Research Article

Convergence Results on the Boundary Conditions for 2D Large-Scale Primitive Equations in Oceanic Dynamics

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In this paper, the initial boundary value problem for the two-dimensional large-scale primitive equations of large-scale oceanic motion in geophysics is considered, which are fundamental models for weather prediction. By establishing rigorous a priori bounds with coefficients and deriving some useful inequalities, the convergence result for the boundary conditions is obtained.

1. Introduction

The primitive equations are fundamental models for weather prediction, which are derived from the Boussinesq system of incompressible flow (see, e.g., [1–5]). Due to their importance, many authors have considered the primitive equations analytically by using many new methods (see Zeng [6], Lions and Temam [7, 8], Sun and Cui [9], Hieber et al. [10], and You and Li [11]). For more papers, one can see [9, 12–14] and the references therein. It is very obvious that the papers in the literature mainly concern the well posedness of the 2D or 3D primitive equations and the properties of solutions.

Different from the results above, the aim of this paper is to establish the convergence result of the solution when the boundary data tend to zero. It is very important to know whether a small change in the equation can cause a large change in the solution. By taking advantage of the mathematical analysis to study these equations, it is helpful to know their applicability in physics. Since some inevitable errors will appear in reality, the study of continuous dependence or convergence results becomes more and more significant. There have been many papers in the literature to study the continuous dependence or convergence for varieties of equations (e.g., Brinkman, Darcy, and Forchheimer equations) (see [15–21]). For some type of primitive equations, one can see [22, 23].

In this paper, the two-dimensional large-scale primitive equations (see [24]) are considered

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} + w \frac{\partial u}{\partial x_2} - \frac{1}{\varepsilon'} v + \frac{1}{\varepsilon'} \frac{\partial p}{\partial x_1} = \gamma_1 \Delta u,$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x_1} + w \frac{\partial v}{\partial x_2} + \frac{1}{\varepsilon'} u = \gamma_2 \Delta v,$$

$$\frac{\partial p}{\partial x_2} + \rho = 0,$$

$$\frac{\partial u}{\partial x_1} + \frac{\partial w}{\partial x_2} = 0,$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x_1} + w \frac{\partial T}{\partial x_2} = \gamma_3 \Delta T,$$

$$\rho = \rho_{\text{ref}} (1 - \beta_T (T - T_{\text{ref}})), \quad (1)$$

in a cylindrical domain $\Omega = (0, 1) \times (-h, 0)$, $h > 0$. In (1), the unknown functions (u, v) , w , ρ , p , and T are the horizontal velocity field, the vertical velocity, the density, the pressure, and the temperature, respectively; ε' is the Rossby number; $\gamma_i > 0$ ($i = 1, 2, 3$) are the viscosity coefficients; ρ_{ref} and T_{ref} are the reference values of the density and the temperature; β_T is the expansion coefficient (constants); $\Delta = \partial_{x_1}^2 + \partial_{x_2}^2$.

The boundary of Ω is defined by $\partial\Omega$ which can be partitioned into

$$\begin{aligned}\Gamma_0 &= \{(x_1, x_2) \in \overline{\Omega}: 0 < x_1 < 1, x_2 = 0\}, \\ \Gamma_{-h} &= \{(x_1, x_2) \in \overline{\Omega}: 0 < x_1 < 1, x_2 = -h\}, \\ \Gamma_s &= \{(x_1, x_2) \in \overline{\Omega}: x_1 = 0 \text{ or } x_1 = 1, -h \leq x_2 \leq 0\}.\end{aligned}\quad (2)$$

System (1) also has the following boundary conditions:

$$\begin{aligned}\frac{\partial u}{\partial x_2} &= \alpha_1 g_1(x_1, t), \frac{\partial v}{\partial x_2} = \alpha_2 g_2(x_1, t), w = 0, \frac{\partial T}{\partial x_2} = \beta T^*(x_1, t), \\ &\text{on } \Gamma_0, \\ \frac{\partial u}{\partial x_2} &= \frac{\partial v}{\partial x_2} = 0, w = 0, \frac{\partial T}{\partial x_2} = 0, \quad \text{on } \Gamma_{-h}, \\ u &= v = w = 0, \frac{\partial T}{\partial x_1} = 0, \quad \text{on } \Gamma_s,\end{aligned}\quad (3)$$

where $g_1(x_1, t)$ and $g_2(x_1, t)$ are the wind stress on the ocean surface, α_1, α_2 , and β are the positive constants, and $T^*(x_1, t)$ is the typical temperature distribution of the top surface of the ocean. $g_1(x_1, t)$, $g_2(x_1, t)$, and $T^*(x_1, t)$ also satisfy the compatibility boundary conditions:

$$\begin{aligned}g_1(0, t) &= g_2(0, t) = T^*(0, t) = g_1(1, t) = g_2(1, t) \\ &= T^*(1, t) = 0.\end{aligned}\quad (4)$$

In addition, the initial conditions can be written as

$$\begin{aligned}u(x_1, x_2, 0) &= u_0(x_1, x_2), \\ v(x_1, x_2, 0) &= v_0(x_1, x_2), \\ T(x_1, x_2, 0) &= T_0(x_1, x_2), \\ &\text{in } \Omega.\end{aligned}\quad (5)$$

The present paper is organized as follows. In Section 2, some preliminaries of the problem and some well-known inequalities which will be used in the whole paper are given. Inspired by [25–27], rigorous a priori bounds with coefficients are established. Finally, the convergence result on the boundary data of our problem is derived in Section 4.

2. Preliminaries of the Problem

Equations (1)–(5) are formulated as in [7–9]. Realizing the boundary conditions (4), equation (1)₄ is integrated from $-h$ to x_2 to obtain

$$\begin{aligned}w(x_1, x_2, t) &= w(x_1, -h, t) - \int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t) d\zeta \\ &= -\frac{\partial}{\partial x_1} \int_{-h}^{x_2} u(x_1, \zeta, t) d\zeta,\end{aligned}\quad (6)$$

$$\int_{-h}^0 \frac{\partial}{\partial x_1} u(x_1, \zeta, t) d\zeta = -\frac{\partial}{\partial x_1} \int_{-h}^0 u(x_1, \zeta, t) d\zeta = 0. \quad (7)$$

By integrating (1)₃ and (1)₄, the following is obtained:

$$\begin{aligned}p(x_1, x_2, t) &= p_s - \int_{x_2}^0 \rho(x_1, \zeta, t) d\zeta = p_s \\ &\quad - \rho_{\text{ref}} \int_{x_2}^0 [1 - \beta_T(T(x_1, \zeta, t) - T_{\text{ref}})] d\zeta,\end{aligned}\quad (8)$$

where $p_s = p(x_1, 0, t)$ is the pressure on the surface of the ocean. For convenience, suppose $\rho_{\text{ref}}\beta_T = \epsilon' = \gamma_1 = \gamma_2 = \gamma_3 = 1$. Inserting (6) and (7) into (1)–(5), the problem can be rewritten as

$$\begin{aligned}\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t) d\zeta \right) \frac{\partial u}{\partial x_2} - v \\ + \frac{\partial p_s}{\partial x_1} - \left(\int_{x_2}^0 \frac{\partial}{\partial x_1} T(x_1, \zeta, t) d\zeta \right) = \Delta u,\end{aligned}\quad (9)$$

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t) d\zeta \right) \frac{\partial v}{\partial x_2} + u = \Delta v,$$

$$\frac{\partial T}{\partial t} + u \frac{\partial T}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t) d\zeta \right) \frac{\partial T}{\partial x_2} = \Delta T,$$

with the following boundary conditions:

$$\begin{aligned}\frac{\partial u}{\partial x_2} \Big|_{x_2=0} &= \alpha_1 g_1(x_1, t), \\ \frac{\partial v}{\partial x_2} \Big|_{x_2=0} &= \alpha_2 g_2(x_1, t), \\ \frac{\partial u}{\partial x_2} \Big|_{x_2=-h} &= \frac{\partial v}{\partial x_2} \Big|_{x_2=-h} = 0, \\ (u, v) \Big|_{\Gamma_s} &= 0, \\ \frac{\partial T}{\partial x_2} \Big|_{x_2=0} &= -\beta T^*(x_1, t), \\ \frac{\partial T}{\partial x_2} \Big|_{x_2=-h} &= 0, \\ \frac{\partial T}{\partial x_1} \Big|_{\Gamma_s} &= 0,\end{aligned}\quad (10)$$

and the initial conditions:

$$(u, v, T) \Big|_{t=0} = (u_0, v_0, T_0). \quad (11)$$

In this paper, some well-known inequalities are used throughout this paper.

Lemma 1. *If $\omega(x_1) \in C^1(0, 1)$ and $\omega(0) = \omega(1) = 0$, then*

$$\int_0^1 \omega^2 dx_1 \leq \frac{1}{\pi^2} \int_0^1 \left(\frac{\partial \omega}{\partial x_1} \right)^2 dx_1. \quad (12)$$

Lemma 2. If $\omega(x_2) \in C^1(-h, 0)$ and $\omega(-h) = \omega(0) = 0$, then

$$\int_{-h}^0 \omega^2 dx_2 \leq \frac{h^2}{\pi^2} \int_{-h}^0 \left(\frac{\partial \omega}{\partial x_2} \right)^2 dx_2. \quad (13)$$

For proofs of these inequalities, see references [28, 29].

