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Research Article

Dynamics of a Stochastic SIRS Epidemic Model with Regime Switching and Specific Functional Response

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The purpose of this work is to investigate the dynamic behaviors of the SIRS epidemic model with nonlinear incident rate under regime switching. We establish the existence of a unique positive solution of our system. Furthermore, we obtain the conditions for the extinction of diseases, and we show the existence of the stationary distribution for our stochastic SIRS model under regime switching. Numerical simulations are employed to illustrate our theoretical analysis.

1. Introduction

Several of mathematicians have developed various epidemic models to prevent and control the spread of transmissible diseases in the community.

The classical SIR model presented by Kermack and McKendrick [1] has played an important role in mathematical epidemiology. The SIR model are used to study the disease spread between three groups of population to know the susceptible S, the infective I, and the recovered R.

In this work, we introduce a switched stochastic SIRS epidemic model with specific functional response. Then, we consider the following deterministic SIRS epidemic model with specific functional response:

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= \Lambda - \mu S - \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI} + \gamma R,\\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI} - (\mu+\lambda+\delta)I, \qquad (1)\\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \lambda I - (\mu+\gamma)R,
\end{aligned}$$

where $S(t)$ denotes the number of susceptible individuals, $I(t)$ denotes the number of infective individuals, and $R(t)$ represents the number of removed individuals. Λ is the recruitment rate of the population, μ is the natural death rate of the population, γ is the rate at which recovered individuals loss immunity and return to the susceptible class, λ denotes the natural recovery rate of the infectious individuals, and δ denotes the disease inducing death rate. The infection transmission process in (1) is modeled by the specific functional response $(\beta SI/1+\alpha_1 S+\alpha_2 I+\alpha_3 SI)$, where β is the transmission coefficient between compartments S and I, and α_1, α_2, $\alpha_3 \geq 0$ are the saturation factors measuring the psychological or inhibitory effect. In addition, this functional response generalizes many common types existing in the literature such as the Crowley–Martin functional response introduced in [2] and used in [3] when $\alpha_3 = \alpha_1\alpha_2$ and the Beddington–DeAngelis functional response proposed in [4] and used in [5] when $\alpha_3 = 0$.

Environmental fluctuations have been indicated to play an important role in the propagation of disease [6, 7]. In effect, disease infestation is highly stochastic, and stochastic noise can raise the probability of disease extinction in the early phase of epidemics. By running an ODE system, we can get only a certain sample solution, whereas by running an SDE system, we can obtain the stochastic distribution of disease dynamics [8]. Lately, dynamic modeling of infectious diseases based on stochastic differential equations (SDE) has received considerable attention from experts and academics [9–11]. The SIRS epidemic model with white noise is expressed by

$$
\begin{aligned}
\mathrm{d}S &= \left(\Lambda - \mu S - \frac{\beta SI}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI} + \gamma R\right)\mathrm{d}t + \sigma_1 S\mathrm{d}B_1(t),\\
\mathrm{d}I &= \left(\frac{\beta SI}{1+\alpha_1 S+\alpha_2 I+\alpha_3 SI} - (\mu+\lambda+\delta)I\right)\mathrm{d}t + \sigma_2 I\mathrm{d}B_2(t),\\
\mathrm{d}R &= (\lambda I - (\mu+\gamma)R)\mathrm{d}t + \sigma_3 R\mathrm{d}B_3(t),
\end{aligned}
\tag{2}
$$

where $B_1(t), B_2(t),$ and $B_3(t)$ are independent Brownian motions and $\sigma_1, \sigma_2,$ and σ_3 are their intensities. Besides environment white noise, in this paper, we will also consider another noise, namely, telegraph noise ([12–15]). The latter can be described as a switching between two or more regimes of environment, which differ in terms of factors such as nutrition, climatic characteristics, or sociocultural factors. The switching among different environments is memoryless and the waiting time for the next switch is exponentially distributed. The regime switching can hence be modeled by a finite-state Markov chain $\{\varsigma(t), t\geq 0\}$ taking values in a finite-state space $\mathbb{M} = \{1, 2, \ldots, N\}$. The stochastic system (1) with regime switching can be described by the following model:

$$
\begin{aligned}
\mathrm{d}S &= \left[\Lambda(\varsigma(t)) - \mu(\varsigma(t))S - \frac{\beta(\varsigma(t))SI}{1+\alpha_1(\varsigma(t))S+\alpha_2(\varsigma(t))I+\alpha_3(\varsigma(t))SI} + \gamma(\varsigma(t))R\right]\mathrm{d}t\\
&\quad + \sigma_1(\varsigma(t))S\mathrm{d}B_1(t),\\
\mathrm{d}I &= \left[\frac{\beta(\varsigma(t))SI}{1+\alpha_1(\varsigma(t))S+\alpha_2(\varsigma(t))I+\alpha_3(\varsigma(t))SI} - (\mu(\varsigma(t))+\lambda(\varsigma(t))+\delta(\varsigma(t)))I\right]\mathrm{d}t\\
&\quad + \sigma_2(\varsigma(t))I\mathrm{d}B_2(t),\\
[\mathrm{d}R &= \lambda(\varsigma(t))I - (\mu(\varsigma(t))+\gamma(\varsigma(t)))R]\mathrm{d}t + \sigma_3(\varsigma(t))R\mathrm{d}B_3(t).
\end{aligned}
\tag{3}
$$

Throughout this paper, we let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual conditions (i.e. it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets). Let $\{\varsigma(t), t\geq 0\}$ be a right-continuous Markov chain on the probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ taking values in a finite-state space $\mathbb{M} = \{1, 2, \ldots, N\}$ with the generator $\Phi = (\phi_{uv})_{1\leq u,v\leq N}$ given, for $\delta > 0$, by

$$
P(\varsigma(t+\delta) = v \mid \varsigma(t) = u) = \begin{cases} \phi_{uv}\delta + o(\delta), & \text{if } u \neq v,\\ 1 + \phi_{uu}\delta + o(\delta), & \text{if } u = v. \end{cases}
\tag{4}
$$

Here, ϕ_{uv} is the transition rate from u to v and $\phi_{uv}\geq 0$ if $u\neq v$, while

$$
\phi_{uu} = -\sum_{u\neq v}\phi_{uv}.
\tag{5}
$$

Suppose that the Markov chain $\varsigma(t)$ is independent of the Brownian motion $B(\cdot)$ and it is irreducible. Under this condition, the Markov chain has a unique stationary (probability) distribution $\pi = (\pi_1, \ldots, \pi_N)$, which can be determined by solving the linear equation $\pi\Phi = 0$, subject to $\sum_{i=1}^N \pi_i = 1$, and $\pi_i > 0$, $\forall i \in \mathbb{M}$. Thereafter, for any vector $h = (h(1), \ldots, h(N))^T$, let $\hat{h} = \min_{i\in\mathbb{M}}\{h(i)\}$ and $\check{h} = \max_{i\in\mathbb{M}}\{h(i)\}$.

The rest of the paper is organized as follows. In Section 2, we show that there exists a unique global positive solution of system (3). In Section 3, we give sufficient conditions for the extinction of the disease. In Section 4, sufficient conditions for the existence of the ergodic stationary distribution are established for model (3). Finally, numerical simulations are carried out to support the theoretical results.

2. Existence and Uniqueness of the Global Positive Solution

In this section, we will prove that model (3) has a unique global positive solution. We also denote

$$
\mathbb{R}_+^3 = \left\{(x_1, x_2, x_2) \mid x_i > 0, i = 1, 2, 3\right\}.
\tag{6}
$$

Thus, we established the following theorem.

