Research Article

Study of a Class of Generalized Multiterm Fractional Differential Equations with Generalized Fractional Integral Boundary Conditions

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The aim of this work is to study the new generalized fractional differential equations involving generalized multiterms and equipped with multipoint boundary conditions. The nonlinear term is taken from Orlicz space. The existence and uniqueness results, with the help of contemporary tools of fixed point theory, are investigated. The Ulam stability of our problem is also presented. The obtained results are well illustrated by examples.

1. Introduction

Fractional differential equations (FDEs) have a considerable interest both in mathematics and in applications. They were used in modeling in various disciplines of engineering, physics, chemical technology, population dynamics, biotechnology, economics, etc. (see, for example, [1–10], and the references cited therein). Recently, some authors explored FDEs with various types of conditions as multipoint, nonlocal, antiperiodic, and integral boundary conditions [11–15].

Furthermore, there has been a significant development in fractional derivatives and integrals due to the necessity of a better model for real phenomena. In [16], Katugampola suggested a new fractional integral that combines the Riemann–Liouville and Hadamard integrals into a single integral. In 2015, Caputo and Fabrizio [17] defined the so-called Caputo–Fabrizio fractional derivatives by imposing a nonsingular exponential kernel. Later, the exponential kernel was replaced by the Mittag–Leffler function [18, 19]. Another definition of generalized proportional fractional derivatives-generated Caputo and Riemann–Liouville

involving exponential functions in their kernels was introduced by Jarad et al. [20]. Another new type of new fractional derivative can be found in [21]. An application on such derivatives can be seen in [22]. More recently, generalized fractional derivatives that contain kernels depending on the function on the space of absolutely continuous functions were investigated [23]. Following the above points, many authors established various generalized fractional boundary value problems (GFBVPs) (see, for example, [24–27]). It is worth to mention that some authors worked on the existence of solutions and studied the stability analysis for nonlinear singular fractional differential equations. For more details, we refer to [28–30].

On the other hand, Birnbaum and Orlicz [31] proposed a more general setting of function spaces in 1931, which were called Orlicz spaces. These spaces are the generalization of the classical Lebesgue spaces L^p ($1 < p < \infty$), where the kernel is given by a convex function instead of x^p . Some results that deal with differential equations in the framework of Orlicz spaces can be found in [32, 33].

In this paper, we investigate the boundary value problem as follows:

$$\left\{ \begin{array}{l} {}^C D_{0^+}^{\alpha, \rho} u(t) = f(t, u(t), {}^C D_{0^+}^{\alpha-1, \rho} u(t), {}^C D_{0^+}^{\alpha-2, \rho} u(t), {}^C D_{0^+}^{\alpha-3, \rho} u(t)), \quad t \in [0, 1], \rho > 0, \\ u(0) = \delta u(0) = \delta^2 u(0) = 0, \\ \int_0^1 u(s) ds = \sum_{i=1}^m \delta_i I_0^{\beta, \rho} u(\xi_i), \quad 0 < \xi_i < 1, \delta = t^{1-\rho} \frac{d}{dt}, \end{array} \right. \quad (1)$$

where ${}^C D_{0^+}^{\alpha, \rho}$ is the generalized fractional derivative of Caputo type of order α , $3 < \alpha \leq 4$, $I_0^{\beta, \rho}$ denotes the Katugampola-type fractional integral of order $0 < \beta < 1$, f is defined on an Orlicz space $L_F([0, 1])$, and $\delta_i \in \mathbb{R}$, $(i = 1, \dots, m)$.

The rest of the paper is organized as follows. In Section 2, we will briefly recall some preliminary materials related to our problem. In Section 3, we discuss the existence and uniqueness results of GFBVP using some fixed point theorems and support the obtained results by using examples to well illustrate. The Ulam stability of our problem is given in Section 4.

2. Preliminaries

For convenience of the reader, we present some basic definitions about generalized fractional calculus theory, which can be found in [16, 34, 35]. Also, we introduce some necessary concepts for Orlicz spaces which are used throughout this paper. For more details about Orlicz spaces, one can see [36, 37].

Definition 1. Let $Q: [0, \infty) \rightarrow [0, \infty)$ be a right continuous, monotone, increasing function with the following:

- (i) $Q(0) = 0$
- (ii) $\lim_{t \rightarrow \infty} Q(t) = \infty$
- (iii) $Q(t) > 0$ whenever $t > 0$

Then, the function defined by

$$F(x) = \int_0^x Q(t) dt, \quad x \geq 0, \quad (2)$$

is called N -function. Alternatively, the function F is an N -function iff F is continuous, even, and convex with the following:

- (i) $\lim_{x \rightarrow 0} (F(x)/x) = 0$
- (ii) $\lim_{x \rightarrow 0} (F(x)/x) = \infty$
- (iii) $F(x) > 0$ if $x > 0$

Definition 2. For an N -function, we define

$$F^*(x) = \int_0^x Q^{-1}(t) dt, \quad x \geq 0, \quad (3)$$

where Q^{-1} is the right inverse of the right derivative of F and is called the complementary of F and it satisfies the following condition:

$$F^*(x) = \sup\{tx - F(t): t \geq 0\}, \quad \forall x \geq 0. \quad (4)$$

Remark 1. Note that the function F^* is also N -function and the complementary pairs F and F^* satisfy the following Young inequality:

$$xt \leq F(x) + F^*(t), \quad \forall x, t \geq 0. \quad (5)$$

Definition 3. A function $F: [0, \infty) \rightarrow [0, \infty)$ is called a Young function if it is convex and satisfies the conditions $F(0) = \lim_{x \rightarrow 0^+} F(x) = 0$ and $\lim_{x \rightarrow \infty} F(x) = \infty$.

Definition 4 (Orlicz space). For an N -function F , the Orlicz space $L_F([0, 1])$ is the space of measurable functions $u: [0, 1] \rightarrow \mathbb{R}$ such that $\int_0^1 F(|u(x)|) dx < \infty$. This space endowed with the Luxemburg norm, i.e.,

$$\|u\|_F = \inf \left\{ \lambda > 0: \int_0^1 F\left(\frac{|u(x)|}{\lambda}\right) dx \leq 1 \right\}, \quad (6)$$

and the pair $(L_F([0, 1]), \|u\|_F)$ is a Banach space.

Remark 2. For an Orlicz space, the Hölder inequality holds, that is,

$$\int_0^1 uv \, dx \leq 2 \|u\|_F \|v\|_{F^*}, \quad (7)$$

where $u \in L_F([0, 1])$ and $v \in L_{F^*}([0, 1])$.

For generalized fractional calculus, Katugampola [16, 34, 38] introduced the following definitions.

Definition 5 (Katugampola generalized fractional integral). The generalized fractional integral of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $\rho > 0$ for $-\infty < a < t < \infty$ is defined by

$$({}^I_{a^+}^{\alpha, \rho} u)(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_a^t \frac{s^{\rho-1}}{(t-s)^{1-\alpha}} u(s) ds, \quad (8)$$

where u belongs to the space $X_c^p(a, b)$, which denotes the space of all complex-valued Lebesgue measurable functions f on $[a, b]$ for which $\|f\|_{X_c^p} < \infty$, where the norm is defined by

$$\|f\|_{X_c^p} = \left(\int_a^b |t^c f(t)|^p \frac{dt}{t} \right)^{1/p} < \infty, \quad 1 \leq p < \infty, c \in \mathbb{R}. \quad (9)$$

Note that integral (8) is called the left-sided generalized fractional integral.

