Research Article

Asymptotic Stability of Neutral Set-Valued Functional Differential Equation by Fixed Point Method

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This paper studies a class of nonlinear neutral set-valued functional differential equations. The globally asymptotic stability theorem with necessary and sufficient conditions is obtained via the fixed point method. Meanwhile, we give an example to illustrate the obtained result.

1. Introduction

It is well known that Lyapunov's direct method is an important technique to consider the stability of various differential equations. However, this method is not always valid for stability analysis in the functional differential equation when the delay is unbounded or when the equation has unbounded terms [1–3]. Burton and many researchers found a way to these difficulties by using various fixed point theorems; we can refer to the literature studies [4–14].

Recently, the study of qualitative analysis of the set-valued differential equation has attracted much attention. The stability results of various set-valued differential equations were obtained by applying Lyapunov's direct method. The results can be found in the monograph [15], the papers for the set-valued differential equation [16–24], set-valued functional differential equation [25–29], and other equations [6, 9, 30–32]. However, according to what we know so far, there are few stability results for the set-valued differential equation via the fixed point method. Inspired by the application of the fixed point method mentioned above, in this paper, we study a class of nonlinear neutral set-valued functional differential equations:

$$\begin{cases} D_H X(t) = -a(t)X(t) + b(t)D_H X(t - \tau) \\ \quad + F(t, X(t), X(t - \tau)), \quad t \in I, \\ X(t) = \Phi(t), \quad t \in I_0, \end{cases} \quad (1)$$

where $X \in C^1(I, K_c(\mathbb{E})) \cup C(I_0, K_c(\mathbb{E}))$, $F \in C(I \times K_c(\mathbb{E}) \times K_c(\mathbb{E}), K_c(\mathbb{E}))$, and $F(t, \theta, \theta) = \theta$; $\Phi \in C(I_0, K_c(\mathbb{E}))$, $a, b \in C(I, \mathbb{R})$, $I = [t_0, +\infty)$, and $I_0 = [t_0 - \tau, t_0]$. $K_c(\mathbb{E})$ denote the collection of all nonempty, compact convex subsets of Banach space $\mathbb{E}$; θ denotes the null set-valued function $\theta: I \longrightarrow K_c(\mathbb{E})$, and $\theta(t) = \{0\}$ for $t \in I$, $\tau > 0$ is a constant.

The aim of this paper is to obtain an asymptotic stability theorem with a necessary and sufficient condition via the fixed point method. In addition, an application of the main result is presented.

2. Preliminaries

To get the desired result, we first give some notations, definitions, and propositions briefly; for the details, see the literature [15].

Let $K_c(\mathbb{E})$ denote the collection of all nonempty, compact convex subsets of Banach space $\mathbb{E}$, given $A, B \in K_c(\mathbb{E})$, defining the Hausdorff metric between A and B as follows:

$$D[A, B] = \max \left\{ \sup_{b \in B} d(b, A), \sup_{a \in A} d(a, B) \right\}, \quad (2)$$

where $d(b, A) = \inf\{d(b, a): a \in A\}$ and $d(a, B) = \inf\{d(a, b): b \in B\}$. The Hausdorff metric satisfies the properties as follows:

$$\begin{aligned}
D[A, B] &\geq 0 \text{ with } D[A, B] = 0 \text{ if and only if } A = B, \\
D[A, B] &= D[B, A], \\
D[\lambda A, \lambda B] &= |\lambda|D[A, B], \\
D[A, B] &\leq D[A, C] + D[C, B],
\end{aligned} \tag{3}$$

for all $A, B, C \in K_c(\mathbb{E})$ and $\lambda \in \mathbb{R}$.

In the sense of the above metric D , the set $K_c(\mathbb{E})$ is a complete metric space.

Definition 1 (see [15]). The set-valued function $F: [a, b] \rightarrow K_c(\mathbb{E})$ is Hukuhara differentiable at $t_0 \in [a, b]$ if the limits

$$\begin{aligned}
\lim_{h \rightarrow 0^+} \frac{F(t_0 + h) - F(t_0)}{h}, \\
\lim_{h \rightarrow 0^+} \frac{F(t_0) - F(t_0 - h)}{h},
\end{aligned} \tag{4}$$

exist in $K_c(\mathbb{E})$ and equal to $D_H F(t_0)$. $D_H F(t_0) \in K_c(\mathbb{E})$ is called the Hukuhara derivative of $F(t)$ at $t_0 \in [a, b]$.

