Research Article

Regulator-Based Risk Statistics for Portfolios

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Risk statistic is a critical factor not only for risk analysis but also for financial application. However, the traditional risk statistics may fail to describe the characteristics of regulator-based risk. In this paper, we consider the regulator-based risk statistics for portfolios. By further developing the properties related to regulator-based risk statistics, we are able to derive dual representation for such risk.

1. Introduction

Risk measure is a popular topic in both financial application and theoretical research. The quantitative calculation of risk involves two problems: choosing an appropriate risk model and allocating risk to individual institutions. This has led to further research on risk statistics. In a seminal paper, Sun et al. and Liu et al. [1, 2] first introduced the class of natural risk statistics with representation results. Furthermore, Ahmed et al. [3] derived an alternative proof for the natural risk statistics. Later, Tian and Jiang and Tian and Suo [4, 5] obtained the representation results for convex risk statistics and quasiconvex risk statistics, respectively.

However, traditional risk statistics may fail to describe the characteristics of regulator-based risk. Therefore, the study of regulator-based risk statistics is particularly interesting. On the other hand, in the abovementioned research on risk statistics, the set-valued risk was never studied. Jouini et al. [6] pointed out that a set-valued risk measure is more appropriate than a scalar risk measure especially in the case where several different kinds of currencies are involved when one is determining capital requirements for the portfolio. Indeed, a natural set-valued risk statistic can be considered as an empirical (or a data-based) version of a set-valued risk measure. More recent studies of set-valued risk measures include those of [7–14] and the references therein.

The main focus of this paper is regulator-based risk statistics for portfolios. In this context, both empirical

versions and data-based versions of regulator-based risk measures are discussed. By further developing the properties related to regulator-based risk statistics, we are able to derive their dual representations. Indeed, this class of risk statistics can be considered as an extension of those introduced in [15–17].

The remainder of this paper is organized as follows. In Section 2, we briefly introduce some preliminaries. In Section 3, we state the main results of regulator-based risk statistics, including the dual representations. Section 4 investigates the data-based versions of regulator-based risk measures. Finally, in Section 5, the main proofs in this paper are discussed.

2. Preliminary Information

In this section, we briefly introduce some preliminaries that are used throughout this paper. Let $d \geq 1$ be a fixed positive integer. The space $\mathbb{R}^{d \times n}$ represents the set of financial risk positions. With positive values of $X \in \mathbb{R}^{d \times n}$, we denote the gains while the negative denotes the losses. Let n_j be the sample size of $D = (X_1, \dots, X_d)$ in the j^{th} scenario, $j = 1, \dots, l$. Let $n := n_1 + \dots + n_l$. More precisely, suppose that the behavior of D is represented by a collection of data $X = (X_1, \dots, X_d) \in \mathbb{R}^n \times \dots \times \mathbb{R}^n$, where $X_i = (X^{i,1}, \dots, X^{i,l}) \in \mathbb{R}^n$ and $X^{i,j} = (x_1^{i,j}, \dots, x_{n_j}^{i,j}) \in \mathbb{R}^{n_j}$ is the data subset that corresponds to the j^{th} scenario with respect to X_i . For each $j = 1, \dots, l$, $h = 1, \dots, n_j$, $X_h^j :=$

$(x_h^{1,j}, x_h^{2,j}, \dots, x_h^{d,j})$ is the data subset that corresponds to the h^{th} observation of D in the j^{th} scenario.