Theorem 1. *For any given initial value, $X(0) = (S(0), I(0), R(0), \varsigma(0)) \in \mathbb{R}_+^3\times\mathbb{M}$, there is a unique positive solution $X(t) = (S(t), I(t), R(t), \varsigma(t)) \in \mathbb{R}_+^3\times\mathbb{M}$ of model (3) on $t\geq 0$ and the solution will remain in $\mathbb{R}_+^3\times\mathbb{M}$ with probability 1, namely, $(S(t), I(t), R(t), \varsigma(t)) \in \mathbb{R}_+^3\times\mathbb{M}$ for all $t\geq 0$ almost surely.*

Proof. Since the coefficients of system (3) are locally Lipshitz continuous, for any initial value $X(0)\in\mathbb{R}_+^3$, there exists a unique local solution $X(t)$ on $t\in[0, \tau_e)$, where τ_e is the explosion time. We need to show that this solution is global almost surely that is, $\tau_e = \infty$ a.s. Let m_0 be sufficiently large such that every component of $X(0)$ lies within the interval

$[(1/m_0), m_0]$. For each integer $m \geq m_0$, define a sequence of stopping times by

$$\tau_m = \inf\left\{t \in [0, \tau_e): \ S(t) \notin \left(\frac{1}{m}, m\right) \text{ or } I(t) \notin \left(\frac{1}{m}, m\right) \text{ or } R(t) \notin \left(\frac{1}{m}, m\right)\right\}. \tag{7}$$

We set $\inf\{\varnothing\} = \infty$, where $\varnothing$ denotes the empty set. Obviously, τ_m is increasing as $m \longrightarrow \infty$. Set $\tau_\infty = \lim_{m\longrightarrow\infty}\tau_m$ with $\tau_\infty \leq \tau_e$ $(a.s)$. Now, we need to show $\tau_\infty = \infty\, a.s.$ If this statement is violated, then there exist $T > 0$ and $\varepsilon \in (0, 1)$ such that

$$\mathbb{P}(\tau_\infty \leq T) > \varepsilon. \tag{8}$$

Hence, there is an integer $m_1 \geq m_0$ such that

$$\mathbb{P}(\tau_\infty \leq T) \geq \varepsilon \text{ for all } m \geq m_1. \tag{9}$$

Define a C^2–function $V_1\colon \mathbb{R}_+^3 \longrightarrow \mathbb{R}_+$ as

$$V_1(S, I, R) = (S - 1 - \ln S) + (I - 1 - \ln I) + (R - 1 - \ln R). \tag{10}$$

By Itô's formula, we have

$$\mathrm{d}V_1 = \mathscr{L}V_1\mathrm{d}t + (S-1)\mathrm{d}B_1(t) + (I-1)\mathrm{d}B_2(t) + (R-1)\mathrm{d}B_3(t), \tag{11}$$

where

$$\begin{aligned}\mathscr{L}V_1 = {} & \left(1 - \frac{1}{S}\right)\left[\Lambda(k) - \mu(k)S - \frac{\beta(k)SI}{1 + \alpha_1(k)S + \alpha_2(k)I + \alpha_3(k)SI} + \gamma(k)R\right] \\ & + \left(1 - \frac{1}{I}\right)\left[\frac{\beta(k)SI}{1 + \alpha_1(k)S + \alpha_2(k)I + \alpha_3(k)SI} - (\mu(k) + \lambda(k) + \delta(k))I\right] \\ & + \left(1 - \frac{1}{R}\right)[\lambda(k)I - (\mu(k) + \gamma(k))R] + \frac{\sigma_1^2(k)}{2} + \frac{\sigma_2^2(k)}{2} + \frac{\sigma_3^2(k)}{2},\end{aligned} \tag{12}$$

which implies that

$$\begin{aligned}\mathscr{L}V_1 = {} & \Lambda(k) - \mu(k)S - \frac{\Lambda(k)}{S} + \mu(k) + \frac{\beta(k)I}{1 + \alpha_1(k)S + \alpha_2(k)I + \alpha_3(k)SI} \\ & - \gamma(k)\frac{R}{S} + \frac{\sigma_1^2(k)}{2} - (\mu(k) + \delta(k))I - \frac{\beta(k)S}{1 + \alpha_1(k)S + \alpha_2(k)I + \alpha_3(k)SI} \\ & + (\mu(k) + \lambda(k) + \delta(k)) + \frac{\sigma_2^2(k)}{2} - \mu(k)R - \lambda(k)\frac{I}{R} + (\mu(k) + \gamma(k)) \\ & + \frac{\sigma_3^2(k)}{2} \\ \leq {} & \Lambda(k) + 3\mu(k) + \lambda(k) + \delta(k) + \gamma(k) + \frac{\sigma_1^2(k) + \sigma_2^2(k) + \sigma_3^2(k)}{2} \frac{\beta(k)I}{1 + \alpha_1(k)S + \alpha_2(k)I + \alpha_3(k)SI} \\ \leq {} & \breve{\Lambda} + 3\breve{\mu} + \breve{\lambda} + \breve{\delta} + \breve{\gamma} + \frac{\breve{\sigma}_1^2 + \breve{\sigma}_2^2 + \breve{\sigma}_3^2}{2} + \frac{\breve{\beta}}{\hat{\alpha}_2} := \mathscr{K}.\end{aligned} \tag{13}$$

Hence,

$$\mathrm{d}V_1(S, I, R) \leq \mathscr{K}\mathrm{d}t + (S-1)\mathrm{d}B_1(t) + (I-1)\mathrm{d}B_2(t) + (R-1)\mathrm{d}B_3(t). \tag{14}$$

Integrating both sides of the above inequality from 0 to $\tau_m \wedge T$, and taking the expectations, we get

$$\begin{aligned}& \mathbb{E}V_1(S(\tau_m \wedge T), I(\tau_m \wedge T), R(\tau_m \wedge T)) \\ & \leq V_1(S(0), I(0), R(0)) + \mathscr{K}T.\end{aligned} \tag{15}$$

Set $\Omega_m = \{\tau_m \leq T\}$ for $m \geq m_1$ and by (4), we have $\mathbb{P}(\tau_\infty \leq T) \geq \varepsilon$ for each $m \geq m_1$. For every $\omega \in \Omega_m$, we have

$$
\begin{aligned}
&V_1(S(\tau_m \wedge T), I(\tau_m \wedge T), R(\tau_m \wedge T)) \\
&\quad \geq \min\left\{(m-1-\ln m), \left(\frac{1}{m}-1-1\ln\frac{1}{m}\right)\right\} := L_m.
\end{aligned} \tag{16}
$$

Then, we obtain

$$
\begin{aligned}
V_1(S(0), I(0), R(0)) + \mathscr{K}T &\geq \mathbb{E}\left[1_{\Omega_m} V_1(S(\tau_m \wedge T),\right. \\
&\quad \left. I(\tau_m \wedge T), R(\tau_m \wedge T))\right] \\
&\geq \varepsilon L_m,
\end{aligned} \tag{17}
$$

where 1_{Ω_m} is the indicator function of Ω_m. Letting $m \longrightarrow +\infty$ leads to the contradiction $\infty = V_1(S(0), I(0), R(0)) + \mathscr{K}T < \infty$. So, we must therefore have $\tau_\infty = \infty$ a.s. This completes the proof. □

3. Extinction

Our goal in this section is to study the extinction and give the extinction threshold of system (3). Then, the following theorem gives a sufficient condition for extinction of the disease.