By embedding $K_c(\mathbb{E})$ as a complete cone in a corresponding Banach space and taking into account the differentiation of the Bochner integral, we can find that if

$$F(t) = X_0 + \int_a^t \Psi(s) ds, \quad X_0 \in K_c(\mathbb{E}), \tag{5}$$

then $D_H F(t)$ exists and $D_H F(t) = \Psi(t)$ a.e. on $[a, b]$ holds, where $\Psi: [a, b] \rightarrow K_c(\mathbb{E})$ is integrable in the sense of Bochner.

Proposition 1 (see [15]). Let $X: [a, b] \rightarrow K_c(\mathbb{E})$ be Hukuhara differentiable and have continuous Hukuhara derivative $D_H X$ on $[a, b]$; then, we have

$$X(t) = X(t_0) + \int_{t_0}^t D_H X(s) ds, \quad a \leq t_0 \leq t \leq b. \tag{6}$$

Proposition 2 (see [15]). If $X, Y: [a, b] \rightarrow K_c(\mathbb{E})$ is integrable, then

$$D \left[\int_{t_0}^t X(s) ds, \int_{t_0}^t Y(s) ds \right] \leq \int_{t_0}^t D[X(s), Y(s)] ds. \tag{7}$$

For each $(t_0, \Phi) \in I \times C(I_0, K_c(\mathbb{E}))$, $X(t)$ is said to be a solution of (1) through (t_0, Φ) if $X \in C^1(I, K_c(\mathbb{E})) \cup C(I_0, K_c(\mathbb{E}))$ satisfies (1) on I and $X(t) = \Phi(t)$ on I_0 , and we denote the solution $X(t)$ by $X(t, t_0, \Phi)$.

Let $\mathcal{X} = C^1(I_0 \cup I, K_c(\mathbb{E}))$ and $\mathcal{F} = C^1(I_0, K_c(\mathbb{E}))$ with norm

$$\begin{aligned}
\|X\| &= \sup_{t \in I} D[X(t), \theta], \\
\|\Phi\| &= \max_{t \in I_0} D[\Phi(t), \theta],
\end{aligned} \tag{8}$$

respectively, and $\|X\|_0 = \max\{\|X\|, \|D_H X\|\}$. By the properties of Hausdorff metric D and the definition of $\|\cdot\|_0$, we can get the following:

- (i) $\|X(t)\|_0 \geq 0$, and $\|X(t)\|_0 = 0$ if and only if $X(t) = \theta$,
- (ii) $\|X(t) + Y(t)\|_0 \leq \|X(t)\|_0 + \|Y(t)\|_0$,
- (iii) $|\lambda X(t)| = |\lambda| \|X(t)\|_0$,

where $X(t), Y(t) \in \mathcal{X}$ and $\lambda \in \mathbb{R}$.

Definition 2. The trivial solution of (1) is said to be

- (i) Stable in C^1 if for any $t_0 \in \mathbb{R}^+$ and $\varepsilon > 0$, there exists $\delta = \delta(\varepsilon, t_0)$ such that $\Phi \in \mathcal{F}$, and $\|\Phi\|_0 < \delta$ implies that $\|X\|_0 < \varepsilon$ and $t \geq t_0$.
- (ii) Globally asymptotically stable in C^1 if it is stable, and for any $t_0 \in I$, $\Phi \in \mathcal{F}$ implies that

$$\lim_{t \rightarrow +\infty} X(t, t_0, \Phi) = \lim_{t \rightarrow +\infty} D_H X(t, t_0, \Phi) = \{0\}. \tag{9}$$

For $A, B \in K_c(\mathbb{E})$ and $\lambda \in \mathbb{R}$, we can define the following addition and scalar multiplication as follows: $A + B = \{a + b: a \in A, b \in B\}$, $\lambda A = \{\lambda a: a \in A\}$. Then, with the algebraic operations of addition and nonnegative scalar multiplication, $(K_c(\mathbb{E}), D)$ becomes a semilinear metric space.