In this paper, an element z of $\mathbb{R}^d$ is denoted by $z := (z_1, \dots, z_d)$. An element X of $\mathbb{R}^{d \times n}$ is denoted by $X := (X_1, \dots, X_d) := (x_1^{1,1}, \dots, x_n^{1,1}, \dots, x_1^{1,l}, \dots, x_n^{1,l}, \dots, x_1^{d,1}, \dots, x_n^{d,1}, \dots, x_1^{d,l}, \dots, x_n^{d,l})$. Let K be a closed convex polyhedral cone of $\mathbb{R}^d$ where $K \supseteq \mathbb{R}_+^d := \{(x_1, \dots, x_d) \in \mathbb{R}^d; x_i \geq 0, 1 \leq i \leq d\}$ and $K \cap \mathbb{R}_-^d = \emptyset$ where $\mathbb{R}_-^d := \{(x_1, \dots, x_d) \in \mathbb{R}^d; x_i \leq 0, 1 \leq i \leq d\}$. Let K^+ be the positive dual cone of K , that is, $K^+ := \{u \in \mathbb{R}^d; u^{\text{tr}} v \geq 0 \text{ for any } v \in K\}$, where u^{tr} means the transpose of u . For any $X = (X_1, \dots, X_d)$, $Y = (Y_1, \dots, Y_d) \in \mathbb{R}^{d \times n}$, $X + Y$ stands for $(X_1 + Y_1, \dots, X_d + Y_d)$ and aX stands for $(aX_1, \dots, aX_d)$ for $a \in \mathbb{R}$. Denote $K1_n := \{(z_1 1_n, z_2 1_n, \dots, z_d 1_n) : z \in K\}$ and $z1_n := \{(z, z, \dots, z) : z \in \mathbb{R}\} \in \mathbb{R}^n$ where $1_n := (1, \dots, 1) \in \mathbb{R}^n$. By $(K1_n)^+$, and we denote the positive dual cone of $K1_n$ in $\mathbb{R}^{d \times n}$, i.e., $(K1_n)^+ := \{w \in \mathbb{R}^{d \times n}; wz^{\text{tr}} \geq 0 \text{ for any } z \in K\}$. The partial order with respect to K is defined as $a \leq_K b$, which means $b - a \in K$ where $a, b \in \mathbb{R}^d$, and $X \leq_{K1_n} Y$ means $Y - X \in K1_n$ where $X, Y \in \mathbb{R}^{d \times n}$.

Let $M := \mathbb{R}^m \times \{0\}^{d-m}$ be the linear subspace of $\mathbb{R}^d$ for $1 \leq m \leq d$. The introduction of M was considered in [6, 9]. Denote $M_+ := M \cap \mathbb{R}_+^d$, where $\mathbb{R}_+^d := \{(x_1, \dots, x_d) \in \mathbb{R}^d; x_i \geq 0, 1 \leq i \leq d\}$ and $M^\perp := \{0\}^m \times \mathbb{R}^{d-m}$. Therefore, a regulator can only accept security deposits in the first m reference instruments. Denoting $K_M := K \cap M$ by the closed convex polyhedral cone in M , $K_M^+ := \{u \in M; u^{\text{tr}} z \geq 0 \text{ for any } z \in K_M\}$ the positive dual cone of K_M in M and $\text{int } K_M$ is the interior of K_M in M . We denote $Q_M^* := \{A \subset M; A = \text{clco}(A + K_M)\}$ and $Q_{M^+}^* := \{A \subset K_M; A = \text{clco}(A + K_M)\}$, where the $\text{clco}(A)$ represents the closed convex hull of A .

By [18], a set-valued risk statistic is any map ρ ,

$$\rho : \mathbb{R}^{d \times n} \longrightarrow 2^M, \quad (1)$$

which can be considered as an empirical (or a data-based) version of a set-valued risk measure. The axioms related to this set-valued risk statistic are organized as follows:

[A0] Normalization: $K_M \subseteq \rho(0)$ and $\rho(0) \cap -\text{int } K_M = \emptyset$

[A1] Monotonicity: for any $X, Y \in \mathbb{R}^{d \times n}$, $X - Y \in K1_n$ implies that $\rho(X) \supseteq \rho(Y)$

[A2] M-translative invariance: for any $X \in \mathbb{R}^{d \times n}$ and $z \in \mathbb{R}^d$, $\rho(X - z1_n) = \rho(X) + z$

[A3] Convexity: for any $X, Y \in \mathbb{R}^{d \times n}$ and $\lambda \in [0, 1]$, $\rho(\lambda(X) + (1 - \lambda)Y) \supseteq \lambda\rho(X) + (1 - \lambda)\rho(Y)$

