Hindawi
Discrete Dynamics in Nature and Society
Volume 2020, Article ID 6489532, 16 pages
https://doi.org/10.1155/2020/6489532

*Research Article*

# Optimal Reinsurance-Investment Problem under Mean-Variance Criterion with $n$ Risky Assets

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Received 28 August 2019; Accepted 11 May 2020; Published 1 June 2020

Academic Editor: Emilio Jiménez Macías



Based on the mean-variance criterion, this paper investigates the continuous-time reinsurance and investment problem. The insurer's surplus process is assumed to follow Cramér–Lundberg model. The insurer is allowed to purchase reinsurance for reducing claim risk. The reinsurance pattern that the insurer adopts is combining proportional and excess of loss reinsurance. In addition, the insurer can invest in financial market to increase his wealth. The financial market consists of one risk-free asset and $n$ correlated risky assets. The objective is to minimize the variance of the terminal wealth under the given expected value of the terminal wealth. By applying the principle of dynamic programming, we establish a Hamilton–Jacobi–Bellman (HJB) equation. Furthermore, we derive the explicit solutions for the optimal reinsurance-investment strategy and the corresponding efficient frontier by solving the HJB equation. Finally, numerical examples are provided to illustrate how the optimal reinsurance-investment strategy changes with model parameters.

## 1. Introduction

Reinsurance is an effective way to reduce claim risk, while investment is the most common way to increase wealth. Therefore, reinsurance and investment are two core problems of paramount importance in insurance and actuarial science. Many scholars have studied this subject. For example, Browne [1] and Chen and Yang [2] studied the optimal strategy to maximize the expected exponential utility of the terminal wealth, where the surplus process is modeled by a Brownian motion with drift; Yang and Zhang [3] and Zhao et al. [4] studied the optimal strategy to maximize the expected exponential utility of the terminal wealth with a jump-diffusion model; Asmussen and Taksar [5] and Chen et al. [6] investigated the optimal strategy to maximize the expected value of discounted dividends paid until time of ruin; Belkina and Luo [7] and Sun [8] considered the optimal strategy to minimize the ruin probability.

The mean-variance (MV) criterion for portfolio selection pioneered by Markowitz [9] refers to the selection of an optimal portfolio balancing the gain and the risk, which are measured by the expectation and variance of random returns, respectively. Since the pioneer work of Markowitz [9], the MV portfolio selection problem has become a main research topic in finance. Zhou and Li [10] and Li and Ng [11] extended Markowitz's model to the multiperiod setting and continuous-time setting, respectively, and they derived the analytical optimal investment strategy and efficient frontier. Bielecki et al. [12] studied the continuous-time MV portfolio selection problem under bankruptcy prohibition. However, these studies did not consider reinsurance. Under the MV criterion, the optimal reinsurance-investment problem has also been studied in many papers. Bäuerle [13] considered the optimal reinsurance problem under the MV criterion, where the surplus process of an insurer is described by the Cramér–Lundberg model. Yang [14] investigated the MV reinsurance-investment strategy with dependence between the finance market and the insurance market. Wang et al. [15] studied the MV reinsurance-investment strategy under default risk. Sun et al. [16] studied the MV reinsurance-investment strategy in a class of dependence insurance model. Yang et al. [17] investigated the MV reinsurance-investment strategy under a new interaction mechanism and a general investment framework.

The above-mentioned studies on reinsurance focused primarily on pure proportional reinsurance or pure excess of loss reinsurance. Proportional reinsurance means that no matter how large the amount of claims that the insurance company encounters is, it will seek the protection of the reinsurance company. Excess of loss reinsurance means that no reinsurance will be conducted when the claim amount is lower than a certain set value, and the excess amount will be distributed to reinsurance when the claim amount is larger than the set value. When reinsurance is implemented, the insurance company must pay a certain amount of expenses for the reinsurance company because it bears part of the claim for the insurance company. Therefore, insurance companies are very cautious about reinsurance. The pure excess of loss reinsurance or the pure proportional reinsurance has been widely studied in literature and insurance practice. However, very few scholars discuss the optimal combinational reinsurance problem, which is more difficult to deal with than the pure proportional reinsurance problem or the pure excess of loss reinsurance problem. Liang and Guo [18] considered the optimal combination of proportional and excess of loss reinsurance to maximize the expected utility. Hu et al. [19] considered the problem of minimizing the probability of ruin by controlling the combination of proportional and excess of loss reinsurance. However, Liang and Guo [18] and Hu et al. [19] did not consider investment. It is well known that the insurer can increase wealth through investment.

Motivated by the above-mentioned studies, this paper investigates an optimal combination of proportional and excess of loss reinsurance and investment problem under the MV criterion. There exist very often many risky assets in a financial market; however, few papers consider multiple risky assets in reinsurance-investment problem. There is a well-known proverb in economics: "do not put all eggs in one basket." This shows that insurer can effectively reduce his investment risks by investing in multiple risky assets. In view of the present status, we assume that the financial market consists of one risk-free asset and $n$ correlated risky assets. Furthermore, we assume that the reinsurance takes the form of combining proportional and excess of loss reinsurance. We derive the explicit optimal reinsurance-investment strategy and corresponding efficient frontier. Numerical examples are also provided to illustrate how the optimal reinsurance strategy and investment strategy vary with model parameters.

The main contributions of this paper include the following:

(i) We first study combining proportional and excess of loss reinsurance and investment problem under the MV criterion and multiple risky assets. Although Liang and Guo [18] and Hu et al. [19] also considered the combining proportional and excess of loss reinsurance, they did not consider investment. It is well known that the insurer can increase wealth through investment. Investment is very important for insurance companies and cannot be ignored.

(ii) Our objective is different from Liang and Guo [18] and Hu et al. [19]. The objective of Liang and Guo [18] is to maximize the expected utility, and the objective of Hu et al. [19] is to minimize the ultimate ruin probability. Our objective is the MV criterion; that is, we find an optimal reinsurance-investment strategy to minimize the variance of the terminal wealth under given expected value of the terminal wealth. Liang and Guo [18] and Hu et al. [19] did not directly consider the claim risk and the investment risk. However, these risks significantly affect the insurer's reinsurance-investment behavior. The MV criterion that we consider can directly measure the claim risk and the investment risk through variance.

(iii) We compare the pure excess of loss reinsurance with the pure proportional reinsurance. We find that the pure proportional reinsurance might be better than the pure excess of loss reinsurance.

The rest of the paper is organized as follows. In Section 2, we describe the models and assumptions. In Section 3, we introduce the portfolio selection problem under MV criterion. The optimal reinsurance-investment strategy to an auxiliary problem is obtained in Section 4. Section 5 derives the explicit expressions for the optimal strategy and the efficient frontier. In Section 6, numerical examples illustrate the results obtained in Section 5. The final section summarizes the paper.

## 2. Model Setting and Assumptions

In this section, we present a continuous-time reinsurance-investment model and introduce some basic assumptions. We follow the standard assumption in continuous-time financial models: continuous trading is allowed, there is no transaction cost or tax, and all assets are infinitely divisible. We also assume that all processes and random variables are defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$ satisfying the usual conditions; that is, $\mathcal{F} := \{\mathcal{F}_t, t \geq 0\}$ is right continuous and $P$-complete; $\mathcal{F}_t$ stands for the information available until time $t$.

*2.1. Risk Model.* Suppose that an insurer's surplus process follows the Cramér–Lundberg model:

$$\mathrm{d}X_t = c\mathrm{d}t - d\sum_{j=1}^{N(t)} Y_j, \tag{1}$$

where $c > 0$ is the rate of premium per unit time; $\{Y_j, j = 1, 2, \ldots, +\infty\}$ is a sequence of independent and identically distributed nonnegative random variables with a common distribution function $F(y)$ with $F(0) = 0$ and the density function $f(y)$ with finite mean $\overline{\mu}_1 = \mathrm{E}(Y_j)$ and second moment $\overline{\mu}_2 = \mathrm{E}(Y_j^2)$; $\{N(t), t \geq 0\}$ is a Poisson process with intensity being $\lambda > 0$, representing the number of claims up to time $t$; and $X_t$ is the surplus of the insurer at time $t$.

We assume that $F(y)$, the distribution function of $Y_j$, satisfies

$$\begin{cases} 0 < F(y) < 1, & \text{for } 0 < y < \overline{N}, \\ F(y) = 1, & \text{for } y \geq \overline{N}, \end{cases} \tag{2}$$

with $\overline{N} := \sup\{y\colon F(y) \leq 1\} < +\infty$. Note that $P(Y_j \leq \overline{N}) = 1$; this is because, for $y \geq \overline{N}$, we have $F(y) = 1$, so $F(\overline{N}) = 1$. That is, $P(Y_j \leq \overline{N}) = F(\overline{N}) = 1$.

