenvalue of $K(b_0^j)$, and its algebraic multiplicity is 1

- (ii) The index of $I - K(b) - W$ changes when b crosses b_0^j

Now, we will prove (i). Suppose $\Psi = \begin{pmatrix} \xi \\ \psi \end{pmatrix}$, $\xi = \sum a_j \varphi_j$, $\psi = \sum c_j \varphi_j$. Let

$$(K(b) - I)\Psi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (61)$$

i.e.,

$$\begin{pmatrix} \frac{1}{d_1} \left[-(b+1)f'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} \right] + A & \frac{[f^{-1}(a)]^2}{d_1} \\ \frac{1}{d_2} \left[bf'(f^{-1}(a)) - \frac{2ab}{f^{-1}(a)} \right] & -\frac{[f^{-1}(a)]^2}{d_2} + A \end{pmatrix} \Psi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (62)$$

Thus,

$$\sum_{j=0}^{\infty} L_j \begin{pmatrix} a_j \\ c_j \end{pmatrix} \varphi_j = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (63)$$

where

$$L_j = \begin{pmatrix} \frac{1}{d_1} \left[-(b+1)f'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} \right] - \mu_j & \frac{[f^{-1}(a)]^2}{d_1} \\ \frac{1}{d_2} \left[bf'(f^{-1}(a)) - \frac{2ab}{f^{-1}(a)} \right] & -\frac{[f^{-1}(a)]^2}{d_2} - \mu_j \end{pmatrix}. \quad (64)$$

By computation, $\det L_j = 0$ if and only if $b = b_0^j$; taking $b = b_0^j$ leads to

$$L_j \begin{pmatrix} a_j \\ c_j \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ f'(f^{-1}(a)) + d_1 \mu_j & d_2 \mu_j \end{pmatrix} \begin{pmatrix} a_j \\ c_j \end{pmatrix}. \quad (65)$$

Then, $\ker(K(b_0^j) - I) = \text{span}(\Psi)$, $\Psi = \begin{pmatrix} d_2 \mu_j \\ f'(f^{-1}(a)) + d_1 \mu_j \end{pmatrix} \varphi_j$. This implies that 1 is the

eigenvalue of $K = K(b_0^j)$ and $\dim \ker(K - I) = 1$. The algebraic multiplicity of eigenvalue 1 is the dimension of the generalized null space $\cup_{i=1}^{\infty} \ker(K - I)^i$; therefore, $\ker(K - I) \cap \text{Im}(K - I) = \{\theta^T\}$.

Let K^T be the transposed matrix of K :

$$K^T = \begin{pmatrix} \frac{1}{d_1} \left[-bf'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} \right] H_1 & \frac{1}{d_2} \left[bf'(f^{-1}(a)) - \frac{2ab}{f^{-1}(a)} \right] H_2 \\ \frac{[f^{-1}(a)]^2}{d_1} H_1 & 0 \end{pmatrix}, \quad (66)$$

and $\Psi^* = \begin{pmatrix} \xi^* \\ \psi^* \end{pmatrix}$, $\xi^* = \sum a_j^* \varphi_j$, $\psi^* = \sum c_j^* \varphi_j$. Suppose $\Psi^* \in \ker(K^T - I)$. Then,

$$\begin{cases} \frac{1}{d_1} \left[-bf'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} \right] H_1(\xi^*) + \frac{1}{d_2} \left[bf'(f^{-1}(a)) - \frac{2ab}{f^{-1}(a)} \right] H_2(\psi^*) = \xi^*, \\ \frac{[f^{-1}(a)]^2}{d_1} H_1(\xi^*) = \psi^*. \end{cases} \tag{67}$$

From the definition of H_1 and H_2 , (67) can also be written as

$$\begin{cases} -d_1 d_2 \Delta^2 \xi^*(x-1) = \left(d_2 \left[-bf'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} \right] - [f^{-1}(a)]^2 d_1 \right) \xi^* - \frac{d_2 f'(f^{-1}(a))}{[f^{-1}(a)]^2} \left[-bf'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} \right] \psi^*, \\ -d_1 \Delta^2 \psi^*(x-1) = [f^{-1}(a)]^2 \xi^*(x) - f'(f^{-1}(a)) \psi^*(x). \end{cases} \tag{68}$$