[A4] Positive homogeneity: $\rho(\lambda X) = \lambda\rho(X)$ for any $X \in \mathbb{R}^{d \times n}$ and $\lambda > 0$

[A5] Subadditivity: $\rho(X + Y) \supseteq \rho(X) + \rho(Y)$ for any $X, Y \in \mathbb{R}^{d \times n}$

We end this section with more notations. A function $\rho : \mathbb{R}^{d \times n} \longrightarrow 2^M$ is said to be proper if $\text{dom } \rho := \{X \in \mathbb{R}^{d \times n}; \rho(X) \neq \emptyset\} \neq \emptyset$ and $\rho(X) \neq M$ for all $X \in \text{dom } \rho$. ρ is said to be closed if $\text{graph } \rho$ is a closed set. For the properties of the graphs, see [19–21].

3. Empirical Versions of Regulator-Based Risk Measures

In this section, we state the dual representations of regulator-based risk statistics, which are the empirical versions of regulator-based risk measures. Firstly, for any $X \in \mathbb{R}^{d \times n}$, $X \wedge_{K1_n} 0$ is defined as follows:

$$X \wedge_{K1_n} 0 := \begin{cases} X, & X \notin K1_n, \\ 0, & X \in K1_n. \end{cases} \quad (2)$$

Therefore, the position that belongs to K regarded is as 0 position. Next, we derive the properties related to regulator-based risk statistics.

Definition 1. A regulator-based risk statistic is a function $\varrho : \mathbb{R}^{d \times n} \longrightarrow Q_{M^+}^*$ that satisfies the following properties:

[P0] Normalization: $K_M \subseteq \varrho(0)$ and $\varrho(0) \cap -\text{int } K_M = \emptyset$

[P1] Cash cover: for any $z \in K_M$, $z \in \varrho(-z1_n)$

[P2] Monotonicity: for any $X, Y \in \mathbb{R}^{d \times n}$, $X - Y \in \mathbb{R}^{d \times n} \cap K1_n$ implies that $\varrho(X) \supseteq \varrho(Y)$

[P3] Regulator-dependence: for any $X \in \mathbb{R}^{d \times n}$, $\varrho(X) = \varrho(X \wedge_{K1_n} 0)$

[P4] Convexity: for any $X, Y \in \mathbb{R}^{d \times n}$ and $\lambda \in [0, 1]$, $\varrho(\lambda(X) + (1 - \lambda)Y) \supseteq \lambda\varrho(X) + (1 - \lambda)\varrho(Y)$

Remark 1. The property of [P1] means any fixed negative risk position $-z$ can be canceled by its positive quality z ; [P2] says that if X_1 is bigger than X_2 for the partial order in K , then the X_1 need less capital requirement than X_2 , so $\varrho(X_1)$ contains $\varrho(X_2)$; [P3] means the regulator-based risk statistics start only from the viewpoint of regulators which only care about the positions that need to pay capital requirements, while the positions that belong to K regarded as 0 position.

We now construct an example for regulator-based risk statistics.

Example 1. The coherent risk measure $AV@R$ was studied by Föllmer and Schied [22] in detail. They have given several representations and many properties such as law invariance and the Fatou property [12]. First, they introduced set-valued $AV@R$, where the representation result is derived. Moreover, they also proved that it is a set-valued coherent risk measure. We now define the regulator-based average value at risk. For any $X \in \mathbb{R}^{d \times n}$ and $0 < \alpha < 1$, we define $\varrho(X)$ as

$$\begin{aligned} \varrho(X) &:= AV@R_\alpha^{\text{loss}}(X), \\ &:= \inf_{z \in \mathbb{R}^d} \left\{ \frac{1}{\alpha} \left(- (X \wedge_{K1_n} 0) \Big|_M + z \right)^+ - z \right\} + \mathbb{R}_+^m, \end{aligned} \quad (3)$$

It is clear that ϱ satisfies the cash cover, monotonicity, regulator dependence properties, and convexity, so ϱ is a regulator-based risk statistic.