The insurer often takes reinsurance in order to transfer claim risk. We assume that the insurer can reinsure his claim risk by combining proportional and excess of loss reinsurance. For each $t \in [0, T]$, the reinsurance level is associated with the parameters $a_t$ and $b_t$. Here, $a_t \in [0, 1]$ is the decision variable representing the retention at time $t$, and $b_t \in [0, \overline{N}]$ is the decision variable representing the excess of loss retention limit at time $t$. Assume $\{(a_t, b_t)\}_{t \geq 0}$, denoted by $(a, b)$ for simplicity. After the combination of a proportional reinsurance with an excess of loss reinsurance, the insurer will retain, from the $j$th claim, $Y_j(a, b) = \min\{aY_j, b\} = aY_j \wedge b$, $j = 1, 2, \ldots, +\infty$. We assume that both the premium income rate $c$ of the insurer and the premium income rate $\widetilde{c}$ of the reinsurer are calculated according to the expected value premium principle. Under the expected value premium principle, we can obtain that $c = (1 + \theta)\lambda\overline{\mu}_1$ and $\widetilde{c} = (1 + \eta)\lambda[\overline{\mu}_1 - aE(Y_j \wedge (b/a))]$. Note that when $a = 0$, we have $\widetilde{c} = (1 + \eta)\lambda\overline{\mu}_1$. On the other hand, $\lim_{a \longrightarrow 0} \{(1 + \eta)\lambda[\overline{\mu}_1 - aE(Y_j \wedge (b/a))]\} = (1 + \eta)\lambda[\overline{\mu}_1 - \lim_{a \longrightarrow 0} a \lim_{a \longrightarrow 0} E(Y_j \wedge (b/a))] = (1 + \eta)\lambda\overline{\mu}_1$. Therefore, we do not need to discuss whether $a$ is 0. Here $\theta \geq 0$ and $\eta \geq 0$ are the safety loadings of the insurer and the reinsurer, respectively. $\theta \geq 0$ implies that $c \geq \lambda\overline{\mu}_1$. This is a natural assumption in insurance practice, because the premium income must be greater than the claim expenditure. Otherwise, the insurer will not accept the client's insurance business. Many references, for example, Wang et al. [15], Zhao et al. [20], and Huang et al. [21], also assume $\theta \geq 0$. Without loss of generality, we assume that $\eta > \theta$; that is, the reinsurance is more expensive than the original insurance, which is reasonable in actuarial practice. To this end, the premium income rate $c(a, b)$ with the combinational reinsurance strategy $(a, b)$ becomes

$$c(a, b) = c - \widetilde{c} = c_1 + (1 + \eta)\lambda aE\left(Y_j \wedge \frac{b}{a}\right), \tag{3}$$

where $c_1 = (\theta - \eta)\lambda\overline{\mu}_1$.

Let $Z_j = Y_j \wedge (b/a)$; it is not difficult to get that

$$\begin{aligned} E(Z_j) &= \int_0^{b/a} [1 - F(y)]\mathrm{d}y := \mu(a, b), \\ E(Z_j^2) &= \int_0^{b/a} 2y[1 - F(y)]\mathrm{d}y := \sigma^2(a, b). \end{aligned} \tag{4}$$

Then, the corresponding surplus process $X_t^{a,b}$ of the insurer after combinational reinsurance becomes

$$\mathrm{d}X_t^{a,b} = [c_1 + (1 + \eta)\lambda a\mu(a, b)]\mathrm{d}t - a\mathrm{d}\sum_{j=1}^{N(t)} Z_j. \tag{5}$$

*2.2. Financial Market.* The financial market consists of one risk-free asset, whose price at time $t$ is denoted by $P(t)$, and $n$ risky assets with common dependence, whose price at time $t$ is denoted by $S_i(t)$, $i = 1, 2, \ldots, n$. $P(t)$ is assumed as follows:

$$\mathrm{d}P(t) = rP(t)\mathrm{d}t, \tag{6}$$

where $r > 0$ is the interest rate of the risk-free asset. The price process $S_i(t)$ of the $i$th risky asset satisfies the following stochastic differential equation (SDE):

$$\mathrm{d}S_i(t) = S_i(t)[\mu_i\mathrm{d}t + \sigma_i\mathrm{d}B_i(t) + \overline{\sigma}_i\mathrm{d}\overline{B}(t)], \tag{7}$$

where $\mu_i \geq r$; $\sigma_i$ and $\overline{\sigma}_i$ are positive constants. $\mu_i$ is the appreciation rate; $\sigma_i$ and $\overline{\sigma}_i$ are the volatilities rate of the risky asset $i$. $\{B_i(t), t \geq 0\}$, $i = 1, 2, \ldots, n$, and $\{\overline{B}(t), t \geq 0\}$ are $n + 1$ mutually independent standard Brownian motions. The Brownian motion $\overline{B}(t)$ induces a correlation among the prices of $n$ risky assets.

*2.3. Wealth Process.* Insurer can invest in financial market to increase his wealth. Let $\pi_1(t), \pi_2(t), \ldots, \pi_n(t)$ denote the dollar amounts that the insurer invests in risky assets $1, 2, \ldots, n$ at time $t$; the rest of his wealth is then invested in the risk-free asset. Denote $\pi(t) = (\pi_1(t), \pi_2(t), \ldots, \pi_n(t))$ by $\pi = (\pi_1, \pi_2, \ldots, \pi_n)$ for simplicity. At any time $t \geq 0$, $(a, b) = (a(t), b(t))$ and $\pi = \pi(t)$ are chosen by the insurer as control strategies, and we denote them as $u(\cdot) = (a(\cdot), b(\cdot), \pi(\cdot))$. With respect to each strategy $u(\cdot)$, the wealth process $X_t^u$ of the insurer with reinsurance and investment can be described as the following SDE:

$$\begin{aligned} \mathrm{d}X_t^u &= \left[c_1 + (1 + \eta)\lambda a\mu(a, b) + rX_t^u + \sum_{i=1}^n \pi_i(\mu_i - r)\right]\mathrm{d}t \\ &\quad + \sum_{i=1}^n \pi_i\sigma_i\mathrm{d}B_i(t) + \sum_{i=1}^n \pi_i\overline{\sigma}_i\mathrm{d}\overline{B}(t) - a\mathrm{d}\sum_{j=1}^{N(t)} Z_j, \end{aligned} \tag{8}$$

with the initial capital being $X_0^u = x_0$.

*Definition 1.* A control strategy $u(\cdot) = (a(\cdot), b(\cdot), \pi(\cdot))$ is said to be admissible if $a(\cdot)$, $b(\cdot)$, and $\pi(\cdot)$ are predictable with respect to $\mathscr{F}_t$ and, for each $t \geq 0$, the processes $a(\cdot)$, $b(\cdot)$, and $\pi(\cdot)$ satisfy the following conditions: (i) $0 \leq a(t) \leq 1$, (ii) $0 \leq b(t) \leq \overline{N}$, (iii) $P\{\int_0^\infty \sum_{i=1}^n \pi_i^2(t)\mathrm{d}t < \infty\} = 1$, and (iv) the SDE (8) with respect to $u$ has a unique strong solution.

The set of all admissible strategies is denoted by $U$.

## 3. Problem Formulation

In this section, we introduce the MV portfolio selection problem. Let $X_T^u$ denote the terminal wealth under the strategy $u(\cdot)$. For simplicity of notation, let $\mathrm{E}_{t,x}[\cdot] = \mathrm{E}[|\cdot|X_t^u = x]$ and $\mathrm{Var}_{t,x}[\cdot] = \mathrm{Var}[|\cdot|X_t^u = x]$.

The MV portfolio selection problem aims to maximize the expected terminal wealth $\mathrm{E}_{0,x_0}(X_T^u)$ and at the same time to minimize the variance of the terminal wealth $\mathrm{Var}_{0,x_0}(X_T^u)$. This is a biobjective optimization problem with two conflicting criteria. Concretely, we have the following problem:

$$\begin{aligned} \max \quad & \left(-\mathrm{Var}_{0,x_0}(X_T^u), \mathrm{E}_{0,x_0}(X_T^u)\right) \\ \text{s.t.} \quad & X_t^u \text{ satisfies } (8), \quad u \in U, \end{aligned} \tag{9}$$

where $\max(\mathrm{Var}_{0,x_0}(X_T^u), \mathrm{E}_{0,x_0}(X_T^u))$ means that we simultaneously maximize $-\mathrm{Var}_{0,x_0}(X_T^u)$ and $\mathrm{E}_{0,x_0}(X_T^u)$. Problem (9) is equivalent to the following problem:

$$\begin{aligned} \min \quad & \left(\mathrm{Var}_{0,x_0}(X_T^u), -\mathrm{E}_{0,x_0}(X_T^u)\right) \\ \text{s.t.} \quad & X_t^u \text{ satisfies } (8), \quad u \in U, \end{aligned} \tag{10}$$

where $\min(\mathrm{Var}_{0,x_0}(X_T^u), -\mathrm{E}_{0,x_0}(X_T^u))$ means that we simultaneously minimize $\mathrm{Var}_{0,x_0}(X_T^u)$ and $-\mathrm{E}_{0,x_0}(X_T^u)$.

Let $J_1(u(\cdot)) = \mathrm{Var}_{0,x_0}(X_T^u)$ and $J_2(u(\cdot)) = \mathrm{E}_{0,x_0}(X_T^u)$. Moreover, an admissible strategy $u^*$ is called an efficient strategy if there exists no admissible strategy $u(\cdot)$ such that

$$\begin{aligned} J_1(u(\cdot)) &\le J_1(u^*(\cdot)), \\ J_2(u(\cdot)) &\ge J_2(u^*(\cdot)), \end{aligned} \tag{11}$$

and at least one of the inequalities holds strictly. In this case, we call $(J_1(u^*(\cdot)), J_2(u^*(\cdot))) \in R^2$ an efficient point. The set of all efficient points forms the efficient frontier.

We consider the problem of finding an admissible reinsurance-investment strategy such that the expected terminal wealth satisfies $\mathrm{E}_{0,x_0}(X_T^u) = k$; here $k$ is a constant, while the risk measured by the variance of the terminal wealth,

$$\mathrm{Var}_{0,x_0}(X_T^u) = \mathrm{E}_{0,x_0}\left\{\left[X_T^u - \mathrm{E}_{0,x_0}(X_T^u)\right]^2\right\} = \mathrm{E}_{0,x_0}\left[(X_T^u - k)^2\right], \tag{12}$$

is minimized. Concretely, we have the following problem.

Problem (10) can be formulated as the following optimization problem:

$$\begin{aligned} \min \quad & \mathrm{Var}_{0,x_0}(X_T^u) = \min \mathrm{E}_{0,x_0}\left[(X_T^u - k)^2\right] \\ \text{s.t.} \quad & \mathrm{E}_{0,x_0}(X_T^u) = k, \\ & X_t^u \text{ satisfies } (8), \quad u \in U. \end{aligned} \tag{13}$$

The optimal strategy for problem (13) (corresponding to a fixed $k$) is called a variance minimizing portfolio, and the set of all points $(\mathrm{Var}_{0,x_0}(X_T^{u^*}), k)$, where $\mathrm{Var}_{0,x_0}(X_T^{u^*})$ denotes the optimal value of problem (13), is called the variance minimizing frontier.