Definition 6 (Katugampola generalized Caputo fractional derivative). The left generalized fractional derivative of order $\alpha \in \mathbb{C}$ with $\Re(\alpha) > 0$ and $\rho > 0$ of $u \in AC_\delta^n[a, b]$, where $-\infty < a < t < \infty$, is defined by

$$\begin{aligned} ({}^C D_{a^+}^{\alpha, \rho} u)(t) &= \frac{1}{\Gamma(n - \alpha)} \int_a^t \left(\frac{t^\rho - s^\rho}{\rho} \right)^{n - \alpha - 1} \frac{(\delta^n u)(s) ds}{s^{1 - \rho}} \\ &= I_{a^+}^{n - \alpha, \rho} (\delta^n u)(s), \end{aligned} \quad (10)$$

$$AC_\delta^n[a, b] = \left\{ f: [a, b] \longrightarrow \mathbb{C} \text{ and } \delta^{n-1} f \in AC[a, b], \delta = t^{1-\rho} \frac{d}{dt} \right\}, \quad (11)$$

$$C_{\delta, \epsilon}^n[a, b] = \left\{ f: [a, b] \longrightarrow \mathbb{C} \text{ and } \delta^{n-1} f \in C[a, b], \delta^n f \in C_{\epsilon, \rho}[a, b], \delta = t^{1-\rho} \frac{d}{dt} \right\}, \quad (12)$$

for $0 \leq \epsilon \leq 1$, endowed with the norms $\|f\|_{C_\delta^n} = \sum_{k=0}^n \|\delta^k f\|_C$ and $\|f\|_{C_{\delta, \epsilon}^n} = \sum_{k=0}^{n-1} \|\delta^k f\|_C + \|\delta^n f\|_{C_{\epsilon, \rho}}$, respectively.

Next, we recall some basic properties of generalized fractional integral and derivative [39].

Lemma 1. Let $\Re(\alpha) \geq 0, n = [\Re(\alpha) + 1]$, and $f \in AC_\delta^n[a, b]$, with $0 < a < b < \infty$. For $k, m \in \mathbb{N}$,

$$\begin{aligned} ({}_a I^{\alpha, \rho})^k ({}_a^C D^{\alpha, \rho})^m f(t) \\ = \frac{({}_a^C D^{\alpha, \rho})^m f(\tau) \left(\frac{t^\rho - a^\rho}{\rho} \right)^{k\alpha}}{\Gamma(k\alpha + 1)}, \quad \tau \in (a, t). \end{aligned} \quad (13)$$

Theorem 1. Let $\Re(\alpha) \geq 0, n = [\Re(\alpha) + 1]$, and $f \in AC_\delta^n[a, b]$, where $0 < a < b < \infty$. Then,

- (i) If $\alpha \notin \mathbb{N}_0$, $({}_a^C D^{\alpha, \rho} f)(x) = {}_a I^{n - \alpha, \rho} (\delta^n f)(x)$
- (ii) If $\alpha \in \mathbb{N}$, $({}_a^C D^{\alpha, \rho} f)(x) = (\delta^n f)(x)$
- (iii) ${}_a^C D^{0, \rho} f = f$

Theorem 2. Let $f(x) \in C_\delta^n[a, b]$, $0 < a < b < \infty$, and $\alpha, \beta \in \mathbb{C}$ such that $\Re(\alpha) \geq 0$ and $\Re(\beta) \geq 0$. Then,

$${}_a^C D^{\alpha, \rho} {}_a I^{\beta, \rho} f(x) = {}_a I^{\beta - \alpha, \rho} f(x). \quad (14)$$

Our main results are based on utilizing the following fixed point theorems.

Theorem 3 (Schaefer's fixed point theorem [40]). Let X be a Banach space. Assume that $\mathcal{L}: X \longrightarrow X$ is a completely continuous operator and the set $T = \{u \in X: u = \lambda \mathcal{L}u, 0 < \lambda < 1\}$ is bounded. Then, $\mathcal{L}$ has a fixed point in X .

Theorem 4 (Krasnoselskii's fixed point theorem [40]). Let $\mathcal{N}$ be a closed, convex, bounded, and nonempty subset of a Banach space X . Let T_1 and T_2 be operators such that

- (i) $T_1(u_1) + T_2(u_2)$ belong to $\mathcal{N}$ whenever $u_1, u_2 \in \mathcal{N}$

where $n = [\alpha] + 1$ and $AC_\delta^n[a, b]$ denotes the space of all absolutely continuous real valued functions on $[a, b]$, where

(ii) T_1 is a compact and continuous and T_2 is a contraction mapping

Then, there exists $u_0 \in \mathcal{N}$ such that $u_0 = T_1(u_0) + T_2(u_0)$.

For computational convenience, we introduce the following notations:

$$\psi = \frac{1}{\rho^3(3\rho + 1)} - \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\beta+3)} \Gamma(4)}{\rho^{\beta+3} \Gamma(\beta + 4)}. \quad (15)$$

Lemma 2. Let $h \in L_F([0, 1])$ and ψ be given by (15). Then, the solution of the GFBVP

$$\begin{cases} {}^C D_{0^+}^{\alpha, \rho} u(t) = h(t), t \in [0, 1], \quad \rho > 0, \\ u(0) = \delta u(0) = \delta^2 u(0) = 0, \\ \int_0^1 u(s) ds = \sum_{i=1}^m \delta_i I^{\beta, \rho} u(\xi_i), \quad \xi_i \in (0, 1), \delta = t^{1-\rho} \frac{d}{dt}, \end{cases} \quad (16)$$

is given by

$$u(t) = I^{\alpha, \rho} h(t) + \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i I^{\alpha+\beta, \rho} h(\xi_i) - \int_0^1 I^{\alpha, \rho} h(s) ds \right]. \quad (17)$$

Proof. By using Lemma 1, we obtain

$$u(t) = I^{\alpha, \rho} h(t) + C_0 + C_1 \frac{t^\rho}{\rho} + C_2 \frac{t^{2\rho}}{\rho^2} + C_3 \frac{t^{3\rho}}{\rho^3}, \quad (18)$$

where $C_i \in \mathbb{R}$, $(i = 0, 1, 2, 3)$ are the arbitrary constants. Using the boundary conditions $u(0) = \delta u(0) = \delta^2 u(0) = 0$, we get $C_0 = C_1 = C_2 = 0$. Then, from $\int_0^1 u(s) ds = \sum_{i=1}^m \delta_i I^{\beta, \rho} u(\xi_i)$, it follows that

$$C_3 = \frac{1}{\psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} h(s) ds \right. \\ \left. - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\int_0^s (s^\rho - \tau^\rho)^\alpha \tau^{\rho-1} h(\tau) d\tau \right) ds \right]. \quad (19)$$

Substituting the values of C_0, C_1, C_2 , and C_3 in (18), we get (17). $\square$

3. Main Results

In this section, we prove the existence and uniqueness of solutions of GFBVP (1). We assume that f belongs to an

Orlicz space $L_E[0, 1]$. For $3 < \alpha \leq 4$, let $X = \{u: u, {}^C D_{0+}^{\alpha-1, \rho} u, {}^C D_{0+}^{\alpha-2, \rho} u, {}^C D_{0+}^{\alpha-3, \rho} u \in C([0, 1], \mathbb{R})\}$ denote the Banach space of all continuous functions defined on $[0, 1]$ into $\mathbb{R}$ endowed with the norm

$$\|u\| = \sup \left\{ |u(t)| + \left| {}^C D_{0+}^{\alpha-1, \rho} u(t) \right| + \left| {}^C D_{0+}^{\alpha-2, \rho} u(t) \right| + \left| {}^C D_{0+}^{\alpha-3, \rho} u(t) \right|, \quad t \in [0, 1] \right\}. \quad (20)$$

Relative to GFBVP (1), in view of Lemma 2, we define an operator $\mathcal{L}: X \rightarrow X$ as

$$\mathcal{L}u(t) = \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \\ + \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \right. \\ \left. - \int_0^1 t^{\alpha\rho} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \right], \quad t \in [0, 1]. \quad (21)$$

Notice that GFBVP (1) has solutions if and only if the operator $\mathcal{L}$ has fixed points.

The following result plays a major role in our analysis.

Lemma 3. *Let F be a Young function which has a Young complement F^* satisfying*

$$\int_0^t F^*(s^{\alpha-1}) ds < \infty, \\ \int_0^t F^*(s^{\alpha+\beta-1}) ds < \infty, \quad (22) \\ t > 0,$$

for $\alpha \in (3, 4]$ and $\beta \in (0, 1)$. Then, the operator $\mathcal{L}$ exists and is well defined.