That is to say,

$$\sum_{j=0}^{\infty} B_j \begin{pmatrix} a_j^* \\ c_j^* \end{pmatrix} \varphi_j = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \tag{69}$$

where

$$B_j = \begin{pmatrix} d_2 \left[-bf'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} \right] - [f^{-1}(a)]^2 d_1 - d_1 d_2 \mu_j & -\frac{d_2 f'(f^{-1}(a))}{[f^{-1}(a)]^2} \left[-bf'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} \right] \\ [f^{-1}(a)]^2 & -d_1 \mu_j - f'(f^{-1}(a)) \end{pmatrix}. \tag{70}$$

Similarly, $\det B_j = 0$ if and only if $b = b_j^j$; taking $b = b_j^j$ leads to

$$B_j \begin{pmatrix} a_j^* \\ c_j^* \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ [f^{-1}(a)]^2 & -d_1 \mu_j - f'(f^{-1}(a)) \end{pmatrix} \begin{pmatrix} a_j^* \\ c_j^* \end{pmatrix}. \tag{71}$$

Then, $\ker(K^* - I) = \text{span} \left(\begin{pmatrix} d_1 \mu_j + f'(f^{-1}(a)) \\ [f^{-1}(a)]^2 \end{pmatrix} \right) \varphi_j$.
According to $\mu_j \neq [f^{-1}(a)]^2/d_2$, we obtain

$$\begin{aligned} (\Psi, \Psi^*)_Y &= \sum_{x=1}^T (-d_2 \mu_j) (d_1 \mu_j + f'(f^{-1}(a))) \varphi_j(x) + \sum_{x=1}^T (f'(f^{-1}(a)) + d_1 \mu_j) [f^{-1}(a)]^2 \varphi_j^2(x) \\ &= (d_1 \mu_j + f'(f^{-1}(a))) \left([f^{-1}(a)]^2 - d_2 \mu_j \right) \sum_{x=1}^T \varphi_j^2(x) \\ &= (d_1 \mu_j + f'(f^{-1}(a))) \left([f^{-1}(a)]^2 - d_2 \mu_j \right) \sum_{x=1}^T \cos^2 \left(\frac{j\pi}{T} x \right) \\ &= (d_1 \mu_j + f'(f^{-1}(a))) \left([f^{-1}(a)]^2 - d_2 \mu_j \right) \cdot \frac{1}{2} \sum_{x=1}^T \left(\cos \left(\frac{2j\pi}{T} x \right) + 1 \right) \\ &= (d_1 \mu_j + f'(f^{-1}(a))) \left([f^{-1}(a)]^2 - d_2 \mu_j \right) \left[\frac{1}{2} \cdot \frac{\cos((2T+2)j\pi/T) \sin \frac{2\pi}{T} + \frac{T}{2}}{\sin(j\pi/T)} \right] \neq 0. \end{aligned} \tag{72}$$

This suggests that $\Psi \notin (\ker(K^* - I))^\perp = \text{Im}(K - I)$, and so (i) is proved.

Now, we will prove (ii). From (i), for any $b > 0$, $b \neq b_0^j$ and b belongs to a small neighborhood of b_0^j , and $K(b) - I: X \rightarrow X$ is a bijection. Fix $b > 0$; then, θ is a solution of (60), and θ is isolated. From Leray-Schauder fixed point theory, we can get

$$\text{index}(I - K(b) - W, (b, \theta)) = \text{deg}(I - K(b), B, \theta) = (-1)^\gamma, \quad (73)$$

where B is a sufficiently small ball centered at θ , γ is the sum of the algebraic multiplicity of the eigenvalues of $K(b)$, and $\gamma > 1$.