An efficient portfolio is one for which there does not exist another strategy that has higher mean and no higher variance and/or has less variance and no less mean at the terminal time $T$. In other words, an efficient portfolio is one that is Pareto optimal. From problem (9) and problem (10), we know that the efficient frontier is a subset of the variance minimizing frontier. In the following, we will only discuss the variance minimization problem (13).

Since problem (13) is a convex programming problem, the equality constraint $\mathrm{E}_{0,x_0}(X_T^u) = k$ can be dealt with by introducing a Lagrange multiplier $q \in R$. In this way, problem (13) can be solved via the following optimization (for every fixed $q$):

$$\begin{aligned} \min \quad & \mathrm{E}_{0,x_0}\left[(X_T^u - k)^2\right] + 2q\mathrm{E}_{0,x_0}(X_T^u - k) \\ \text{s.t.} \quad & X_t^u \text{ satisfies } (8), \quad u \in U, \end{aligned} \tag{14}$$

where factor 2 in the front of $q$ is just for convenience. To obtain the optimal value and optimal strategy for problem (13), we need to maximize the optimal value of problem (14) with respect to $q \in R$, according to the Lagrange duality theory (see Luenberger [22]). Clearly, problem (14) is equivalent to

$$\begin{aligned} \min \quad & \mathrm{E}_{0,x_0}\left\{\left[X_T^u - (k-q)\right]^2\right\} \\ \text{s.t.} \quad & X_t^u \text{ satisfies } (8), \quad u \in U, \end{aligned} \tag{15}$$

in the sense that the two problems have exactly the same optimal control for fixed $q$.

To solve problem (15), we firstly solve an auxiliary problem. Consider the following SDE:

$$\begin{aligned} \mathrm{d}L_t^u = {} & \left[m + (1+\eta)\lambda a\mu(a,b) + rL_t^u + \sum_{i=1}^n \pi_i(\mu_i - r)\right]\mathrm{d}t \\ & + \sum_{i=1}^n \pi_i\sigma_i \mathrm{d}B_i(t) + \sum_{i=1}^n \pi_i\bar{\sigma}_i \mathrm{d}\bar{B}(t) - ad\sum_{j=1}^{N(t)} Z_j, \end{aligned} \tag{16}$$

with the initial capital being $L_0^u = l_0$ and the corresponding optimization problem

$$\begin{aligned} \min \quad & \mathrm{E}_{0,l_0}\left[\frac{1}{2}(L_T^u)^2\right] \\ \text{s.t.} \quad & L_t^u \text{ satisfies } (16), \quad u \in U. \end{aligned} \tag{17}$$

Note that if we set $L_t^u = X_t^u - (k-q)$, $m = c_1 + (k-q)r$, and $L_0^u = X_0^u - (k-q)$, equation (8) can be derived from equation (16).

To apply the dynamic programming technique to solve problem (17), we define the optimal value function $J(t,l)$ at time $t$ as

$$J(t,l) = \inf_{u\in U} J^u(t,l) = \inf_{u\in U} \mathrm{E}_{t,l}\left[\frac{1}{2}(L_T^u)^2\right], \tag{18}$$

where $\mathrm{E}_{t,l}[\cdot] = \mathrm{E}[L_t^u = l]$.

Obviously, when $t = 0$, $J(0,l_0)$ is the optimal value of problem (17). We start with the associated Hamilton–Jacobi–Bellman (HJB) equation for the optimal value function $J(t,l)$.

**Theorem 1.** *Assume that $J(t,l)$ is continuously differentiable in $t$ on $[0,T]$ and twice continuously differentiable in $l$ on $R$;*

*that is to say,* $J(t,l) \in C^{1,2}([0,T]\times R)$*. Then,* $J(t,l)$ *satisfies the following HJB equation:*

$$\inf_{u\in U}\Big\{ J_t(t,l) + \Big[m + (1+\eta)\lambda a\mu(a,b) + rl + \sum_{i=1}^{n}\pi_i(\mu_i - r)\Big] J_l(t,l) + \frac{1}{2}\Big[\sum_{i=1}^{n}\pi_i^2\sigma_i^2 + \Big(\sum_{i=1}^{n}\pi_i\overline{\sigma}_i\Big)^2\Big] J_{ll}(t,l) + \lambda \mathrm{E}[J(t,l-aZ) - J(t,l)]\Big\} = 0,$$
$$J(T,l) = \frac{1}{2}l^2. \tag{19}$$

The proof of this theorem is standard; one can refer to the proof of Lemma 4.2 by Fleming and Soner [23].

Theorem 1 requires $J(t,l) \in C^{1,2}([0,T]\times R)$. But in most of the examples this is not the case, so we study the viscosity solution of problem (18). Next, we will give the definition of viscosity solution according to that of Definition 3.1 in Bi and Guo [24].

*Definition 2.* Let $V \in C([0,T]\times R)$, which denotes the set of continuous functions on $[0,T]\times R$.

(1) We say that $V$ is a viscosity subsolution of problem (18) in $(t,l) \in [0,T]\times R$, if, for each $\psi \in C^{1,2}([0,T]\times R)$,

$$\inf_{u\in U}\Big\{ \psi_{\bar t}(\bar t,\bar l) + \Big[m + (1+\eta)\lambda a\mu(a,b) + r\bar l + \sum_{i=1}^{n}\pi_i(\mu_i - r)\Big]\psi_{\bar l}(\bar t,\bar l) + \frac{1}{2}\Big[\sum_{i=1}^{n}\pi_i^2\sigma_i^2 + \Big(\sum_{i=1}^{n}\pi_i\overline{\sigma}_i\Big)^2\Big]\psi_{\bar l\bar l}(\bar t,\bar l) + \lambda \mathrm{E}[\psi(\bar t,\bar l - aZ) - \psi(\bar t,\bar l)]\Big\} \ge 0, \tag{20}$$

at every $(\bar t,\bar l) \in [0,T]\times R$ which is a maximizer of $V-\psi$ on $[0,T]\times R$ with $V(\bar t,\bar l) = \psi(\bar t,\bar l)$.

(2) We say that $V$ is a viscosity supersolution of problem (18) in $(t,l) \in [0,T]\times R$, if, for each $\psi \in C^{1,2}([0,T]\times R)$,

$$\inf_{u\in U}\Big\{ \psi_{\bar t}(\bar t,\bar l) + \Big[m + (1+\eta)\lambda a\mu(a,b) + r\bar l + \sum_{i=1}^{n}\pi_i(\mu_i - r)\Big]\psi_{\bar l}(\bar t,\bar l) + \frac{1}{2}\Big[\sum_{i=1}^{n}\pi_i^2\sigma_i^2 + \Big(\sum_{i=1}^{n}\pi_i\overline{\sigma}_i\Big)^2\Big]\psi_{\bar l\bar l}(\bar t,\bar l) + \lambda \mathrm{E}[\psi(\bar t,\bar l - aZ) - \psi(\bar t,\bar l)]\Big\} \le 0, \tag{21}$$

at every $(\bar t,\bar l) \in [0,T]\times R$ which is a minimizer of $V-\psi$ on $[0,T]\times R$ with $V(\bar t,\bar l) = \psi(\bar t,\bar l)$.

(3) We say that $V$ is a viscosity solution of problem (18) in $(t,l) \in [0,T]\times R$, if it is both a viscosity subsolution and a viscosity supersolution of problem (18) in $(t,l) \in [0,T]\times R$.

## 4. Solution to the Auxiliary Problem

In this section, using the stochastic control technique, we solve problem (18). According to the boundary condition in Theorem 1, we will try to find a solution of problem (18) with the following parametric form:

$$V(t,l) = \frac{1}{2}Q(t)l^2 + W(t)l + K(t). \tag{22}$$

Here, $Q(t)$, $W(t)$, and $K(t)$, to be determined later on, respectively, satisfy the boundary conditions $Q(T) = 1$, $W(T) = 0, K(T) = 0$.

To simplify our description, we shall use the following notations:

$$\begin{cases} \Delta_i = \left[\dfrac{\mu_i - r}{\sigma_i^2} - \dfrac{\overline{\sigma}_i}{\sigma_i^2\left(1+\sum_{i=1}^{n}\overline{\sigma}_i^2/\sigma_i^2\right)}\displaystyle\sum_{i=1}^{n}\dfrac{\overline{\sigma}_i\left(\mu_i - r\right)}{\sigma_i^2}\right], \quad i = 1,2,\ldots,n, \\ \rho_1 = \displaystyle\sum_{i=1}^{n}\Delta_i^2\sigma_i^2 + \left(\sum_{i=1}^{n}\Delta_i\overline{\sigma}_i\right)^2, \\ \rho_2 = \displaystyle\sum_{i=1}^{n}\Delta_i\left(\mu_i - r\right), \\ \rho_3 = \dfrac{\lambda\eta^2\overline{\mu}_1^2}{\overline{\mu}_2}. \end{cases} \tag{23}$$

By (22), we have

$$\begin{cases} V_t = \dfrac{1}{2}Q'(t)l^2 + W'(t)l + K'(t), \\ V_l = Q(t)l + W(t), \\ V_{ll} = Q(t), \\ \mathrm{E}\left[V(t, l - aZ) - V(t,l)\right] = \dfrac{1}{2}Q(t)a^2\sigma^2(a,b) - \left[Q(t)l + W(t)\right]a\mu(a,b), \end{cases} \tag{24}$$

where $V_t$, $V_l$, and $V_{ll}$ represent the partial derivatives of $V(t,l)$ with respect to the corresponding variables.