In order to apply the fixed point method, we first need to prove the following lemma.

Lemma 1. Let $\mathcal{X}_0 = \{X(t) \in \mathcal{X}: \lim_{t \rightarrow +\infty} X(t) = \lim_{t \rightarrow +\infty} D_H X(t) = \{0\}\}$. Then, $(\mathcal{X}_0, \|\cdot\|_0)$ is a semilinear complete metric space.

Proof. Firstly, from the previous discussion, we know that $(\mathcal{X}_0, \|\cdot\|_0)$ is a semilinear metric space. Next, we prove the space is complete. Assume that $\{X_m(t)\}$ is a Cauchy sequence; then, for any $\varepsilon > 0$, there exists positive integer N such that, for all $m, n > N$, we have $\|X_n(t) - X_m(t)\|_0 < \varepsilon$, $t \geq t_0 - \tau$. Hence,

$$D[X_m(t), X_n(t)] \leq \|X_m(t) - X_n(t)\|_0 < \varepsilon, \quad t \geq t_0 - \tau,$$

$$D[D_H X_m(t), D_H X_n(t)] \leq \|X_m(t) - X_n(t)\|_0 < \varepsilon, \quad t \geq t_0 - \tau. \tag{10}$$

Then, $\{X_n(t)\}$ and $\{D_H X_n(t)\}$ are Cauchy sequences in $K_c(\mathbb{E})$, respectively. Since $(K_c(\mathbb{E}), D)$ is a complete metric space, there exist set-valued functions $X(t)$ and $Z(t)$ such that

$$\lim_{m \rightarrow \infty} X_m(t) = X(t), \quad \lim_{m \rightarrow \infty} D_H X_m(t) = Z(t). \tag{11}$$

First, we prove that $X(t)$ is a continuous function on $I_0 \cup I$. From (11), there exists a positive integer M such that $D[X_m(t), X(t)] < (\varepsilon/3)$ for all $m > M$ and $t \geq t_0 - \tau$. In addition, since $X_m(t)$ is a continuous set-valued function, there exists $\delta > 0$ such that $|t - s| < \delta$ implies that $D[X_m(t), X_m(s)] < (\varepsilon/3)$ for $s, t \geq t_0 - \tau$. Then, for any $t \geq t_0 - \tau$ with $|t - s| < \delta$ and $m > M$, we have

$$\begin{aligned}
D[X(t), X(s)] &\leq D[X(t), X_m(t)] + D[X_m(t), X_m(s)] \\
&\quad + D[X_m(s), X(s)] < \varepsilon.
\end{aligned} \tag{12}$$

Using the same way, we can also prove that $Z(t)$ is a continuous function.

Next, we prove that $D_H X(t) = Z(t)$. In fact, for any $t_0, t \in I$ and $t > t_0$, we have

$$X_m(t) = X_m(t_0) + \int_{t_0}^t D_H X_m(s) ds. \quad (13)$$

By (11) and (13), $X(t) = X(t_0) + \int_{t_0}^t Z(s) ds$ holds when $m \rightarrow \infty$. Using Proposition 1, we get $D_H X(t) = Z(t)$. Meanwhile, we can prove that $\lim_{t \rightarrow +\infty} X(t) = \lim_{t \rightarrow +\infty} Z(t) = \{0\}$. Therefore, $X(t) \in \mathcal{X}_0$. This completes the proof of Lemma 1. $\square$

3. Main Result

In this section, we establish the necessary and sufficient conditions for the global asymptotic stability of trivial solution of equation (1) by using the fixed point method.

Theorem 1. Assume that the following conditions hold:

(A₁): the function $a(t)$ is bounded and $\lim_{t \rightarrow +\infty} \inf \int_{t_0}^t a(s) ds > -\infty$.