Theorem 2. *If $\mathscr{R}_0^s = \sum_{k=1}^N \pi_k \beta(k)/\widehat{\alpha}_1 \sum_{k=1}^N \pi_k (\mu(k) + \lambda(k) + \delta(k) + (\sigma_2(k)/2)) < 1$, then the disease I tends to zero exponentially with probability one, i.e.,*

$$
\limsup_{t\longrightarrow\infty} \frac{\ln I(t)}{t} \leq \sum_{k=1}^N \pi_k \left(\mu(k) + \lambda(k) + \delta(k) + \frac{\sigma_2(k)}{2}\right)\left[\mathscr{R}_0^s - 1\right]. \tag{18}
$$

Proof. Applying Itô's formula, we can get

$$
\begin{aligned}
\mathrm{d}\ln I(t) = &\frac{\beta(\varsigma(t))S}{1 + \alpha_1(\varsigma(t))S + \alpha_2(\varsigma(t))I + \alpha_3(\varsigma(t))SI} \\
&- \left(\mu(\varsigma(t)) + \lambda(\varsigma(t)) + \delta(\varsigma(t)) + \frac{\sigma_2(\varsigma(t))}{2}\right) \\
&+ \sigma_2(\varsigma(t))\mathrm{d}B_2(t).
\end{aligned} \tag{19}
$$

Integrating (19) from 0 to t and then dividing by t into both sides leads to

$$
\begin{aligned}
\frac{\ln I(t) - \ln I(0)}{t} &= \frac{1}{t}\int_0^t \frac{\beta(\varsigma(s))S}{1 + \alpha_1(\varsigma(s))S + \alpha_2(\varsigma(s))I(s) + \alpha_3(\varsigma(t))S(s)I(s)}\mathrm{d}s \\
&\quad - \frac{1}{t}\int_0^t \left[\mu(\varsigma(s)) + \lambda(\varsigma(s)) + \delta(\varsigma(s)) + \frac{\sigma_2(\varsigma(s))}{2}\right]\mathrm{d}s \\
&\quad + \frac{1}{t}\int_0^t \sigma_2(\varsigma(s))\mathrm{d}B_2(s) \\
&\leq \frac{1}{t}\int_0^t \left[\frac{\beta(\varsigma(s))}{\widehat{\alpha}_1} - \left(\mu(\varsigma(s)) + \lambda(\varsigma(s)) + \delta(\varsigma(s)) + \frac{\sigma_2(\varsigma(s))}{2}\right)\right]\mathrm{d}s \\
&\quad + \frac{1}{t}\int_0^t \sigma_2(\varsigma(s))\mathrm{d}B_2(s) \\
&= \frac{1}{t}\int_0^t \left[\frac{\beta(\varsigma(s))}{\widehat{\alpha}_1} - \left(\mu(\varsigma(s)) + \lambda(\varsigma(s)) + \delta(\varsigma(s)) + \frac{\sigma_2(\varsigma(s))}{2}\right)\right]\mathrm{d}s \\
&\quad + \frac{M(t)}{t},
\end{aligned} \tag{20}
$$

where $M(t)$ is a local martingale defined by $M(t) = \int_0^t \sigma_2(\varsigma(s))\mathrm{d}B_2(s)$, whose quadratic variation is $\langle M, M\rangle_t = \int_0^t \sigma_2^2(\varsigma(s))\mathrm{d}s \leq \check{\sigma}_2^2 t$. Making use of the strong law of large numbers for martingales ([16]) yields

$$
\lim_{t\longrightarrow\infty} \frac{M(t)}{t} = 0, \quad \text{a.s.} \tag{21}
$$

Taking the superior limit on both sides of (20) and applying the ergodicity of Markov chain $\varsigma(t)$, we get

$$
\begin{aligned}
\limsup_{t\longrightarrow\infty} \frac{\ln I(t)}{t} &\leq \sum_{k=1}^N \pi_k \left[\frac{\beta(k)}{\widehat{\alpha}_1} - \left(\mu(k) + \lambda(k) + \delta(k) + \frac{\sigma_2(k)}{2}\right)\right] \\
&= \sum_{k=1}^N \pi_k \left(\mu(k) + \lambda(k) + \delta(k) + \frac{\sigma_2(k)}{2}\right) \\
&\quad \times \left[\frac{\sum_{k=1}^N \pi_k \beta(k)}{\widehat{\alpha}_1 \sum_{k=1}^N \pi_k (\mu(k) + \lambda(k) + \delta(k) + (\sigma_2(k)/2))} - 1\right] \\
&= \sum_{k=1}^N \pi_k \left(\mu(k) + \lambda(k) + \delta(k) + \frac{\sigma_2(k)}{2}\right)\left[\mathscr{R}_0^s - 1\right] < 0,
\end{aligned} \tag{22}
$$

which implies that $\lim_{t\longrightarrow\infty} I(t) = 0$. This completes the proof of theorem. □

4. Existence of Ergodic Stationary Distribution

In this section, we shall discuss sufficient conditions for the existence of an ergodic stationary distribution to model (3). The following lemma gives a criterion for positive recurrence in terms of lyapunov function [17].

Let $(X(t), \varsigma(t))$ is the diffusion Markov process and satisfy the following equation

$$\begin{aligned} \mathrm{d}X(t) &= f(X(t), \varsigma(t))\mathrm{d}t + \sigma(X(t), \varsigma(t))\mathrm{d}B(t), \\ X(0) &= X_0, \quad \varsigma(0) = \varsigma, \end{aligned} \tag{23}$$

where $f\colon \mathbb{R}^n \times \mathbb{M} \longrightarrow \mathbb{R}^n$, $\sigma\colon \mathbb{R}^n \times \mathbb{M} \longrightarrow \mathbb{R}^{n\times d}$ satisfying $\sigma(x,k)\sigma^T(x,k) = (d_{ij}(x,k))$. For each $k \in \mathbb{M}$, and for any twice continuously differentiable function $V(.,k)$, the operator $\mathscr{L}$ can be defined by

$$\begin{aligned} \mathscr{L}V(x,k) &= \sum_{i=1}^{n} f_i(x,k)\frac{\partial V(x,k)}{\partial x_i} \\ &\quad + \frac{1}{2}\sum_{i,j=1}^{n} d_{ij}(x,k)\frac{\partial^2 V(x,k)}{\partial x_i \partial x_j} + \sum_{l=1}^{n}\phi_{kl}V(x,l). \end{aligned} \tag{24}$$

Lemma 1. *If the following conditions are satisfied:*

(1) $\phi_{ij} > 0$ for any $i \neq j$.

(2) For each $k \in \mathbb{M}$, $D(x,k) = (d_{ij}(x,k))$ is symmetric and satisfies

$$\lambda|\zeta|^2 \leq \langle D(x,k)\zeta, \zeta\rangle \leq \lambda^{-1}|\zeta|^2 \quad \text{for all } \zeta \in \mathbb{R}^n, \tag{25}$$

with some constant $\lambda \in (0,1]$ for all $x \in \mathbb{R}^n$.

(3) There exists a nonempty open set $\mathscr{D}$ with compact closure, satisfying that, for each $k \in \mathbb{M}$, there is a nonnegative function $V(.,k)\colon \mathscr{D}^c \longrightarrow \mathbb{R}$ such that $V(.,k)$ is twice continuously differential and that for some $\alpha > 0$,

$$LV(x,k) \leq -\alpha \quad \text{for all } (x,k) \in \mathscr{D}^c \times \mathbb{M}. \tag{26}$$

Then, $(X(t), \varsigma(t))$ of system (23) is positive recurrent and ergodic. That is to say, there exists a unique stationary distribution $\mu(.,.)$ such that for any Borel measurable function $f\colon \mathbb{R}^n \times \mathbb{M} \longrightarrow \mathbb{R}$ satisfying

$$\sum_{k=1}^{N}\int_{\mathbb{R}^n}|f(x,k)|\mu(dx,k) < \infty, \tag{27}$$

we have

$$\mathbb{P}\left(\lim_{t\longrightarrow\infty}\frac{1}{t}\int_0^t f(X(s),\varsigma(s))\mathrm{d}s = \sum_{k=1}^{N}\int_{\mathbb{R}^n}|f(x,k)|\mu(dx,k)\right) = 1. \tag{28}$$

Theorem 3. *Assume that $\mathscr{R}_0^s > 1$, then for any initial value $(S(0), I(0), R(0), \varsigma(0)) \in \mathbb{R}_+^3 \times \mathbb{M}$, the solution $(S(t), I(t), R(t), \varsigma(t))$ of system (3) admits a unique ergodic stationary distribution.*