Substituting (24) into (19), we have after simplification that

$$\frac{1}{2}Q'(t)l^2 + W'(t)l + K'(t) + (m + rl)\left[Q(t)l + W(t)\right] + \inf_{\pi\in U} g(\pi) + \inf_{(a,b)\in U}\overline{f}(a,b) = 0, \tag{25}$$

with

$$g(\pi) = \frac{1}{2}Q(t)\left[\sum_{i=1}^{n}\pi_i^2\sigma_i^2 + \left(\sum_{i=1}^{n}\pi_i\overline{\sigma}_i\right)^2\right] + \left[Q(t)l + W(t)\right]\sum_{i=1}^{n}\pi_i\left(\mu_i - r\right), \tag{26}$$

$$\overline{f}(a,b) = \lambda\eta\left(Q(t)l + W(t)\right)a\int_0^{b/a}\left[1 - F(y)\mathrm{d}y\right] + \frac{1}{2}\lambda Q(t)a^2\int_0^{b/a}2y\left[1 - F(y)\right]\mathrm{d}y. \tag{27}$$

Setting $\partial g(\pi)/\partial\pi_i = 0$, we obtain

$$\pi_i\left(\sigma_i^2 + \overline{\sigma}_i^2\right) + \overline{\sigma}_i\sum_{j\neq i}\pi_j\overline{\sigma}_j = -\left(\mu_i - r\right)\left[l + \frac{W(t)}{Q(t)}\right], \quad i = 1,2,\ldots,n. \tag{28}$$

The following lemma is essential for deriving the explicit optimal investment strategy.

**Lemma 1.** *Equation (28) has a unique solution $\widehat{\pi}_i$, which is given by*

$$\widehat{\pi}_i = -\Delta_i\left[l + \frac{W(t)}{Q(t)}\right], \quad i = 1,2,\ldots,n. \tag{29}$$

*Proof.* Equation (28) can be rewritten as

$$\pi_i + \frac{\overline{\sigma}_i}{\sigma_i^2}\sum_{i=1}^{n}\overline{\sigma}_i\pi_i = -\frac{\mu_i - r}{\sigma_i^2}\left[l + \frac{W(t)}{Q(t)}\right], \quad i = 1,2,\ldots,n. \tag{30}$$

Multiplying both sides of (30) by $\overline{\sigma}_i$ and then averaging over $i = 1,2,\ldots,n$, we obtain

$$\sum_{i=1}^{n}\overline{\sigma}_i\pi_i = -\left[l + \frac{W(t)}{Q(t)}\right]\left(1 + \sum_{i=1}^{n}\frac{\overline{\sigma}_i^2}{\sigma_i^2}\right)^{-1}\sum_{i=1}^{n}\frac{\overline{\sigma}_i\left(\mu_i - r\right)}{\sigma_i^2}. \tag{31}$$

By substituting (31) into (30), we can deduce that the unique solution of equation (28) is given by (29).

Substituting (29) into (25), we obtain

$$l^2\left\{\frac{1}{2}Q'(t)+\left(r+\frac{1}{2}\rho_1-\rho_2\right)Q(t)\right\} + l\{W'(t)+(r+\rho_1-2\rho_2)W(t)+mQ(t)\} + K'(t)+mW(t)+\left(\frac{1}{2}\rho_1-\rho_2\right)\frac{W^2(t)}{Q(t)}+\inf_{(a,b)\in U}\overline{f}(a,b)=0. \quad (32)$$

Differentiating $\overline{f}(a,b)$ with respect to $b$ yields

$$\frac{\partial\overline{f}(a,b)}{\partial b}=\lambda\left[1-F\left(\frac{b}{a}\right)\right]\{bQ(t)+\eta[Q(t)l+W(t)]\}. \quad (33)$$

Let

$$\widehat{b}=-\eta\left(l+\frac{W(t)}{Q(t)}\right),$$
$$m_1=\lambda\eta\int_0^{\widehat{b}}[1-F(y)]\mathrm{d}y, \quad (34)$$
$$m_2=\frac{1}{2}\lambda\int_0^{\widehat{b}}2y[1-F(y)]\mathrm{d}y,$$

and we will discuss the excess of loss reinsurance strategy $\widehat{b}$ for the following three cases, that is, $\widehat{b}\le 0$, $0<\widehat{b}<a\overline{N}$, and $\widehat{b}\ge a\overline{N}$. □

*4.1.* $0<\widehat{b}<a\overline{N}$. In this section, we discuss the case $0<\widehat{b}<a\overline{N}$. We first give two lemmas and then present the solution to problem (18).

**Lemma 2.** *For any $t\ge 0$, we have $Q(t)>0$.*

*The proof is similar to that of Theorem 8.3.1 in Zhang [25], so we omit it here.*

When $0<\widehat{b}<a\overline{N}$, it is easy to see that $\partial\overline{f}(a,b)/\partial b<0$ for any $b\in(0,\widehat{b})$, which means that $\overline{f}(a,b)$ is a decreasing function with respect to $b$ when $b\in(0,\widehat{b})$. Therefore, the optimal excess of loss reinsurance strategy is $b^*=\widehat{b}$.

Plugging $b^*=\widehat{b}$ into (27) and differentiating it with respect to $a$ yield

$$\frac{\partial\overline{f}(a,\widehat{b})}{\partial a}=-\lambda Q(t)\int_0^{\widehat{b}/a}[1-F(y)][-2ya+\widehat{b}]\mathrm{d}y. \quad (35)$$

**Lemma 3.** *$\int_0^{\widehat{b}/a}[1-F(y)][-2ya+\widehat{b}]dy>0$ holds for any $0<a\le 1$.*

*Proof.* Let $\overline{g}(y)=-2ya+\widehat{b}$, $y\in[0,\widehat{b}/a]$. We have $\overline{g}'(y)=-2a<0$ for any $y\in[0,\widehat{b}/a]$; thus $\overline{g}(y)$ is a strictly decreasing and continuous function with $\overline{g}(0)=\widehat{b}>0$ and $\overline{g}(\widehat{b}/a)=-\widehat{b}<0$. Therefore, there exists a unique $y_0\in(0,\widehat{b}/a)$ which satisfies $\overline{g}(y_0)=0$; moreover, $\overline{g}(y)>0$ for $0\le y<y_0$, and $\overline{g}(y)<0$ for $y_0<y\le\widehat{b}/a$.

In addition, $1-F(y)$ is a decreasing function; that is, $1-F(y)>1-F(y_0)$ for $0\le y<y_0$, and $1-F(y)<1-F(y_0)$ for $y_0<y\le\widehat{b}/a$. Then, $(1-F(y))g(y)>(1-F(y_0))\overline{g}(y)$ holds for $0\le y\le\widehat{b}/a$.

Since $\int_0^{\widehat{b}/a}\overline{g}(y)\mathrm{d}y=0$, we have

$$\int_0^{\widehat{b}/a}[1-F(y)]\overline{g}(y)\mathrm{d}y>\int_0^{\widehat{b}/a}[1-F(y_0)]\overline{g}(y)\mathrm{d}y =[1-F(y_0)]\int_0^{\widehat{b}/a}\overline{g}(y)\mathrm{d}y=0, \quad (36)$$

which completes the proof. □

**Theorem 2.** *When $0<\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta<\overline{N}$, the solution to problem (18) is as follows. The optimal excess of loss reinsurance strategy is given by*

$$b^*=\left\{\frac{m+m_1}{r}\left[e^{r(t-T)}-1\right]-l\right\}\eta, \quad (37)$$

*the optimal proportional reinsurance strategy is $a^*=1$, the optimal investment strategy is given by*

$$\pi_i^*=\Delta_i\left\{\frac{m+m_1}{r}\left[e^{r(t-T)}-1\right]-l\right\},\quad i=1,2,\ldots,n, \quad (38)$$

*and the candidate optimal value function $V(t,l)$ is given by*

$$V(t,l)=\frac{1}{2}e^{(2\rho_2-\rho_1-2r)(t-T)}\left\{l-\frac{m+m_1}{r}\left[e^{r(t-T)}-1\right]\right\}^2 +\frac{m_2}{2\rho_2-\rho_1-2r}\left[1-e^{(2\rho_2-\rho_1-2r)(t-T)}\right]. \quad (39)$$

*Proof.* For any $0<a\le 1$, we obtain from Lemmas 2 and 3 that $\partial\overline{f}(a,\widehat{b})/\partial a<0$. Therefore, the candidate optimal proportional reinsurance strategy is 1.

From (27), we have

$$\frac{\partial^2\overline{f}(a,b)}{\partial a^2}=\lambda Q(t)\int_0^{b/a}y^2f(y)\mathrm{d}y-\frac{\lambda b^2}{a^3}f\left(\frac{b}{a}\right)\cdot\{bQ(t)+\eta[Q(t)l+W(t)]\}. \quad (40)$$

From (33), we have

$$\frac{\partial^2\overline{f}(a,b)}{\partial b\,\partial a}=\frac{\lambda b}{a^2}f\left(\frac{b}{a}\right)\{bQ(t)+\eta[Q(t)l+W(t)]\},$$
$$\frac{\partial^2\overline{f}(a,b)}{\partial b^2}=\lambda\left[1-F\left(\frac{b}{a}\right)\right]Q(t)-\frac{\lambda}{a}f\left(\frac{b}{a}\right)\cdot\{bQ(t)+\eta[Q(t)l+W(t)]\}. \quad (41)$$

Since $a\ne 0$, it is easy to prove that $\partial^2\overline{f}(a,b)/\partial b\,\partial a$ and $\partial^2\overline{f}(a,b)/\partial a\,\partial b$ are continuous. Hence, $\partial^2\overline{f}(a,b)/$