Definition 2. Let $Y \in \mathbb{R}^{d \times n}$, $u \in M$. Define a function $S_{(Y,u)}(X) : \mathbb{R}^{d \times n} \rightarrow 2^M$ as

$$S_{(Y,u)}(X) := \{z \in M : X^{\text{tr}} Y \leq u^{\text{tr}} z\}. \quad (4)$$

In fact, the $S_{(Y,u)}(X)$ is the support function of X . Before we derive the dual representations of regulator-based risk statistics, the Legendre–Fenchel conjugate theory ([9]) should be recalled.

Lemma 1 (see [9], Theorem 2). *Let $R : \mathbb{R}^{d \times n} \rightarrow Q_M^t$ be a set-valued closed convex function. Then, the Legendre–Fenchel conjugate and the biconjugate of R can be defined, respectively, as*

$$\begin{aligned} -R^*(Y, u) &:= \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} (R(X) + S_{(Y,u)}(-X)), \\ Y &\in \mathbb{R}^{d \times n}, u \in \mathbb{R}^d, \end{aligned} \quad (5)$$

$$\begin{aligned} R(X) = R^{**}(X) &:= \bigcap_{(Y,u) \in \mathbb{R}^{d \times n} \times K_M^+ \setminus \{0\}} \\ &[-R^*(Y, u) + S_{(Y,u)}(X)], \quad X \in \mathbb{R}^{d \times n}. \end{aligned} \quad (6)$$

Definition 3 (indicator function). For any $Z \subseteq \mathbb{R}^{d \times n}$, the Q_M^t -valued indicator function $I_Z : \mathbb{R}^{d \times n} \rightarrow Q_M^t$ is defined as

$$I_Z(X) := \begin{cases} \text{cl} K_M, & X \in Z, \\ \phi, & X \notin Z. \end{cases} \quad (7)$$

Remark 2. The conjugate of Q_M^t -valued indicator function I_Z is

$$-(I_Z)^*(Y, u) := \text{cl} \bigcup_{X \in Z} S_{(Y,u)}(-X). \quad (8)$$

Remark 3. It is easy to see that the regulator-based risk statistic ϱ does not have cash additivity, see [9]. However, ϱ has cash subadditivity introduced in [23, 24]. Indeed, from

Theorem 2 of [10], ϱ satisfies the Fatou property. Then, considering any $X \in \mathbb{R}^{d \times n}$ and $z \in K_M$, for any $\varepsilon \in (0, 1)$, we have

$$\begin{aligned} \varrho((1-\varepsilon)X - z1_n) &= \varrho\left((1-\varepsilon)X + \varepsilon\left(-\frac{z}{\varepsilon}\right)1_n\right), \\ &\supseteq (1-\varepsilon)\varrho(X) + \varepsilon\varrho\left(-\frac{z}{\varepsilon}1_n\right), \\ &\supseteq (1-\varepsilon)\varrho(X) + z, \end{aligned} \quad (9)$$

where the last inclusion is due to the property [P1]. Using the arbitrariness of ε , we have the following lemma.

Lemma 2. *Assume that ϱ is a regulator-based risk statistic. For any $z \in \mathbb{R}_+^d$, $X \in \mathbb{R}^{d \times n}$,*

$$\varrho(X - z1_n) \supseteq \varrho(X) + z, \quad (10)$$

which also implies

$$\varrho(X + z1_n) \subseteq \varrho(X) - z. \quad (11)$$

Proposition 1. *Let $\varrho : \mathbb{R}^{d \times n} \rightarrow Q_{M^+}^t$ be a proper closed convex regulator-based risk statistic with $u \in \{(-\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{1,j}, \dots, -\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{d,j}) + M^\perp\} \cap K_M^+ \setminus \{0\}$. Then,*

$$-\varrho^*(Y, u) = \begin{cases} \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X), & Y \in -\mathbb{R}_+^{d \times n} \cap (K^+ 1_n), \\ M, & \text{elsewhere.} \end{cases} \quad (12)$$

Now, we state the main result of this paper, the dual representations of regulator-based risk statistics.