Proof. Let $u \in X$. Define a function as follows:

$$\varrho_1(s) = \begin{cases} (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1}, & \text{if } s \in [0, t], t > 0, \\ 0, & \text{otherwise.} \end{cases} \quad (23)$$

At the beginning, we will show that $\varrho_1 \in L_{F^*}[0, 1]$. By using appropriate substitution and properties of the Young functions, one obtains

$$\int_0^1 F^*\left(\frac{|\varrho_1(s)|}{\kappa}\right) ds = \int_0^t F^*\left(\frac{(t^\rho - s^\rho)^{\alpha-1}}{\kappa}\right) s^{\rho-1} ds \\ = \frac{1}{\rho} \left(\frac{1}{\kappa}\right)^{1/(1-\alpha)} \int_0^{\kappa^{1/(1-\alpha)} t^\rho} F^*(s^{\alpha-1}) ds, \quad (24)$$

by the assumption of the theorem, and we get $\varrho_1 \in L_{F^*}[0, 1]$. Similarly, setting

$$\varrho_2(s) = \begin{cases} (t^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1}, & \text{if } s \in [0, t], t > 0, \\ 0, & \text{otherwise,} \end{cases} \quad (25)$$

one can get $\varrho_2 \in L_{F^*}[0, 1]$. Next, we show that $\mathcal{L}$ is well defined, i.e., $\mathcal{L}u(t) \in C([0, 1], \mathbb{R})$. Let $0 \leq \tau < t \leq 1$. Thus, we have

$$\begin{aligned}
|(\mathcal{L}u)(t) - (\mathcal{L}u)(\tau)| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left[\int_0^\tau \left| (t^\rho - s^\rho)^{\alpha-1} - (\tau^\rho - s^\rho)^{\alpha-1} \right| s^{\rho-1} \right. \\
&\quad \cdot \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\quad + \int_\tau^t \left| (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right| \left\| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right\| ds \Big] \\
&\quad + \frac{(t^{3\rho} - \tau^{3\rho})}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} \left| (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right| \right. \\
&\quad \cdot \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\int_0^s \left| (s^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \right| \right. \\
&\quad \cdot \left| f\left(\tau, u(\tau), {}^C D_{0+}^{\alpha-1, \rho} u(\tau), {}^C D_{0+}^{\alpha-2, \rho} u(\tau), {}^C D_{0+}^{\alpha-3, \rho} u(\tau)\right) \right| d\tau \Big) ds \Big] \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \left[\int_0^1 [\mathcal{G}_1(s) + \mathcal{G}_2(s)] \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \right. \\
&\quad + \frac{(t^{3\rho} - \tau^{3\rho})}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} \left| (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right| \right. \\
&\quad \cdot \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\quad - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\int_0^s \left| (s^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \right| \right. \\
&\quad \cdot \left| f\left(\tau, u(\tau), {}^C D_{0+}^{\alpha-1, \rho} u(\tau), {}^C D_{0+}^{\alpha-2, \rho} u(\tau), {}^C D_{0+}^{\alpha-3, \rho} u(\tau)\right) \right| d\tau \Big) ds \Big],
\end{aligned} \tag{26}$$

where

$$\mathcal{G}_1(s) = \begin{cases} \left| (t^\rho - s^\rho)^{\alpha-1} - (\tau^\rho - s^\rho)^{\alpha-1} \right| s^{\rho-1}, & \text{if } s \in [0, \tau], \\ 0, & \text{otherwise,} \end{cases} \tag{27}$$

$$\mathcal{G}_2(s) = \begin{cases} \left| (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right|, & \text{if } s \in [\tau, t], \\ 0, & \text{otherwise.} \end{cases} \tag{28}$$

The functions $\mathcal{G}_i$, $i = 1, 2$, belong to $L_{F^*}[0, 1]$ with $\|\mathcal{G}_i\|_{F^*} \leq g(|t^\rho - \tau^\rho|)$, $i = 1, 2$, where $g: \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is a continuous and increasing function with $g(0) = 0$. Using the Hölder inequality, we have

$$\begin{aligned}
|(\mathcal{L}u)(t) - (\mathcal{L}u)(\tau)| &\leq \|f\|_F \left\{ \frac{2}{\rho^{\alpha-1} \Gamma(\alpha)} [\|\mathcal{G}_1\|_{F^*} + \|\mathcal{G}_2\|_{F^*}] \right. \\
&\quad \left. + \frac{|t^{3\rho} - \tau^{3\rho}|}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\}.
\end{aligned} \tag{29}$$

Then, $0 < |t^\rho - \tau^\rho| < \epsilon$ and by the continuity of g , we see that $\mathcal{L}u$ is continuous, which completes the proof.

Our first existence result relies on Schaefer's fixed point theorem. $\square$

Theorem 5. Assume that there exists $\mathcal{M} \in C([0, 1], \mathbb{R})$ such that

$$\begin{aligned} (H_1) \left| f\left(t, u(t), {}^C D_{0^+}^{\alpha-1, \rho} u(t), {}^C D_{0^+}^{\alpha-2, \rho} u(t), {}^C D_{0^+}^{\alpha-3, \rho} u(t)\right) \right| \\ \leq \mathcal{M}(t), \quad \text{for } t \in [0, 1], \end{aligned} \quad (30)$$

with $\|\mathcal{M}\| = \max_{t \in [0, 1]} |\mathcal{M}(t)|$. Then, the GFBVP (1) has at least one solution on $[0, 1]$.

Proof. The proof proceeds in few steps as follows:

Step 1: we will prove the operator $\mathcal{L}$ maps bounded sets into bounded sets in $C([0, 1], \mathbb{R})$. Let $E \subset X$ be a bounded set. Then, for all $u \in E$, by using the assumption (H_1) , for $t \in [0, 1]$, we are able to obtain

$$\begin{aligned} |\mathcal{L}u(t)| &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\ &\quad + \frac{|t^{3\rho}|}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\ &\quad \cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\ &\quad \left. - \int_0^1 I^{\alpha, \rho} \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \right] \\ &\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} |\mathcal{M}(s)| ds \\ &\quad + \frac{1}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} |\mathcal{M}(s)| ds \right. \\ &\quad \left. - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\int_0^s (s^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} |\mathcal{M}(\tau)| d\tau \right) ds \right] \\ &\leq \frac{\rho^{\alpha-1} \|\mathcal{M}\|}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} ds \\ &\quad + \frac{1}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i (\alpha + \beta) \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} ds \right. \\ &\quad \left. - \frac{\|\mathcal{M}\|}{\rho^\alpha \Gamma(\alpha+1) (\rho\alpha+1)} \right]. \end{aligned} \quad (31)$$

Thus, we obtain

$$\begin{aligned} \|\mathcal{L}u\| &\leq \|\mathcal{M}\| \left\{ \frac{1}{\rho^\alpha \Gamma(\alpha+1)} + \frac{1}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1) (\rho\alpha+1)} \right] \right\} \\ &= \mu_1. \end{aligned} \quad (32)$$

Now, with some efforts of computation, we have

$$\begin{aligned}
& \left({}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}u \right)(t) = \left(I^{1, \rho} f \right)(t) \\
& + {}^C D_{0^+}^{\alpha-1, \rho} \left\{ \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \right. \\
& \cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
& \left. \left. - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \right] \right\} \\
& = \int_0^t s^{\rho-1} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
& + \frac{6t^{\rho(4-\alpha)}}{\rho^{4-\alpha} \psi \Gamma(5-\alpha)} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
& \cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
& \left. - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \right] \\
& \leq \int_0^t s^{\rho-1} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
& + \frac{6t^{\rho(4-\alpha)}}{\rho^{4-\alpha} \psi \Gamma(5-\alpha)} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
& \cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
& \left. - \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^1 \left(\int_0^s (s^\rho - \tau^\rho)^{\alpha-1} \tau^{\rho-1} \right. \right. \\
& \cdot \left. \left. \left| f\left(\tau, u(\tau), {}^C D_{0^+}^{\alpha-1, \rho} u(\tau), {}^C D_{0^+}^{\alpha-2, \rho} u(\tau), {}^C D_{0^+}^{\alpha-3, \rho} u(\tau)\right) \tau^{\rho-1} \right| d\tau \right) ds \right].
\end{aligned} \tag{33}$$