$\partial b\,\partial a = \partial^2\overline{f}(a,b)/\partial a\,\partial b$. Furthermore, we can obtain the following Hessian matrix:

$$\begin{pmatrix} \left.\dfrac{\partial^2\overline{f}(a,b)}{\partial b^2}\right|_{(1,\widehat{b})} & \left|\dfrac{\partial^2\overline{f}(a,b)}{\partial b\,\partial a}\right|_{(1,\widehat{b})} \\ \left|\dfrac{\partial^2\overline{f}(a,b)}{\partial a\,\partial b}\right|_{(1,\widehat{b})} & \left|\dfrac{\partial^2\overline{f}(a,b)}{\partial a^2}\right|_{(1,\widehat{b})} \end{pmatrix} = \begin{pmatrix} \lambda Q(t)[1-F(\widehat{b})] & 0 \\ 0 & \lambda Q(t)\int_0^{\widehat{b}} y^2 f(y)\mathrm{d}y \end{pmatrix}, \quad (42)$$

is a positive definite matrix. Therefore, 1 and $\widehat{b}$ are optimal proportional and excess of loss reinsurance strategies, respectively. Substituting $a^* = 1$ and $b^* = \widehat{b}$ into (32), we obtain

$$\begin{aligned} &l^2\left\{\frac{1}{2}Q'(t) + \left(r + \frac{1}{2}\rho_1 - \rho_2\right)Q(t)\right\} \\ &+ l\{W'(t) + (r+\rho_1-2\rho_2)W(t) + (m+m_1)Q(t)\} \\ &+ K'(t) + (m+m_1)W(t) \\ &+ \left(\frac{1}{2}\rho_1 - \rho_2\right)\frac{W^2(t)}{Q(t)} + m_2 Q(t) = 0. \end{aligned} \quad (43)$$

In this case, we assume $Q(t) \coloneqq Q_1(t)$, $W(t) \coloneqq W_1(t)$, and $K(t) \coloneqq K_1(t)$. Comparing the coefficients of $l^2$, $l$, and the term without $l$, respectively, and adding to the boundary conditions, we have the following ordinary differential equations (ODEs) with associated terminal conditions:

$$\begin{cases} \dfrac{1}{2}Q_1'(t) + \left(r + \dfrac{1}{2}\rho_1 - \rho_2\right)Q_1(t) = 0, \quad Q_1(T) = 1, \\ W_1'(t) + (r+\rho_1-2\rho_2)W_1(t) + (m+m_1)Q_1(t) = 0, \quad W_1(T) = 0, \\ K_1'(t) + (m+m_1)W_1(t) + \left(\dfrac{1}{2}\rho_1 - \rho_2\right)\dfrac{W_1^2(t)}{Q_1(t)} + m_2Q_1(t) = 0, \quad K_1(T) = 0. \end{cases} \quad (44)$$

Solving these ODEs gives us

$$\begin{cases} Q_1(t) = e^{(2\rho_2-\rho_1-2r)(t-T)}, \\ W_1(t) = -\dfrac{m+m_1}{r}\left[e^{(2\rho_2-\rho_1-r)(t-T)} - e^{(2\rho_2-\rho_1-2r)(t-T)}\right], \\ K_1(t) = \left(\dfrac{m+m_1}{r}\right)^2\left[\dfrac{1}{2}e^{(2\rho_2-\rho_1)(t-T)} + \dfrac{1}{2}e^{(2\rho_2-\rho_1-2r)(t-T)} - e^{(2\rho_2-\rho_1-r)(t-T)}\right] \\ \quad + \dfrac{m_2}{2\rho_2-\rho_1-2r}\left[1 - e^{(2\rho_2-\rho_1-2r)(t-T)}\right]. \end{cases} \quad (45)$$

By substituting $Q_1(t)$, $W_1(t)$, and $K_1(t)$ into $\widehat{b}$, (29), and (22), we can obtain the optimal excess of loss reinsurance strategy, the optimal investment strategy, and the candidate optimal value function $V(t,l)$ given by (37)–(39), respectively. Considering $\widehat{b} = \{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta$ and $a^* = 1$, the inequality $0 < \widehat{b} < aN$ becomes $0 < \{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta < \overline{N}$.

From Theorem 2, the inequalities $\widehat{b} \le 0$ and $\widehat{b} \ge a\overline{N}$ become $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta \le 0$ and $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta \ge \overline{N}$, respectively. In the next two subsections, we will discuss $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta \le 0$ and $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta \ge \overline{N}$, respectively. □

*4.2. $\widehat{b} \le 0$ (that is, $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta \le 0$)*

**Theorem 3.** *When $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta \le 0$, the solution to problem (18) is as follows. The optimal excess of loss reinsurance strategy is $b^* = 0$, the optimal proportional reinsurance strategy is $a^* = 1$, the optimal investment strategy is given by*

$$\pi_i^* = \Delta_i\left\{\frac{m}{r}\left[e^{r(t-T)}-1\right]-l\right\}, \quad i = 1,2,\ldots,n, \quad (46)$$

*and the candidate optimal value function $V(t,l)$ is given by*

$$V(t,l)=\frac{1}{2}e^{(2\rho_2-\rho_1-2r)(t-T)}\left\{l-\frac{m}{r}\left[e^{r(t-T)}-1\right]\right\}^2. \quad (47)$$

*Proof.* When $\hat{b}\le 0$, it is easy to obtain that the optimal excess of loss reinsurance strategy is $b^*=0$. For any $a\in[0,1]$, the insurer will retain, from the $j$th claim, $Y_j(a,b^*)=aY_j\wedge b^*=aY_j\wedge 0=0$. This shows that the insurer transfers all claims to the reinsurer through excess of loss reinsurance. Hence, the insurer will not take proportional reinsurance. Therefore, the corresponding optimal proportional reinsurance strategy is $a^*=1$. By substituting them into (32), we obtain

$$\begin{aligned}&l^2\left\{\frac{1}{2}Q'(t)+\left(r+\frac{1}{2}\rho_1-\rho_2\right)Q(t)\right\}\\&\quad+l\{W'(t)+(r+\rho_1-2\rho_2)W(t)+mQ(t)\}\\&\quad+K'(t)+mW(t)+\left(\frac{1}{2}\rho_1-\rho_2\right)\frac{W^2(t)}{Q(t)}=0.\end{aligned} \quad (48)$$

In this case, we assume $Q(t):=Q_2(t)$, $W(t):=W_2(t)$, and $K(t):=K_2(t)$. To ensure that (48) always holds, we have that the coefficients of $l^2$, $l$, and the term without $l$, respectively, should be equal to 0. From this and the boundary conditions, we obtain the following ODEs with associated terminal conditions:

$$\begin{cases}\frac{1}{2}Q_2'(t)+\left(r+\frac{1}{2}\rho_1-\rho_2\right)Q_2(t)=0, & Q_2(T)=1,\\ W_2'(t)+(r+\rho_1-2\rho_2)W_2(t)+mQ_2(t)=0, & W_2(T)=0,\\ K_2'(t)+mW_2(t)+\left(\frac{1}{2}\rho_1-\rho_2\right)\frac{W_2^2(t)}{Q_2(t)}=0, & K_2(T)=0.\end{cases} \quad (49)$$

Solving these ODEs, we obtain

$$\begin{cases}Q_2(t)=e^{(2\rho_2-\rho_1-2r)(t-T)},\\ W_2(t)=-\frac{m}{r}\left[e^{(2\rho_2-\rho_1-r)(t-T)}-e^{(2\rho_2-\rho_1-2r)(t-T)}\right],\\ K_2(t)=\left(\frac{m}{r}\right)^2\left[\frac{1}{2}e^{(2\rho_2-\rho_1)(t-T)}+\frac{1}{2}e^{(2\rho_2-\rho_1-2r)(t-T)}-e^{(2\rho_2-\rho_1-r)(t-T)}\right].\end{cases} \quad (50)$$

By substituting $Q_2(t)$, $W_2(t)$, and $K_2(t)$ into (29) and (22), we can obtain the optimal investment strategy and the candidate optimal value function $V(t,l)$ given by (46) and (47), respectively. □

*4.3. $\hat{b}\ge a\overline{N}$ (that is, $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta\ge\overline{N}$).* We first define $\hat{a}$ as

$$\hat{a}=-\frac{\eta\overline{\mu}_1}{\overline{\mu}_2}\left[l+\frac{W(t)}{Q(t)}\right]. \quad (51)$$

**Theorem 4.** *When $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta\ge\overline{N}$, the solution to problem (18) can be classified into the following three cases:*

*(i) When $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta\ge\overline{N}$ and $\eta\overline{\mu}_1/\overline{\mu}_2\{(m/r)[e^{r(t-T)}-1]-l\}\le 0$, the optimal proportional reinsurance strategy is 0, and the optimal excess of loss reinsurance strategy is $\overline{N}$. The optimal investment strategy and the candidate optimal value function $V(t,l)$ are the same as those in Theorem 3, so we will not repeat them here.*

*(ii) When $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta\ge\overline{N}$ and $0<(\eta\overline{\mu}_1/\overline{\mu}_2)\{(m/r)[e^{r(t-T)}-1]-l\}<1$, the optimal proportional reinsurance strategy is given by*

$$a^*=\frac{\eta\overline{\mu}_1}{\overline{\mu}_2}\left\{\frac{m}{r}\left[e^{r(t-T)}-1\right]-l\right\}, \quad (52)$$

*the optimal excess of loss reinsurance strategy is $b^*=\overline{N}$, the optimal investment strategy is given by*

$$\pi_i^*=\Delta_i\left\{\frac{m}{r}\left[e^{r(t-T)}-1\right]-l\right\},\quad i=1,2,\ldots,n, \quad (53)$$

*and the candidate optimal value function $V(t,l)$ is given by*

$$V(t,l)=\frac{1}{2}e^{(2\rho_2-\rho_1+\rho_3-2r)(t-T)}\left\{l-\frac{m}{r}\left[e^{r(t-T)}-1\right]\right\}^2. \quad (54)$$

*(iii) When $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta\ge\overline{N}$ and $\eta\overline{\mu}_1/\overline{\mu}_2\{m/r[e^{r(t-T)}-1]-l\}\ge 1$, the optimal proportional reinsurance strategy is 1, the optimal excess of loss reinsurance strategy is $\overline{N}$, the optimal investment strategy is given by*

$$\pi_i^*=\Delta_i\left\{\frac{m+\lambda\eta\overline{\mu}_1}{r}\left[e^{r(t-T)}-1\right]-l\right\},\quad i=1,2,\ldots,n, \quad (55)$$