(A₂): there exist functions $\lambda_1, \lambda_2 \in C(I, \mathbb{R}^+)$ such that

$$\begin{aligned} & \|F(t, X_1, Y_1) - F(t, X_2, Y_2)\| \\ & \leq \lambda_1(t) \|X_1 - X_2\| + \lambda_2(t) \|Y_1 - Y_2\|. \end{aligned} \quad (14)$$

(A₃): there exists a constant $\alpha \in (0, 1)$ such that

$$\begin{aligned} & \int_{t_0}^t e^{-\int_s^t a(v) dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \leq \alpha, \\ & |a(t)| \int_{t_0}^t e^{-\int_s^t a(v) dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \\ & + |b(t)| + \lambda_1(t) + \lambda_2(t) \leq \alpha. \end{aligned} \quad (15)$$

Then, the trivial solution of equation (1) is globally asymptotically stable in C^1 if and only if

$$\int_{t_0}^{+\infty} a(s) ds = +\infty. \quad (16)$$

Proof. We will prove this conclusion in two steps.

3.1. Proof of Sufficiency. Let $\mathcal{D} = \{X \in \mathcal{X}_0: X(t) = \Phi(t), t \in I_0\}$. Then, $\mathcal{D}$ is a nonempty, closed subset of $\mathcal{X}_0$. At the same time, we can get equation (1) with initial condition $X(t_0) = \Phi(t)$ which is equivalent to the following integral equation:

$$\begin{cases} X(t) = \Phi(t_0) e^{-\int_{t_0}^t a(v) dv} + \int_{t_0}^t e^{-\int_s^t a(v) dv} [b(s) D_H X(s - \tau) \\ \quad + F(s, X(s), X(s - \tau))] ds, t \in I, \\ X(t) = \Phi(t), t \in I_0. \end{cases} \quad (17)$$

We define the mapping $Q: \mathcal{D} \rightarrow C^1(I_0 \cup I, K_c(\mathbb{E}))$ as follows:

$$(QX)(t) = \begin{cases} \Phi(t_0) e^{-\int_{t_0}^t a(v) dv} + \int_{t_0}^t e^{-\int_s^t a(v) dv} [b(s) D_H X(s - \tau) \\ \quad + F(s, X(s), X(s - \tau))] ds, t \in I, \\ \Phi(t), t \in I_0. \end{cases} \quad (18)$$

Firstly, we prove that Q is a mapping of $\mathcal{D} \rightarrow \mathcal{D}$. In fact, we just need to prove that $(QX)(t) \rightarrow \{0\}$ and $D_H(QX)(t) \rightarrow \{0\}$ as $t \rightarrow +\infty$. Since $\lim_{t \rightarrow +\infty} X(t) = \lim_{t \rightarrow +\infty} D_H X(t) = \{0\}$, for any $\varepsilon > 0$, there exists $T > 0$ such that $t > T$ implies

$$\max\{\|X(t)\|, \|X(t - \tau)\|, \|D_H X(t - \tau)\|\} < \varepsilon. \quad (19)$$

Thus, from (18), (19), (A₂), and (A₃), we have

$$\begin{aligned} \|(QX)(t)\| & \leq \|\Phi(t_0)\| e^{-\int_{t_0}^t a(v) dv} \\ & + \int_{t_0}^T e^{-\int_s^t a(v) dv} \|b(s) D_H X(s - \tau) + F(s, X(s), X(s - \tau))\| ds \\ & + \int_T^t e^{-\int_s^t a(v) dv} \|b(s) D_H X(s - \tau) + F(s, X(s), X(s - \tau)) - F(s, \theta, \theta)\| ds \leq e^{-\int_{t_0}^t a(v) dv} \left[\|\Phi(t_0)\| \right. \\ & + \int_{t_0}^T e^{\int_{t_0}^s a(v) dv} \|b(s) D_H X(s - \tau) + F(s, X(s), X(s - \tau))\| ds \Big] \\ & + \int_T^t e^{-\int_s^t a(v) dv} [|b(s)| \|D_H X(s - \tau)\| + \lambda_1(s) \|X(s)\| + \lambda_2(s) \|X(s - \tau)\|] ds \end{aligned}$$