Lemma 3. If $\omega(x_1, x_2, t)$ is a sufficiently smooth function in $\Omega = (0, 1) \times (-h, 0)$ and $\omega(0, x_2, t) = \omega(1, x_2, t) = 0$, then

$$\left(\int_{\Omega} \omega^4 dA \right)^{1/2} \leq C \left[\left(\int_{\Omega} \omega^2 dA \right)^{1/2} \left(\int_{\Omega} |\nabla \omega|^2 dA \right)^{1/2} + \left(\int_{\Omega} \omega^2 dA \right)^{1/4} \left(\int_{\Omega} |\nabla \omega|^2 dA \right)^{3/4} \right], \quad (14)$$

or

$$\left(\int_{\Omega} \omega^4 dA \right)^{1/2} \leq C \left[\int_{\Omega} \omega^2 dA + \delta \int_{\Omega} |\nabla \omega|^2 dA \right], \quad (15)$$

where $\nabla = (\partial_{x_1}, \partial_{x_2})$, C is a positive computable constant, and δ is a positive arbitrary constant.

Proof. By the Hölder inequality, one can write

$$\int_{\Omega} \omega^4 dA \leq \int_{-h}^0 \left(\int_0^1 \omega^6 dx_1 \right)^{1/2} \left(\int_0^1 \omega^2 dx_1 \right)^{1/2} dx_2. \quad (16)$$

Since $\omega(0, x_2, t) = \omega(1, x_2, t) = 0$, the following is obtained:

$$\begin{aligned} \omega^3 &= 3 \int_0^{x_1} \omega^2(\xi, x_2, t) \frac{\partial \omega(\xi, x_2, t)}{\partial \xi} d\xi \\ &= -3 \int_{x_1}^1 \omega^2(\xi, x_2, t) \frac{\partial \omega(\xi, x_2, t)}{\partial \xi} d\xi. \end{aligned} \quad (17)$$

Therefore,

$$|\omega|^3 \leq \frac{3}{2} \int_0^1 \omega^2(x_1, x_2, t) \left| \frac{\partial \omega(x_1, x_2, t)}{\partial x_1} \right| dx_1. \quad (18)$$

Then,

$$\left(\int_0^1 \omega^6 dx_1 \right)^{1/2} \leq \frac{3}{2} \left(\int_0^1 \omega^2 \left| \frac{\partial \omega}{\partial x_1} \right| dx_1 \right). \quad (19)$$

Inserting (19) into (16), the following is obtained:

$$\begin{aligned} \int_{\Omega} \omega^4 dA &\leq \frac{3}{2} \int_{-h}^0 \left(\int_0^1 \omega^2 \left| \frac{\partial \omega}{\partial x_1} \right| dx_1 \right) \left(\int_0^1 \omega^2 dx_1 \right)^{1/2} dx_2 \\ &\leq \frac{3}{2} \max_{-h \leq x_2 \leq 0} \left\{ \left(\int_0^1 \omega^2 dx_1 \right)^{1/2} \right\} \int_{\Omega} \omega^2 \left| \frac{\partial \omega}{\partial x_1} \right| dA. \end{aligned} \quad (20)$$

Obviously,

$$\begin{aligned} \omega^2 &= 2 \int_{-h}^{x_2} \omega(x_1, \zeta, t) \frac{\partial \omega(x_1, \zeta, t)}{\partial \zeta} d\zeta + \omega^2(x_1, -h, t) \\ &= -2 \int_{x_2}^0 \omega(x_1, \zeta, t) \frac{\partial \omega(x_1, \zeta, t)}{\partial \zeta} d\zeta + \omega^2(x_1, 0, t), \end{aligned} \quad (21)$$

so,

$$\omega^2 \leq \int_{-h}^0 |\omega| \left| \frac{\partial \omega}{\partial x_2} \right| dx_2 + \frac{1}{2} [\omega^2(x_1, 0, t) + \omega^2(x_1, -h, t)]. \quad (22)$$

To bound the last term of (22), a new known function $f(x_2)$ is defined, satisfying

$$\begin{aligned} f(0) &> 0, \\ f(-h) &< 0, \\ |f'(x_2)| &\leq m_1, \\ |f(x_2)| &\leq m_2, \\ \text{for } -h &\leq x_2 \leq 0, \end{aligned} \quad (23)$$

where m_1 and m_2 are the positive constants. For example, $f(x_2) = (m_1/2)((x_2 + (h/2)))$, $m_1 h < 4m_2$, satisfies all the conditions in (23). Using the above estimates and employing the divergence theorem allow one to write

$$\begin{aligned} \min\{f(0), -f(-h)\} &([\omega^2(x_1, 0, t) + \omega^2(x_1, -h, t)]) \\ &\leq f(0)\omega^2(x_1, 0, t) - f(-h)\omega^2(x_1, -h, t) \\ &= \int_{-h}^0 \frac{\partial}{\partial x_2} (f\omega^2) dx_2 = \int_{-h}^0 f'(x_2)\omega^2 dx_2 + 2 \int_{-h}^0 f\omega \frac{\partial \omega}{\partial x_2} dx_2 \\ &\leq m_1 \int_{-h}^0 \omega^2 dx_2 + 2m_2 \int_{-h}^0 |\omega| \left| \frac{\partial \omega}{\partial x_2} \right| dx_2. \end{aligned} \quad (24)$$

Inserting (24) into (22),

$$\omega^2 \leq m_3 \int_{-h}^0 \omega^2 dx_2 + m_4 \int_{-h}^0 |\omega| \left| \frac{\partial \omega}{\partial x_2} \right| dx_2, \quad (25)$$

where

$$\begin{aligned} m_3 &= \frac{m_1}{2 \min\{f(0), -f(-h)\}}, \\ m_4 &= 1 + \frac{m_2}{\min\{f(0), -f(-h)\}}. \end{aligned} \quad (26)$$

Therefore,

$$\max_{-h \leq x_2 \leq 0} \left\{ \left(\int_0^1 \omega^2 dx_1 \right)^{1/2} \right\} \leq \left(m_3 \int_{\Omega} \omega^2 dA + m_4 \int_{\Omega} |\omega| \left| \frac{\partial \omega}{\partial x_2} \right| dA \right)^{1/2}. \quad (27)$$

Thus, from (20) and (27) and by the Hölder inequality, one has

$$\begin{aligned} \int_{\Omega} \omega^4 dA &\leq \frac{3}{2} \left[m_3 \int_{\Omega} \omega^2 dA + m_4 \left(\int_{\Omega} \omega^2 dA \right)^{1/2} \right. \\ &\quad \cdot \left. \left(\int_{\Omega} \left| \frac{\partial \omega}{\partial x_2} \right|^2 dA \right)^{1/2} \right]^{1/2} \left(\int_{\Omega} \omega^4 dA \right)^{1/2} \\ &\quad \cdot \left(\int_{\Omega} \left| \frac{\partial \omega}{\partial x_1} \right|^2 dA \right)^{1/2}. \end{aligned} \quad (28)$$

After simplification,

$$\begin{aligned} \left(\int_{\Omega} \omega^4 dA \right)^{1/2} &\leq C \left[\left(\int_{\Omega} \omega^2 dA \right)^{1/2} \left(\int_{\Omega} |\nabla \omega|^2 dA \right)^{1/2} \right. \\ &\quad \left. + \left(\int_{\Omega} \omega^2 dA \right)^{1/4} \left(\int_{\Omega} |\nabla \omega|^2 dA \right)^{3/4} \right]. \end{aligned} \quad (29)$$

□

3. A Priori Estimates

Now, some a priori estimates for the solutions of (9)–(11) are derived.

Lemma 4. Assume T be the solutions to (9)–(11) with $T_0 \in L_2(\Omega)$, $T^* \in H^1(\Omega)$. Then,

$$\int_0^t \int_{\Omega} |\nabla T|^2 dA d\eta \leq F_1(t), \quad (30)$$

where $F_1(t)$ will be defined later.

Proof. Taking the inner product of equation (9)₃ with T , in $L^2(\Omega)$, the following is obtained:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} T^2 dA + \int_{\Omega} |\nabla T|^2 dA &= \int_0^1 \frac{\partial T}{\partial x_2} \Big|_{x_2=0} T(x_1, 0, t) dx_1 \\ &\quad - \int_{\Omega} \left[u \frac{\partial T}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t) d\zeta \right) \frac{\partial T}{\partial x_2} \right] T dA. \end{aligned} \quad (31)$$

A function $H(x_1, x_2, t)$ is defined as

$$\frac{\partial H}{\partial x_2} = \frac{\beta(x_2 + h)}{h} T^*(x_1, t). \quad (32)$$

Therefore,

$$\begin{aligned} \int_0^1 \frac{\partial T}{\partial x_2} \Big|_{x_2=0} T(x_1, 0, t) dx_1 &= \int_0^1 \frac{\partial H}{\partial x_2} \Big|_{x_2=0} T(x_1, 0, t) dx_1 \\ &= \int_{\Omega} \frac{\partial}{\partial x_2} \left(\frac{\partial H}{\partial x_2} T \right) dA \\ &= \int_{\Omega} \frac{\partial^2 H}{\partial x_2^2} T dA + \int_{\Omega} \frac{\partial H}{\partial x_2} \frac{\partial T}{\partial x_2} dA \\ &= \int_{\Omega} \frac{\partial^2 H}{\partial x_2^2} T dA + \int_{\Omega} \frac{\partial H}{\partial x_2} \frac{\partial T}{\partial x_2} dA \\ &\quad + \frac{1}{2} \int_{\Omega} \left(\frac{\partial H}{\partial x_2} \right)^2 dA + \frac{1}{2} \int_{\Omega} \left(\frac{\partial T}{\partial x_2} \right)^2 dA. \end{aligned} \quad (33)$$