Therefore, we have

$$\begin{aligned}
\|{}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}u\| &\leq \|\mathcal{M}\| \int_0^t s^{\rho-1} ds \\
&+ \frac{6t^{\rho(4-\alpha)}}{|\psi|\rho^{4-\alpha}\Gamma(5-\alpha)} \left[\sum_{i=1}^m \delta_i \frac{\|\mathcal{M}\|\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \right. \\
&\cdot \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} ds - \frac{\|\mathcal{M}\|}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \Big] \\
&\leq \frac{\|\mathcal{M}\|t^\rho}{\rho} + \frac{6t^{\rho(4-\alpha)}}{|\psi|\rho^{4-\alpha}\Gamma(5-\alpha)} \left[\sum_{i=1}^m \delta_i \frac{\|\mathcal{M}\|\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right. \\
&\quad \left. - \frac{\|\mathcal{M}\|}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \\
&\leq \|\mathcal{M}\| \left\{ \frac{1}{\rho} + \frac{6}{|\psi|\rho^{4-\alpha}\Gamma(5-\alpha)} \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right. \right. \\
&\quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\} = \mu_2.
\end{aligned} \tag{34}$$

In a similar way, we arrive at

$$\begin{aligned}
\|{}^C D_{0^+}^{\alpha-2} \mathcal{L}u\| &\leq \|\mathcal{M}\| \left\{ \frac{1}{\rho^2 \Gamma(3)} + \frac{6}{|\psi|\rho^{5-\alpha}\Gamma(6-\alpha)} \right. \\
&\cdot \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right. \\
&\quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\} \\
&= \mu_3.
\end{aligned} \tag{35}$$

$$\begin{aligned}
\|{}^C D_{0^+}^{\alpha-3} \mathcal{L}u\| &\leq \|\mathcal{M}\| \left\{ \frac{1}{\rho^3 \Gamma(4)} + \frac{6}{|\psi|\rho^{6-\alpha}\Gamma(7-\alpha)} \right. \\
&\cdot \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right. \\
&\quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\} \\
&= \mu_4.
\end{aligned} \tag{36}$$

Step 2: from (32)–(36), we can immediately get the operator $\mathcal{L}: X \rightarrow X$ to map bounded sets into bounded sets. Let $0 < t_1 < t_2 < 1$, and for all $u \in E$, we get

$$\begin{aligned}
|(\mathcal{L}u)(t_2) - (\mathcal{L}u)(t_1)| &= \left| \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_2} (t_2^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right. \\
&\cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&+ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} (t_1^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \\
&\cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&+ \frac{t_2^{3\rho} - t_1^{3\rho}}{\psi \rho^3} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
&\cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&\left. - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \right] \Big| \\
&\leq \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^{t_1} [(t_2^\rho - s^\rho)^{\alpha-1} - (t_1^\rho - s^\rho)^{\alpha-1}] s^{\rho-1} \\
&\cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&+ \int_{t_1}^{t_2} (t_2^\rho - s^\rho)^{\alpha-1} \\
&\cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&+ \frac{(t_2^{3\rho} - t_1^{3\rho})}{|\psi|\rho^3} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
&\cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\left. - \int_0^1 I^{\alpha, \rho} \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \right].
\end{aligned} \tag{37}$$

Thus, we have

$$\begin{aligned}
|(\mathcal{L}u)(t_2) - (\mathcal{L}u)(t_1)| &\leq \|\mathcal{M}\| \left\{ \frac{1}{\rho^\alpha \Gamma(\alpha+1)} [2(t_2^\rho - t_1^\rho)^\alpha - (t_2^{\alpha\rho} - t_1^{\alpha\rho})] \right. \\
&+ \frac{(t_2^{3\rho} - t_1^{3\rho})}{|\psi|\rho^3} \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} \right. \\
&\quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\}.
\end{aligned} \tag{38}$$

Similarly, it can be easily shown that

$$\left| \left({}^C D_{0+}^{\alpha-1,\rho} \mathcal{L}u \right)(t_2) - \left({}^C D_{0+}^{\alpha-1,\rho} \mathcal{L}u \right)(t_1) \right| \leq \| \mathcal{M} \| \left\{ \frac{(t_2^\rho - t_1^\rho)}{\rho} + \frac{6(t_2^{\rho(4-\alpha)} - t_1^{\rho(4-\alpha)})}{|\psi|\rho^{4-\alpha}\Gamma(5-\alpha)} \right. \\ \left. \cdot \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha\Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\}. \quad (39)$$

$$\left| \left({}^C D_{0+}^{\alpha-2,\rho} \mathcal{L}u \right)(t_2) - \left({}^C D_{0+}^{\alpha-2,\rho} \mathcal{L}u \right)(t_1) \right| \leq \| \mathcal{M} \| \left\{ \frac{(t_2^{2\rho} - t_1^{2\rho})}{\rho^2\Gamma(3)} + \frac{6(t_2^{\rho(5-\alpha)} - t_1^{\rho(5-\alpha)})}{|\psi|\rho^{5-\alpha}\Gamma(6-\alpha)} \right. \\ \left. \cdot \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha\Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\}. \quad (40)$$

$$\left| \left({}^C D_{0+}^{\alpha-3,\rho} \mathcal{L}u \right)(t_2) - \left({}^C D_{0+}^{\alpha-3,\rho} \mathcal{L}u \right)(t_1) \right| \leq \| \mathcal{M} \| \left\{ \frac{(t_2^{3\rho} - t_1^{3\rho})}{\rho^3\Gamma(4)} + \frac{6(t_2^{\rho(6-\alpha)} - t_1^{\rho(6-\alpha)})}{|\psi|\rho^{6-\alpha}\Gamma(7-\alpha)} \right. \\ \left. \cdot \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha\Gamma(\alpha+1)(\rho\alpha+1)} \right] \right\}. \quad (41)$$

The right side of (38)–(41) goes to zero as $(t_2 - t_1) \rightarrow 0$. In view of steps I and II, it follows that by Arzela–Ascoli theorem, the sets $\{\mathcal{L}(u): u \in E\}$, $\{{}^C D_{0+}^{\alpha-1,\rho} \mathcal{L}u: u \in E\}$, $\{{}^C D_{0+}^{\alpha-2,\rho} \mathcal{L}u: u \in E\}$, and $\{{}^C D_{0+}^{\alpha-3,\rho} \mathcal{L}u: u \in E\}$ are relatively compact in $C([0, 1])$. Therefore, $\mathcal{L}(E)$ is a relatively compact set in X . We consider the set $T = \{u \in X; u = \lambda \mathcal{L}u, 0 < \lambda < 1\}$. Then, T is bounded. Indeed, let $u \in T$. So, $u = \lambda \mathcal{L}u$, $0 < \lambda < 1$, for any $t \in [0, 1]$, and it follows that

$$\|u\| \leq \frac{\| \mathcal{M} \|}{\rho^\alpha\Gamma(\alpha+1)} + \frac{1}{\rho^3|\psi|} \left[\sum_{i=1}^m \delta_i \frac{\| \mathcal{M} \| \xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} - \frac{\| \mathcal{M} \|}{\rho^\alpha\Gamma(\alpha+1)(\rho\alpha+1)} \right]. \quad (42)$$

Thus, all the hypotheses of Theorem 3 are satisfied. Therefore, we can conclude that the operator $\mathcal{L}$ has at least

one fixed point. Hence, the GFBVP (1) has at least one solution on $[0, 1]$.

For computation convenience, we set

$$V = \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}\Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha\Gamma(\alpha+1)(\rho\alpha+1)}, \quad (43)$$

$$K = \frac{6}{\rho^{4-\alpha}\psi\Gamma(5-\alpha)}, \quad (44)$$

$$\sigma = \frac{1}{\rho}. \quad (45)$$

Now, we make use of Theorem 4 to prove the existence of solutions of GFBVP (1). $\square$

Theorem 6. Let $f: [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ be a continuous function such that the following assumptions hold:

$$(H_2) |f(t, u, p, q, w) - f(t, v, \tilde{p}, \tilde{q}, \tilde{w})| \leq L(|u - v| + |p - \tilde{p}| + |q - \tilde{q}| + |w - \tilde{w}|),$$

$$\text{for all } t \in [0, 1], u, v, p, q, w, \tilde{p}, \tilde{q}, \tilde{w} \in \mathbb{R} \text{ and } L > 0,$$

$$(H_3) \left| f(t, u(t), {}^C D_{0+}^{\alpha-1,\rho} u(t), {}^C D_{0+}^{\alpha-2,\rho} u(t), {}^C D_{0+}^{\alpha-3,\rho} u(t)) \right| \leq \phi(t), \quad (46)$$

$$\text{for } t \in [0, 1] \text{ and } \phi \in C([0, 1], \mathbb{R}) \text{ with } \|\phi\| = \max_{t \in [0, 1]} |\phi(t)|.$$

