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Research Article

Lower Bound for the Blow-Up Time for the Nonlinear Reaction-Diffusion System in High Dimensions

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In this paper, we study the blow-up phenomenon for a nonlinear reaction-diffusion system with time-dependent coefficients under nonlinear boundary conditions. Using the technique of a first-order differential inequality and the Sobolev inequalities, we can get the energy expression which satisfies the differential inequality. The lower bound for the blow-up time could be obtained if blow-up does really occur in high dimensions.

1. Introduction

During the past decades, the blow-up phenomena for the solutions to the parabolic problems have been widely concerned. It is important in practice that how to determine the bound of the blow-up time t^* of the solutions about the parabolic equations and systems. Their applications are included in physics, chemistry, astronomy, biology, and population dynamics [1, 2]. Actually, when the blow-up occurs at t^*, it is difficult to get the exact value of t^*. We mainly focus on estimating its bounds. At present, the studies on the blow-up phenomena of parabolic problems mainly focus on homogeneous Dirichlet boundary condition and homogeneous Neumann and Robin boundary conditions [3–12]. There are also some works under nonlinear boundary conditions [13–15]. Most of these articles are focused on R^3. There are only a few papers dealing with a lower bound for the blow-up time in high dimensions (see [16–18]). Recently, some scholars have started to investigate the blow-up problems with time-dependent coefficients [19–21]. In paper [21], the authors considered the following nonlinear reaction-diffusion system with time-dependent coefficients:

$$\begin{aligned}
&u_t = \triangle u + k_1(t)u^p v^q, \quad (x,t) \in \Omega \times (0,t^*),\\
&v_t = \triangle v + k_2(t)v^r u^s, \quad (x,t) \in \Omega \times (0,t^*),\\
&u(x,t) = v(x,t) = 0, \quad (x,t) \in \partial\Omega \times (0,t^*),\\
&u(x,0) = u_0(x),\ v(x,0) = v_0(x),\ x \in \Omega.
\end{aligned} \tag{1}$$

The authors obtained the lower and upper bounds for the blow-up time when the blow-up occurred. In this paper, we further consider the blow-up phenomena for the following system with time-dependent coefficients under nonlinear boundary conditions in high dimensions:

$$\begin{aligned}
&u_t = \Delta u^m + k_1(t)u^p v^q,\\
&v_t = \Delta v^l + k_2(t)v^r u^s,\\
&\quad \cdot (x,t) \in \Omega \times (0,t^*),\\
&\frac{\partial u}{\partial n} = g_1(u),\ \frac{\partial v}{\partial n} = g_2(v),\\
&\quad (x,t) \in \partial\Omega \times (0,t^*),\\
&u(x,0) = u_0(x) \ge 0,\ v(x,0) = v_0(x) \ge 0,\ x \in \Omega.
\end{aligned} \tag{2}$$

We assume that $g_i(\zeta)$ are continuous, and $\alpha, \beta, p, q, r, s, a_1, a_2, m, l, \sigma$, and ζ satisfy

$$\begin{gathered} g_1(\zeta) \le a_1 \zeta^{\alpha},\ g_2(\zeta) \le a_2 \zeta^{\beta}, \\ \zeta > 0,\ \alpha > 1, \\ \beta > 1,\ p > 1,\ q > 0,\ r > 1,\ s > 0,\ a_1 > 0,\ a_2 > 0, \\ \max\left\{\frac{\sigma(p+q-1)}{\sigma-q} - \frac{2\sigma}{n} + 1, 1\right\} < m < \frac{\sigma(p+q)-q}{\sigma-q}, \\ \max\left\{\frac{\sigma(r+s-1)}{\sigma-s} - \frac{2\sigma}{n} + 1, 1\right\} < l < \frac{\sigma(r+s)-s}{\sigma-s}, \end{gathered} \tag{3}$$

where σ is a positive constant to be defined later.

Our goal in this paper is to obtain a lower bound for the blow-up time of the solutions to systems (2) and (3) in R^n for any $n > 3$. The nonlinear terms Δu^m and Δv^l and the boundary conditions are difficult to tackle. We cannot get the result by following the method proposed in [21], so we must use a new method to overcome these difficulties. To the best of our knowledge, no results exist in that direction, and we think our result is new and interesting.