Integrating by parts,

$$-\int_{\Omega} \left[u \frac{\partial T}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t) d\zeta \right) \frac{\partial T}{\partial x_2} \right] T dA = 0. \quad (34)$$

By the above results, the following is obtained:

$$\frac{d}{dt} \int_{\Omega} T^2 dA + \int_{\Omega} |\nabla T|^2 dA = \int_{\Omega} T^2 dA + a_1(t), \quad (35)$$

where $a_1(t) = \int_{\Omega} (\partial^2 H / \partial x_2^2)^2 dA + \int_{\Omega} (\partial H / \partial x_2)^2 dA$. Using inequality (35) and the Gronwall inequality, the following is obtained:

$$\int_{\Omega} T^2 dA \leq \int_{\Omega} T_0^2 dA \cdot e^t + \int_0^t a_1(\eta) e^{t-\eta} d\eta \doteq a_2(t). \quad (36)$$

Moreover,

$$\int_0^t \int_{\Omega} |\nabla T|^2 dA d\eta \leq F_1(t), \quad (37)$$

where $F_1(t) = \int_0^t [a_1(\eta) + a_2(\eta)] d\eta + \int_{\Omega} T_0^2 dA$. □

Lemma 5. Let (u, v) be the solution of (9)–(11) with $u_0, v_0, T_0 \in L_2(\Omega)$. Then,

$$\begin{aligned} \int_{\Omega} u^2 dA + \int_{\Omega} v^2 dA &\leq F_2(t), \\ \int_0^t \int_{\Omega} |\nabla u|^2 dA d\eta + \int_0^t \int_{\Omega} |\nabla v|^2 dA d\eta &\leq F_3(t), \end{aligned} \quad (38)$$

where $F_2(t)$ and $F_3(t)$ will be defined later.

Proof. Taking the inner product of equation (9)₁ with u , in $L^2(\Omega)$, the following is obtained:

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \int_{\Omega} u^2 dA + \int_{\Omega} |\nabla u|^2 dA &= \int_0^1 \frac{\partial u}{\partial x_2} \Big|_{x_2=0} u(x_1, 0, t) dx_1 \\ &\quad - \int_{\Omega} uv dA \\ &\quad + \int_{\Omega} \left(\int_{x_2}^0 \frac{\partial}{\partial x_1} T(x_1, \zeta, t) d\zeta \right) u dA. \end{aligned} \quad (39)$$

A function $S_1(x_1, x_2, t)$ is defined as

$$\frac{\partial S_1}{\partial x_2} = \frac{\alpha_1(x_2 + h)}{h} g_1(x_1, t). \quad (40)$$

Obviously, S_1 has the same boundary conditions of u . Therefore,

$$\begin{aligned}
\int_0^1 \frac{\partial u}{\partial x_2} \Big|_{x_2=0} u(x_1, 0, t) dx_1 &= \int_0^1 \frac{\partial S_1}{\partial x_2} \Big|_{x_2=0} u(x_1, 0, t) dx_1 = \int_{\Omega} \frac{\partial}{\partial x_2} \left(\frac{\partial S_1}{\partial x_2} u \right) dA \\
&= \int_{\Omega} \frac{\partial^2 S_1}{\partial x_2^2} u dA + \int_{\Omega} \frac{\partial S_1}{\partial x_2} \frac{\partial u}{\partial x_2} dA \\
&\leq \frac{1}{2} \int_{\Omega} \left(\frac{\partial^2 S_1}{\partial x_2^2} \right)^2 dA + \frac{1}{2} \int_{\Omega} u^2 dA \\
&\quad + \frac{1}{2} \int_{\Omega} \left(\frac{\partial S_1}{\partial x_2} \right)^2 dA + \frac{1}{2} \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^2 dA.
\end{aligned} \tag{41}$$

By the Cauchy-Schwarz inequality, the following is obtained:

$$\begin{aligned}
\int_{\Omega} \left(\int_{x_2}^0 \frac{\partial}{\partial x_1} T(x_1, \zeta, t) d\zeta \right) u dA &\leq h \left[\int_{\Omega} \left(\frac{\partial T}{\partial x_1} \right)^2 dA \right]^{1/2} \cdot \left[\int_{\Omega} u^2 dA \right]^{1/2} \\
&\leq \frac{h}{2} \int_{\Omega} \left(\frac{\partial T}{\partial x_1} \right)^2 dA \\
&\quad + \frac{h}{2} \int_{\Omega} u^2 dA.
\end{aligned} \tag{42}$$

By the above results, the following is obtained:

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} u^2 dA + \int_{\Omega} |\nabla u|^2 dA &\leq -2 \int_{\Omega} uv dA + (1+h) \int_{\Omega} u^2 dA \\
&\quad + h \int_{\Omega} \left(\frac{\partial T}{\partial x_1} \right)^2 dA + \int_{\Omega} \left(\frac{\partial^2 S_1}{\partial x_2^2} \right)^2 dA + \int_{\Omega} \left(\frac{\partial S_1}{\partial x_2} \right)^2 dA.
\end{aligned} \tag{43}$$

Similarly, from (9)₂,

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} v^2 dA + \int_{\Omega} |\nabla v|^2 dA &\leq 2 \int_{\Omega} uv dA + \int_{\Omega} v^2 dA \\
&\quad + \int_{\Omega} \left(\frac{\partial^2 S_2}{\partial x_2^2} \right)^2 dA + \int_{\Omega} \left(\frac{\partial S_2}{\partial x_2} \right)^2 dA,
\end{aligned} \tag{44}$$

where

$$\frac{\partial S_2}{\partial x_2} = \frac{\alpha_1(x_2 + h)}{h} g_2(x_1, t). \tag{45}$$

Combining (43) and (44), the following is obtained:

$$\begin{aligned}
&\frac{d}{dt} \left(\int_{\Omega} u^2 dA + \int_{\Omega} v^2 dA \right) + \int_{\Omega} |\nabla u|^2 dA + \int_{\Omega} |\nabla v|^2 dA \\
&\leq (1+h) \left(\int_{\Omega} u^2 dA + \int_{\Omega} v^2 dA \right) + h \int_{\Omega} \left(\frac{\partial T}{\partial x_1} \right)^2 dA + a_3(t),
\end{aligned} \tag{46}$$

where $a_3(t) = \int_{\Omega} (\partial^2 S_1 / \partial x_2^2)^2 dA + \int_{\Omega} (\partial S_1 / \partial x_2)^2 dA + \int_{\Omega} (\partial^2 S_2 / \partial x_2^2)^2 dA + \int_{\Omega} (\partial S_2 / \partial x_2)^2 dA$. By the Gronwall inequality and Lemma 2, one gets

$$\int_{\Omega} u^2 dA + \int_{\Omega} v^2 dA \leq F_2(t), \tag{47}$$

where $F_2(t) = h e^{(1+h)t} \left(\int_{\Omega} u_0^2 dA + \int_{\Omega} v_0^2 dA + F_1(t) \right) + \int_0^t a_3(\eta) e^{(1+h)(t-\eta)} d\eta$. Moreover,

$$\int_0^t \int_{\Omega} |\nabla u|^2 dA d\eta + \int_0^t \int_{\Omega} |\nabla v|^2 dA d\eta \leq F_3(t), \tag{48}$$

where $F_3(t) = (1+h) \int_0^t F_2(\eta) d\eta + h F_1(t) + \int_0^t a_3(\eta) d\eta + \int_{\Omega} u_0^2 dA + \int_{\Omega} v_0^2 dA$. $\square$

Lemma 6. If $T_0, T^* \in L_{\infty}(\Omega)$, then the solution T of (2.4)₃ satisfies

$$\sup_{\Omega} |T| \leq T_m, \tag{49}$$

where $T_m = \sup_{\Omega} \{ \|T^*\|_{\infty}, \|T_0\|_{\infty} \}$.

Proof. Multiplying (2.4)₃ by T^{p-1} and integrating by parts, one gets

$$\begin{aligned}
&\frac{1}{p} \frac{d}{dt} \int_{\Omega} T^p dA + \frac{p-1}{p^2} \int_{\Omega} |\nabla T^{p/2}|^2 dA \\
&= \int_0^1 \frac{\partial T}{\partial x_2} \Big|_{x_2=0} T^{p-1}(x_1, 0, t) dx_1.
\end{aligned} \tag{50}$$

By the Hölder inequality and the Cauchy-Schwarz inequality, the following is obtained:

$$\begin{aligned}
\int_0^1 \frac{\partial T}{\partial x_2} \Big|_{x_2=0} T^{p-1}(x_1, 0, t) dx &= \int_0^1 \frac{\partial H}{\partial x_2} T^{p-1} dx_1 \\
&= \int_{\Omega} \frac{\partial}{\partial x_2} \left(\frac{\partial H}{\partial x_2} T^{p-1} \right) dA \\
&= \int_{\Omega} \frac{\partial^2 H}{\partial x_2^2} T^{p-1} dA \\
&\quad + (p-1) \int_{\Omega} \frac{\partial H}{\partial x_2} T^{p-2} \frac{\partial T}{\partial x_2} dA \\
&\leq \frac{1}{p} \int_{\Omega} \left(\frac{\partial^2 H}{\partial x_2^2} \right)^p dA \\
&\quad + \frac{(p-1)}{p} \int_{\Omega} T^p dA \\
&\quad + \frac{2(p-1)}{p} \int_{\Omega} \left(\frac{\partial H}{\partial x_2} \right)^p dA \\
&\quad + \frac{(p-1)(p-2)}{p} \int_{\Omega} T^p dA \\
&\quad + \frac{(p-1)}{p^2} \int_{\Omega} |\nabla T^{p/2}|^2 dA.
\end{aligned} \tag{51}$$