Then, GFBVP (1) has at least one solution on $[0, 1]$, provided that

$$LKV < 1. \quad (47)$$

Proof. We define $B_r = \{u \in X: \|u\| \leq r\}$, where $r \geq \|\phi\|(\sigma + KV)$. We split the operator $\mathcal{L}$ defined by (21) on B_r as $\mathcal{L} = \mathcal{L}_1 + \mathcal{L}_2$, where $\mathcal{L}_1$ and $\mathcal{L}_2$ are given by

$$\begin{aligned}
(\mathcal{L}_1 u)(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds. \\
(\mathcal{L}_2 u)(t) &= \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
&\quad \cdot f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \\
&\quad \left. - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \right].
\end{aligned} \tag{48}$$

For $u, v \in B_r$, we find that

$$\begin{aligned}
\|\mathcal{L}_1(u) + \mathcal{L}_2(u)\| &\leq \max_{t \in [0,1]} \left\{ \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \right. \\
&\quad \cdot \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\quad + \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
&\quad \cdot \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
&\quad \left. - \int_0^1 I^{\alpha, \rho} \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \right] \Big\} \\
&\leq \|\phi\| \left[\frac{1}{\rho^\alpha \Gamma(\alpha+1)} + \frac{1}{\rho^3 |\psi|} \left(\sum_{i=1}^m \delta_i \frac{\xi_i^\rho (\alpha+\beta)}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right. \right. \\
&\quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1) (\rho\alpha+1)} \right) \right] \\
&\leq \|\phi\| (\sigma + KV) \leq r.
\end{aligned} \tag{49}$$

Furthermore,

$$\begin{aligned}
\left\| \left({}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}_1 \right) (u) + \left({}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}_2 \right) (u) \right\| &\leq \|\phi\| \left[\frac{1}{\rho} + \frac{6}{|\psi| \rho^{4-\alpha} \Gamma(5-\alpha)} \right. \\
&\quad \cdot \left. \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right] \\
&\leq \|\phi\| (\sigma + KV) \leq r, \\
\left\| \left({}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}_1 \right) (u) + \left({}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}_2 \right) (u) \right\| &\leq \|\phi\| \left[\frac{1}{\rho^2 \Gamma(3)} + \frac{6}{|\psi| \rho^{5-\alpha} \Gamma(6-\alpha)} \right. \\
&\quad \cdot \left. \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right] \quad (50) \\
&\leq \|\phi\| (\sigma + KV) \leq r, \\
\left\| \left({}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}_1 \right) (u) + \left({}^C D_{0^+}^{\alpha-3, \rho} T_3 \right) (u) \right\| &\leq \|\phi\| \left[\frac{1}{\rho^3 \Gamma(4)} + \frac{6}{\|\psi\| \rho^{6-\alpha} \Gamma(7-\alpha)} \right. \\
&\quad \cdot \left. \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right] \\
&\leq \|\phi\| (\sigma + KV) \leq r.
\end{aligned}$$

Consequently, we obtain

$$\|\mathcal{L}_1(u) + \mathcal{L}_2(u)\|_X \leq \|\phi\| (\sigma + KV) \leq r, \quad (51)$$

which shows that $\mathcal{L}_1(u) + \mathcal{L}_2(u) \in B_r$. In what follows, we prove that $\mathcal{L}_2$ is a contraction. Let $u, v \in B_r$ and for all $t \in [0, 1]$, we get

$$\begin{aligned}
\|\mathcal{L}_2(u) - \mathcal{L}_2(v)\| &= \sup_{t \in [0, 1]} |\mathcal{L}_2(u) - \mathcal{L}_2(v)| \\
&= \sup_{t \in [0, 1]} \left| \frac{t^3 \rho}{\rho^3 \psi} \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}} \int_0^{\xi_i} (\xi_i^p - s^p)^{\alpha+\beta+1} s^{p-1} \right. \\
&\quad \cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&\quad - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&\quad - \frac{t^3 \rho}{\rho^3 \psi} \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}} \int_0^{\xi_i} (\xi_i^p - s^p)^{\alpha+\beta+1} s^{p-1} \\
&\quad \cdot f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \\
&\quad \left. + \int_0^1 I^{\alpha, \rho} f\left(s, v(s), {}^C D_{0^+}^{\alpha-1, \rho} v(s), {}^C D_{0^+}^{\alpha-2, \rho} v(s), {}^C D_{0^+}^{\alpha-3, \rho} v(s)\right) ds \right| \\
&= \sup_{t \in [0, 1]} \left| \frac{t^3 \rho}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta}} \int_0^{\xi_i} (\xi_i^p - s^p)^{\alpha+\beta+1} s^{p-1} \right. \right.
\end{aligned}$$

$$\begin{aligned}
& \cdot \left(f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right. \right. \\
& \left. \left. - f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right) ds \\
& - \int_0^1 I^{\alpha, \rho} \left(f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right. \\
& \left. \left. - f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right) ds \right] \\
& \leq \sup_{t \in [0,1]} \left[L \left(|u(t) - v(t)| + \left| {}^C D_{0^+}^{\alpha-1, \rho} u(t) - {}^C D_{0^+}^{\alpha-1, \rho} v(t) \right| \right. \right. \\
& \quad \left. \left. + \left| {}^C D_{0^+}^{\alpha-2, \rho} u(t) - {}^C D_{0^+}^{\alpha-2, \rho} v(t) \right| + \left| {}^C D_{0^+}^{\alpha-3, \rho} u(t) - {}^C D_{0^+}^{\alpha-3, \rho} v(t) \right| \right) \right. \\
& \quad \cdot \left(\frac{t^3 \rho}{\rho^3 \psi} \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta+1} s^{\rho-1} ds - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \\
& \leq L \left[\frac{1}{\rho^3 \psi} \sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right] \|u - v\| \\
& \leq LKV \|u - v\|. \tag{52}
\end{aligned}$$

With the same process, one has

$$\begin{aligned}
& \left\| {}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}_2(u) - {}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}_2(v) \right\| \leq LKV \|u - v\|, \\
& \left\| {}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}_2(u) - {}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}_2(v) \right\| \leq LKV \|u - v\|, \tag{53} \\
& \left\| {}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}_2(u) - {}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}_2(v) \right\| \leq LKV \|u - v\|.
\end{aligned}$$

Hence, we get

$$\left\| \mathcal{L}_2(u) - \mathcal{L}_2(v) \right\|_X \leq LKV \|u - v\|_X. \tag{54}$$

It remains to show that $\mathcal{L}_1$ is continuous and compact. The continuity of the function f implies that the operator $\mathcal{L}_1$ is continuous. To achieve the compactness of the operator $\mathcal{L}_1$, we first prove that $\mathcal{L}_1$ is uniformly bounded on B_r as follows:

$$\begin{aligned}
\left\| \mathcal{L}_1(u) \right\| &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \sup_{t \in [0,1]} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \\
& \quad \cdot \left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right| ds \\
& \leq \frac{\|\phi\|}{\rho^\alpha \Gamma(\alpha+1)} \leq \|\phi\| \sigma, \\
\left\| ({}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}_1)(u) \right\| &= \sup_{t \in [0,1]} \left(\frac{\|\phi\| t^\rho}{\rho} \right) \leq \frac{\|\phi\|}{\rho} \leq \|\phi\| \sigma, \\
\left\| ({}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}_1)(u) \right\| &= \sup_{t \in [0,1]} \left(\frac{\|\phi\| t^{2\rho}}{\rho^2 \Gamma(3)} \right) \leq \frac{\|\phi\|}{\rho^2 \Gamma(3)} \leq \|\phi\| \sigma, \\
\left\| ({}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}_1)(u) \right\| &= \sup_{t \in [0,1]} \left(\frac{\|\phi\| t^{3\rho}}{\rho^3 \Gamma(4)} \right) \leq \frac{\|\phi\|}{\rho^3 \Gamma(4)} \leq \|\phi\| \sigma. \tag{55}
\end{aligned}$$