Proof. In order to prove Theorem 3, it is sufficient to prove conditions (1), (2) and (3) in Lemma 1. Assumption $\phi_{ij} > 0$ for any $i \neq j$, in Section 1 implies that condition (1) in Lemma 1 is satisfied. To verify condition (2), consider the bounded open subset

$$\mathscr{D} = \left(\frac{1}{\eta}, \eta\right) \times \left(\frac{1}{\eta}, \eta\right) \times \left(\frac{1}{\eta}, \eta\right) \subset \mathbb{R}_+^3, \tag{29}$$

where η is a sufficiently large number. Then, $\overline{\mathscr{D}} \subset \mathbb{R}_+^3$. We have $D(S,I,R,k) = Q(S,I,R,k)Q^T(S,I,R,k)$, where $Q(S,I,R,k) = \mathrm{diag}\{S\sigma_1(k), I\sigma_2(k), R\sigma_3(k)\}$, $k \in \mathbb{M}$. Then, $D(S,I,R,k)$ is positive semi-definite, and since $R(S,I,R,k)$ is a nonsingular matrix, we deduce that $D(S,I,R,k)$ is positive definite. Hence,

$$\lambda_{\max}(D(S,I,R,k)) \geq \lambda_{\min}(D(S,I,R,k)) > 0, \tag{30}$$

in addition, we have for all $\zeta \in \mathscr{D}$,

$$\begin{aligned} \lambda_{\min}(D(S,I,R,k))|\zeta|^2 &\leq \zeta^T(D(S,I,R,k))\zeta \\ &\leq \lambda_{\max}(D(S,I,R,k))|\zeta|^2. \end{aligned} \tag{31}$$

It is easy to see that $\lambda_{\min}(D(S,I,R,k))$ and $\lambda_{\max}(D(S,I,R,k))$ are two continuous functions of S, I, and R. Therefore, $\widehat{\lambda} = \min \overline{\mathscr{D}} \times \mathbb{M}\lambda_{\min}(D(S,I,R,k)) > 0$ and $\widehat{\lambda} = \max \overline{\mathscr{D}} \times \mathbb{M}\lambda_{\max}(D(S,I,R,k)) > 0$, which implies that

$$\lambda|\zeta|^2 \leq \zeta^T(D(S,I,R,k))\zeta \leq \lambda^{-1}|\zeta|^2, \tag{32}$$

where $\lambda = \min\left\{\widehat{\lambda}, \check{\lambda}^{-1}\right\}$. Then, condition (2) in Lemma 1 is verified.

Now, we verify condition (3). Define a C^2–function $\widetilde{V}\colon \mathbb{R}_+^3 \times \mathbb{M} \longrightarrow \mathbb{R}$ by

$$\widetilde{V}(S,I,R,k) = -M(\ln I + \omega(k)) - \ln S - \ln R + (S + I + R). \tag{33}$$

In addition, $\widetilde{V}(S,I,R,k)$ is a continuous function on with respect to (S,I,R). So there is a unique minimum value point $\widetilde{V}(\overline{S},\overline{I},\overline{R},\overline{k})$ of $\widetilde{V}(S,I,R,k)$ in the interior of $\mathbb{R}_+^3 \times \mathbb{M}$; then, we define a nonnegative C^2-function $\mathscr{H}$: $\mathbb{R}_+^3 \times \mathbb{M} \longrightarrow \mathbb{R}_+$ as follows:

$$\begin{aligned} \mathscr{H}(S,I,R,k) &= \widetilde{V}(S,I,R,k) - \widetilde{V}(\overline{S},\overline{I},\overline{R},\overline{k}) \\ &= -M(\ln I + \omega(k)) - \ln S - \ln R + (S + I + R) \\ &\quad - \widetilde{V}(\overline{S},\overline{I},\overline{R},\overline{k}) \\ &:= -MV_2 - V_3 - V_4 + V_5, \end{aligned} \tag{34}$$

where $(S,I,R,k) \in ((1/n),n) \times ((1/n),n) \times ((1/n),n) \times \mathbb{M}$ and $n > 1$ is a sufficiently large number, $V_2 = -M(\ln I + \omega(k))$, $V_3 = -\ln S$, $V_4 = -\ln R$, $V_5 = S + I + R - \widetilde{V}(\overline{S},\overline{I},\overline{R},\overline{k})$, and $M > 0$ satisfies the following condition:

$$
\begin{aligned}
&-M\sum_{k=1}^{N}\pi_k\left(\mu(k)+\lambda(k)+\delta(k)+\frac{\sigma_2(k)}{2}\right)(\mathscr{R}_0^s-1)\\
&+M\frac{\breve{\beta}}{\widehat{\alpha}_1}+2\breve{\mu}+\breve{\gamma}+\breve{\Lambda}+\frac{\breve{\sigma}_1}{2}+\frac{\breve{\sigma}_3}{2}\leq -2.
\end{aligned} \tag{35}
$$

Applying Itô's formula to V_2 leads to

$$
\begin{aligned}
\mathscr{L}V_2 &= \frac{-\beta(k)S}{1+\alpha_1(k)S+\alpha_2(k)I+\alpha_3(k)SI}+\mu(k)+\lambda(k)+\delta(k)\\
&\quad+\frac{\sigma_2(k)}{2}-\sum_{l=1}^{N}\phi_{kl}\omega(l)=\frac{-\beta(k)}{\widehat{\alpha}_1}+\mu(k)+\lambda(k)+\delta(k)\\
&\quad+\frac{\sigma_2(k)}{2}-\sum_{l=1}^{N}\phi_{kl}\omega(l)+\beta(k)\left(\frac{1}{\widehat{\alpha}_1}-\frac{S}{1+\alpha_1(k)S+\alpha_2(k)I+\alpha_3(k)SI}\right)\\
&\leq\frac{-\beta(k)}{\widehat{\alpha}_1}+\mu(k)+\lambda(k)+\delta(k)+\frac{\sigma_2(k)}{2}-\sum_{l=1}^{N}\phi_{kl}\omega(l)+\frac{\beta(k)}{\widehat{\alpha}_1}\\
&\coloneqq -R_0(k)-\sum_{l=1}^{N}\phi_{kl}\omega(l)+\frac{\beta(k)}{\widehat{\alpha}_1},
\end{aligned} \tag{36}
$$

where $R_0(k)=(\beta(k)/\breve{\alpha}_1)-\mu(k)-\lambda(k)-\delta(k)-(\sigma_2(k)/2)$. Since the generator matrix Φ is irreducible, then for $R_0=(R_0(1),R_0(2),\ldots,R_0(N))^T$, there exists $\omega=(\omega(1),\omega(2),\ldots,\omega(N))^T$ solution of the Poisson system ([18]):

$$
\Phi\omega=-R_0+\left(\sum_{l=1}^{N}\pi_l R_0(l)\right)\vec{F}, \tag{37}
$$

where $\vec{F}$ denotes the column vector with all its entries equal to 1. Then,

$$
\begin{aligned}
\mathscr{L}V_2&\leq-\sum_{k=1}^{N}\pi_k R_0(k)+\frac{\beta(k)}{\breve{\alpha}_1}\\
&=-\sum_{k=1}^{N}\pi_k\left(\mu(k)+\lambda(k)+\delta(k)+\frac{\sigma_2(k)}{2}\right)(\mathscr{R}_0^s-1)+\frac{\breve{\beta}}{\widehat{\alpha}_1}.
\end{aligned} \tag{38}
$$

Next, we calculate $\mathscr{L}V_3$

$$
\begin{aligned}
\mathscr{L}V_3&=\frac{-\Lambda(k)}{S}+\mu(k)+\frac{\beta(k)I}{1+\alpha_1(k)S+\alpha_2(k)I+\alpha_3(k)SI}\\
&\quad-\gamma(k)\frac{R}{S}+\frac{\sigma_1(k)}{2}\leq\frac{-\widehat{\Lambda}}{S}+\breve{\mu}+\frac{\breve{\beta}I}{1+\widehat{\alpha}_1S+\widehat{\alpha}_2I+\widehat{\alpha}_3SI}\\
&\quad-\widehat{\gamma}\frac{R}{S}+\frac{\breve{\sigma}_1}{2},
\end{aligned} \tag{39}
$$

and $\mathscr{L}V_4$

$$
\mathscr{L}V_4=-\lambda(k)\frac{I}{R}+\mu(k)+\gamma(k)+\frac{\sigma(k)_3^2}{2}\leq-\widehat{\lambda}\frac{I}{R}+\breve{\mu}+\breve{\gamma}+\frac{\breve{\sigma}_3^2}{2}. \tag{40}
$$