*and the candidate optimal value function $V(t,l)$ is given by*

$$V(t,l) = \frac{1}{2}e^{(2\rho_2-\rho_1-2r)(t-T)}\left\{l - \frac{m+\lambda\eta\bar{\mu}_1}{r}\left[e^{r(t-T)}-1\right]\right\}^2 + \frac{1}{2\rho_2-\rho_1-2r}\left[e^{(2\rho_2-\rho_1-2r)(t-T)}-1\right]. \quad (56)$$

*Proof.* When $\hat{b} \geq a\overline{N}$, that is, $\{(m+m_1/r)[e^{r(t-T)}-1]-l\}\eta \geq \overline{N}$, we have $\partial\overline{f}(a,b)/\partial b < 0$ for any $b \in (0,\overline{N})$ and $\partial\overline{f}(a,b)/\partial b = 0$ for all $b \geq \overline{N}$. That is to say, $\overline{f}(a,b)$ is a decreasing function with respect to $b$ when $b \in (0,\overline{N})$ and is flat for all $b \geq \overline{N}$. Therefore, the optimal excess of loss reinsurance strategy is $b^* = \overline{N}$. In this case,

$$Y_j(a,b^*) = \min\{aY_j, b^*\} = aY_j \wedge b^* = aY_j \wedge \overline{N} = aY_j, \quad (57)$$

and here, $j = 1, 2, \ldots, +\infty$. Therefore, the problem of combining proportional and excess of loss reinsurance is then equivalent to the problem of pure proportional reinsurance. Then, equation (32) becomes

$$l^2\left\{\frac{1}{2}Q'(t) + \left(r + \frac{1}{2}\rho_1 - \rho_2\right)Q(t)\right\} + l\{W'(t) + (r+\rho_1-2\rho_2)W(t) + mQ(t)\} + K'(t) + mW(t) + \left(\frac{1}{2}\rho_1 - \rho_2\right)\frac{W^2(t)}{Q(t)} + \inf_{a\in U}\left\{\lambda\eta\bar{\mu}_1 a[Q(t)l + W(t)] + \frac{1}{2}\lambda\bar{\mu}_2 Q(t)a^2\right\} = 0. \quad (58)$$

According to the first-order optimality conditions, $a$ solving the inner infimum problem in equation (58) is given by (51). We discuss the value of $\hat{a}$ in the following three cases: $\hat{a} \leq 0$, $0 < \hat{a} < 1$, and $\hat{a} \geq 1$.

(i) When $\hat{a} \leq 0$, the optimal proportional strategy is $a^* = 0$. By substituting $a^* = 0$ into (58), we obtain that equation (58) is equivalent to equation (48). Therefore, the optimal investment strategy and the candidate optimal value function $V(t,l)$ are the same as those in Theorem 3, so we will not repeat them here.

(ii) When $0 < \hat{a} < 1$, the optimal proportional reinsurance strategy is

$$a^* = \hat{a} = -\frac{\eta\bar{\mu}_1}{\bar{\mu}_2}\left[l + \frac{W(t)}{Q(t)}\right]. \quad (59)$$

In this case, we assume $Q(t) \coloneqq Q_3(t)$, $W(t) \coloneqq W_3(t)$, and $K(t) \coloneqq K_3(t)$. By substituting $Q_3(t), W_3(t), K_3(t)$, and $a^* = \hat{a}$ into (58), we obtain

$$l^2\left\{\frac{1}{2}Q_3'(t) + \left[r - \frac{1}{2}(2\rho_2 - \rho_1 + \rho_3)\right]Q_3(t)\right\} + l\{W_3'(t) + [r - (2\rho_2 - \rho_1 + \rho_3)]W_3(t) + mQ_3(t)\} + K_3'(t) + mW_3(t) - \frac{1}{2}(2\rho_2 - \rho_1 + \rho_3)\frac{W_3^2(t)}{Q_3(t)} = 0. \quad (60)$$

To ensure that (60) always holds, we have that the coefficients of $l^2$, $l$, and the term without $l$, respectively, should be equal to 0. From this and the boundary conditions, we obtain the following ODEs with associated terminal conditions:

$$\begin{cases} \frac{1}{2}Q_3'(t) + \left[r - \frac{1}{2}(2\rho_2 - \rho_1 + \rho_3)\right]Q_3(t) = 0, & Q_3(T) = 1, \\ W_3'(t) + [r - (2\rho_2 - \rho_1 + \rho_3)]W_3(t) + mQ_3(t) = 0, & W_3(T) = 0, \\ K_3'(t) + mW_3(t) - \frac{1}{2}(2\rho_2 - \rho_1 + \rho_3)\frac{W_3^2(t)}{Q_3(t)} = 0, & K_3(T) = 0. \end{cases} \quad (61)$$

Solving these ODEs, we obtain

$$\begin{cases} Q_3(t) = e^{(2\rho_2-\rho_1+\rho_3-2r)(t-T)}, \\ W_3(t) = -\frac{m}{r}\left[e^{(2\rho_2-\rho_1+\rho_3-r)(t-T)} - e^{(2\rho_2-\rho_1-2r)(t-T)}\right], \\ K_3(t) = \left(\frac{m}{r}\right)^2\left[\frac{1}{2}e^{(2\rho_2-\rho_1+\rho_3)(t-T)} + \frac{1}{2}e^{(2\rho_2-\rho_1+\rho_3-2r)(t-T)} - e^{(2\rho_2-\rho_1+\rho_3-r)(t-T)}\right]. \end{cases} \quad (62)$$

By substituting $Q_3(t)$, $W_3(t)$, and $K_3(t)$ into (59), (29), and (22), we can obtain the optimal proportional reinsurance strategy, the optimal investment strategy, and the candidate optimal value function $V(t,l)$ given by (52)–(54), respectively.

(iii) When $\widehat{a} \geq 1$, the optimal proportional reinsurance strategy $a^*$ is 1. Similar to (ii), by substituting $a^* = 1$ into (58), we can obtain $Q(t)$, $W(t)$, and $K(t)$. Furthermore, we can obtain $\pi_i^*$ and $V(t,l)$ given by (55) and (56), respectively. □

*Remark 1.* The candidate optimal value function $V(t,l)$ given by Theorems 2–4 is a viscosity solution of problem (18). In fact, $V(t,l)$ is the optimal value function of problem (18); that is, $V(t,l) = J(t,l)$. Similar to that of Theorem 2 in Bi and Guo [24], we can prove this conclusion. Since the methods are similar, we omit ours here.

## 5. Efficient Strategy and Efficient Frontier

In this section, we apply the auxiliary results established in the previous section to the original MV problem (13). We will obtain the optimal reinsurance-investment strategy and the efficient frontier of problem (13).

To simplify the following description, we define the cases discussed in Theorems 2–4 as follows:

$$\begin{cases} \text{case I: } 0 < \left\{\dfrac{m+m_1}{r}\left[e^{r(t-T)}-1\right]-l\right\}\eta < \overline{N}, \\ \text{case II: } \left\{\dfrac{m+m_1}{r}\left[e^{r(t-T)}-1\right]-l\right\}\eta \leq 0, \\ \text{case III: } \left\{\left[\dfrac{m+m_1}{r}\left[e^{r(t-T)}-1\right]-l\right]\eta \geq \overline{N}\right\} \cap \left\{\dfrac{\eta\overline{\mu}_1}{\overline{\mu}_2}\left[\dfrac{m}{r}\left[e^{r(t-T)}-1\right],-l\right] \leq 0\right\}, \\ \text{case IV: } \left\{\left[\dfrac{m+m_1}{r}\left[e^{r(t-T)}-1\right]-l\right]\eta \geq \overline{N}\right\} \cap \left\{0 < \dfrac{\eta\overline{\mu}_1}{\overline{\mu}_2}\left[\dfrac{m}{r}\left[e^{r(t-T)}-1\right]-l\right] < 1\right\}, \\ \text{case V: } \left\{\left[\dfrac{m+m_1}{r}\left[e^{r(t-T)}-1\right]-l\right]\eta \geq \overline{N}\right\} \cap \left\{\dfrac{\eta\overline{\mu}_1}{\overline{\mu}_2}\left[\dfrac{m}{r}\left[e^{r(t-T)}-1\right]-l\right] \geq 1\right\}. \end{cases} \tag{63}$$

First of all, we derive the optimal value of problem (14). Set $L_t^u = X_t^u - (k-q)$ (then, $X_t^u = L_t^u + (k-q)$, $X_0^u = L_0^u + (k-q)$) and $m = c_1 + (k-q)r$ in (16); we can get (8) from (16). Note that

$$\begin{aligned} E_{0,l_0}\left[\frac{1}{2}(L_T^u)^2\right] &= E_{0,x_0}\left\{\frac{1}{2}\left[X_T^u-(k-q)\right]^2\right\} \\ &= E_{0,x_0}\left[\frac{1}{2}(X_T^u-k)^2\right] + qE_{0,x_0}(X_T^u-k) + \frac{1}{2}q^2. \end{aligned} \tag{64}$$

Hence, for every fixed $q$, we have

$$\begin{aligned} &\min_{u\in U} E_{0,x_0}\left[\frac{1}{2}(X_T^u-k)^2\right] + qE_{0,x_0}(X_T^u-k) \\ &\quad = \min_{u\in U} E_{0,l_0}\left[\frac{1}{2}(L_T^u)^2\right] - \frac{1}{2}q^2 = V(0,l_0) - \frac{1}{2}q^2, \end{aligned} \tag{65}$$

that is

$$\min_{u\in U} \quad E_{0,x_0}\left[(X_T^u-k)^2\right] + 2qE_{0,x_0}(X_T^u-k) = 2V(0,l_0) - q^2. \tag{66}$$