So,

$$\|\mathcal{L}_1(u)\|_X \leq \|\phi\|\sigma. \quad (56)$$

Now, let $\sup_{t \in [0,1] \times B_r^4} |f(t, u, p, q, w)| = \eta$. Consequently, for $t_1, t_2 \in [0, 1]$, $t_1 < t_2$, we have

$$|(\mathcal{L}_1 u)(t_2) - (\mathcal{L}_1 u)(t_1)| \leq \frac{\eta}{\rho} [2(t_2^\rho - t_1^\rho)^\alpha - (t_2^{\alpha\rho} - t_1^{\alpha\rho})]. \quad (57)$$

Furthermore,

$$\begin{aligned} & \left| ({}^C D_{0+}^{\alpha-1, \rho} \mathcal{L}_1 u)(t_2) - ({}^C D_{0+}^{\alpha-1, \rho} \mathcal{L}_1 u)(t_1) \right| \leq \frac{\eta}{\rho} |t_2^\rho - t_1^\rho|, \\ & \left| ({}^C D_{0+}^{\alpha-2, \rho} \mathcal{L}_1 u)(t_2) - ({}^C D_{0+}^{\alpha-2, \rho} \mathcal{L}_1 u)(t_1) \right| \leq \frac{\eta}{\rho} |t_2^{2\rho} - t_1^{2\rho}|, \\ & \left| ({}^C D_{0+}^{\alpha-3, \rho} \mathcal{L}_1 u)(t_2) - ({}^C D_{0+}^{\alpha-3, \rho} \mathcal{L}_1 u)(t_1) \right| \leq \frac{\eta}{\rho} |t_2^{3\rho} - t_1^{3\rho}|, \end{aligned} \quad (58)$$

Therefore, as $(t_2 - t_1) \rightarrow 0$, the right-hand sides of the above inequalities tend to zero independently of $u \in B_r$. Thus, $\mathcal{L}_1$ is equicontinuous and so it is relatively compact on B_r . According to the Arzela–Ascoli theorem, the operator $\mathcal{L}_1$ is compact. By using Theorem 4, there exists at least one solution of GFBVP (1) on $[0, 1]$.

In the next result, we prove the existence of solutions for GFBVP (1) by applying Banach's fixed point theorem. $\square$

Theorem 7. Assume that the condition (H_2) holds. Then, GFBVP (1) has a unique solution on $[0, 1]$, provided that

$$LA < 1, \quad (59)$$

where $A = \sigma + KV$ and V, K , and σ are given by (43)–(45).

Proof. Consider the operator $\mathcal{L}: X \rightarrow X$ defined by

$$\begin{aligned} (\mathcal{L}u)(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \\ &+ \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\ &\cdot f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) ds \\ &\left. - \int_0^1 I^{\alpha, \rho} \left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s) \right) ds \right]. \end{aligned} \quad (60)$$

Set

$$\begin{aligned} \sup_{t \in [0,1]} |f(t, 0, 0, 0, 0)| &= f_0, \\ r &\geq \frac{f_0 A}{1 - LA}. \end{aligned} \quad (61)$$

We shall show that $\mathcal{L}B_r \subset B_r$, where $B_r = \{u \in C([0, 1], \mathbb{R}) : \|u\| \leq r\}$. For $u \in B_r$, we have

$$\begin{aligned} |\mathcal{L}u(t)| &\leq I^{\alpha, \rho} \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| \\ &+ \frac{|t^{3\rho}|}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i I^{\alpha+\beta, \rho} \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| \right. \\ &\left. - \int_0^1 I^{\alpha, \rho} \left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) \right| ds \right] \\ &\leq I^{\alpha, \rho} \left[\left| f\left(s, u(s), {}^C D_{0+}^{\alpha-1, \rho} u(s), {}^C D_{0+}^{\alpha-2, \rho} u(s), {}^C D_{0+}^{\alpha-3, \rho} u(s)\right) - f(s, 0, 0, 0, 0) \right| \right] \end{aligned}$$

$$\begin{aligned}
& + [|f(s, 0, 0, 0, 0)|] + \frac{t^{3\rho}}{\rho^3|\psi|} \left\{ \sum_{i=1}^m \delta_i I^{\alpha+\beta, \rho} \right. \\
& \cdot \left[\left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) - f(s, 0, 0, 0, 0) \right| \right. \\
& + [|f(s, 0, 0, 0, 0)|] \\
& - \int_0^1 I^{\alpha, \rho} \left[\left| f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) - f(s, 0, 0, 0, 0) \right| \right. \\
& \left. \left. + \{ [|f(s, 0, 0, 0, 0)|] ds \} \right] \right\}. \tag{62}
\end{aligned}$$

Therefore,

$$\begin{aligned}
|\mathcal{L}u(t)| & \leq \sup_{t \in [0, 1]} \left[L \left(|u(t)| + \left| {}^C D_{0^+}^{\alpha-1, \rho} u(t) \right| + \left| {}^C D_{0^+}^{\alpha-2, \rho} u(t) \right| + \left| {}^C D_{0^+}^{\alpha-3, \rho} u(t) \right| \right) + f_0 \right] \\
& \cdot \left[\frac{t^{\alpha\rho}}{\rho^\alpha \Gamma(\alpha+1)} + \frac{t^{3\rho}}{\rho^3 |\psi|} \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right] \\
& \leq (L\|u\| + f_0) \left[\frac{1}{\rho^\alpha \Gamma(\alpha+1)} + \frac{1}{\rho^3 |\psi|} \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right. \right. \\
& \quad \left. \left. \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right] \\
& \leq (L\|u\| + f_0)A \leq (Lr + f_0)A \leq r. \tag{63}
\end{aligned}$$

Similarly,

$$\begin{aligned}
\left| \left({}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}u \right)(t) \right| & \leq (L\|u\| + f_0)A \leq (Lr + f_0)A \leq r, \\
\left| \left({}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}u \right)(t) \right| & \leq (L\|u\| + f_0)A \leq (Lr + f_0)A \leq r, \\
\left| \left({}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}u \right)(t) \right| & \leq (L\|u\| + f_0)A \leq (Lr + f_0)A \leq r, \tag{64}
\end{aligned}$$

By taking the norm from $t \in [0, 1]$, it yields $\|\mathcal{L}u\| \leq r$, $\|{}^C D_{0^+}^{\alpha-1, \rho} \mathcal{L}u\| \leq r$, $\|{}^C D_{0^+}^{\alpha-2, \rho} \mathcal{L}u\| \leq r$, and $\|{}^C D_{0^+}^{\alpha-3, \rho} \mathcal{L}u\| \leq r$. Consequently, we have

$$\|\mathcal{L}u\|_X \leq (Lr + f_0)A \leq r, \tag{65}$$

which shows that $\mathcal{L}$ maps B_r into itself. In order to show that the operator $\mathcal{L}$ is a contraction, let $u, v \in C([0, 1], \mathbb{R})$. Then, for each $t \in [0, 1]$, we obtain

$$\begin{aligned}
\|\mathcal{L}u - \mathcal{L}v\| & = \sup_{t \in [0, 1]} |\mathcal{L}u(t) - \mathcal{L}v(t)| \\
& = \sup_{t \in [0, 1]} \left| I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right. \\
& \quad + \frac{t^{3\rho}}{\rho^3 |\psi|} \left[\sum_{i=1}^m \delta_i I^{\alpha+\beta, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) \right. \\
& \quad \left. \left. - \int_0^1 I^{\alpha, \rho} f\left(s, u(s), {}^C D_{0^+}^{\alpha-1, \rho} u(s), {}^C D_{0^+}^{\alpha-2, \rho} u(s), {}^C D_{0^+}^{\alpha-3, \rho} u(s)\right) ds \right] \right|
\end{aligned}$$