Theorem 1. *If $\varrho : \mathbb{R}^{d \times n} \rightarrow Q_{M^+}^t$ is a proper closed convex regulator-based risk statistic, then there is a $-\alpha : (-\mathbb{R}_+^{d \times n} \cap K^+ 1_n) \times K_M^+ \setminus \{0\} \rightarrow Q_{M^+}^t$, which is not identically M of the set*

$$W = \left\{ (Y, u) \in (-\mathbb{R}_+^{d \times n} \cap K^+ 1_n) \times K_M^+ \setminus \{0\} : u \in \left(-\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{1,j}, \dots, -\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{d,j} \right) + M^\perp \right\}, \quad (13)$$

such that for any $X \in \mathbb{R}^{d \times n}$,

$$\varrho(X) = \bigcap_{(Y,u) \in W} \{-\alpha(Y, u) + S_{(Y,u)}(X \wedge_{K1_n} 0)\}. \quad (14)$$

4. Alternative Data-Based Versions of Regulator-Based Risk Measures

In this section, we develop another framework, the data-based versions of regulator-based risk measures. This framework is a little different from the previous one. However, almost all the arguments are the same as those in the previous section.

Therefore, we only state the corresponding notations and results and omit all the proofs and relevant explanations.

We replace M by $\tilde{M} \in \mathbb{R}^{d \times n}$ that is a linear subspace of $\mathbb{R}^{d \times n}$. We also replace K by $\tilde{K} \in \mathbb{R}^{d \times n}$ that is a closed convex polyhedral cone where $\tilde{K} \supseteq \mathbb{R}_{++}^{d \times n}$. The partial order with respect to $\tilde{K}$ is defined as $X \leq_{\tilde{K}} Y$, which means $Y - X \in \tilde{K}$. Let $\tilde{M}_+ := \tilde{M} \cap \mathbb{R}_+^{d \times n}$. Denoting $\tilde{K}_{\tilde{M}} := \tilde{K} \cap \tilde{M}$ by the closed convex polyhedral cone in $\tilde{M}$, $\tilde{K}_{\tilde{M}}^+ := \{\tilde{u} \in \tilde{M} : \tilde{u}^{\text{tr}} \tilde{z} \geq 0 \text{ for any } \tilde{z} \in \tilde{K}_{\tilde{M}}\}$ is the positive dual cone of $\tilde{K}_{\tilde{M}}$ in $\tilde{M}$ and $\text{int } \tilde{K}_{\tilde{M}}^+$ is the interior of $\tilde{K}_{\tilde{M}}^+$ in $\tilde{M}$. We denote $Q_{\tilde{M}}^t := \{\tilde{A} \subset \tilde{M} : \tilde{A} = \text{clco}(\tilde{A} + \tilde{K}_{\tilde{M}}^+)\}$ and

$Q_{\tilde{M}}^t := \{\tilde{A} \subset \tilde{K}_{\tilde{M}} : \tilde{A} = \text{clco}(\tilde{A} + \tilde{K}_{\tilde{M}})\}$. We still start from the viewpoint of regulators which only care about the positions that need to pay capital requirements. Therefore, for any $X \in \mathbb{R}^{d \times n}$, we define $X \wedge_{\tilde{K}} 0$ as

$$X \wedge_{\tilde{K}} 0 := \begin{cases} X, & X \notin \tilde{K}, \\ 0, & X \in \tilde{K}, \end{cases} \quad (15)$$

Then, we state the axioms related to regulator-based risk statistics.