We have

$$
\begin{aligned}
\mathscr{L}V_5&=\Lambda(k)-\mu(k)(S+I+R)-\delta(k)I\\
&\leq\Lambda(k)-\mu(k)(S+I+R)\\
&\leq\breve{\Lambda}-\widehat{\mu}(S+I+R).
\end{aligned} \tag{41}
$$

The differential operator $\mathscr{L}$ acting on the function $\mathscr{H}$ yields

$$
\begin{aligned}
\mathscr{L}\mathscr{H}&\leq-M\sum_{k=1}^{N}\pi_k\left(\mu(k)+\lambda(k)+\delta(k)+\frac{\sigma_2(k)}{2}\right)(\mathscr{R}_0^s-1)+M\frac{\breve{\beta}}{\widehat{\alpha}_1}-\frac{\widehat{\Lambda}}{S}\\
&\quad+\frac{\breve{\beta}I}{1+\widehat{\alpha}_1S+\widehat{\alpha}_2I+\widehat{\alpha}_3SI}-\widehat{\gamma}\frac{R}{S}-\widehat{\mu}(S+I+R)-\widehat{\lambda}\frac{I}{R}+\breve{\Lambda}+2\breve{\mu}\\
&\quad+\breve{\gamma}+\frac{\breve{\sigma}_3^2}{2}+\frac{\breve{\sigma}_1^2}{2}.
\end{aligned} \tag{42}
$$

Defining the following compact set

$$
\mathscr{D}_\varepsilon=\left\{(S,I,R)\in\mathbb{R}_+^3:\ \varepsilon\leq S\leq\frac{1}{\varepsilon},\varepsilon^2\leq I\leq\frac{1}{\varepsilon^2},\varepsilon^3\leq R\leq\frac{1}{\varepsilon^3}\right\}, \tag{43}
$$

where ε is a sufficiently small number. In the set $(\mathbb{R}_+^3/\mathscr{D}_\varepsilon)$, we can choose ε sufficiently small such that the following conditions hold:

$$
\begin{aligned}
&-\frac{\widehat{\Lambda}}{\varepsilon}+C_1 \le -1,\\
&\varepsilon \le \frac{\widehat{\alpha}_1}{\check{\beta}},\\
&-\frac{\widehat{\lambda}}{\varepsilon}+C \le -1,\\
&-\frac{\widehat{\mu}}{\varepsilon}+C \le -1,\\
&-\frac{\widehat{\mu}}{\varepsilon^2}+C \le -1,\\
&-\frac{\widehat{\mu}}{\varepsilon^3}+C \le -1,\\
&C=\frac{\widehat{\beta}}{\widehat{\alpha}_2}+M\frac{\check{\beta}}{\widehat{\alpha}_1}+\check{\Lambda}+2\check{\mu}+\check{\gamma}+\frac{\check{\sigma}_3^2}{2}+\frac{\check{\sigma}_1^2}{2}.
\end{aligned} \tag{44}
$$

Next, we can divide $(\mathbb{R}_+^3/\mathscr{D}_\varepsilon)$ into the following six domains:

$$
\begin{aligned}
&\mathscr{D}_1=\left\{(S,I,R)\in\mathbb{R}_+^3: S\le\varepsilon\right\},\quad \mathscr{D}_2=\left\{(S,I,R)\in\mathbb{R}_+^3: \varepsilon\le S, \le I\le\varepsilon^2\right\},\\
&\% \ \mathscr{D}_3=\left\{(S,I,R)\in\mathbb{R}_+^3: \varepsilon^2\le I, R\le\varepsilon^3\right\},\quad \mathscr{D}_4=\left\{(S,I,R)\in\mathbb{R}_+^3: S>\frac{1}{\varepsilon}\right\},\\
&\mathscr{D}_5=\left\{(S,I,R)\in\mathbb{R}_+^3: I>\frac{1}{\varepsilon^2}\right\},\quad \mathscr{D}_6=\left\{(S,I,R)\in\mathbb{R}_+^3: R>\frac{1}{\varepsilon^3}\right\}.
\end{aligned} \tag{45}
$$

Clearly, $\mathscr{D}_\varepsilon^c=\mathscr{D}_1\cup\mathscr{D}_2\cup\mathscr{D}_3\cup\mathscr{D}_4\cup\mathscr{D}_5\cup\mathscr{D}_6$. Next, we will prove that

$$
\mathscr{L}\mathscr{H}(S,I,R,k)\le -1 \quad \text{on}\left(\mathbb{R}_+^3/\mathscr{D}_\varepsilon\right)\times\mathbb{M}. \tag{46}
$$

□

Case 1. If $(S,I,R,k)\in\mathscr{D}_1\times\mathbb{M}$, we have that

$$
\begin{aligned}
\mathscr{L}\mathscr{H} &\le -\frac{\widehat{\Lambda}}{S}+\frac{\check{\beta}I}{1+\widehat{\alpha}_1S+\widehat{\alpha}_2I+\widehat{\alpha}_3SI}+M\frac{\check{\beta}}{\widehat{\alpha}_1}\\
&\quad+\check{\Lambda}+2\check{\mu}+\check{\gamma}+\frac{\check{\sigma}_3^2}{2}+\frac{\check{\sigma}_1^2}{2}\\
&\le -\frac{\widehat{\Lambda}}{S}+C\\
&\le -\frac{\widehat{\Lambda}}{\varepsilon}+C,
\end{aligned} \tag{47}
$$

which together with (9) implies that

$$
\mathscr{L}\mathscr{H}\le -1 \quad \text{for all}\,(S,I,R,k)\in\mathscr{D}_1\times\mathbb{M}. \tag{48}
$$

Case 2. If $(S,I,R,k)\in\mathscr{D}_2\times\mathbb{M}$, we have

$$
\begin{aligned}
\mathscr{L}\mathscr{H} &\le -M\sum_{k=1}^{N}\pi_k\left(\mu(k)+\lambda(k)+\delta(k)+\frac{\sigma_2(k)}{2}\right)\left(\mathscr{R}_0^s-1\right)+M\frac{\check{\beta}}{\widehat{\alpha}_1}\\
&\quad+\frac{\check{\beta}I}{1+\widehat{\alpha}_1S+\widehat{\alpha}_2I+\widehat{\alpha}_3SI}+\check{\Lambda}+2\check{\mu}+\check{\gamma}+\frac{\check{\sigma}_3^2}{2}+\frac{\check{\sigma}_1^2}{2}\\
&\le -M\sum_{k=1}^{N}\pi_k\left(\mu(k)+\lambda(k)+\delta(k)+\frac{\sigma_2(k)}{2}\right)\left(\mathscr{R}_0^s-1\right)+M\frac{\check{\beta}}{\widehat{\alpha}_1}\\
&\quad+2\check{\mu}+\check{\gamma}+\check{\Lambda}+\frac{\check{\sigma}_1}{2}+\frac{\check{\sigma}_3}{2}+\frac{\check{\beta}\varepsilon}{\widehat{\alpha}_1}\\
&\le -2+\frac{\check{\beta}\varepsilon}{\widehat{\alpha}_1}.
\end{aligned} \tag{49}
$$

Combining with (10), it can be achieved that

$$
\mathscr{L}\mathscr{H}\le -1 \quad \text{for all}\,(S,I,R,k)\in\mathscr{D}_2\times\mathbb{M}. \tag{50}
$$

Case 3. If $(S,I,R,k)\in\mathscr{D}_3\times\mathbb{M}$, we get

$$
\begin{aligned}
\mathscr{L}\mathscr{H} &\le -\widehat{\lambda}\frac{I}{R}+\frac{\check{\beta}I}{1+\widehat{\alpha}_1S+\widehat{\alpha}_2I+\widehat{\alpha}_3SI}+M\frac{\check{\beta}}{\widehat{\alpha}_1}\\
&\quad+\check{\Lambda}+2\check{\mu}+\check{\gamma}+\frac{\check{\sigma}_3^2}{2}+\frac{\check{\sigma}_1^2}{2}\\
&\le -\widehat{\lambda}\frac{I}{R}+C\\
&\le -\frac{\widehat{\lambda}}{\varepsilon}+C\\
&\le -1,
\end{aligned} \tag{51}
$$

which follows from (11).