In the further discussions, we will use the following Hölder inequality:

$$\int_{\Omega} w^{x_1+x_2}\mathrm{d}x \le \left(\int_{\Omega} w^{x_1/n_1}\mathrm{d}x\right)^{1/n_1}\left(\int_{\Omega} w^{x_2/n_2}\mathrm{d}x\right)^{1/n_2}, \tag{4}$$

where w is a nonnegative function and x_1, x_2, n_1, and n_2 are positive constants satisfying $(1/n_1) + (1/n_2) = 1$.

We also need the following Sobolev inequality [22]:

$$\begin{aligned} &\int_{\Omega} u^{(\sigma+m-1)n/(n-2)}\mathrm{d}x \le C^{2n/(n-2)} 2^{(n/(n-2))-1} \\ &\quad \cdot \left[\left(\int_{\Omega} u^{\sigma+m-1}\mathrm{d}x\right)^{n/(n-2)} + \left(\int_{\Omega}\left|\nabla u^{(\sigma+m-1)/2}\right|^2\mathrm{d}x\right)^{n/(n-2)}\right], \end{aligned} \tag{5}$$

$$\begin{aligned} &\int_{\Omega} v^{(\sigma+l-1)n/(n-2)}\mathrm{d}x \le C^{2n/(n-2)} 2^{(n/(n-2))-1} \\ &\quad \cdot \left[\left(\int_{\Omega} v^{\sigma+l-1}\mathrm{d}x\right)^{n/(n-2)} + \left(\int_{\Omega}\left|\nabla v^{(\sigma+l-1)/2}\right|^2\mathrm{d}x\right)^{n/(n-2)}\right], \end{aligned} \tag{6}$$

with $C = C(n, \Omega)$ which is a Sobolev embedding constant depending on n and Ω.

And the classical (or elementary) inequality is

$$(a+b)^w \le a^w + b^w, \tag{7}$$

where a, b, and w are positive constants, and w satisfies $0 < w \le 1$.

2. Lower Bound for the Blow-Up Time

In this part, we define an auxiliary function of the form

$$\phi(t) = k_1^{\delta}(t)\int_{\Omega} u^{\sigma}\mathrm{d}x + k_2^{\chi}(t)\int_{\Omega} v^{\sigma}\mathrm{d}x, \tag{8}$$

where $\delta = (\sigma-q)(mn-n+2\sigma)/2\sigma(p+q-1)$, $\chi = (\sigma-s)(\ln - n + 2\sigma)/2\sigma(r+s-1)$, and $\sigma > \max\{(\alpha-1)n, (\beta-1)n, q, s\}$.

We establish the following theorem:

Theorem 1. *Let $u(x,t)$ be the weak solution of problems (1)–(3) in a bounded convex domain $\Omega(\Omega \in R^n(n>3))$. Then, the quantity $\phi(t)$ defined in (8) satisfies the integral inequality*

$$\Theta(t^*) \ge \int_{\phi}^{\infty} \frac{1}{\eta + \eta^{\xi_1} + \eta^{\xi_2} + \eta^{\xi_3} + \eta^{\xi_4}} = S, \tag{9}$$

which follows that the blow-up time t^* is bounded below. We have

$$t^* \ge \Theta^{-1}(S), \tag{10}$$

where Θ, ξ_1, ξ_2, ξ_3, and ξ_4 will be defined later.

Now, we prove Theorem 1. For simplicity, assume that the solution is classical of problems (1)–(3). The general case can be done by approximation. Differentiating $\phi(t)$, we have

$$\begin{aligned} \phi'(t) &= \delta k_1^{\delta-1}(t)k_1'(t)\int_{\Omega} u^{\sigma}\mathrm{d}x + \sigma k_1^{\delta}(t)\int_{\Omega} u^{\sigma-1}u_t\mathrm{d}x \\ &\quad + \chi k_2^{\chi-1}(t)k_2'(t)\int_{\Omega} v^{\sigma}\mathrm{d}x + \sigma k_2^{\chi}(t)\int_{\Omega} v^{\sigma-1}v_t\mathrm{d}x \\ &\le L\phi(t) + \sigma k_1^{\delta}(t)\int_{\Omega} u^{\sigma-1}\triangle u^m\mathrm{d}x + \sigma k_1^{\delta+1}(t)\int_{\Omega} u^{\sigma+p-1}v^q\mathrm{d}x \\ &\quad + \sigma k_2^{\chi}(t)\int_{\Omega} v^{\sigma-1}\triangle v^l\mathrm{d}x + \sigma k_2^{\chi+1}(t)\int_{\Omega} v^{\sigma+r-1}u^s\mathrm{d}x, \end{aligned} \tag{11}$$

where $L = \max\{\delta|k_1'(t)|/k_1(t), \chi|k_2'(t)|/k_2(t)\}$.