Therefore, from (50) and (51), one has

$$\begin{aligned}
\frac{d}{dt} \int_{\Omega} T^p dA &\leq (p-1)^2 \int_{\Omega} T^p dA + 2(p-1) \int_{\Omega} \left(\frac{\partial H}{\partial x_2} \right)^p dA \\
&\quad + (p-1)(p-2) \int_{\Omega} \left(\frac{\partial^2 H}{\partial x_2^2} \right)^p dA.
\end{aligned} \tag{52}$$

By the Gronwall inequality, the following is obtained:

$$\begin{aligned}
\int_{\Omega} T^p dA &\leq \int_{\Omega} T_0^p dA \cdot e^{(p-1)^2 t} + \int_0^t e^{(p-1)^2(t-\eta)} \\
&\quad \cdot \left[2(p-1) \int_{\Omega} \left(\frac{\partial H}{\partial x_2} \right)^p dA + (p-1)(p-2) \int_{\Omega} \left(\frac{\partial^2 H}{\partial x_2^2} \right)^p dA \right] d\eta.
\end{aligned} \tag{53}$$

Therefore,

$$\begin{aligned}
\left(\int_{\Omega} T^p dA \right)^{1/p} &\leq \left\{ \int_{\Omega} T_0^p dA \cdot e^{(p-1)^2 t} + \int_0^t e^{(p-1)^2(t-\eta)} \right. \\
&\quad \cdot \left[2(p-1) \int_{\Omega} \left(\frac{\partial H}{\partial x_2} \right)^p dA + (p-1)(p-2) \int_{\Omega} \left(\frac{\partial^2 H}{\partial x_2^2} \right)^p dA \right] d\eta \Big\}^{1/p}.
\end{aligned} \tag{54}$$

Letting now $p \rightarrow \infty$ in (54), one obtains

$$\sup_{\Omega} |T| \leq T_m. \tag{55}$$

□

Lemma 7. Let (u, v) be the solution of (9)–(11) with $u_0, v_0, T_0 \in H^1(\Omega)$. Then,

$$\begin{aligned}
&\int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^2 dA + \int_{\Omega} \left(\frac{\partial v}{\partial x_2} \right)^2 dA + \int_0^t \int_{\Omega} \left| \nabla \frac{\partial u}{\partial x_2} \right|^2 dAd\eta \\
&\quad + \int_0^t \int_{\Omega} \left| \nabla \frac{\partial v}{\partial x_2} \right|^2 dAd\eta \leq F_5(t).
\end{aligned} \tag{56}$$

Moreover,

$$\int_0^t \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^4 dAd\eta + \int_0^t \int_{\Omega} \left(\frac{\partial v}{\partial x_2} \right)^4 dAd\eta \leq F_6(t), \tag{57}$$

where $F_5(t)$ and $F_6(t)$ will be defined later.

Proof. Using (2.4)₁, one starts with

$$\begin{aligned}
&\int_0^t \int_{\Omega} \frac{\partial}{\partial x_2} \left\{ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t) d\zeta \right) \frac{\partial u}{\partial x_2} \right. \\
&\quad \left. - v + \frac{\partial p_s}{\partial x_1} - \left(\int_{x_2}^0 \frac{\partial}{\partial x_1} T(x_1, \zeta, t) d\zeta \right) - \Delta u \right\} \frac{\partial u}{\partial x_2} dAd\eta = 0.
\end{aligned} \tag{58}$$

Integrating by parts, the following is obtained:

$$\begin{aligned}
& \frac{1}{2} \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^2 dA + \int_0^t \int_{\Omega} \left| \nabla \frac{\partial u}{\partial x_2} \right|^2 dAd\eta - \int_0^t \int_{\Omega} \frac{\partial T}{\partial x_1} \frac{\partial u}{\partial x_2} dAd\eta \leq \left(\int_0^t \int_{\Omega} \left(\frac{\partial T}{\partial x_1} \right)^2 dAd\eta \right)^{1/2} \\
& = \int_0^t \int_0^1 \frac{\partial u}{\partial x_2} \frac{\partial^2 u}{\partial x_2^2} dx_1 d\eta + \frac{1}{2} \int_{\Omega} \left(\frac{\partial u_0}{\partial x_2} \right)^2 dA \quad (59) \\
& \quad + \int_0^t \int_{\Omega} \frac{\partial v}{\partial x_2} \frac{\partial u}{\partial x_2} dAd\eta - \int_0^t \int_{\Omega} \frac{\partial T}{\partial x_1} \frac{\partial u}{\partial x_2} dAd\eta. \\
& \leq \sqrt{F_1(t)F_3(t)}. \quad (60)
\end{aligned}$$

By Lemmas 4 and 5 and the Hölder inequality, one has

By (40) and using the divergence theorem,

$$\begin{aligned}
& \int_0^t \int_0^1 \frac{\partial u}{\partial x_2} \frac{\partial^2 u}{\partial x_2^2} dx_1 d\eta = \int_0^t \int_{\Omega} \frac{\partial}{\partial x_2} \left(\frac{\partial S_1}{\partial x_2} \frac{\partial^2 u}{\partial x_2^2} \right) dAd\eta \\
& = \int_0^t \int_{\Omega} \frac{\partial^2 S_1}{\partial x_2^2} \frac{\partial^2 u}{\partial x_2^2} dAd\eta + \int_0^t \int_{\Omega} \frac{\partial S_1}{\partial x_2} \frac{\partial^3 u}{\partial x_2^3} dAd\eta \\
& \leq \frac{1}{2} \int_0^t \int_{\Omega} \left(\frac{\partial^2 S_1}{\partial x_2^2} \right)^2 dAd\eta + \frac{1}{2} \int_0^t \int_{\Omega} \left(\frac{\partial^2 u}{\partial x_2^2} \right)^2 dAd\eta \\
& \quad + \int_0^t \int_{\Omega} \frac{\partial S_1}{\partial x_2} \frac{\partial}{\partial x_2} \left\{ \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t) d\zeta \right) \frac{\partial u}{\partial x_2} - v \right. \\
& \quad \left. + \frac{\partial p_s}{\partial x_1} - \left(\int_{x_2}^0 \frac{\partial}{\partial x_1} T(x_1, \zeta, t) d\zeta \right) - \frac{\partial^2}{\partial x_1^2} u \right\} dAd\eta \quad (61) \\
& = \frac{1}{2} \int_0^t \int_{\Omega} \left(\frac{\partial^2 S_1}{\partial x_2^2} \right)^2 dAd\eta + \frac{1}{2} \int_0^t \int_{\Omega} \left| \nabla \frac{\partial u}{\partial x_2} \right|^2 dAd\eta \\
& \quad + \int_{\Omega} \frac{\partial S_1}{\partial x_2} \frac{\partial u}{\partial x_2} dA + \int_0^t \int_{\Omega} \frac{\partial (S_1)_t}{\partial x_2} \frac{\partial u}{\partial x_2} dAd\eta + \int_0^t \int_{\Omega} \frac{\partial S_1}{\partial x_2} \frac{\partial v}{\partial x_2} dAd\eta \\
& \quad + \int_0^t \int_{\Omega} \frac{\partial S_1}{\partial x_2} \left[u \frac{\partial^2 u}{\partial x_1 \partial x_2} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, \eta) d\zeta \right) \frac{\partial^2 u}{\partial x_2^2} \right] dAd\eta \\
& \quad + \int_0^t \int_{\Omega} \frac{\partial S_1}{\partial x_2} \frac{\partial T}{\partial x_1} dAd\eta + \int_0^t \int_{\Omega} \frac{\partial^2 S_1}{\partial x_1 \partial x_2} \frac{\partial^2 u}{\partial x_1 \partial x_2} dAd\eta.
\end{aligned}$$

By the Cauchy–Schwarz inequality and Lemmas 4 and 5, the following is obtained:

$$\begin{aligned}
 \int_{\Omega} \frac{\partial S_1}{\partial x_2} \frac{\partial u}{\partial x_2} dA &\leq \int_{\Omega} \left(\frac{\partial S_1}{\partial x_2} \right)^2 dAd\eta + \frac{1}{4} \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^2 dA, \\
 \int_0^t \int_{\Omega} \frac{\partial(S_1)_t}{\partial x_2} \frac{\partial u}{\partial x_2} dAd\eta &\leq \sqrt{\int_0^t \int_{\Omega} \left(\frac{\partial(S_1)_t}{\partial x_2} \right)^2 dAd\eta \int_0^t \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^2 dAd\eta} \\
 &\leq \sqrt{F_3(t) \int_0^t \int_{\Omega} \left(\frac{\partial(S_1)_t}{\partial x_2} \right)^2 dAd\eta}, \\
 \int_0^t \int_{\Omega} \frac{\partial S_1}{\partial x_2} \frac{\partial v}{\partial x_2} dAd\eta &\leq \sqrt{\int_0^t \int_{\Omega} \left(\frac{\partial S_1}{\partial x_2} \right)^2 dAd\eta \int_0^t \int_{\Omega} \left(\frac{\partial v}{\partial x_2} \right)^2 dAd\eta} \\
 &\leq \sqrt{F_3(t) \int_0^t \int_{\Omega} \left(\frac{\partial S_1}{\partial x_2} \right)^2 dAd\eta}, \\
 \int_0^t \int_{\Omega} \frac{\partial S_1}{\partial x_2} \frac{\partial T}{\partial x_1} dAd\eta &\leq \sqrt{\int_0^t \int_{\Omega} \left(\frac{\partial S_1}{\partial x_2} \right)^2 dAd\eta \int_0^t \int_{\Omega} \left(\frac{\partial T}{\partial x_1} \right)^2 dAd\eta} \\
 &\leq \sqrt{F_1(t) \int_0^t \int_{\Omega} \left(\frac{\partial S_1}{\partial x_2} \right)^2 dAd\eta}, \\
 \int_0^t \int_{\Omega} \frac{\partial^2 S_1}{\partial x_1 \partial x_2} \frac{\partial^2 u}{\partial x_1 \partial x_2} dAd\eta &\leq 2 \int_0^t \int_{\Omega} \left(\frac{\partial^2 S_1}{\partial x_1 \partial x_2} \right)^2 dAd\eta + \frac{1}{8} \int_0^t \int_{\Omega} \left| \nabla \frac{\partial u}{\partial x_2} \right|^2 dAd\eta,
 \end{aligned} \tag{62}$$