$$\begin{aligned}
&\leq e^{-\int_{t_0}^t a(v)dv} \left[\|\Phi(t_0)\| + \int_{t_0}^T e^{\int_{t_0}^s a(v)dv} \|b(s)D_H X(s-\tau) + F(s, X(s), X(s-\tau))\| ds \right] \\
&\quad + \varepsilon \int_T^t e^{-\int_s^t a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \\
&\leq e^{-\int_{t_0}^t a(v)dv} \left[\|\Phi(t_0)\| + \int_{t_0}^T e^{\int_{t_0}^s a(v)dv} \|b(s)D_H X(s-\tau) + F(s, X(s), X(s-\tau))\| ds \right] + \alpha\varepsilon, \quad t > T.
\end{aligned} \tag{20}$$

Then, by condition (16), there exists $T_1 > T$ such that, for $t > T_1$ implies

$$\begin{aligned}
&e^{-\int_{t_0}^t a(v)dv} \left[\|\Phi(t_0)\| + \int_{t_0}^T e^{\int_{t_0}^s a(v)dv} \|b(s)D_H X(s-\tau) \right. \\
&\quad \left. + F(s, X(s), X(s-\tau))\| ds \right] < \varepsilon.
\end{aligned} \tag{21}$$

Thus, we have $\|(QX)(t)\| < (1 + \alpha)\varepsilon$ for $t > T_1$, and we can obtain $(QX)(t) \rightarrow \{0\}$ as $t \rightarrow +\infty$. In addition, we can also get

$$\begin{aligned}
D_H(QX)(t) &= -a(t)\Phi(t_0)e^{-\int_{t_0}^t a(v)dv} \\
&\quad - a(t) \int_{t_0}^t e^{-\int_s^t a(v)dv} [b(s)D_H X(s-\tau) \\
&\quad + F(s, X(s), X(s-\tau))] ds \\
&\quad + [b(t)D_H X(t-\tau) + F(t, X(t), X(t-\tau))] \\
&= -a(t)(QX)(t) + b(t)D_H X(t-\tau) \\
&\quad + F(t, X(t), X(t-\tau)) \\
&= -a(t)(QX)(t) + D_H X(t) + a(t)X(t).
\end{aligned} \tag{22}$$

Furthermore, from (22), we can get

$$\|D_H(QX)(t)\| \leq |a(t)| \|(QX)(t)\| + \|D_H X(t)\| + |a(t)| \|X(t)\|. \tag{23}$$

From (A₁) and $\lim_{t \rightarrow +\infty} (QX)(t) = \lim_{t \rightarrow +\infty} X(t) = \{0\}$, we have $\lim_{t \rightarrow +\infty} D_H(QX)(t) = \{0\}$ for $X \in \mathcal{D}$. Therefore, $(QX)(t) \in \mathcal{D}$ for $X \in \mathcal{D}$, that is, $Q: \mathcal{D} \rightarrow \mathcal{D}$.

Secondly, we prove that the mapping Q has a unique fixed point.

Let $X, Y \in \mathcal{D}$, and by (18), (A₂), and (A₃), we have

$$\begin{aligned}
&\|(QX)(t) - (QY)(t)\| \\
&\leq \int_{t_0}^t e^{-\int_s^t a(v)dv} [|b(s)| \|D_H X(s) - D_H Y(s)\| \\
&\quad + \|F(s, X(s), X(s-\tau)) - F(s, Y(s), Y(s-\tau))\|] ds \\
&\leq \|X(t) - Y(t)\|_0 \int_{t_0}^t e^{-\int_s^t a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \\
&\leq \alpha \|X(t) - Y(t)\|_0, \quad \alpha < 1, \quad t \geq t_0.
\end{aligned} \tag{24}$$

Similarly, from (18), (24), (A₂), and (A₃), we can also get

$$\begin{aligned}
&\|D_H(QX)(t) - D_H(QY)(t)\| \\
&\leq |a(t)| \|(QX)(t) - (QY)(t)\| \\
&\quad + |b(t)| \|D_H X(t) - D_H Y(t)\| \\
&\quad + \|F(t, X(t), X(t-\tau)) - F(t, Y(t), Y(t-\tau))\| \\
&\leq \|X(t) - Y(t)\|_0 \left\{ |a(t)| \int_{t_0}^t e^{-\int_s^t a(v)dv} [|b(s)| + \lambda_1(s) \right. \\
&\quad \left. + \lambda_2(s)] ds + |b(t)| + \lambda_1(t) + \lambda_2(t) \right\} \\
&\leq \alpha \|X(t) - Y(t)\|_0, \quad t \geq t_0.
\end{aligned} \tag{25}$$