Definition 4. A regulator-based risk statistic is a function $\tilde{q} : \mathbb{R}^{d \times n} \rightarrow Q_{\tilde{M}}^t$ that satisfies the following properties:

- [Q0] Normalization: $\tilde{K}_{\tilde{M}} \subseteq \tilde{q}(0)$ and $\tilde{q}(0) \cap -\text{int} \tilde{K}_{\tilde{M}} = \emptyset$
- [Q1] Cash cover: for any $\tilde{z} \in \tilde{K}_{\tilde{M}}$, $\tilde{z} \in \tilde{q}(-\tilde{z})$
- [Q2] Monotonicity: for any $X_1, X_2 \in \mathbb{R}^{d \times n}$, $X_1 - X_2 \in \mathbb{R}^{d \times n} \cap \tilde{K}$ implies that $\tilde{q}(X_1) \supseteq \tilde{q}(X_2)$
- [Q3] Regulator-dependence: for any $X \in \mathbb{R}^{d \times n}$, $\tilde{q}(X) = \tilde{q}(X \wedge_{\tilde{K}} 0)$
- [Q4] Convexity: for any $X, Y \in \mathbb{R}^{d \times n}$, $\lambda \in [0, 1]$, $\tilde{q}(\lambda(X) + (1 - \lambda)Y) \supseteq \lambda \tilde{q}(X) + (1 - \lambda) \tilde{q}(Y)$

We need more notations. Let $Y \in \mathbb{R}^{d \times n}$, $\tilde{u} \in \tilde{M}$. Define a function $S_{(Y, \tilde{u})}(X) : \mathbb{R}^{d \times n} \rightarrow 2^{\tilde{M}}$ as

$$S_{(Y, \tilde{u})}(X) := \{\tilde{z} \in \tilde{M} : X^{\text{tr}} Y \leq \tilde{u}^{\text{tr}} \tilde{z}\}, \quad (16)$$

let $\tilde{R} : \mathbb{R}^{d \times n} \rightarrow Q_{\tilde{M}}^t$ be a set-valued closed convex function. Then, the Legendre–Fenchel conjugate and the biconjugate of $\tilde{R}$ can be defined, respectively, as

$$-\tilde{R}^*(Y, u) := \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} (\tilde{R}(X) + S_{(Y, \tilde{u})}(-X)), \quad (17)$$

$$Y \in \mathbb{R}^{d \times n}, \tilde{u} \in \mathbb{R}^{d \times n},$$

$$\tilde{R}(X) = \tilde{R}^{**}(X) := \bigcap_{(Y, \tilde{u}) \in \mathbb{R}^{d \times n} \times \tilde{K}_{\tilde{M}}^+ \setminus \{0\}} [-\tilde{R}^*(Y, \tilde{u}) + S_{(Y, \tilde{u})}(X)], \quad X \in \mathbb{R}^{d \times n}. \quad (18)$$

For any $\tilde{Z} \subseteq \mathbb{R}^{d \times n}$, the $Q_{\tilde{M}}^t$ -valued indicator function $I_{\tilde{Z}} : \mathbb{R}^{d \times n} \rightarrow Q_{\tilde{M}}^t$ is defined as

$$I_{\tilde{Z}}(X) := \begin{cases} \text{cl} \tilde{K}_{\tilde{M}}, & X \in \tilde{Z}, \\ \emptyset, & X \notin \tilde{Z}. \end{cases} \quad (19)$$

The conjugate of $Q_{\tilde{M}}^t$ -valued indicator function $I_{\tilde{Z}}$ is

$$-(I_{\tilde{Z}})^*(Y, \tilde{u}) := \text{cl} \bigcup_{X \in \tilde{Z}} S_{(Y, \tilde{u})}(-X). \quad (20)$$

Assume that $\tilde{q}$ is a regulator-based risk statistic. For any $\tilde{z} \in \mathbb{R}_+^{d \times n}$, $X \in \mathbb{R}^{d \times n}$,

$$\tilde{q}(X - \tilde{z}) \supseteq \tilde{q}(X) + \tilde{z}, \quad (21)$$

which also implies

$$\tilde{q}(X + \tilde{z}) \subseteq \tilde{q}(X) - \tilde{z}. \quad (22)$$

Next, we state the dual representations of regulator-based risk statistics.