and by (13) with $\delta = 1$ and Lemma 5,

$$\begin{aligned}
& \int_0^t \int_{\Omega} \frac{\partial S_1}{\partial x_2} \left[u \frac{\partial^2 u}{\partial x_1 \partial x_2} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, \eta) d\zeta \right) \frac{\partial^2 u}{\partial x_2^2} \right] dA d\eta \\
&= - \int_0^t \int_{\Omega} \frac{\partial^2 S_1}{\partial x_1 \partial x_2} u \frac{\partial u}{\partial x_2} dA d\eta + \int_0^t \int_{\Omega} \frac{\partial^2 S_1}{\partial x_2^2} \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, \eta) d\zeta \right) \frac{\partial u}{\partial x_2} dA d\eta \\
&\leq \left[\int_0^t \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^2 dA \right]^{1/2} \left[\int_0^t \left(\int_{\Omega} u^4 dA \right)^{1/2} \left(\int_{\Omega} \left| \frac{\partial^2 S_1}{\partial x_1 \partial x_2} \right|^4 dA \right)^{1/2} d\eta \right]^{1/2} \\
&\quad + \left[\int_0^t \int_{\Omega} \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, \eta) d\zeta \right)^2 dA \right]^{1/2} \left[\int_0^t \left(\int_{\Omega} \frac{\partial^2 S_1}{\partial x_2^2} \right)^4 dA \right]^{1/2} \left(\int_{\Omega} \left| \frac{\partial u}{\partial x_2} \right|^4 dA \right)^{1/2} d\eta \right]^{1/2} \\
&\leq \max_t \left\{ \sqrt[4]{\int_{\Omega} \left| \frac{\partial^2 S_1}{\partial x_1 \partial x_2} \right|^4 dA} \right\} \sqrt{CF_3(t)} \left[\int_0^t \left(\int_{\Omega} u^2 dA + \int_{\Omega} |\nabla u|^2 dA \right) d\eta \right]^{1/2} \\
&\quad + \frac{h}{\pi} \max_t \left\{ \sqrt[4]{\int_{\Omega} \left| \frac{\partial^2 S_1}{\partial x_2^2} \right|^4 dA} \right\} \sqrt{CF_3(t)} \left[\int_0^t \left(\int_{\Omega} \left| \frac{\partial u}{\partial x_2} \right|^2 dA + \int_{\Omega} \left| \nabla \frac{\partial u}{\partial x_2} \right|^2 dA \right) d\eta \right]^{1/2} \\
&\leq \max_t \left\{ \sqrt[4]{\int_{\Omega} \left| \frac{\partial^2 S_1}{\partial x_1 \partial x_2} \right|^4 dA} \right\} \sqrt{CF_3(t)} \left[\int_0^t F_2(\eta) d\eta + F_3(t) \right]^{1/2} \\
&\quad + \frac{2h^2}{\pi^2} \max_t \left\{ \sqrt[4]{\int_{\Omega} \left| \frac{\partial^2 S_1}{\partial x_2^2} \right|^4 dA} \right\} CF_3(t) + \frac{1}{8} \left(F_3(t) + \int_0^t \int_{\Omega} \left| \nabla \frac{\partial u}{\partial x_2} \right|^2 dA d\eta \right).
\end{aligned} \tag{63}$$

Inserting the above two inequalities into (61), one obtains

$$\int_0^t \int_0^1 \frac{\partial u}{\partial x_2} \frac{\partial^2 u}{\partial x_2^2} dx_1 d\eta \leq \frac{1}{4} \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^2 dA + \frac{3}{4} \int_0^t \int_{\Omega} \left| \nabla \frac{\partial u}{\partial x_2} \right|^2 dA d\eta + a_4(t), \tag{64}$$

where

$$\begin{aligned}
 a_4(t) = & \frac{1}{2} \int_0^t \int_{\Omega} \left(\frac{\partial^2 S_1}{\partial x_2^2} \right)^2 dAd\eta + \int_{\Omega} \left(\frac{\partial S_1}{\partial x_2} \right)^2 dAd\eta + \sqrt{F_3(t) \int_0^t \int_{\Omega} \left(\frac{\partial(S_1)_t}{\partial x_2} \right)^2 dAd\eta} \\
 & + \sqrt{F_3(t) \int_0^t \int_{\Omega} \left(\frac{\partial S_1}{\partial x_2} \right)^2 dAd\eta} + \sqrt{F_1(t) \int_0^t \int_{\Omega} \left(\frac{\partial S_1}{\partial x_2} \right)^2 dAd\eta} + 2 \int_0^t \int_{\Omega} \left(\frac{\partial^2 S_1}{\partial x_1 \partial x_2} \right)^2 dAd\eta \\
 & + \max_t \left\{ \sqrt{\int_{\Omega} \left| \frac{\partial^2 S_1}{\partial x_1 \partial x_2} \right|^4 dA} \right\} \sqrt{CF_3(t) \left[\int_0^t F_2(\eta) d\eta + F_3(t) \right]^{1/2}} \\
 & + \frac{2h^2}{\pi^2} \max_t \left\{ \sqrt{\int_{\Omega} \left| \frac{\partial^2 S_1}{\partial x_2^2} \right|^4 dA} \right\} CF_3(t) + \frac{1}{8} F_3(t) + \frac{1}{2} \int_{\Omega} \left(\frac{\partial u_0}{\partial x_2} \right)^2 dA.
 \end{aligned} \tag{65}$$

Combining the inequalities (59), (60), and (64), the following is obtained:

$$\begin{aligned}
 & \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^2 dA + \int_0^t \int_{\Omega} \left| \nabla \frac{\partial u}{\partial x_2} \right|^2 dAd\eta \\
 & \leq 4 \int_0^t \int_{\Omega} \frac{\partial v}{\partial x_2} \frac{\partial u}{\partial x_2} dAd\eta + a_5(t),
 \end{aligned} \tag{66}$$

where $a_5(t) = 4a_4(t) + 4\sqrt{F_1(t)F_3(t)}$. Similar to (58) and (59), by (2.4)₂, one has

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega} \left(\frac{\partial v}{\partial x_2} \right)^2 dA + \int_{\Omega} \left| \nabla \frac{\partial v}{\partial x_2} \right|^2 dAd\eta = \int_0^t \int_0^1 \frac{\partial v}{\partial x_2} \frac{\partial^2 v}{\partial x_2^2} dx_1 d\eta - \int_{\Omega} \frac{\partial v}{\partial x_2} \frac{\partial u}{\partial x_2} dAd\eta \\
 & + \frac{1}{2} \int_{\Omega} \left(\frac{\partial v_0}{\partial x_2} \right)^2 dA + \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \frac{\partial v}{\partial x_1} - \frac{\partial u}{\partial x_1} \frac{\partial v}{\partial x_2} \right) \frac{\partial v}{\partial x_2} dA \\
 & - \int_{\Omega} \left[u \frac{\partial^2 v}{\partial x_1 x_2} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u(x_1, \zeta, t) d\zeta \right) \frac{\partial^2 v}{\partial x_2^2} \right] \frac{\partial v}{\partial x_2} dAd\eta.
 \end{aligned} \tag{67}$$

In a similar way,

$$\begin{aligned}
 & \int_{\Omega} \left(\frac{\partial v}{\partial x_2} \right)^2 dA + \int_0^t \int_{\Omega} \left| \nabla \frac{\partial v}{\partial x_2} \right|^2 dAd\eta \\
 & \leq -4 \int_0^t \int_{\Omega} \frac{\partial v}{\partial x_2} \frac{\partial u}{\partial x_2} dAd\eta + a_6(t),
 \end{aligned} \tag{68}$$

for some computable positive function $a_6(t)$. Combining (66) and (67), one obtains

$$\begin{aligned}
 & \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^2 dA + \int_{\Omega} \left(\frac{\partial v}{\partial x_2} \right)^2 dA + \int_0^t \int_{\Omega} \left| \nabla \frac{\partial u}{\partial x_2} \right|^2 dAd\eta \\
 & + \int_0^t \int_{\Omega} \left| \nabla \frac{\partial v}{\partial x_2} \right|^2 dAd\eta \leq F_5(t),
 \end{aligned} \tag{69}$$

where $F_5(t) = a_5(t) + a_6(t)$. Using Lemma 3 with $\delta = 1$, Lemma 5, and (69),

$$\begin{aligned}
& \int_0^t \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^4 dAd\eta + \int_0^t \int_{\Omega} \left(\frac{\partial v}{\partial x_2} \right)^4 dAd\eta \leq C \left[\int_0^t \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^2 dAd\eta \right. \\
& \quad \left. + \int_0^t \int_{\Omega} \left(\frac{\partial v}{\partial x_2} \right)^2 dAd\eta + \int_0^t \int_{\Omega} \left| \nabla \frac{\partial u}{\partial x_2} \right|^2 dAd\eta + \int_0^t \int_{\Omega} \left| \nabla \frac{\partial v}{\partial x_2} \right|^2 dAd\eta \right]^2 \\
& \leq C (F_3(t) + F_5(t))^2 \doteq F_6(t).
\end{aligned} \tag{70}$$