For the second term on the right side of (11), we apply the divergence theorem, the L^1 trace embedding, and (3) to get

$$\begin{aligned}\int_\Omega u^{\sigma-1}\triangle u^m \mathrm{d}x &= \int_{\partial\Omega} u^{\sigma-1}\frac{\partial u^m}{\partial \mathbf{n}}\mathrm{d}A - \int_\Omega \nabla u^{\sigma-1}\cdot\nabla u^m \mathrm{d}x \\ &\le m\int_{\partial\Omega} u^{\sigma+m-2}\frac{\partial u}{\partial \mathbf{n}}\mathrm{d}A - m(\sigma-1)\int_\Omega u^{\sigma+m-3}|\nabla u|^2\mathrm{d}x \\ &\le ma_1\int_{\partial\Omega} u^{\sigma+m+\alpha-2}\mathrm{d}A - m(\sigma-1)\int_\Omega u^{\sigma+m-3}|\nabla u|^2\mathrm{d}x \\ &\le \frac{ma_1 n}{\rho_0}\int_\Omega u^{\sigma+m+\alpha-2}\mathrm{d}x + \frac{ma_1(\sigma+m+\alpha-2)d}{\rho_0}\int_\Omega u^{\sigma+m+\alpha-3}|\nabla u|\mathrm{d}x \\ &\quad - \frac{4m(\sigma-1)}{(\sigma+m-1)^2}\int_\Omega \left|\nabla u^{\sigma+m-1/2}\right|^2\mathrm{d}x,\end{aligned} \tag{12}$$

where $\rho_0 := \min_{\partial\Omega}|x\cdot\vec{n}|$, $\vec{n}$ is the outward normal vector of $\partial\Omega$ and $d := \max_{\partial\Omega}|x|$.

For the second term on the right side of (12), using (11), we obtain

$$\int_\Omega u^{\sigma+m+\alpha-3}|\nabla u|\mathrm{d}x \le \frac{1}{2\varepsilon_1}\int_\Omega u^{\sigma+m+2\alpha-3}\mathrm{d}x + \frac{2\varepsilon_1}{(\sigma+m-1)^2}\cdot\int_\Omega\left|\nabla u^{(\sigma+m-1)/2}\right|^2\mathrm{d}x, \tag{13}$$

where ε_1 is a positive constant which will be defined later. Using (4), we have

$$\begin{aligned}\int_\Omega u^{\sigma+m+\alpha-2}\mathrm{d}x &\le \left(\int_\Omega u^{\sigma+m+2\alpha-3}\mathrm{d}x\right)^{x_{10}}\left(\int_\Omega u^\sigma \mathrm{d}x\right)^{x_{20}} \\ &\le x_{10}\int_\Omega u^{\sigma+m+2\alpha-3}\mathrm{d}x + x_{20}\int_\Omega u^\sigma\mathrm{d}x,\end{aligned} \tag{14}$$

where $x_{10} = (m+\alpha-2)/(m+2\alpha-3)$ and $x_{20} = (\alpha-1)/(m+2\alpha-3)$.

Choosing $x_{11} = (m+2\alpha-2)(n-2)/(m-1)n+2\sigma$ and $x_{21} = ((m-1)n+2\sigma-(m+2\alpha-2)(n-2))/((m-1)n+2\sigma)$ and using (4), (5), and (7), we have

$$\begin{aligned}\int_\Omega u^{\sigma+m+2\alpha-3}\mathrm{d}x &\le \left(\int_\Omega u^{(\sigma+m-1)n/(n-2)}\mathrm{d}x\right)^{x_{11}}\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{x_{21}} \\ &\le \frac{r_1 n x_{11}}{n-2}\int_\Omega u^{\sigma+m-1}\mathrm{d}x \\ &\quad + \frac{r_1(n-2-nx_{11})}{n-2}\left(1+\varepsilon_2^{-nx_{11}/(n-2-nx_{11})}\right) \\ &\quad \cdot\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{x_{21}(n-2)/(n-2-nx_{11})} + \frac{r_1 n x_{11}}{n-2}\varepsilon_2 \\ &\quad \cdot\int_\Omega\left|\nabla u^{(\sigma+m-1)/2}\right|^2\mathrm{d}x,\end{aligned} \tag{15}$$

where $r_1 = (C^{2n/(n-2)}2^{(n/(n-2))-1})^{x_{11}}$ and ε_2 is a positive constant which will be defined later.