Case 4. If $(S,I,R,k)\in\mathscr{D}_4\times\mathbb{M}$, it follows that

$$
\begin{aligned}
\mathscr{L}\mathscr{H} &\le -\widehat{\mu}S+\frac{\check{\beta}I}{1+\widehat{\alpha}_1S+\widehat{\alpha}_2I+\widehat{\alpha}_3SI}+M\frac{\check{\beta}}{\widehat{\alpha}_1}+\check{\Lambda}\\
&\quad+2\check{\mu}+\check{\gamma}+\frac{\check{\sigma}_3^2}{2}+\frac{\check{\sigma}_1^2}{2}\\
&\le -\widehat{\mu}S+C\\
&\le -\frac{\widehat{\mu}}{\varepsilon}+C,
\end{aligned} \tag{52}
$$

which together with (44) implies that

$$
\mathscr{L}\mathscr{H}\le -1 \quad \text{for all}\,(S,I,R,k)\in\mathscr{D}_4\times\mathbb{M}. \tag{53}
$$

Case 5. If $(S, I, R, k) \in \mathscr{D}_5 \times \mathbb{M}$, we obtain

$$\begin{aligned}\mathscr{L}\mathscr{H} &\leq -\widehat{\mu}I + \frac{\check{\beta}I}{1+\widehat{\alpha}_1 S + \widehat{\alpha}_2 I + \widehat{\alpha}_3 SI} + M\frac{\check{\beta}}{\widehat{\alpha}_1} \\ &\quad + \check{\Lambda} + 2\check{\mu} + \check{\gamma} + \frac{\check{\sigma}_3^2}{2} + \frac{\check{\sigma}_1^2}{2} \\ &\leq -\widehat{\mu}I + C \\ &\leq -\frac{\widehat{\mu}}{\varepsilon^2} + C \\ &\leq -1,\end{aligned} \tag{54}$$

which follows from (44). Hence,

$$\mathscr{L}\mathscr{H} \leq -1 \quad \text{for all}\,(S, I, R, k) \in \mathscr{D}_5 \times \mathbb{M}. \tag{55}$$

Case 6. If $(S, I, R, k) \in \mathscr{D}_6 \times \mathbb{M}$, we have

$$\begin{aligned}\mathscr{L}\mathscr{H} &\leq -\widehat{\mu}R + \frac{\check{\beta}I}{1+\widehat{\alpha}_1 S + \widehat{\alpha}_2 I + \widehat{\alpha}_3 SI} + M\frac{\check{\beta}}{\widehat{\alpha}_1} \\ &\quad + \check{\Lambda} + 2\check{\mu} + \check{\gamma} + \frac{\check{\sigma}_3^2}{2} + \frac{\check{\sigma}_1^2}{2} \\ &\leq -\widehat{\mu}R + C \\ &\leq -\frac{\widehat{\mu}}{\varepsilon^3} + C.\end{aligned} \tag{56}$$

In view of (44), we arrive at

$$\mathscr{L}\mathscr{H} \leq -1 \quad \text{for all}\,(S, I, R, k) \in \mathscr{D}_6 \times \mathbb{M}. \tag{57}$$

Therefore, we have proof that

$$\mathscr{L}\mathscr{H} \leq -1 \quad \text{for all}\ (S, I, R, k) \in \left(\mathbb{R}_+^3 / \mathscr{D}_\varepsilon\right) \times \mathbb{M}. \tag{58}$$

Thus, condition 3 in Lemma 1 has been satisfied, and system (3) has a unique stationary distribution and ergodicity holds. This completes the proof.

Remark 1. Assume the condition $\mathscr{R}_0^s < 1$ holds. Disease I goes to extinction exponentially with probability one, Theorem 2, and if $\mathscr{R}_0^s > 1$ there is a unique ergodic stationary distribution $\mu(\cdot,\cdot)$ of system (3), which implies that disease I persists Theorem 3. Then, the number $\mathscr{R}_0^s$ can be considered as a threshold to identifying the stochastic extinction and persistence of system (3).

5. Simulations

Numerical solutions of stochastic differential equations are very important in the study of real examples of epidemic. In this section, we present some numerical results to illustrate the theoretical one. For numerical simulations of the SDEMS model (3), we use the Euler–Maruyama (EM) method ([19]).

Let $\varsigma(t)$ be a right-continuous Markov chain taking values on the state space $\mathscr{M} = \{1, 2, 3\}$ with the generator

$$\Gamma = \begin{pmatrix} -5 & 1 & 4 \\ 1 & -1 & 0 \\ 1 & 3 & -4 \end{pmatrix}. \tag{59}$$

Obviously, the Markov chain $\varsigma(t)$ has a unique stationary distribution $\pi = (\pi_1, \pi_2, \pi_3) = (0.1666, 0.6666, 0.1666)$. Given a step size $\Delta = 0.0001$, the Markov chain can be simulated by computing the one-step transition probability matrix $P = e^{\Delta\Gamma}$ ([20]), and the transition probability matrix is given by

$$P = \begin{pmatrix} 0.9995 & 0.0001 & 0.0004 \\ 0.0001 & 0.9999 & 0 \\ 0.0001 & 0.0003 & 0.9996 \end{pmatrix}. \tag{60}$$

Figure 1 shows a result of one simulation run of the Markov chain $\varsigma(t)$.

Example 1. To illustrate Theorem 2, we choose the parameter values in system (3) as follows:

$$\begin{gathered}\Lambda(1) = 1, \\ \mu(1) = 0.2, \\ \beta(1) = 0.1, \\ \gamma(1) = 0.25, \\ \lambda(1) = 0.1, \\ \delta(1) = 0.05, \\ (\alpha_1(1), \alpha_2(1), \alpha_3(1)) = (0.1, 0.1, 0.1), \\ (\sigma_1(1), \sigma_2(1), \sigma_3(1)) = (0.5, 0.7, 0.8), \\ \Lambda(2) = 1.1, \\ \mu(2) = 0.21, \\ \beta(2) = 0.12, \\ \gamma(2) = 0.26, \\ \lambda(2) = 0.09, \\ \delta(2) = 0.056, \\ (\alpha_1(2), \alpha_2(2), \alpha_3(2)) = (0.1, 0.1, 0.1), \\ (\sigma_1(2), \sigma_2(2), \sigma_3(2)) = (0.7, 0.8, 0.6), \\ \Lambda(3) = 1.3, \\ \mu(3) = 0.19, \\ \beta(3) = 0.13, \\ \gamma(3) = 0.24, \\ \lambda(3) = 0.11, \\ \delta(3) = 0.058, \\ (\alpha_1(3), \alpha_2(3), \alpha_3(3)) = (0.1, 0.1, 0.1), \\ (\sigma_1(3), \sigma_2(3), \sigma_3(3)) = (0.8, 0.6, 0.7).\end{gathered} \tag{61}$$

Simple computations result,

$$\mathscr{R}_0^s = \frac{\sum_{i=1}^N \pi_i \beta(i)}{\widehat{\alpha}_1 \sum_{i=1}^N \pi_i \left(\mu(i) + \lambda(i) + \delta(i) + (\sigma_2(i)/2)\right)} = 0.9505 < 1, \tag{62}$$

as a consequence result of Theorem 2. Disease I dies out exponentially with probability one. Figure 2 confirms this.