4. Convergence Results on Boundary Conditions

Supposing (u^*, v^*, T^*) also be the solutions of (9)–(11) with the boundary conditions $g_1(x_1, t) = g_2(x_1, t) = T^*(x_1, t) = 0$. Let

$$\tilde{u} = u - u^*, \tilde{v} = v - v^*, \tilde{T} = T - T^*, \pi_s = p_s - p_s^*, \quad (71)$$

then $(\tilde{u}, \tilde{v}, \tilde{T})$ satisfies

$$\begin{aligned}
& \frac{\partial \tilde{u}}{\partial t} + \tilde{u} \frac{\partial u}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, t) d\zeta \right) \frac{\partial u}{\partial x_2} + u^* \frac{\partial \tilde{u}}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u^*(x_1, \zeta, t) d\zeta \right) \frac{\partial \tilde{u}}{\partial x_2} \\
& \quad - \tilde{v} + \frac{\partial \pi_s}{\partial x_1} - \left(\int_{x_2}^0 \frac{\partial}{\partial x_1} \tilde{T}(x_1, \zeta, t) d\zeta \right) = \Delta \tilde{u}, \\
& \frac{\partial \tilde{v}}{\partial t} + \tilde{u} \frac{\partial v}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, t) d\zeta \right) \frac{\partial v}{\partial x_2} + u^* \frac{\partial \tilde{v}}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u^*(x_1, \zeta, t) d\zeta \right) \frac{\partial \tilde{v}}{\partial x_2} - \tilde{u} = \Delta \tilde{v}, \\
& \frac{\partial \tilde{T}}{\partial t} + \tilde{u} \frac{\partial T}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, t) d\zeta \right) \frac{\partial T}{\partial x_2} + u^* \frac{\partial \tilde{T}}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u^*(x_1, \zeta, t) d\zeta \right) \frac{\partial \tilde{T}}{\partial x_2} = \Delta \tilde{T},
\end{aligned} \tag{72}$$

with the following boundary conditions:

$$\begin{aligned}
& \left. \frac{\partial \tilde{u}}{\partial x_2} \right|_{x_2=0} = \alpha_1 g_1(x_1, t), \\
& \left. \frac{\partial \tilde{v}}{\partial x_2} \right|_{x_2=0} = \alpha_2 g_2(x_1, t), \\
& \left. \frac{\partial \tilde{u}}{\partial x_2} \right|_{x_2=-h} = \left. \frac{\partial \tilde{v}}{\partial x_2} \right|_{x_2=-h} = 0, \\
& (\tilde{u}, \tilde{v})|_{\Gamma_s} = 0, \\
& \left. \frac{\partial \tilde{T}}{\partial x_2} \right|_{x_2=0} = -\beta T^*(x_1, t), \\
& \left. \frac{\partial \tilde{T}}{\partial x_2} \right|_{x_2=-h} = 0, \\
& \left. \frac{\partial \tilde{T}}{\partial x_1} \right|_{\Gamma_s} = 0,
\end{aligned} \tag{73}$$

and the initial conditions:

$$(\tilde{u}, \tilde{v}, \tilde{T})|_{t=0} = (0, 0, 0). \tag{74}$$

The following theorem is obtained.

Theorem 1. Let $(\tilde{u}, \tilde{v}, \tilde{T})$ be the solutions of (71)–(73) with $g_1, g_2, T^* \in H^1(\Omega)$, $T^*, T_0 \in L_{\infty}(\Omega)$, and $u_0, v_0 \in L_2(\Omega)$. Then, $(\tilde{u}, \tilde{v}, \tilde{T})$ satisfies the inequality

$$\begin{aligned}
& \frac{3h^2}{\pi^2} T_m^2 \int_{\Omega} \tilde{u}^2 dA + \frac{3h^2}{\pi^2} T_m^2 \int_{\Omega} \tilde{v}^2 dA + \frac{h^2}{\pi^2} T_m^2 \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dAd\eta \\
& \quad + \frac{3h^2}{\pi^2} T_m^2 \int_0^t \int_{\Omega} |\nabla \tilde{v}|^2 dAd\eta + \int_{\Omega} \tilde{T}^2 dA \\
& \quad + \frac{1}{2} \int_0^t \int_{\Omega} |\nabla \tilde{T}|^2 dAd\eta \\
& \leq \int_0^t \int_0^1 \left[\frac{3h^2 \alpha_1}{\pi^2} T_m^2 g_1^2(x_1, \eta) + \frac{3h^2 \alpha_2}{\pi^2} T_m^2 g_2^2(x_1, \eta) \right. \\
& \quad \left. + \beta (T^*)^2(x_1, \eta) \right] dx_1 d\eta \\
& \quad + b(t) \int_0^t \int_0^1 \int_0^1 e^{\int_s^t b(\eta) d\eta} \left[\frac{3h^2 \alpha_1}{\pi^2} T_m^2 g_1^2(x_1, \eta) \right. \\
& \quad \left. + \frac{3h^2 \alpha_2}{\pi^2} T_m^2 g_2^2(x_1, \eta) + \beta (T^*)^2(x_1, \eta) \right] dx_1 d\eta ds,
\end{aligned} \tag{75}$$

which are the convergence results on boundary conditions g_1, g_2, T^* .

Proof. Now taking the inner product of equation (72)₁ with $\tilde{u}$, in $L_2(\Omega)$, the following is obtained:

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \tilde{u}^2 dA + \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dAd\eta &= \int_0^t \int_{\Omega} \tilde{u} \tilde{v} dAd\eta - \int_0^t \int_{\Omega} \frac{\partial \pi_s}{\partial x_1} \tilde{u} dAd\eta \\ &+ \int_0^t \int_0^1 \frac{\partial \tilde{u}}{\partial x_2} \tilde{u} dx_1 d\eta + \int_0^t \int_{\Omega} \left(\int_{x_2}^0 \frac{\partial}{\partial x_1} \tilde{T}(x_1, \zeta, \eta) d\zeta \right) \tilde{u} dAd\eta \\ &- \int_0^t \int_{\Omega} \left[\tilde{u} \frac{\partial u}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, \eta) d\zeta \right) \frac{\partial u}{\partial x_2} \right] \tilde{u} dAd\eta. \end{aligned} \quad (76)$$

By the Hölder inequality, Lemmas 1, 5, and 7, and the A-G mean inequality, the following is obtained:

$$\begin{aligned} & - \int_0^t \int_{\Omega} \left[\tilde{u} \frac{\partial u}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, \eta) d\zeta \right) \frac{\partial u}{\partial x_2} \right] \tilde{u} dAd\eta \leq \left[\int_0^t \int_{\Omega} \left(\frac{\partial u}{\partial x_1} \right)^2 dA \right]^{1/2} \left[\int_0^t \int_{\Omega} \tilde{u}^4 dAd\eta \right]^{1/2} \\ & + \left[\int_0^t \int_{\Omega} \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, \eta) d\zeta \right)^2 dAd\eta \right]^{1/2} \left[\int_0^t \int_{\Omega} \left(\frac{\partial u}{\partial x_2} \right)^4 dAd\eta \right]^{1/2} \left[\int_0^t \int_{\Omega} \tilde{u}^4 dAd\eta \right]^{1/4} \\ & \leq \sqrt{F_3(t)} C \left[\int_0^t \int_{\Omega} \tilde{u}^2 dAd\eta + \delta_1 \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dAd\eta \right] \\ & + \frac{\sqrt{C}h}{\pi} \sqrt[4]{F_6(t)} \left[\int_0^t \int_{\Omega} \left(\frac{\partial \tilde{u}}{\partial x_1} \right)^2 dAd\eta \right]^{1/2} \left[\int_0^t \int_{\Omega} \tilde{u}^2 dAd\eta + \delta_1 \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dAd\eta \right]^{1/2} \\ & \leq \sqrt{F_3(t)} C \left[\int_0^t \int_{\Omega} \tilde{u}^2 dAd\eta + \delta_1 \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dAd\eta \right] \\ & + \frac{\sqrt{C}h}{\pi} \sqrt[4]{F_6(t)} \left[\int_0^t \int_{\Omega} \left(\frac{\partial \tilde{u}}{\partial x_1} \right)^2 dAd\eta \right]^{1/2} \left[\int_0^t \int_{\Omega} \tilde{u}^2 dAd\eta \right]^{1/2} \\ & + \frac{\sqrt{C}\delta_1 h}{\pi} \sqrt[4]{F_6(t)} \left[\int_0^t \int_{\Omega} \left(\frac{\partial \tilde{u}}{\partial x_1} \right)^2 dAd\eta \right]^{1/2} \left[\int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dAd\eta \right]^{1/2} \\ & \leq \left[\sqrt{F_3(t)} C + \frac{Ch^2}{4\delta_2 \pi^2} \sqrt{F_6(t)} \right] \int_0^t \int_{\Omega} \tilde{u}^2 dAd\eta \\ & + \left[\sqrt{F_3(t)} C \delta_1 + \delta_2 + \frac{\sqrt{C}\delta_1 h}{\pi} \sqrt[4]{F_6(t)} \right] \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dAd\eta, \end{aligned} \quad (77)$$