For the first term on the right side of (15), using (4) and Young's inequality, we have

$$\begin{aligned}\int_\Omega u^{\sigma+m-1}\mathrm{d}x &\le \left(\int_\Omega u^{\sigma+m+2\alpha-3}\mathrm{d}x\right)^{x_{12}}\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{x_{22}} \\ &\le x_{12}\varepsilon_3\int_\Omega u^{\sigma+m+2\alpha-3}\mathrm{d}x + x_{22}\varepsilon_3^{-((m-1)/(2\alpha-2))}\int_\Omega u^\sigma\mathrm{d}x,\end{aligned} \tag{16}$$

where $x_{12} = (m-1)/(m+2\alpha-3)$, $x_{22} = (2\alpha-2)/(m+2\alpha-3)$, and ε_3 is a positive constant which will be defined later.

Combining (15) and (16), if we choose suitable ε_3 such that $r_1 n x_{11}x_{12}\varepsilon_3/(n-2) = 1/2$, we have

$$\begin{aligned}\int_\Omega u^{\sigma+m+2\alpha-3}\mathrm{d}x &\le r_2\int_\Omega u^\sigma\mathrm{d}x + r_3\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{(x_{21}(n-2))/(n-2-nx_{11})} \\ &\quad + r_4\int_\Omega\left|\nabla u^{(\sigma+m-1)/2}\right|^2\mathrm{d}x,\end{aligned} \tag{17}$$

where $r_2 = (2r_1 n x_{11}/(n-2))x_{22}\varepsilon_3^{-((m-1)/(2\alpha-2))}$, $r_3 = 2r_1(n-2-nx_{11})/(n-2)(1+\varepsilon_2^{-(nx_{11}/(n-2-nx_{11}))})$, and $r_4 = (2r_1 n x_{11}/(n-2))\varepsilon_2$.

Combining (12), (13), (14), and (17), we have

$$\begin{aligned}\int_\Omega u^{\sigma-1}\triangle u^m\mathrm{d}x &\le \left(r_5 r_2 + \frac{ma_1 n x_{20}}{\rho_0}\right)\int_\Omega u^\sigma\mathrm{d}x + r_5 r_3 \\ &\quad \cdot\left(\int_\Omega u^\sigma\mathrm{d}x\right)^{x_{21}(n-2)/(n-2-nx_{11})} \\ &\quad + r_6\int_\Omega\left|\nabla u^{(\sigma+m-1)/2}\right|^2\mathrm{d}x,\end{aligned} \tag{18}$$

where $r_5 = (ma_1 n x_{10}/\rho_0) + (ma_1(\sigma+m+\alpha-2)d/2\varepsilon_1\rho_0)$ and $r_6 = (ma_1(\sigma+m+\alpha-2)d/\rho_0)\cdot(2\varepsilon_1/(\sigma+m-1)^2) - (4m(\sigma-1)/(\sigma+m-1)^2)$.

Similarly, for the fourth term on the right side of (11), using the divergence theorem and (3), we have

$$\int_\Omega v^{\sigma-1}\triangle v^l \mathrm{d}x = \int_{\partial\Omega} v^{\sigma-1}\frac{\partial v^l}{\partial \mathbf{n}}\mathrm{d}A - \int_\Omega \nabla v^{\sigma-1}\cdot\nabla v^l \mathrm{d}x$$
$$\le \frac{la_2 n}{\rho_0}\int_\Omega v^{\sigma+l+\beta-2}\mathrm{d}x + \frac{la_2(\sigma+l+\beta-2)d}{\rho_0}\cdot\int_\Omega v^{\sigma+l+\beta-3}|\nabla v|\mathrm{d}x - \frac{4l(\sigma-1)}{(\sigma+l-1)^2}\int_\Omega\left|\nabla v^{(\sigma+v-1)/2}\right|^2\mathrm{d}x. \tag{19}$$