Let $\widetilde{f}(q) = 2V(0,l_0) - q^2$; we have

$$\tilde{f}(q)=\begin{cases}\text{for case I}: \\ e^{-T(2\rho_2-\rho_1-2r)}\left\{x_0-k+q-\frac{c_1+(k-q)r+m_1}{r}\left[e^{-rT}-1\right]\right\}^2 \\ \quad+\dfrac{2m_2}{2\rho_2-\rho_1-2r}\left[1-e^{-T(2\rho_2-\rho_1-2r)}\right]-q^2, \\ \text{for cases II and III}: \\ e^{-T(2\rho_2-\rho_1-2r)}\left\{x_0-k+q-\frac{c_1+(k-q)r}{r}\left[e^{-rT}-1\right]\right\}^2-q^2, \\ \text{for case IV}: \\ e^{-T(2\rho_2-\rho_1+\rho_3-2r)}\left\{x_0-k+q-\frac{c_1+(k-q)r}{r}\left[e^{-rT}-1\right]\right\}^2-q^2, \\ \text{for case V}: \\ e^{-T(2\rho_2-\rho_1-2r)}\left\{x_0-k+q-\frac{c_1+(k-q)r+\lambda\eta\overline{\mu}_1}{r}\left[e^{-rT}-1\right]\right\}^2 \\ \quad+\dfrac{2}{2\rho_2-\rho_1-2r}\left[e^{-T(2\rho_2-\rho_1-2r)}-1\right]-q^2.\end{cases} \tag{67}$$

To obtain the optimal value (i.e., the minimum variance $\text{Var}_{0,x_0}(X_T^{u^*})$) and the optimal strategy for portfolio selection problem (13), we need to maximize the function $\tilde{f}(q)$ in (67) over $q \in R$ according to the Lagrange duality theorem.

We can see from (67) that $\tilde{f}(q)$ is a concave function. A simple calculation shows that $\tilde{f}(q)$ attains its maximum value at $q^*$:

$$q^*=\begin{cases}\text{for case I: } q_1^*=\dfrac{\left[x_0-k-\left(c_1+kr+m_1/r\right)\left(e^{-rT}-1\right)\right]e^{-(2\rho_2-\rho_1-r)T}}{1-e^{-(2\rho_2-\rho_1)T}}, \\ \text{for cases II and III: } q_2^*=\dfrac{\left[x_0-k-\left(c_1+kr/r\right)\left(e^{-rT}-1\right)\right]e^{-T(2\rho_2-\rho_1-r)}}{1-e^{-(2\rho_2-\rho_1)T}}, \\ \text{for case IV: } q_3^*=\dfrac{\left[x_0-k-\left(c_1+kr/r\right)\left(e^{-rT}-1\right)\right]e^{-T(2\rho_2-\rho_1+\rho_3-r)}}{1-e^{-T(2\rho_2-\rho_1+\rho_3)}}, \\ \text{for case V: } q_4^*=\dfrac{\left[x_0-k-\left(c_1+kr+\lambda\eta\overline{\mu}_1/r\right)\left(e^{-rT}-1\right)\right]e^{-T(2\rho_2-\rho_1-r)}}{1-e^{-T(2\rho_2-\rho_1)}}.\end{cases} \tag{68}$$

From the above discussion, we obtain the following theorem.

**Theorem 5.** *According to the above five cases, the solution to problem (13) is as follows.*

(1) *For case I, the optimal proportional reinsurance strategy is $a^*=1$, the optimal excess of loss reinsurance strategy is given by*

$$b^*=\left\{\frac{c_1+kr+m_1}{r}\left[e^{r(t-T)}-1\right]-x_0+k-q_1^*e^{r(t-T)}\right\}\eta, \tag{69}$$

*and the optimal investment strategy is given by*

$$\pi_i^*=\Delta_i\left\{\frac{c_1+kr+m_1}{r}\left[e^{r(t-T)}-1\right]-x_0+k-q_1^*e^{r(t-T)}\right\}, \quad i=1,2,\ldots,n. \tag{70}$$

*Moreover, the efficient frontier is given by*

$$\mathrm{Var}_{0,x_0}\left(X_T^{u^*}\right) = \frac{\left[x_0 - \left(c_1 + m_1/r\right)\left(e^{-rT} - 1\right) - ke^{-rT}\right]^2 e^{-\left(2\rho_2 - \rho_1 - 2r\right)T}}{1 - e^{-\left(2\rho_2 - \rho_1\right)T}} + \frac{2m_2}{2\rho_2 - \rho_1 - 2r}\left[1 - e^{-\left(2\rho_2 - \rho_1 - 2r\right)T}\right]. \quad (71)$$

(2) *For case II, the optimal proportional reinsurance strategy is $a^* = 1$, the optimal excess of loss reinsurance strategy is $b^* = 0$, and the optimal investment strategy is given by*

$$\pi_i^* = \Delta_i\left\{\frac{c_1 + kr}{r}\left[e^{r(t-T)} - 1\right] - x_0 + k - q_2^* e^{r(t-T)}\right\}, \quad i = 1, 2, \ldots, n. \quad (72)$$

*Moreover, the efficient frontier is given by*

$$\mathrm{Var}_{0,x_0}\left(X_T^{u^*}\right) = \frac{\left[x_0 - \left(c_1/r\right)\left(e^{-rT} - 1\right) - ke^{-rT}\right]^2 e^{-\left(2\rho_2 - \rho_1 - 2r\right)T}}{1 - e^{-\left(2\rho_2 - \rho_1\right)T}}. \quad (73)$$

(3) *For case III, the optimal proportional reinsurance strategy is $a^* = 0$, and the optimal excess of loss reinsurance strategy is $b^* = \overline{N}$. The optimal investment strategy and the efficient frontier are the same as those in case II.*

(4) *For case IV, the optimal proportional reinsurance strategy is given by*

$$a^* = -\frac{\eta\overline{\mu}_1}{\overline{\mu}_2}\left\{\frac{c_1 + kr}{r}\left[e^{r(t-T)} - 1\right] - x_0 + k - q_3^* e^{r(t-T)}\right\}, \quad (74)$$

*the optimal excess of loss reinsurance strategy is $b^* = \overline{N}$, and the optimal investment strategy is given by*

$$\pi_i^* = \Delta_i\left\{\frac{c_1 + kr}{r}\left[e^{r(t-T)} - 1\right] - x_0 + k - q_3^* e^{r(t-T)}\right\}, \quad i = 1, 2, \ldots, n. \quad (75)$$

*Moreover, the efficient frontier is given by*

$$\mathrm{Var}_{0,x_0}\left(X_T^{u^*}\right) = \frac{\left[x_0 - \left(c_1/r\right)\left(e^{-rT} - 1\right) - ke^{-rT}\right]^2 e^{-\left(2\rho_2 - \rho_1 + \rho_3 - 2r\right)T}}{1 - e^{-\left(2\rho_2 - \rho_1 + \rho_3\right)T}}. \quad (76)$$

(5) *For case V, the optimal proportional reinsurance strategy is 1, the optimal excess of loss reinsurance strategy is $\overline{N}$, and the optimal investment strategy is given by*

$$\pi_i^* = \Delta_i\left\{\frac{c_1 + kr + \lambda\eta\overline{\mu}_1}{r}\left[e^{r(t-T)} - 1\right] - x_0 + k - q_4^* e^{r(t-T)}\right\}, \quad i = 1, 2, \ldots, n. \quad (77)$$

*Moreover, the efficient frontier is given by*

$$\mathrm{Var}_{0,x_0}\left(X_T^{u^*}\right) = \frac{\left[x_0 - \left(c_1 + \lambda\eta\overline{\mu}_1/r\right)\left(e^{-rT} - 1\right) - ke^{-rT}\right]^2 e^{-\left(2\rho_2 - \rho_1 - 2r\right)T}}{1 - e^{-\left(2\rho_2 - \rho_1\right)T}} + \frac{2}{2\rho_2 - \rho_1 - 2r}\left[e^{\left(2\rho_2 - \rho_1 - 2r\right)(t-T)} - 1\right]. \quad (78)$$

*Remark 2.* From Theorem 5, we can see that, for the MV portfolio selection problem (13), the optimal combination of proportional and excess of loss reinsurance can be classified into the following five cases: (1) the pure excess of loss

Table 1: Values of model parameters in numerical experiments.

| $\lambda$ | $\theta$ | $\eta$ | $T$ | $t$ | $x_0$ | $k$ | $r$ |
|---|---|---|---|---|---|---|---|
| 2 | 0.3 | 0.8 | 10 | 0 | 10 | 15 | 0.03 |
| $\overline{\mu}_1$ | $\overline{\mu}_2$ | $\mu_1$ | $\mu_2$ | $\sigma_1$ | $\sigma_2$ | $\overline{\sigma}_1$ | $\overline{\sigma}_2$ |
| 1 | 3 | 0.06 | 0.08 | 0.1 | 0.2 | 0.15 | 0.25 |

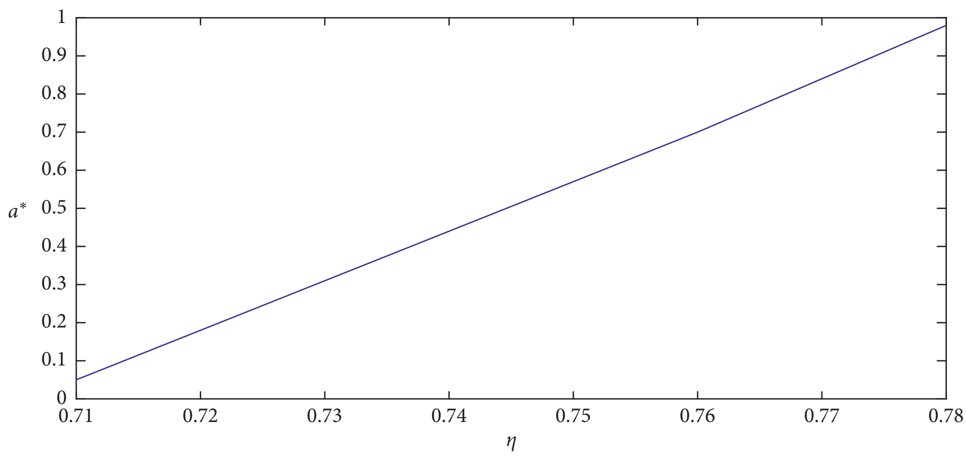


Figure 1: Effect of $\eta$ on $a^*$.

reinsurance case, that is, case I; (2) the case without proportional reinsurance and full excess of loss reinsurance, that is, case II; (3) the case without excess of loss reinsurance and full proportional reinsurance, that is, case III; (4) the pure proportional reinsurance case, that is, case IV; and (5) the case with neither excess of loss reinsurance nor proportional reinsurance, that is, case V. It is clear that cases II, III, and V are three special cases. They maybe occur theoretically, but they hardly occur in practice, because insurers rarely adopt full reinsurance and do not take any reinsurance. In the next section, we will numerically compare case I with case IV.