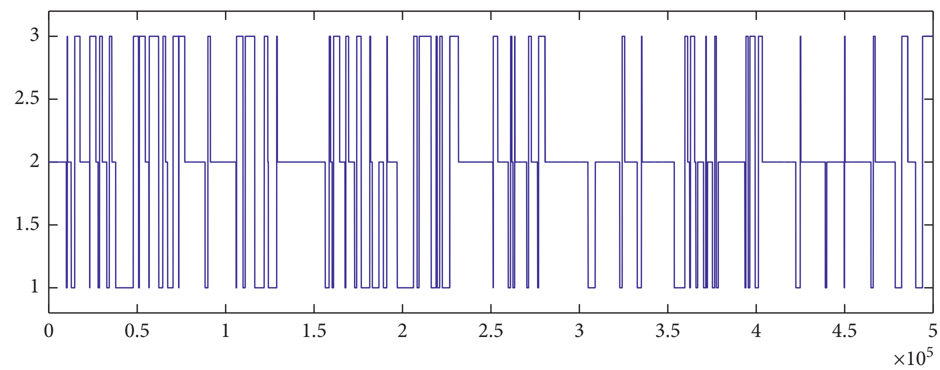


FIGURE 1: Computer simulation of a single path of $\varsigma(t)$ with initial value $\varsigma(0) = 2$.

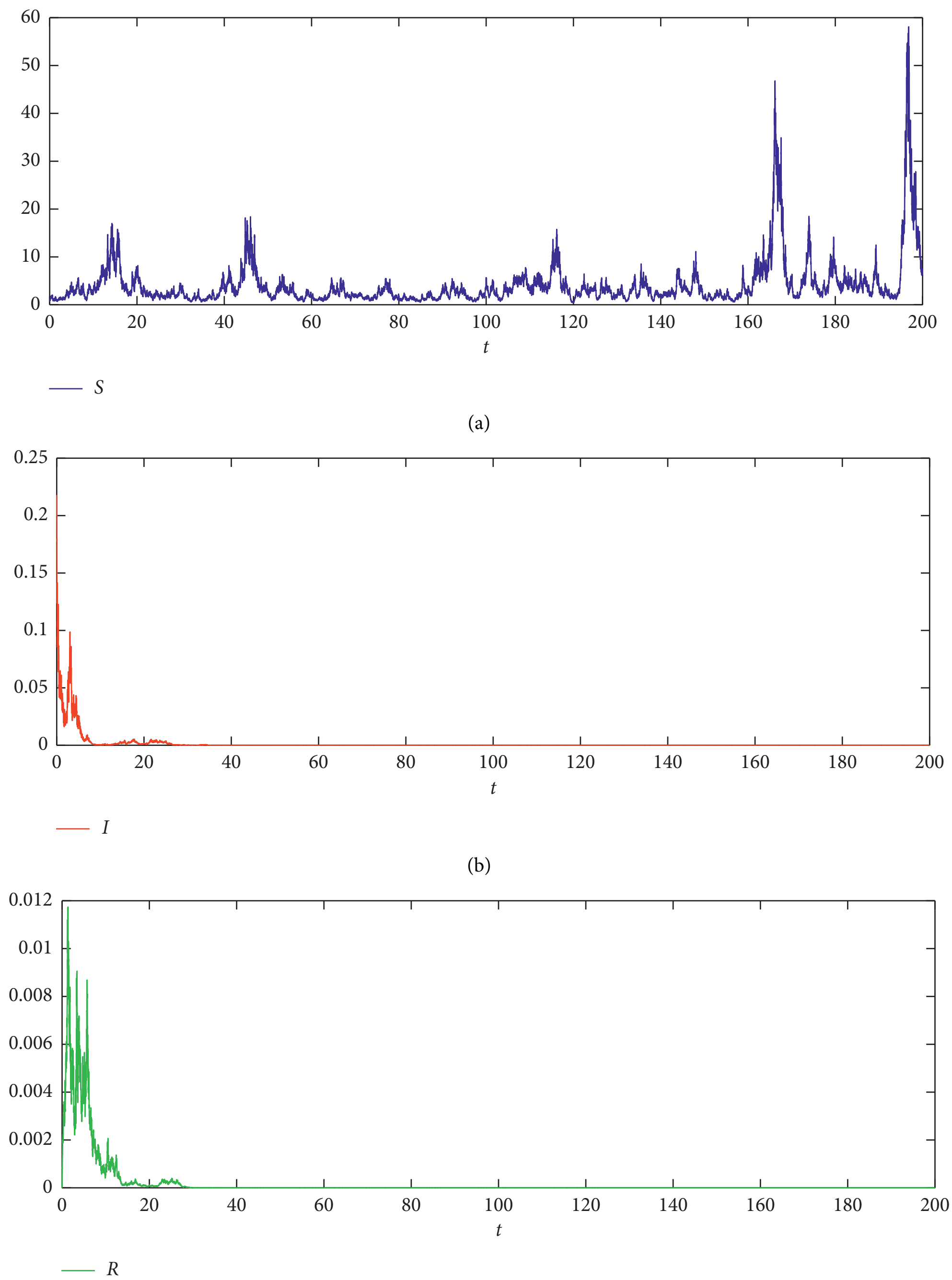
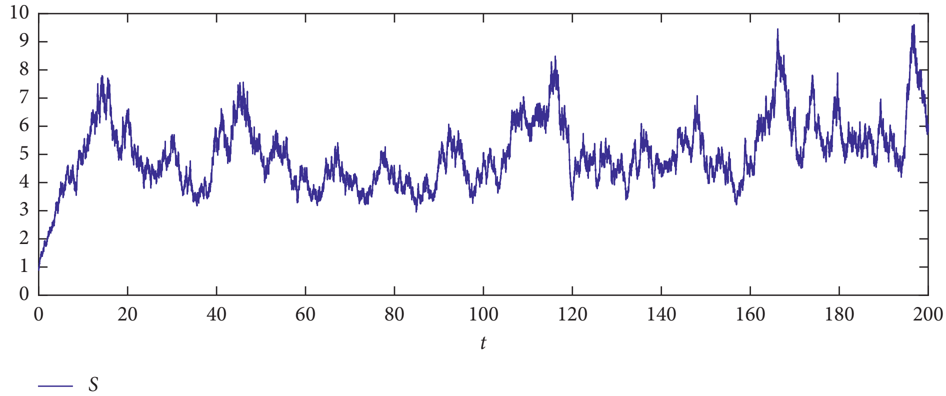
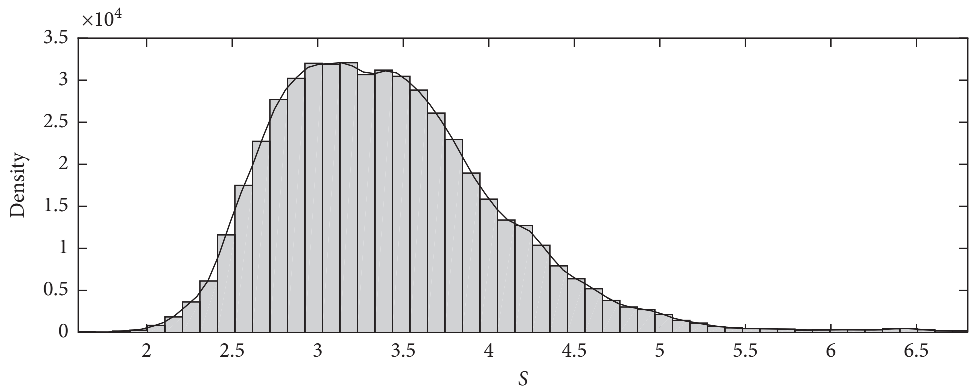


FIGURE 2: The solution (S(t), I(t), and R(t)) of stochastic model (3) with (S(0), I(0), and R(0)=(0.7, 0.2, and 0).