We are going to verify that, for $\varepsilon > 0$ is small enough,

$$\text{index}(I - K(b_0^j - \varepsilon) - W, (b_0^j - \varepsilon, \theta)) \neq \text{index}(I - K(b_0^j + \varepsilon) - W, (b_0^j + \varepsilon, \theta)). \quad (74)$$

If τ is an eigenvalue of $K(b)$ and $\Psi = \begin{pmatrix} \xi \\ \psi \end{pmatrix}$ is the corresponding eigenfunction, then

$$(K(b) - I)\Psi = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad (75)$$

i.e.,

$$\begin{cases} -\tau d_1 \Delta^2 \xi(x-1) = \left(-bf'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} - \tau f'(f^{-1}(a)) \right) \xi(x) + [f^{-1}(a)]^2 \psi(x), & x \in \mathbb{T}, \\ -\tau d_2 \Delta^2 \psi(x-1) = \left(bf'(f^{-1}(a)) - \frac{2ab}{f^{-1}(a)} \right) \xi(x) - [f^{-1}(a)]^2 \tau \psi(x), & x \in \mathbb{T}. \end{cases} \quad (76)$$

By virtue of $\xi = \sum a_j \varphi_j$ and $\psi = \sum c_j \varphi_j$, we can get

$$\sum_{j=0}^{\infty} \begin{pmatrix} \tau \mu_j d_1 + bf'(f^{-1}(a)) - \frac{2ab}{f^{-1}(a)} + \tau f'(f^{-1}(a)) & -[f^{-1}(a)]^2 \\ -bf'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} & \tau \mu_j d_2 + \tau [f^{-1}(a)]^2 \end{pmatrix} \begin{pmatrix} a_j \\ c_j \end{pmatrix} \varphi_j = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (77)$$

Then, the characteristic equation is

$$\begin{aligned} & \left(d_1 d_2 \mu_j^2 + d_2 \mu_j f'(f^{-1}(a)) + d_1 \mu_j [f^{-1}(a)]^2 + f'(f^{-1}(a)) [f^{-1}(a)]^2 \right) \tau^2 \\ & + b \left(d_2 f'(f^{-1}(a)) \mu_j - \frac{2ad_2 \mu_j}{f^{-1}(a)} + f'(f^{-1}(a)) [f^{-1}(a)]^2 - 2af^{-1}(a) \right) \tau + [f^{-1}(a)]^2 \left(-bf'(f^{-1}(a)) + \frac{2ab}{f^{-1}(a)} \right) = 0, \\ & j = 0, 1, 2, \dots, T. \end{aligned} \quad (78)$$

If $\tau = 1$, b can be solved from (78):

$$b = \frac{f^{-1}(a)\{d_1 d_2 \mu_j^2 + d_2 f'(f^{-1}(a))\mu_j + d_1 [f^{-1}(a)]^2 \mu_j + [f^{-1}(a)]^2 f'(f^{-1}(a))\}}{2ad_2 \mu_j - d_2 f^{-1}(a) f'(f^{-1}(a))\mu_j} = b_0^j. \tag{79}$$

Therefore, by calculating the corresponding eigenvalues of (78), we can obtain that when b passes through b_0^j , the number of eigenvalues of $K(b)$ which is greater than 1 is the

same, and their algebraic multiplicity are equal. By plugging (79) into (78), we have

$$\begin{aligned} & \frac{2ad_2 \mu_j - d_2 f^{-1}(a) f'(f^{-1}(a))\mu_j}{f^{-1}(a)} \tau^2 + \left(f'(f^{-1}(a)) (d_2 \mu_j + [f^{-1}(a)]^2) - \frac{2ad_2 \mu_j}{f^{-1}(a)} - 2a f^{-1}(a) \right) \tau \\ & + [f^{-1}(a)]^2 \left(-f'(f^{-1}(a)) + \frac{2a}{f^{-1}(a)} \right) = 0. \end{aligned} \tag{80}$$