$$\begin{aligned}
& -I^{\alpha,\rho} f\left(s, v(s), {}^C D_{0^+}^{\alpha-1,\rho} v(s), {}^C D_{0^+}^{\alpha-2,\rho} v(s), {}^C D_{0^+}^{\alpha-3,\rho} v(s)\right) \\
& -\frac{t^{3\rho}}{\rho^3|\psi|} \left[\sum_{i=1}^m \delta_i I^{\alpha+\beta,\rho} f\left(s, v(s), {}^C D_{0^+}^{\alpha-1,\rho} v(s), {}^C D_{0^+}^{\alpha-2,\rho} v(s), {}^C D_{0^+}^{\alpha-3,\rho} v(s)\right) \right. \\
& \left. + \int_0^1 I^{\alpha,\rho} f\left(s, v(s), {}^C D_{0^+}^{\alpha-1,\rho} v(s), {}^C D_{0^+}^{\alpha-2,\rho} v(s), {}^C D_{0^+}^{\alpha-3,\rho} v(s)\right) ds \right] \\
& \leq \sup_{t \in [0,1]} \left[L \left(|u(t) - v(t)| + \left| {}^C D_{0^+}^{\alpha-1,\rho} u(t) - {}^C D_{0^+}^{\alpha-1,\rho} v(t) \right| \right. \right. \\
& \quad \left. \left. + \left| {}^C D_{0^+}^{\alpha-2,\rho} u(t) - {}^C D_{0^+}^{\alpha-2,\rho} v(t) \right| + \left| {}^C D_{0^+}^{\alpha-3,\rho} u(t) - {}^C D_{0^+}^{\alpha-3,\rho} v(t) \right| \right) \right. \\
& \quad \left. \cdot \left[\frac{1}{\rho^\alpha \Gamma(\alpha+1)} + \frac{1}{\rho^3|\psi|} \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right] \right] \\
& \leq L \|u - v\| \left[\frac{1}{\rho^\alpha \Gamma(\alpha+1)} + \frac{1}{\rho^3|\psi|} \left(\sum_{i=1}^m \delta_i \frac{\xi_i^{\rho(\alpha+\beta)}}{\rho^{\alpha+\beta} \Gamma(\alpha+\beta+1)} \right. \right. \\
& \quad \left. \left. - \frac{1}{\rho^\alpha \Gamma(\alpha+1)(\rho\alpha+1)} \right) \right] \\
& \leq LA \|u - v\|.
\end{aligned} \tag{66}$$

Similarly,

$$\begin{aligned}
& \left\| ({}^C D_{0^+}^{\alpha-1,\rho} \mathcal{L}u) - ({}^C D_{0^+}^{\alpha-1,\rho} \mathcal{L}v) \right\| \leq LA \|u - v\|, \\
& \left\| ({}^C D_{0^+}^{\alpha-2,\rho} \mathcal{L}u) - ({}^C D_{0^+}^{\alpha-2,\rho} \mathcal{L}v) \right\| \leq LA \|u - v\|, \\
& \left\| ({}^C D_{0^+}^{\alpha-3,\rho} \mathcal{L}u) - ({}^C D_{0^+}^{\alpha-3,\rho} \mathcal{L}v) \right\| \leq LA \|u - v\|.
\end{aligned} \tag{67}$$

Consequently,

$$\|(\mathcal{L}u) - (\mathcal{L}v)\|_X \leq LA \|u - v\|_X. \tag{68}$$

Thus, in view of condition (59), it follows that the operator $\mathcal{L}$ is a contraction. Hence, the operator $\mathcal{L}$ has a unique fixed point which corresponds to a unique solution of GFBVP (1).

We conclude this section with some examples showing the applicability of our main results. $\square$

Example 1. Let us consider the following boundary value problem:

$$\begin{cases} {}^C D_{0^+}^{(7/2), (1/3)} u(t) = f(t, u(t), {}^C D_{0^+}^{\alpha-1,\rho} u(t), {}^C D_{0^+}^{\alpha-2,\rho} u(t), {}^C D_{0^+}^{\alpha-3,\rho} u(t)), & t \in [0, 1], \\ u(0) = \delta u(0) = \delta^2 u(0) = 0, \\ \int_0^1 u(s) ds = \sum_{i=1}^2 \delta_i I^{(1/2), (1/3)} u(\xi_i), \end{cases} \tag{69}$$

where $\alpha = 7/2, \beta = 1/2, \rho = 1/3, \delta_1 = 2/3, \delta_2 = 7/6, \xi_1 = 1/3, \xi_2 = 5/6$, and

$$\begin{aligned}
f\left(t, u(t), {}^C D_{0^+}^{(5/2), (1/3)} u(t), {}^C D_{0^+}^{(3/2), (1/3)} u(t), {}^C D_{0^+}^{(1/2), (1/3)} u(t)\right) &= \frac{1}{9} e^t \sin t + \frac{u(t)}{(t+3)^2} \\
&+ \frac{|{}^C D_{0^+}^{(5/2), (1/3)} u(t)|}{t^2 + 20} + \frac{|{}^C D_{0^+}^{(3/2), (1/3)} u(t)|}{5\sqrt{10t+64}} + \frac{|{}^C D_{0^+}^{(1/2), (1/3)} u(t)|}{11(t+1)}.
\end{aligned} \tag{70}$$

To check that the condition (H_2) holds, let $u, v \in C([0, 1], \mathbb{R})$ and $t \in [0, 1]$; then, we get

$$\begin{aligned} & \left| f\left(t, u(t), {}^C D_{0^+}^{(5/2), (1/3)} u(t), {}^C D_{0^+}^{(3/2), (1/3)} u(t), {}^C D_{0^+}^{(1/2), (1/3)} u(t)\right) \right. \\ & \quad \left. - f\left(t, v(t), {}^C D_{0^+}^{(5/2), (1/3)} v(t), {}^C D_{0^+}^{(3/2), (1/3)} v(t), {}^C D_{0^+}^{(1/2), (1/3)} v(t)\right) \right| \\ & \leq \frac{1}{9} |u(t) - v(t)| + \frac{1}{20} \left| {}^C D_{0^+}^{(5/2), (1/3)} u(t) - {}^C D_{0^+}^{(5/2), (1/3)} v(t) \right| + \frac{1}{40} \left| {}^C D_{0^+}^{(3/2), (1/3)} u(t) - {}^C D_{0^+}^{(3/2), (1/3)} v(t) \right| \\ & \quad + \frac{1}{11} \left| {}^C D_{0^+}^{(1/2), (1/3)} u(t) - {}^C D_{0^+}^{(1/2), (1/3)} v(t) \right| \leq \frac{1}{9} \|u - v\|_X. \end{aligned} \quad (71)$$

Hence, the condition (H_2) holds with $L = 1/9$ and it follows that $LA = 0.497230703 < 1$, where $A = \sigma + KV$ and V, K and σ are given by (43)–(45). Then, the hypotheses of Theorem 7 are satisfied and problem (69) has a unique solution on $[0, 1]$.

Example 2. We consider the following boundary value problem:

$$\begin{cases} {}^C D_{0^+}^{(7/2), (1/3)} u(t) = f\left(t, u(t), {}^C D_{0^+}^{\alpha-1, \rho} u(t), {}^C D_{0^+}^{\alpha-2, \rho} u(t), {}^C D_{0^+}^{\alpha-3, \rho} u(t)\right), & t \in [0, 1], \\ u(0) = \delta u(0) = \delta^2 u(0) = 0, \\ \int_0^1 u(s) ds = \sum_{i=1}^2 \delta_i I^{(1/2), (1/3)} u(\xi_i). \end{cases} \quad (72)$$

where $\alpha = 7/2, \beta = 1/2, \rho = 1/3, \delta_1 = 2/3, \delta_2 = 7/6, \xi_1 = 1/3$, and $\xi_2 = 5/6$. Choose

$$\begin{aligned} & f\left(t, u(t), {}^C D_{0^+}^{(5/2), (1/3)} u(t), {}^C D_{0^+}^{(3/2), (1/3)} u(t), {}^C D_{0^+}^{(1/2), (1/3)} u(t)\right) \\ & = \frac{1}{30} e^t + \frac{\tan^{-1} u(t)}{\sqrt{\cos t + 145}} + \frac{t}{8t^3 + 30} \left(\frac{|{}^C D_{0^+}^{(5/2), (1/3)} u(t)|}{|{}^C D_{0^+}^{(5/2), (1/3)} u(t)| + 1} \right) \\ & \quad + \frac{3}{6(t + 10)} \left(\frac{|{}^C D_{0^+}^{(3/2), (1/3)} u(t)|}{|{}^C D_{0^+}^{(3/2), (1/3)} u(t)| + 1} \right) + \frac{1}{\sqrt{\sin t + 169}} \left(\frac{|{}^C D_{0^+}^{(1/2), (1/3)} u(t)|}{|{}^C D_{0^+}^{(1/2), (1/3)} u(t)| + 1} \right). \end{aligned} \quad (73)$$

Set $F(u) = |u|^\alpha / \alpha, \alpha > 1$. In particular, if $\alpha = 2$, then F is an N-function and satisfies

$$\int_0^1 F(|f(u(s))|) ds < \infty, \quad (74)$$

which shows that f belongs to an Orlicz space $L_F([0, 1])$. Furthermore,

$$\begin{aligned} & \left| f\left(t, u(t), {}^C D_{0^+}^{(5/2), (1/3)} u(t), {}^C D_{0^+}^{(3/2), (1/3)} u(t), {}^C D_{0^+}^{(1/2), (1/3)} u(t)\right) \right| \\ & \leq \mathcal{M}(t), \end{aligned} \quad (75)$$

with $\mathcal{M}(t) = (e^t/30) + (\pi/2\sqrt{\cos t + 145}) + (t/(8t^3 + 30)) + (3/6(t + 10)) + (1/\sqrt{\sin t + 169})$ and $\|\mathcal{M}\| = 1.225$. Thus, the condition (H_1) holds and by the conclusion of Theorem 5, problem (72) has at least one solution on $[0, 1]$.