for arbitrary positive constants δ_1 and δ_2 . By the Cauchy-Schwarz inequality again

$$\int_0^t \int_{\Omega} \left(\int_{x_2}^0 \frac{\partial}{\partial x_1} \tilde{T}(x_1, \zeta, \eta) d\zeta \right) \tilde{u} dA d\eta \leq \frac{\pi^2}{12h^2 T_m^2} \int_0^t \int_{\Omega} \left(\frac{\partial}{\partial x_1} \tilde{T} \right)^2 dA d\eta + \frac{3h^4}{\pi^2} T_m^2 \int_0^t \int_{\Omega} \tilde{u}^2 dA d\eta, \quad (78)$$

$$\begin{aligned} \int_0^t \int_0^1 \frac{\partial \tilde{u}}{\partial x_2} \tilde{u} dx_1 d\eta &= \alpha_1 \int_0^t \int_0^1 g_1(x_1, t) \tilde{u}(x_1, 0, t) dx_1 d\eta \\ &\leq \frac{\alpha_1}{2} \int_0^t \int_0^1 g_1^2(x_1, t) dx_1 d\eta + \frac{\alpha_1}{2} \int_0^t \int_0^1 \tilde{u}^2(x_1, 0, t) dx_1 d\eta. \end{aligned} \quad (79)$$

In view of (25), the following is obtained:

$$\begin{aligned} \int_0^t \int_0^1 \frac{\partial \tilde{u}}{\partial x_2} \tilde{u} dx_1 d\eta &\leq \frac{\alpha_1}{2} \int_0^t \int_0^1 g_1^2(x_1, t) dx_1 d\eta + \frac{m_3 \alpha_1}{2} \int_0^t \int_{\Omega} \tilde{u}^2 dA d\eta \\ &\quad + \frac{m_4 \alpha_1}{2} \int_0^t \int_{\Omega} |\tilde{u}| \left| \frac{\partial \tilde{u}}{\partial x_1} \right| dA d\eta \\ &\leq \frac{\alpha_1}{2} \int_0^t \int_0^1 g_1^2 dx_1 d\eta + \frac{(2m_3 + m_4 \delta_3^{-1}) \alpha_1}{4} \int_0^t \int_{\Omega} \tilde{u}^2 dA d\eta \\ &\quad + \frac{m_4 \alpha_1}{4} \delta_3 \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dA d\eta. \end{aligned} \quad (80)$$

Inserting (77)–(80) into (76), one has

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \tilde{u}^2 dA + \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dA d\eta &\leq \int_0^t \int_{\Omega} \tilde{u} \tilde{v} dA d\eta + \frac{\alpha_1}{2} \int_0^t \int_0^1 g_1^2 dx_1 d\eta + \frac{\pi^2}{12h^2 T_m^2} \int_0^t \int_{\Omega} |\nabla \tilde{T}|^2 dA d\eta \\ &\quad + \left[\sqrt{F_3(t)} C + \frac{Ch^2}{4\delta_2 \pi^2} \sqrt{F_6(t)} + \frac{(2m_3 + m_4 \delta_3^{-1}) \alpha_1}{4} + \frac{3h^4}{\pi^2} T_m^2 \right] \int_0^t \int_{\Omega} \tilde{u}^2 dA d\eta \\ &\quad + \left[\sqrt{F_3(t)} C \delta_1 + \delta_2 + \frac{\sqrt{C \delta_1} h}{\pi} \sqrt{F_6(t)} + \frac{m_4 \alpha_1}{4} \delta_3 \right] \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dA d\eta. \end{aligned} \quad (81)$$

Now taking the inner product of the second equation of (72) with $\tilde{v}$, the following is obtained:

$$\begin{aligned} \frac{1}{2} \int_{\Omega} \tilde{v}^2 dA + \int_0^t \int_{\Omega} |\nabla \tilde{v}|^2 dA d\eta &= - \int_0^t \int_{\Omega} \tilde{u} \tilde{v} dA d\eta + \int_0^t \int_0^1 \frac{\partial \tilde{v}}{\partial x_2} \tilde{v} dx_1 d\eta \\ &\quad - \int_0^t \int_{\Omega} \left[\tilde{u} \frac{\partial v}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, \eta) d\zeta \right) \frac{\partial v}{\partial x_2} \right] \tilde{v} dA d\eta. \end{aligned} \quad (82)$$

Computing as (80),

$$\begin{aligned} \int_0^t \int_0^1 \frac{\partial \tilde{v}}{\partial x_2} \tilde{v} dx_1 d\eta &\leq \frac{\alpha_2}{2} \int_0^t \int_0^1 g_2^2 dx_1 d\eta + \frac{(2m_3 + m_4 \delta_4^{-1}) \alpha_2}{4} \int_0^t \int_\Omega \tilde{v}^2 dAd\eta \\ &+ \frac{m_4 \alpha_2}{4} \delta_4 \int_0^t \int_\Omega |\nabla \tilde{v}|^2 dAd\eta. \end{aligned} \quad (83)$$

Computing as (77),

$$\begin{aligned} & - \int_0^t \int_\Omega \left[\tilde{u} \frac{\partial v}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, \eta) d\zeta \right) \frac{\partial v}{\partial x_2} \right] \tilde{v} dAd\eta \\ & \leq \frac{C\sqrt{F_3(t)}}{2} \left(\int_0^t \int_\Omega \tilde{u}^2 dAd\eta + \delta_5 \int_0^t \int_\Omega |\nabla \tilde{u}|^2 dAd\eta \right) \\ & + \frac{C\sqrt{F_3(t)}}{2} \left(\int_0^t \int_\Omega \tilde{v}^2 dAd\eta + \delta_5 \int_0^t \int_\Omega |\nabla \tilde{v}|^2 dAd\eta \right) \\ & + \frac{\sqrt{CF_6(t)}h}{\sqrt{\delta_6}\pi} \left(\int_0^t \int_\Omega \tilde{v}^2 dAd\eta + \delta_6 \int_0^t \int_\Omega |\nabla \tilde{u}|^2 dAd\eta + \delta_6 \int_0^t \int_\Omega |\nabla \tilde{v}|^2 dAd\eta \right), \end{aligned} \quad (84)$$

where δ_5 and δ_6 are the arbitrary constants. Inserting (83) and (84) into (82), the following is obtained:

$$\begin{aligned} \frac{1}{2} \int_\Omega \tilde{v}^2 dA + \int_0^t \int_\Omega |\nabla \tilde{v}|^2 dAd\eta &\leq \frac{\alpha_2}{2} \int_0^t \int_0^1 g_2^2 dx_1 d\eta + \frac{C\sqrt{F_3(t)}}{2} \int_0^t \int_\Omega \tilde{u}^2 dAd\eta \\ &+ \left[\frac{(2m_3 + m_4 \delta_4^{-1}) \alpha_2}{4} + \frac{C\sqrt{F_3(t)}}{2} + \frac{\sqrt{CF_6(t)}h}{\sqrt{\delta_6}\pi} \right] \int_0^t \int_\Omega \tilde{v}^2 dAd\eta \\ &+ \left[\frac{m_4 \alpha_2}{4} \delta_4 + \frac{C\sqrt{F_3(t)}}{2} \delta_5 + \frac{\sqrt{C\delta_6 F_6(t)}h}{\pi} \right] \int_0^t \int_\Omega |\nabla \tilde{v}|^2 dAd\eta \\ &+ \frac{\sqrt{C\delta_6 F_6(t)}h}{\pi} \int_0^t \int_\Omega |\nabla \tilde{u}|^2 dAd\eta. \end{aligned} \quad (85)$$

(81) and (85) are combined and suitable δ_i , ($i = 1, 2, \dots, 6$) are chosen such that

$$\begin{aligned} \sqrt{F_3(t)} C \delta_1 + \delta_2 + \frac{\sqrt{C\delta_1}h}{\pi} \sqrt{F_6(t)} + \frac{m_4 \alpha_1}{4} \delta_3 + \frac{\sqrt{C\delta_6 F_6(t)}h}{\pi} &= \frac{1}{2}, \\ \frac{m_4 \alpha_2}{4} \delta_4 + \frac{C\sqrt{F_3(t)}}{2} \delta_5 + \frac{\sqrt{C\delta_6 F_6(t)}h}{\pi} &= \frac{1}{2}, \end{aligned} \quad (86)$$

to have

$$\begin{aligned}
 & \int_{\Omega} \tilde{u}^2 dA + \int_{\Omega} \tilde{v}^2 dA + \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dAd\eta + \int_0^t \int_{\Omega} |\nabla \tilde{v}|^2 dAd\eta \leq \alpha_1 \int_0^t \int_0^1 g_1^2 dx_1 d\eta \\
 & \quad + \alpha_2 \int_0^t \int_0^1 g_2^2 dx_1 d\eta + \frac{\pi^2}{6h^2 T_m^2} \int_0^t \int_{\Omega} |\nabla \tilde{T}|^2 dAd\eta \\
 & + \left[2\sqrt{F_3(t)} C + \frac{Ch^2}{2\delta_2 \pi^2} \sqrt{F_6(t)} + \frac{(2m_3 + m_4 \delta_3^{-1})\alpha_1}{2} + C\sqrt{F_3(t)} + \frac{6h^4}{\pi^2 T_m^2} \right] \int_0^t \int_{\Omega} \tilde{u}^2 dAd\eta \\
 & \quad + \left[\frac{(2m_3 + m_4 \delta_4^{-1})\alpha_2}{2} + C\sqrt{F_3(t)} + \frac{2\sqrt{CF_6(t)}h}{\sqrt{\delta_6}\pi} \right] \int_0^t \int_{\Omega} \tilde{v}^2 dAd\eta.
 \end{aligned} \tag{87}$$