Then,

$$\begin{aligned} & \left(d_2 f'(f^{-1}(a))\mu_j - \frac{2ad_2 \mu_j}{f^{-1}(a)} + f'(f^{-1}(a)) [f^{-1}(a)]^2 - 2a f^{-1}(a) \right)^2 - 4(2ad_2 \mu_j - d_2 f^{-1}(a) f'(f^{-1}(a))\mu_j) \\ & \cdot (-f'(f^{-1}(a)) f^{-1}(a) + 2a) = \left[\left(d_2 f'(f^{-1}(a))\mu_j - \frac{2ad_2 \mu_j}{f^{-1}(a)} \right) - \left(f'(f^{-1}(a)) [f^{-1}(a)]^2 - 2a f^{-1}(a) \right) \right]^2 > 0, \end{aligned} \tag{81}$$

and so (80) has two different roots $\tau_1 = 1, \tau_2 = [f^{-1}(a)]^2 / d_2 \mu_j$. Thus, two things will happen:

- (a) If $\mu_j > [f^{-1}(a)]^2 / d_2$, then $\tau_1(b_0^j) = 1, \tau_2(b_0^j) < 1$.
- (b) If $\mu_j < [f^{-1}(a)]^2 / d_2$, then $\tau_1(b_0^j) = 1, \tau_2(b_0^j) > 1$.

When scenario (a) occurs, b passes through b_0^j and $\tau_2(b) < 1$. From (78), $\tau_1(b_0^j + \varepsilon) > 1, \tau_1(b_0^j - \varepsilon) < 1$. Therefore, the matrix $K(b_0^j + \varepsilon)$ has exactly one more eigenvalue that is > 1 than $K(b_0^j - \varepsilon)$ does, and its algebraic multiplicity is 1. Then, (74) holds. That is to say, the index jumps as b goes through b_0^j .

When scenario (b) occurs, b passes through b_0^j and $\tau_2(b) > 1$. From (78), $\tau_1(b_0^j + \varepsilon) > 1, \tau_1(b_0^j - \varepsilon) < 1$. Similarly, the index jumps as b goes through b_0^j . Therefore, (ii) is true regardless of (a) or (b).

Thus, by the index jump principle and [24], Theorem 1.3, it follows that there exists a connected component $\widehat{\Gamma}_j$ of nontrivial solutions of (60), and $\widehat{\Gamma}_j$ comes from the bifurcation point (b_0^j, θ) . We know that $\widehat{\Gamma}_j$ is also the connected component Γ_j of the nonconstant solution of (4) from (b_0^j, \overline{w}) . $\widehat{\Gamma}_j$ and Γ_j are both in $\mathbb{R} \times X$. By the Rabinowitz global bifurcation theorem, the connected component Γ_j joins (b_0^j, \overline{w}) to either ∞ or (b_0^k, \overline{w}) in $\mathbb{R} \times X$, where $k \neq j$.

We first prove that the latter situation will not happen. According to Theorem 1, the solution on the connected component sent from (b_0^j, \overline{w}) is related to φ_j , and φ_j has

exactly $j - 1$ simple generalized zeros. In the same way, the solution on the connected component sent from (b_0^k, \overline{w}) is related to φ_k , and φ_k has $k - 1$ simple generalized zeros. If the connected component sent Γ_j joining (b_0^j, \overline{w}) to (b_0^k, \overline{w}) , the solution $(b, w) \in \Gamma_j$ is related to both φ_j and φ_k , which is impossible. On the contrary, Lemma 1 shows that if $b = b_c \in (0, \infty)$, then the solutions u and v of (4) are both bounded. So, the connected component Γ_j will not join (b_0^j, \overline{w}) to (b_c, ∞) . Therefore, the connected component Γ_j can only join (b_0^j, \overline{w}) to either (∞, ∞) or (∞, m) , where $m \in (0, \infty)$. But, in any case, the projection of continuum Γ_j is unbounded on the b -axis. \square

Data Availability

Data sharing is not applicable to this article as no datasets were generated.

Conflicts of Interest

All authors declare no conflicts of interest.

Authors' Contributions

The authors claim that the research was realized in collaboration with the same responsibility. All authors read and approved the last version of the manuscript.

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