## 6. Numerical Experiments and Sensitivity Analysis

In this section, we will conduct a series of numerical experiments to illustrate the results obtained in the previous section. We illustrate how optimal reinsurance and investment strategies vary with model parameters. Here, numerical experiments are provided only for case IV we obtained in Theorem 5. Similar to this case, we can analyze the other four cases, which are thus omitted here. Finally, we compare the pure excess of loss reinsurance with the pure proportional reinsurance.

Without loss of generality, we assume that the financial market consists of one risk-free asset and two risky assets. In the following numerical experiments, unless otherwise stated, the basic model parameters are set as those in Table 1.

Figure 1 demonstrates the effect of $\eta$ on optimal reinsurance strategy $a^*$. Here, $\eta$ is the safety loading of the reinsurer. Hence, $\eta$ is positively correlated with the reinsurance premium. The larger $\eta$ is, the greater the reinsurance premium will be. Hence, as $\eta$ increases, $a^*$ increases rapidly.

In the following, we illustrate the effects of model parameters on optimal investment strategy. We will only report the results for $\pi_1^*$. Similar to $\pi_1^*$, we can analyze $\pi_2^*$.

Figures 2 and 3 show the influences of the appreciation rates $\mu_1$ and $\mu_2$ of risky assets 1 and 2 on optimal investment strategy $\pi_1^*$. From Figure 2, we see that $\pi_1^*$ is an increasing function of $\mu_1$; from Figure 3, we see that $\pi_1^*$ is a decreasing function of $\mu_2$. The larger $\mu_1$ (resp., $\mu_2$) is, the greater the expected income of risky asset 1 (resp., risky asset 2) will be and hence the more the insurer will invest in risky asset 1 (resp., risky asset 2; i.e., the insurer reduces his investment in risky asset 1).

Figures 4 and 5 show the influences of the price volatilities $\sigma_1$ and $\sigma_2$ of risky assets 1 and 2 on optimal investment strategy $\pi_1^*$. From Figure 4, we see that $\pi_1^*$ is a decreasing function of $\sigma_1$; from Figure 5, we see that $\pi_1^*$ is an increasing function of $\sigma_2$. With the increase of $\sigma_1$, the risk of risky asset 1 increases, while the risk of risky asset 2 remains unchanged. Therefore, the insurer gradually reduces his investment in risky asset 1 and increases his investments in risky asset 2. Figure 5 displays a similar phenomenon; that is, when $\sigma_2$ increases, the insurer will invest more money in risky asset 1.

Finally, we compare the pure excess of loss reinsurance with the pure proportional reinsurance, utilizing their efficient frontiers. In this part, we assume that the claim amount obeys the exponential distribution with parameter 1 and $T = 6$, and other model parameters are chosen as those in Table 1.

Table 2 illustrates the effect of the level of wealth $\mathrm{E}_{0,x_0}(X_T^{u^*})$ on $\mathrm{Var}_{0,x_0}(X_T^{u^*})$ for case I and case IV, that is, the pure excess loss reinsurance and the pure proportional reinsurance strategy, respectively. We find that $\mathrm{Var}_{0,x_0}(X_T^{u^*})$

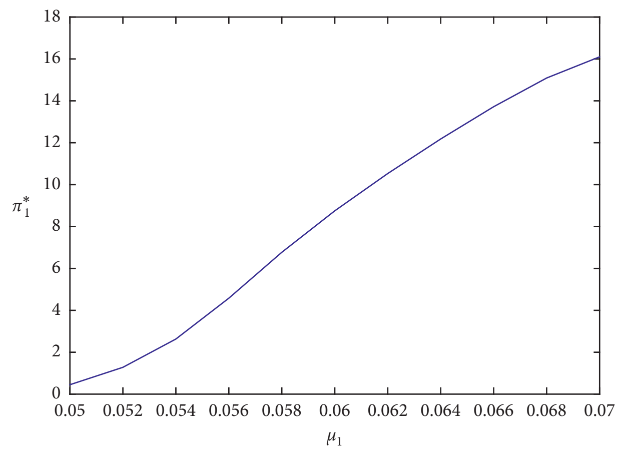


FIGURE 2: Effect of $\mu_1$ on $\pi_1^*$.

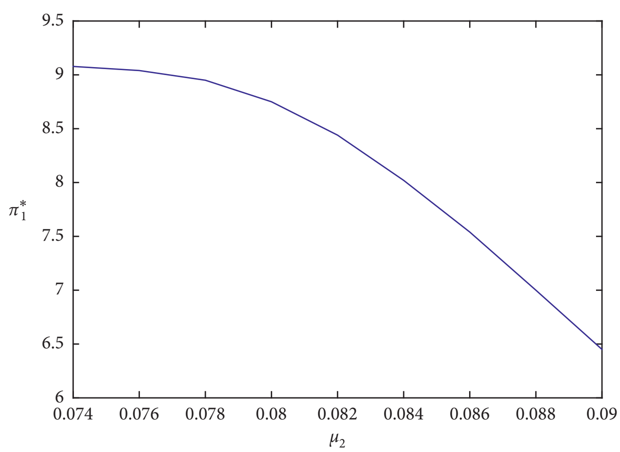


FIGURE 3: Effect of $\mu_2$ on $\pi_1^*$.

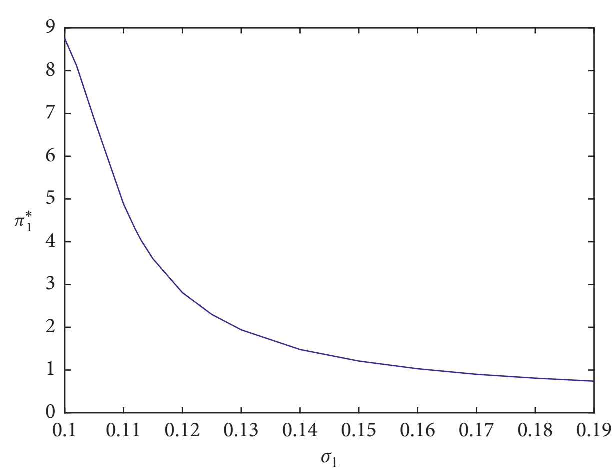


FIGURE 4: Effect of $\sigma_1$ on $\pi_1^*$.

is an increasing function of $E_{0,x_0}(X_T^{u^*})$ in both cases. This shows that the greater the income of the insurer is, the greater the risk will be; this is consistent with the insurance and investment practice. From Table 2, we can see that, for

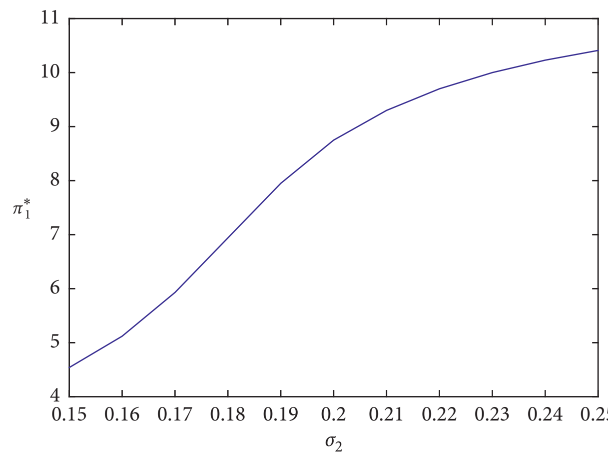


FIGURE 5: Effect of $\sigma_2$ on $\pi_1^*$.

TABLE 2: Values of the efficient frontiers for case I and case IV.

| $\mathrm{Var}_{0,x_0}(X_T^{u^*})$ | $E_{0,x_0}(X_T^{u^*})$ | | | | | | |
|---|---|---|---|---|---|---|---|
| | 6 | 7 | 8 | 9 | 10 | 11 | 12 |
| Case I | 204.8 | 223.8 | 240.8 | 255.8 | 268.8 | 279.7 | 288.6 |
| Case IV | 0.004 | 0.029 | 0.076 | 0.147 | 0.240 | 0.355 | 0.494 |

given wealth level, the insurer bears more risk in the pure excess of loss reinsurance than those in the pure proportional reinsurance. In other words, the pure proportional reinsurance strategy might be better than the pure excess loss reinsurance.

## 7. Conclusion

In this paper, we considered an optimal reinsurance-investment problem under the MV criterion. The insurer's surplus process is governed by the Cramér–Lundberg model. He can transfer the claim risk via combining proportional and excess of loss reinsurance and invests the surplus in a financial market consisting of one risk-free asset and $n$ correlated risky assets. We derive the explicit optimal reinsurance-investment strategy and efficient frontier by using stochastic control technique. Numerical experiments are also provided to illustrate how the optimal reinsurance-investment strategy changes with model parameters.

There are still some issues that need to be investigated in the future. Firstly, it is more meaningful to consider the interaction between the insurance market and the financial market. Secondly, it is also interesting to consider the game between the insurer and the financial market.

## Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

## Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

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