The inner product of equation (72)₃ is taken with $\tilde{T}$ to have

$$\begin{aligned}
 & \frac{1}{2} \int_{\Omega} \tilde{T}^2 dA + \int_0^t \int_{\Omega} |T|^2 dAd\eta = \int_0^t \int_0^1 \frac{\partial \tilde{T}}{\partial x_2} \tilde{T} dx_1 d\eta \\
 & \quad - \int_0^t \int_{\Omega} \left[\tilde{u} \frac{\partial T}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, \eta) d\zeta \right) \frac{\partial T}{\partial x_2} \right] \tilde{T} dAd\eta.
 \end{aligned} \tag{88}$$

Similar to the computation in (79) and (80), the following is obtained:

$$\begin{aligned}
 & \int_0^t \int_0^1 \frac{\partial \tilde{T}}{\partial x_2} \tilde{T} dx_1 d\eta \leq \frac{\beta}{2} \int_0^t \int_0^1 (T^*)^2 dx_1 d\eta \\
 & \quad + \frac{(2m_3 + m_4 \delta_7^{-1})\beta}{4} \int_0^t \int_{\Omega} \tilde{T}^2 dAd\eta \\
 & \quad + \frac{m_4 \beta}{4} \delta_7 \int_0^t \int_{\Omega} |\nabla \tilde{T}|^2 dAd\eta,
 \end{aligned} \tag{89}$$

for an arbitrary positive constant δ_7 . On integrating by parts, one has

$$- \int_0^t \int_{\Omega} \left[u^* \frac{\partial \tilde{T}}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} u^*(x_1, \zeta, \eta) d\zeta \right) \frac{\partial \tilde{T}}{\partial x_2} \right] \tilde{T} dAd\eta = 0. \tag{90}$$

By the Hölder inequality and Lemmas 2 and 6, the following is obtained:

$$\begin{aligned}
& - \int_0^t \int_{\Omega} \left[\tilde{u} \frac{\partial T}{\partial x_1} - \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, \eta) d\zeta \right) \frac{\partial T}{\partial x_2} \right] \tilde{T} dA d\eta \\
& = \int_0^t \int_{\Omega} \tilde{u} T \frac{\partial \tilde{T}}{\partial x_1} dA d\eta - \int_0^t \int_{\Omega} \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, \eta) d\zeta \right) T \frac{\partial \tilde{T}}{\partial x_2} dA d\eta \\
& \leq T_m \left(\int_0^t \int_{\Omega} \left(\frac{\partial \tilde{T}}{\partial x_1} \right)^2 dA d\eta \right)^{1/2} \left(\int_0^t \int_{\Omega} \tilde{u}^2 dA d\eta \right)^{1/2} \\
& \quad + T_m \left(\int_0^t \int_{\Omega} \left(\int_{-h}^{x_2} \frac{\partial}{\partial x_1} \tilde{u}(x_1, \zeta, \eta) d\zeta \right)^2 dA d\eta \right)^{1/2} \left(\int_0^t \int_{\Omega} \left(\frac{\partial \tilde{T}}{\partial x_2} \right)^2 dA d\eta \right)^{1/2} \\
& \leq T_m^2 \int_0^t \int_{\Omega} \tilde{u}^2 dA d\eta + \frac{h^2}{\pi^2} T_m^2 \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dA d\eta + \frac{1}{4} \int_0^t \int_{\Omega} |\nabla \tilde{T}|^2 dA d\eta.
\end{aligned} \tag{91}$$

Inserting (89)–(91) into (88) and choosing $\delta_7 = (1/m_4\beta)$, one gets

$$\begin{aligned}
\int_{\Omega} \tilde{T}^2 dA + \int_0^t \int_{\Omega} |\nabla \tilde{T}|^2 dA d\eta & \leq \beta \int_0^t \int_0^1 (T^*)^2 dx_1 d\eta + \frac{(2m_3 + m_4\delta_7^{-1})\beta}{2} \int_0^t \int_{\Omega} \tilde{T}^2 dA d\eta \\
& \quad + 2T_m^2 \int_0^t \int_{\Omega} \tilde{u}^2 dA d\eta + \frac{2h^2}{\pi^2} T_m^2 \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dA d\eta.
\end{aligned} \tag{92}$$

Combining (87) and (92), the following is obtained:

$$\begin{aligned}
& \frac{3h^2}{\pi^2} T_m^2 \int_{\Omega} \tilde{u}^2 dA + \frac{3h^2}{\pi^2} T_m^2 \int_{\Omega} \tilde{v}^2 dA + \frac{h^2}{\pi^2} T_m^2 \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dA d\eta \\
& \quad + \frac{3h^2}{\pi^2} T_m^2 \int_0^t \int_{\Omega} |\nabla \tilde{v}|^2 dA d\eta + \int_{\Omega} \tilde{T}^2 dA + \frac{1}{2} \int_0^t \int_{\Omega} |\nabla \tilde{T}|^2 dA d\eta \\
& \leq \int_0^t \int_0^1 \left[\frac{3h^2\alpha_1}{\pi^2} T_m^2 g_1^2 + \frac{3h^2\alpha_2}{\pi^2} T_m^2 g_2^2 + \beta (T^*)^2 \right] dx_1 d\eta \\
& \quad + b(t) \left(\frac{3h^2}{\pi^2} T_m^2 \int_0^t \int_{\Omega} \tilde{u}^2 dA d\eta + \frac{3h^2}{\pi^2} T_m^2 \int_0^t \int_{\Omega} \tilde{v}^2 dA d\eta + \int_0^t \int_{\Omega} \tilde{T}^2 dA d\eta \right),
\end{aligned} \tag{93}$$

where

$$b(t) = \max \left\{ 2\sqrt{F_3(t)}C + \frac{Ch^2}{2\delta_2\pi^2} \sqrt{F_6(t)} + \frac{(2m_3 + m_4\delta_3^{-1})\alpha_1}{2} + C\sqrt{F_3(t)} + \frac{6h^4}{\pi^2}T_m^2 + \frac{2\pi^2}{3h^2}, \right. \\ \left. \frac{(2m_3 + m_4\delta_4^{-1})\alpha_2}{2} + C\sqrt{F_3(t)} + \frac{2\sqrt{CF_6(t)}h}{\sqrt{\delta_6}\pi}, \frac{(2m_3 + m_4\delta_7^{-1})\beta}{2} \right\}. \quad (94)$$

Therefore, by the Gronwall inequality,

$$\frac{3h^2}{\pi^2}T_m^2 \int_0^t \int_{\Omega} \tilde{u}^2 dA d\eta + \frac{3h^2}{\pi^2}T_m^2 \int_0^t \int_{\Omega} \tilde{v}^2 dA d\eta + \int_0^t \int_{\Omega} \tilde{T}^2 dA d\eta \\ \leq \int_0^t \int_0^s \int_0^1 e^{\int_s^t b(\eta) d\eta} \left[\frac{3h^2\alpha_1}{\pi^2}T_m^2 g_1^2(x_1, \eta) + \frac{3h^2\alpha_2}{\pi^2}T_m^2 g_2^2(x_1, \eta) + \beta(T^*)^2(x_1, \eta) \right] dx_1 d\eta ds. \quad (95)$$

Then, returning to (93) it can be seen that

$$\frac{3h^2}{\pi^2}T_m^2 \int_{\Omega} \tilde{u}^2 dA + \frac{3h^2}{\pi^2}T_m^2 \int_{\Omega} \tilde{v}^2 dA + \frac{h^2}{\pi^2}T_m^2 \int_0^t \int_{\Omega} |\nabla \tilde{u}|^2 dA d\eta \\ + \frac{3h^2}{\pi^2}T_m^2 \int_0^t \int_{\Omega} |\nabla \tilde{v}|^2 dA d\eta + \int_{\Omega} \tilde{T}^2 dA + \frac{1}{2} \int_0^t \int_{\Omega} |\nabla \tilde{T}|^2 dA d\eta \\ \leq \int_0^t \int_0^1 \left[\frac{3h^2\alpha_1}{\pi^2}T_m^2 g_1^2(x_1, \eta) + \frac{3h^2\alpha_2}{\pi^2}T_m^2 g_2^2(x_1, \eta) + \beta(T^*)^2(x_1, \eta) \right] dx_1 d\eta \\ + b(t) \int_0^t \int_0^s \int_0^1 e^{\int_s^t b(\eta) d\eta} \left[\frac{3h^2\alpha_1}{\pi^2}T_m^2 g_1^2(x_1, \eta) + \frac{3h^2\alpha_2}{\pi^2}T_m^2 g_2^2(x_1, \eta) + \beta(T^*)^2(x_1, \eta) \right] dx_1 d\eta ds. \quad (96)$$

It is obvious that $u \longrightarrow u^*$, $v \longrightarrow v^*$, and $T \longrightarrow T^*$ in $L_2(\Omega)$, as well as $H^1(\Omega)$, when $g_1, g_2, T^* \longrightarrow 0$ from inequality (96). $\square$

Data Availability

All data generated or analyzed during this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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