Proposition 2. Let $\tilde{q} : \mathbb{R}^{d \times n} \rightarrow Q_{\tilde{M}}^t$ be a proper closed convex regulator-based risk statistic with $\tilde{u} \in \{(-\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{1,j}, \dots, -\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{d,j}) + \tilde{M}^\perp\} \cap \tilde{K}_{\tilde{M}}^+ \setminus \{0\}$. Then,

$$-\tilde{q}^*(Y, \tilde{u}) = \begin{cases} \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y, \tilde{u})}(-X), & Y \in -\mathbb{R}_+^{d \times n} \cap (\tilde{K}^+), \\ \tilde{M}, & \text{elsewhere.} \end{cases} \quad (23)$$

Theorem 2. If $\tilde{q} : \mathbb{R}^{d \times n} \rightarrow Q_{\tilde{M}}^t$ is a proper closed convex regulator-based risk statistic, then there is a $-\alpha : (-\mathbb{R}_+^{d \times n} \cap \tilde{K}^+) \times \tilde{K}_{\tilde{M}}^+ \setminus \{0\} \rightarrow Q_{\tilde{M}}^t$, that is not identically $\tilde{M}$ of the set

$$\tilde{W} = \left\{ (Y, \tilde{u}) \in (-\mathbb{R}_+^{d \times n} \cap \tilde{K}^+) \times \tilde{K}_{\tilde{M}}^+ \setminus \{0\} : \tilde{u} \in \left(-\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{1,j}, \dots, -\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{d,j} \right) + \tilde{M}^\perp \right\} \quad (24)$$

such that for any $X \in \mathbb{R}^{d \times n}$,

$$\tilde{q}(X) = \bigcap_{(Y, \tilde{u}) \in \tilde{W}} \{-\alpha(Y, \tilde{u}) + S_{(Y, \tilde{u})}(X \wedge_{\tilde{K}} 0)\}. \quad (25)$$

5. Proofs of Main Results

Proof of Lemma 2. the proof of Lemma 2 is straightforward from Remark 3.

Proof of Proposition 1. if $Y \notin -\mathbb{R}_+^{d \times n} \cap (K^+ 1_n)$, there exists an $\bar{X} \in \mathbb{R}^{d \times n} \cap (K 1_n)$ such that $\bar{X}^{\text{tr}} Y > 0$. Using the definition of

$S_{(Y, \tilde{u})}$, we have $S_{(Y, \tilde{u})}(-t\bar{X}) = \{z \in \tilde{M} : -t\bar{X}^{\text{tr}} Y \leq \tilde{u}^{\text{tr}} z\}$ for $t > 0$. Therefore,

$$\text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y, \tilde{u})}(-X) \supseteq \bigcup_{t > 0} S_{(Y, \tilde{u})}(-t\bar{X}) = \tilde{M}. \quad (26)$$

The last equality is due to $-t\bar{X}^{\text{tr}} Y \rightarrow -\infty$ when $t \rightarrow +\infty$. Using the definition of $S_{(Y, \tilde{u})}$, we conclude that $\text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y, \tilde{u})}(-X) \subseteq \tilde{M}$. Hence,

$$\text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y, \tilde{u})}(-X) = \tilde{M}, \quad Y \notin -\mathbb{R}_+^{d \times n} \cap (K^+ 1_n). \quad (27)$$

It is easy to check that for any $X \in \mathbb{R}^{d \times n}$ and $v \in M$,

$$\begin{aligned} S_{(Y,u)}(-X - v1_n) &= \{z \in M : -X^{\text{tr}}Y \leq u^{\text{tr}}z + Y^{\text{tr}}(v1_n)\}, \\ &= \{z - v \in M : -X^{\text{tr}}Y \leq u^{\text{tr}}(z - v) + (Y + u1_n)^{\text{tr}}(v1_n)\} + v, \\ &= \{z \in M : -X^{\text{tr}}Y \leq u^{\text{tr}}z + (Y + u1_n)^{\text{tr}}(v1_n)\} + v, \end{aligned} \quad (28)$$

when $(-\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{1,j}, \dots, -\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{d,j}) + u \in M^\perp$, and we have $S_{(Y,u)}(-X - v1_n) = S_{(Y,u)}(-X) + v$. However, $u \notin ((-\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{1,j}, \dots, -\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{d,j}) + M^\perp)$. Therefore, $(-\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{1,j}, \dots, -\sum_{j=1}^l \sum_{h=1}^{n_j} Y_h^{d,j}) + u \notin M^\perp$, and we can find $v \in M$ such that for any $z \in M$,

$$-X^{\text{tr}}Y \leq u^{\text{tr}}z + (Y + u1_n)^{\text{tr}}(v1_n). \quad (29)$$