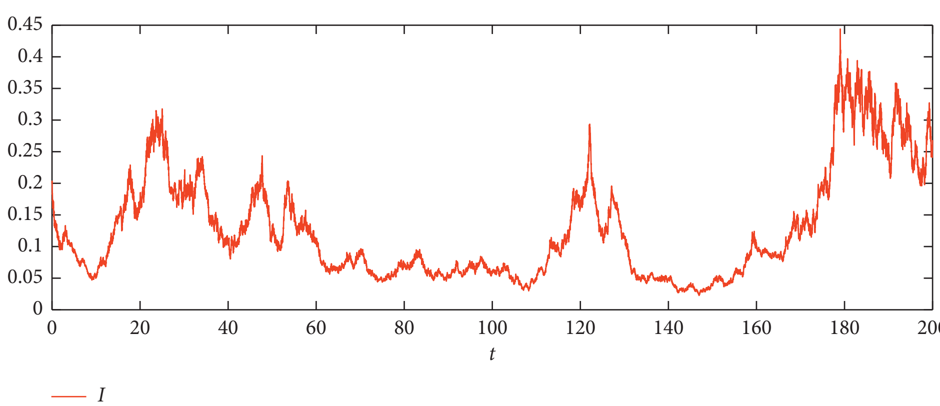


(a)



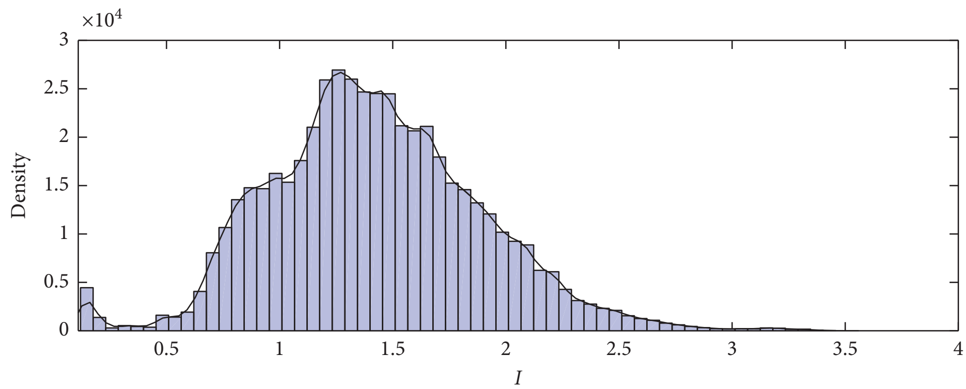
(b)

FIGURE 3: The solution S(t) of stochastic model (3) and its histogram.



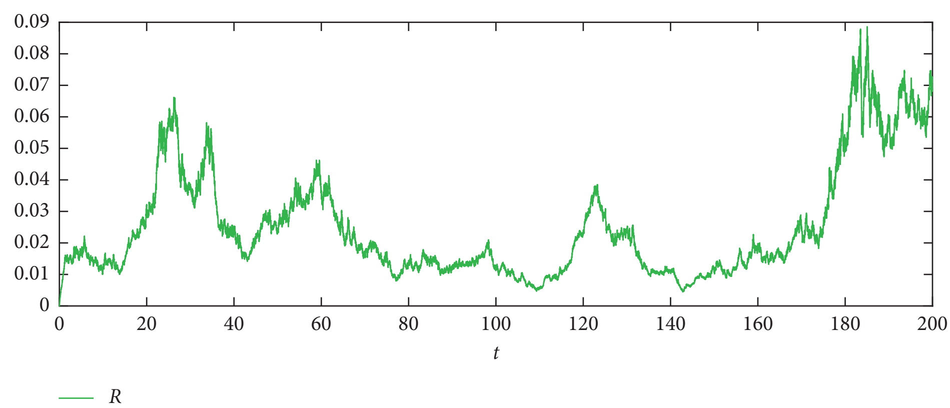
(a)

FIGURE 4: Continued.

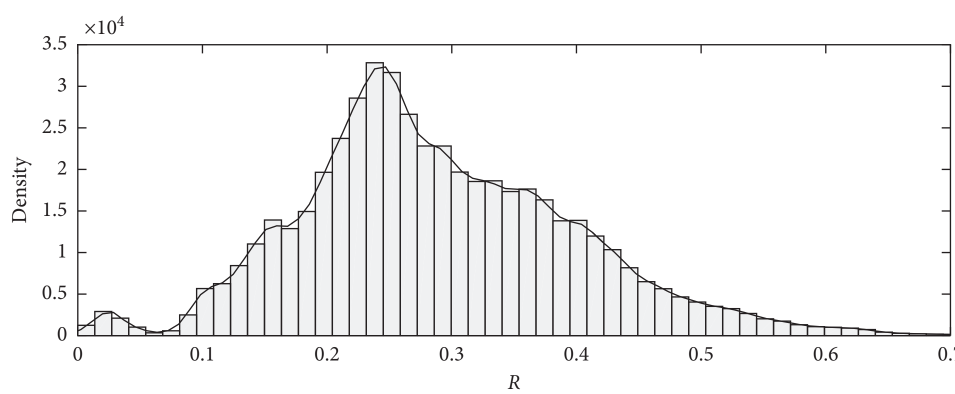


(b)

Figure 4: The solution I(t) of stochastic model (3) and its histogram.



(a)



(b)

Figure 5: The solution R(t) of stochastic model (3) and its histogram.

Example 2. For this example, we have

$$\begin{aligned}
\Lambda(1) &= 1, \\
\mu(1) &= 0.2, \\
\beta(1) &= 0.1, \\
\gamma(1) &= 0.25, \\
\lambda(1) &= 0.1, \\
\delta(1) &= 0.05, \\
(\alpha_1(1), \alpha_2(1), \alpha_3(1)) &= (0.14, 0.1, 0.1), \\
(\sigma_1(1), \sigma_2(1), \sigma_3(1)) &= (0.12, 0.1, 0.12), \\
\Lambda(2) &= 1.1, \\
\mu(2) &= 0.21, \\
\beta(2) &= 0.12, \\
\gamma(2) &= 0.26, \\
\lambda(2) &= 0.09, \\
\delta(2) &= 0.056, \\
(\alpha_1(2), \alpha_2(2), \alpha_3(2)) &= (0.14, 0.1, 0.1), \\
(\sigma_1(2), \sigma_2(2), \sigma_3(2)) &= (0.15, 0.18, 0.19), \\
\Lambda(3) &= 1.3, \\
\mu(3) &= 0.19, \\
\beta(3) &= 0.13, \\
\gamma(3) &= 0.24, \\
\lambda(3) &= 0.11, \\
\delta(3) &= 0.058, \\
(\alpha_1(3), \alpha_2(3), \alpha_3(3)) &= (0.14, 0.1, 0.1), \\
(\sigma_1(3), \sigma_2(3), \sigma_3(3)) &= (0.14, 0.17, 0.13).
\end{aligned} \tag{63}$$

By calculating, we find

$$\mathcal{R}_0^s = \frac{\sum_{i=1}^N \pi_i \beta(i)}{\widehat{\alpha}_1 \sum_{i=1}^N \pi_i (\mu(i) + \lambda(i) + \delta(i) + (\sigma_2(i)/2))} = 1.9624 > 1. \tag{64}$$

Then according to Theorem 3, the solution (S(t), I(t), and R(t)) of system (3) with any initial value (S(0), I(0), and R(0)) = (0.7, 0.6, and 0) has a unique stationary distribution, and it has the ergodic property, that is, the epidemic disease is permanent. Figures 3–5 confirm this.

6. Conclusion

This article discusses the dynamic behavior of a SIRS epidemic model with a regime switching and nonlinear incidence rate. We obtain sufficient conditions for the extinction of system (3) if $\mathcal{R}_0^s < 1$. We prove the stochastic system (3) under regime switching has a unique stationary distribution which is ergodic and positive recurrent by using the Lyapunov function method. In future works, it is interesting to study the effect of Lévy noise and a color noise (telegraph noise) in the stochastic SIRS epidemic model (2). We will investigate this case in our future works.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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