By (24) and (25), we know that Q is a contraction mapping. Therefore, using the principle of contraction mapping, Q has a unique fixed point $X \in \mathcal{D}$ which is a unique solution of equation (1) and satisfies

$$\lim_{t \rightarrow +\infty} X(t) = \lim_{t \rightarrow +\infty} D_H X(t) = \{0\}. \tag{26}$$

Finally, we prove the global asymptotic stability of equation (1). To do this, we first prove that the trivial solution of (1) is stable. Let

$$\begin{aligned}
M_1 &= \sup_{t \in I} \left\{ e^{-\int_{t_0}^t a(v)dv} \right\}, \\
M_2 &= \sup_{t \in I} \{|a(t)|\}.
\end{aligned} \tag{27}$$

For any $\varepsilon > 0$, we choose $\delta > 0$ satisfying $\max\{\delta, M_1\delta + \alpha\varepsilon, M_1M_2\delta + \alpha\varepsilon\} < \varepsilon$. If $X(t) = X(t, t_0, \Phi)$ is a solution of equation (1) with $\|\Phi(t_0)\| < \delta$, we can claim that $\|X(t)\|_0 < \varepsilon$ on $t \in I$. In fact, if this is not true and we notice that $\|X(t)\|_0 < \varepsilon$ on I_0 , then there exists $\bar{t} > t_0$ such that

$$\begin{aligned}
&\max\{\|X(\bar{t})\|, \|D_H X(\bar{t})\|\} = \varepsilon, \\
&\max\{\|X(t)\|, \|D_H X(t)\|\} < \varepsilon, \quad \text{for } t \in [t_0 - \tau, \bar{t}].
\end{aligned} \tag{28}$$

If $\|X(\bar{t})\| = \varepsilon$, then it follows from (A₂) and (A₃) that

$$\begin{aligned} \|X(\bar{t})\| &\leq \|\Phi(t_0)\| e^{-\int_{t_0}^{\bar{t}} a(v)dv} \\ &\quad + \int_{t_0}^{\bar{t}} e^{-\int_s^{\bar{t}} a(v)dv} \|b(s)D_H X(s-\tau) \\ &\quad + F(s, X(s), X(s-\tau))\| ds \\ &\leq M_1 \delta + \varepsilon \int_{t_0}^{\bar{t}} e^{-\int_s^{\bar{t}} a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \\ &\leq M_1 \delta + \alpha \varepsilon < \varepsilon, \end{aligned} \quad (29)$$

which contradicts the definition of $\bar{t}$.

If $\|D_H X(\bar{t})\| = \varepsilon$, then it follows from (A₂) and (A₃) that

$$\begin{aligned} \|D_H X(\bar{t})\| &\leq |a(\bar{t})| \|\Phi(t_0)\| e^{-\int_{t_0}^{\bar{t}} a(v)dv} + |b(\bar{t})| \|D_H X(\bar{t}-\tau)\| \\ &\quad + \|F(\bar{t}, X(\bar{t}), X(\bar{t}-\tau))\| \\ &\quad + |a(\bar{t})| \int_{t_0}^{\bar{t}} e^{-\int_s^{\bar{t}} a(v)dv} \|b(s)D_H X(s-\tau) \\ &\quad + F(s, X(s), X(s-\tau))\| ds \\ &\leq M_1 M_2 \delta + \varepsilon \left\{ |a(t)| \int_{t_0}^t e^{-\int_s^t a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \right. \\ &\quad \left. \leq M_1 M_2 \delta + \alpha \varepsilon < \varepsilon, \right. \end{aligned} \quad (30)$$

which also contradicts the definition $\bar{t}$. Hence, $\|X(t)\|_0 < \varepsilon$ for all $t \geq t_0$, and the trivial solution of (1) is stable. This, together with (26), claims that the trivial solution of (1) is globally asymptotically stable.