Therefore, we have

$$z + v \in S_{(Y,u)}(-X - v1_n). \quad (30)$$

Therefore,

$$\bigcup_{z,v \in M} (z + v) \subset \bigcup_{v \in M} S_{(Y,u)}(-X - v1_n). \quad (31)$$

Therefore,

$$M \subset \bigcup_{v \in M} S_{(Y,u)}(-X - v1_n). \quad (32)$$

From the definition of $S_{(Y,u)}$, the inverse inclusion is always true. So, we conclude that

$$M = \bigcup_{v \in M} S_{(Y,u)}(-X - v1_n). \quad (33)$$

It is also easy to check that

$$\begin{aligned} -\varrho^*(Y, u) &= \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}, v \in M} (\varrho(X + v1_n) + S_{(Y,u)}(-X - v1_n)), \\ &= \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}, v \in M} (\varrho(X + v1_n) + M), \\ &= M, \end{aligned} \quad (34)$$

where the last equality comes from the fact that M is a linear space and $\varrho(X) \subseteq M$. We now derive that $-\varrho^*(Y, u) = \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X)$. In this context, from $-\varrho^*(Y, u) = \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} (\varrho(X) + S_{(Y,u)}(-X))$, we derive it in two cases.

Case 1. When $X \wedge_{K1_n} 0 = 0$, using the definition, we have $\varrho(X) = \varrho(0) \ni 0$. Hence,

$$\text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} (\varrho(X) + S_{(Y,u)}(-X)) \stackrel{\text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X)}{=} \quad (35)$$

Case 2. When $X \wedge_{K1_n} 0 = X$, we can always find an $\alpha \in K_M$ such that $\alpha \in \varrho(X)$. Then,

$$\varrho(X) + S_{(Y,u)}(-X) \supseteq \alpha + S_{(Y,u)}(-X) = S_{(Y,u)}(-X - \alpha 1_n) = S_{(Y,u)}(-\beta), \quad (36)$$

where $\beta = X + \alpha 1_n$. It is relatively simple to check that $\beta \in \mathbb{R}^{d \times n}$. Therefore,

$$\text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} (\varrho(X) + S_{(Y,u)}(-X)) \supseteq \text{cl} \bigcup_{z \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-z), \quad (37)$$

that is,

$$-\varrho^*(Y, u) \supseteq \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X). \quad (38)$$

Consequently, we have

$$-\varrho^*(Y, u) \supseteq \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X). \quad (39)$$

We now need only to derive that $-\varrho^*(Y, u) \subseteq \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X)$. In fact, for any $z \in \varrho(X)$ and $X \in \mathbb{R}^{d \times n}$, $X + z1_n \in \mathbb{R}^{d \times n}$. Therefore,

$$\begin{aligned} \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X) &= \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X) \supseteq S_{(Y,u)}(-X - z1_n) \\ &= S_{(Y,u)}(-X) + z. \end{aligned} \quad (40)$$

Using the arbitrariness of z , we have

$$\varrho(X) + S_{(Y,u)}(-X) \subseteq \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X). \quad (41)$$

Therefore,

$$-\varrho^*(Y, u) \subseteq \text{cl} \bigcup_{X \in \mathbb{R}^{d \times n}} S_{(Y,u)}(-X). \quad (42)$$

Proof of Theorem 1. the proof is straightforward from Lemma 1 and Proposition 1.

Data Availability

No data and code were generated or used during the study.

Disclosure

This manuscript has been released as a preprint at arXiv: 1904.08829v4.

Conflicts of Interest

The authors declare no conflicts of interest.

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