Moreover,

$$\begin{aligned}
& \left| f\left(t, u(t), {}^C D_{0^+}^{(5/2), (1/3)} u(t), {}^C D_{0^+}^{(3/2), (1/3)} u(t), {}^C D_{0^+}^{(1/2), (1/3)} u(t)\right) \right. \\
& \quad \left. - f\left(t, v(t), {}^C D_{0^+}^{(5/2), (1/3)} v(t), {}^C D_{0^+}^{(3/2), (1/3)} v(t), {}^C D_{0^+}^{(1/2), (1/3)} v(t)\right) \right| \\
& \leq \frac{1}{12} |u(t) - v(t)| + \frac{1}{30} \left| {}^C D_{0^+}^{(5/2), (1/3)} u(t) - {}^C D_{0^+}^{(5/2), (1/3)} v(t) \right| + \frac{1}{20} \left| {}^C D_{0^+}^{(3/2), (1/3)} u(t) - {}^C D_{0^+}^{(3/2), (1/3)} v(t) \right| \\
& \quad + \frac{1}{13} \left| {}^C D_{0^+}^{(1/2), (1/3)} u(t) - {}^C D_{0^+}^{(1/2), (1/3)} v(t) \right| \leq \frac{1}{12} \|u - v\|_X.
\end{aligned} \tag{76}$$

which proves the validity of condition (H_2) with $LA = 0.372923027 < 1$, where $A = \sigma + KV$ and V, K , and σ are given by (43)–(45). Hence, Theorem 7 can be applicable and so problem (72) has a unique solution on $[0, 1]$.

4. Ulam Stability

In this section, we develop the criteria for Ulam stability of GFBVP (1) by means of its equivalent integral equation. For simplicity, we set

$$f\left(s, v(s), {}^C D_{0^+}^{\alpha-1, \rho} v(s), {}^C D_{0^+}^{\alpha-2, \rho} v(s), {}^C D_{0^+}^{\alpha-3, \rho} v(s)\right) = \widehat{f}(s). \tag{77}$$

Then,

$$\begin{aligned}
v(t) &= \frac{\rho^{1-\alpha}}{\Gamma(\alpha)} \int_0^t (t^\rho - s^\rho)^{\alpha-1} s^{\rho-1} \widehat{f}(s) ds \\
&+ \frac{t^{3\rho}}{\rho^3 \psi} \left[\sum_{i=1}^m \delta_i \frac{\rho^{1-(\alpha+\beta)}}{\Gamma(\alpha+\beta)} \int_0^{\xi_i} (\xi_i^\rho - s^\rho)^{\alpha+\beta-1} s^{\rho-1} \right. \\
&\quad \left. \cdot \widehat{f}(s) ds - \int_0^1 I^{\alpha, \rho} \widehat{f}(s) ds \right],
\end{aligned} \tag{78}$$

where $v \in X$ and $f: [0, 1] \times \mathbb{R}^4 \rightarrow \mathbb{R}$ is a continuous function. Next, we define a continuous nonlinear operator $\mathcal{F}: X \rightarrow X$ as

$$\mathcal{F}v(t) = {}^C D_{0^+}^{\alpha, \rho} v(t) - \widehat{f}(t). \tag{79}$$

For definitions of Ulam–Hyers, generalized Ulam–Hyers and Ulam–Hyers–Rassias stability, and we refer to [41].

Definition 7. GFBVP (1) is said to be Ulam–Hyers stable if there exists a real number $b > 0$ such that, for each $\epsilon > 0$ and for each solution $v \in X$,

$$\|\mathcal{F}v\| \leq \epsilon, \quad t \in [0, 1], \tag{80}$$

there exists a solution $u \in X$ of (1) satisfying the inequality

$$\|u - v\| \leq b\epsilon_1, \quad t \in [0, 1], \tag{81}$$

where ϵ_1 is a positive real number depending on ϵ .

Definition 8. GFBVP (1) is generalized Ulam–Hyers stable if there exists $\Theta \in C(\mathbb{R}^+, \mathbb{R}^+)$ such that, for each solution $v \in X$ of (1), there exists a solution $u \in X$ of (1) with

$$|u(t) - v(t)| \leq \Theta(\epsilon), \quad t \in [0, 1]. \tag{82}$$

Definition 9. GFBVP (1) is Ulam–Hyers–Rassias stable with respect to $\Psi \in C([0, 1], \mathbb{R})$ if there exists a real number $b > 0$ such that, for each $\epsilon > 0$ and for each solution $v \in X$ of (1),

$$|\mathcal{F}v(t)| \leq \epsilon \Psi(t), \quad t \in [0, 1], \tag{83}$$

we can find a solution $u \in X$ of (1) satisfying the inequality

$$|u(t) - v(t)| \leq b\epsilon_1 \Psi(t), \quad t \in [0, 1], \tag{84}$$

where ϵ_1 is a positive real number depending on ϵ .

Theorem 8. Assume that conditions (47) and (59) hold. Then, GFBVP (1) satisfies both Ulam–Hyers and generalized Ulam–Hyers stability criteria.

Proof. We know that $u \in X$ is a unique solution of (1) (by Theorem 7). Let $v \in X$ be another solution of (1) satisfying $\|\mathcal{F}v\| \leq \epsilon, t \in [0, 1]$. Observe that the operators $\mathcal{F}$ and $\mathcal{L} - \mathcal{F}$ are equivalent for every solution $v \in X$ of (1). Therefore, by the fixed point property of the operator $\mathcal{L}$ together with $u(t) = \mathcal{L}u(t)$ and $\|\mathcal{F}v\| \leq \epsilon, t \in [0, 1]$, we have

$$\begin{aligned}
|v(t) - u(t)| &= |v(t) - \mathcal{L}v(t) + \mathcal{L}v(t) - \mathcal{L}u(t)| \\
&\leq |\mathcal{L}u(t) - \mathcal{L}v(t)| + |\mathcal{L}v(t) - v(t)| \\
&\leq LA\|u - v\|_X + \epsilon,
\end{aligned} \tag{85}$$

where $\epsilon > 0$ and $LA < 1$. Taking the norm for $t \in [0, 1]$ and solving for $\|u - v\|_X$, we obtain

$$\|u - v\|_X \leq \frac{\epsilon}{1 - LA}. \tag{86}$$

Choosing $\epsilon_1 = \epsilon/(1 - LA)$ and $b = 1$, the Ulam–Hyers stability condition is satisfied. More generally, defining $\Theta(\epsilon) = \epsilon/(1 - LA)$, the generalized Ulam–Hyers stability condition is also satisfied which completes the proof. $\square$

Theorem 9. Assume that conditions (47) and (59) hold and there exists a function $z \in C([0, 1], \mathbb{R})$ satisfying condition

$\|u(t) - v(t)\| \leq b\epsilon_1 \Psi(t), t \in [0, 1]$. Then, GFBVP (1) is Ulam–Hyers–Rassias stable with respect to Ψ .

Proof. Similar to the discussion in the proof of Theorem 8, we have

$$\|u - v\|_X \leq \epsilon_1 \Psi(t), \quad (87)$$

with $\epsilon_1 = \epsilon/(1 - LA)$, and we get the desired result. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

Authors' Contributions

Wafa Shammakh, Hadeel Z. Alzumi, and Zahra Albarqi contributed to the design and implementation of the research, to the analysis of the results, and to the writing of the manuscript.

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