3.2. Proof of Necessity. Assume that (16) is not valid. Then, by condition (A₁), there exists a sequence $\{t_n\}$ such that $\lim_{n \rightarrow +\infty} \int_{t_0}^{t_n} a(v)dv = l$ for some $l \in R$, where $t_n \rightarrow +\infty$ as $n \rightarrow \infty$. Furthermore, we also choose a positive constant p satisfying

$$-p \leq \int_{t_0}^{t_n} a(v)dv \leq p, \quad \text{for } n \geq 1. \quad (31)$$

By (A₃), we have

$$I_n = \int_{t_0}^{t_n} e^{\int_{t_0}^s a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \leq \alpha e^{\int_{t_0}^{t_n} a(v)dv}, \quad (32)$$

that is, the sequence of $\{I_n\}$ is bounded. Thus, there exists a convergent subsequence. For convenience, we may assume that $\lim_{n \rightarrow +\infty} I_n = \lambda$. Let

$$K = \sup_{t \in I} \left\{ e^{-\int_{t_0}^t a(v)dv} \right\}, \quad (33)$$

$$A = \sup_{t \in I} \{|a(t)|\}.$$

Then, there exists a sufficient large positive integer k such that, for all $n > k$,

$$|I_n - I_k| = \int_{t_k}^{t_n} e^{\int_{t_0}^s a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds < \frac{1-\alpha}{4Me^{2p}}, \quad (34)$$

where $M = \max\{Ke^p, KAe^p\}$.

For any $\delta > 0$, we consider the solution $X(t, t_k, \Phi)$ of (1) with $\|\Phi(t_k)\| > (1/2)\delta$ and $\|\Phi(t)\| < \delta$, $t \in [t_k - \tau, t_k]$. It follows from (17), (22), (31), (A₂), and (A₃) that

$$\begin{aligned} \|X(t)\| &\leq \|\Phi(t_k)\| e^{-\int_{t_k}^t a(v)dv} \\ &\quad + \int_{t_k}^t e^{-\int_s^t a(v)dv} \|b(s)D_H X(s-\tau) + F(s, X(s), X(s-\tau))\| ds \\ &\leq \|\Phi(t_k)\| e^{-\int_{t_0}^t a(v)dv} e^{\int_{t_0}^{t_k} a(v)dv} \\ &\quad + \|X(t)\|_{0k} \int_{t_k}^t e^{-\int_s^t a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \\ &\leq Ke^p \delta + \alpha \|X(t)\|_{0k}, \quad t \in [t_k, +\infty), \end{aligned}$$

$$\begin{aligned}
\|D_H X(t)\| &\leq |a(t)| \|\Phi(t_k)\| e^{-\int_{t_k}^t a(v)dv} + |b(t)| \|D_H X(t-\tau)\| + \|F(t, X(t), X(t-\tau))\| \\
&\quad + |a(t)| \int_{t_k}^t e^{-\int_s^t a(v)dv} \|b(s) D_H X(s-\tau) + F(s, X(s), X(s-\tau))\| ds \\
&\leq |a(t)| \|\Phi(t_k)\| e^{-\int_{t_0}^t a(v)dv} e^{\int_{t_0}^{t_k} a(v)dv} + \|X(t)\|_{0k} [|b(t)| + \lambda_1(t) + \lambda_2(t)] \\
&\quad + \|X(t)\|_{0k} \int_{t_k}^t e^{-\int_s^t a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \\
&\leq K A e^p \delta + \|X(t)\|_{0k} \left\{ |a(t)| \int_{t_0}^t e^{-\int_s^t a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \right. \\
&\quad \left. + |b(t)| + \lambda_1(t) + \lambda_2(t) \right\} \\
&\leq K A e^p \delta + \alpha \|X(t)\|_{0k}, \quad t \in [t_k, +\infty).
\end{aligned} \tag{35}$$

Hence, we have $\|X(t)\|_{0k} \leq M\delta + \alpha \|X(t)\|_{0k}$, that is, $\|X(t)\|_{0k} \leq (M/1 - \alpha)\delta$, where $\|X(t)\|_{0k} = \max\{\|X(t)\|, \|D_H X(t)\|\}$ for $t \in [t_k, +\infty)$. Furthermore, it follows from (17), (34), and (A_2) that, for $n \geq k$,

$$\begin{aligned}
\|X(t_n)\| &\geq \|\Phi(t_k)\| e^{-\int_{t_k}^{t_n} a(v)dv} - e^{-\int_{t_0}^{t_n} a(v)dv} \int_{t_k}^{t_n} e^{\int_{t_0}^s a(v)dv} \|b(s) D_H X(s-\tau) + F(s, X(s), X(s-\tau))\| ds \\
&\geq \|\Phi(t_k)\| e^{-\int_{t_k}^{t_n} a(v)dv} - \|X(t)\|_{0k} e^{-\int_{t_0}^{t_n} a(v)dv} \int_{t_k}^{t_n} e^{\int_{t_0}^s a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \\
&> \frac{1}{2} \delta e^{-p} - \frac{M}{1-\alpha} \delta e^p \frac{1-\alpha}{4M e^{2p}} = \frac{e^{-p}}{4} \delta,
\end{aligned} \tag{36}$$

which contradicts with (26). Hence, condition (16) is necessary for globally asymptotic stability of the trivial solution of (1). This completes the proof of Theorem 1.

To illustrate our result, we give an example.

Example 1. In equation (1), let $a(t) = (1/\sqrt{1+t})$, $b(t) = (1/8\sqrt{1+t})$, and

$$F(t, X(t), X(t-\tau)) = \frac{1}{8\sqrt{1+t}} [X(t) + X(t-\tau)]. \tag{37}$$

By a straightforward computation, we can obtain

$$|a(t)| \leq 1, \quad \int_{t_0}^{+\infty} a(s)ds = +\infty, \quad \text{for } t \in I. \tag{38}$$

In addition,

$$\begin{aligned}
\|F(t, X_1, Y_1) - F(t, X_2, Y_2)\| &= \frac{1}{8\sqrt{1+t}} \|X_1 + Y_1 - X_2 - Y_2\| \\
&\leq \frac{1}{8\sqrt{1+t}} (\|X_1 - X_2\| + \|Y_1 - Y_2\|).
\end{aligned} \tag{39}$$

Let $\lambda_1(t) = \lambda_2(t) = (1/8\sqrt{1+t})$; then, condition (A_2) of Theorem 1 holds. By direct calculations, we have

$$\begin{aligned}
&\int_{t_0}^t e^{-\int_s^t a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \\
&\leq \frac{3}{8} e^{-2\sqrt{1+t}} \int_{t_0}^t e^{2\sqrt{1+s}} \frac{1}{\sqrt{1+s}} ds \\
&\leq \frac{3}{8} \left[1 - e^{-2\sqrt{1+t}} \right] < \frac{3}{8},
\end{aligned} \tag{40}$$

$$\begin{aligned}
&|a(t)| \int_{t_0}^t e^{-\int_s^t a(v)dv} [|b(s)| + \lambda_1(s) + \lambda_2(s)] ds \\
&\quad + |b(t)| + \lambda_1(t) + \lambda_2(t) \\
&\leq \frac{3}{8\sqrt{1+t}} \left[1 - e^{-2\sqrt{1+t}} \right] + \frac{3}{8\sqrt{1+t}} < \frac{3}{4}.
\end{aligned} \tag{41}$$

Let $\alpha = (3/4)$, and by (38)–(41), we find that conditions (A_1) – (A_3) and (16) hold. Thus, (1) is globally asymptotically stable.

4. Conclusion

Stability is one of the main problems encountered in applications and has recently attracted considerable attention. The fixed point method is an effective method to discuss the stability for the differential equation with unbounded delay or the differential equation with unbounded terms. In this paper, we investigate a class of neutral set-valued functional differential equations and obtain a criterion for the globally asymptotic stability theorem with necessary and sufficient conditions by the fixed point method. Finally, we verify the validity of the result by an example.

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors completed the paper together. All authors read and approved the final manuscript.

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