For the second term on the right side of (19), using (4), we obtain

$$\int_\Omega v^{\sigma+l+\beta-3}|\nabla v|\mathrm{d}x \le \frac{1}{2\varepsilon_4}\int_\Omega v^{\sigma+l+2\beta-3}\mathrm{d}x + \frac{2\varepsilon_4}{(\sigma+l-1)^2}\int_\Omega \cdot\left|\nabla v^{(\sigma+l-1)/2}\right|^2\mathrm{d}x, \tag{20}$$

where ε_4 is a positive constant which will be defined later. Similarly, we have

$$\int_\Omega v^{\sigma+l+\beta-2}\mathrm{d}x \le \left(\int_\Omega v^{\sigma+l+2\beta-3}\mathrm{d}x\right)^{y_{10}}\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{y_{20}} \le y_{10}\int_\Omega v^{\sigma+l+2\beta-3}\mathrm{d}x + y_{20}\int_\Omega v^\sigma\mathrm{d}x, \tag{21}$$

where $y_{10} = (l+\beta-2)/(l+2\beta-3)$ and $y_{20} = (\beta-1)/(l+2\beta-3)$.

Choosing $y_{11} = ((l+2\beta-2)(n-2))/((l-1)n+2\sigma)$ and $y_{21} = ((l-1)n+2\sigma-(l+2\beta-3)(n-2))/((l-1)n+2\sigma)$ and using (4), (6), and (7), we have

$$\int_\Omega v^{\sigma+l+2\beta-3}\mathrm{d}x \le \left(\int_\Omega v^{(\sigma+l-1)n/(n-2)}\mathrm{d}x\right)^{y_{11}}\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{y_{21}}$$
$$\le \frac{r_7 n y_{11}}{n-2}\int_\Omega v^{\sigma+l-1}\mathrm{d}x + \frac{r_7(n-2-ny_{11})}{n-2}\left(1+\varepsilon_5^{-ny_{11}/(n-2-ny_{11})}\right)\cdot\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{y_{21}(n-2)/(n-2-ny_{11})} + \frac{r_7 n y_{11}}{n-2}\varepsilon_5\cdot\int_\Omega\left|\nabla v^{(\sigma+l-1)/2}\right|^2\mathrm{d}x, \tag{22}$$

where $r_7 = (C^{2n/(n-2)}2^{(n/(n-2))-1})^{y_{11}}$ and ε_5 is a positive constant which will be defined later.

For the first term on the right side of (22), using (4), we have

$$\int_\Omega v^{\sigma+l-1}\mathrm{d}x \le \left(\int_\Omega v^{\sigma+l+2\beta-3}\mathrm{d}x\right)^{y_{12}}\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{y_{22}} \le y_{12}\varepsilon_6\int_\Omega v^{\sigma+l+2\beta-3}\mathrm{d}x + y_{22}\varepsilon_6^{-((l-1)/(2\beta-2))}\int_\Omega v^\sigma\mathrm{d}x, \tag{23}$$

where $y_{12} = (l-1)/(l+2\beta-3)$, $y_{22} = (2\beta-2)/(l+2\beta-3)$, and ε_6 is a positive constant which will be defined later.

Combining (22) and (23), if we choose suitable ε_6 such that $r_7 n y_{11} y_{12}\varepsilon_6/(n-2) = 1/2$, we have

$$\int_\Omega v^{\sigma+l+2\beta-3}\mathrm{d}x \le r_8\int_\Omega v^\sigma\mathrm{d}x + r_9\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{y_{21}(n-2)/(n-2-ny_{11})} + r_{10}\int_\Omega\left|\nabla v^{(\sigma+l-1)/2}\right|^2\mathrm{d}x, \tag{24}$$

where $r_8 = (2r_7 n y_{11} y_{22}/(n-2))\varepsilon_6^{-((l-1)/(2\beta-2))}$, $r_9 = (2r_7(n-2-ny_{11})/(n-2))(1+\varepsilon_5^{-(ny_{11}/(n-2-ny_{11}))})$, and $r_{10} = (2r_7 n y_{11}/(n-2))\varepsilon_5$.

Combining (19)–(21) and (24), we have

$$\int_\Omega v^{\sigma-1}\triangle v^l\mathrm{d}x \le \left(r_{11}r_8 + \frac{la_2 n y_{20}}{\rho_0}\right)\int_\Omega v^\sigma\mathrm{d}x + r_{11}r_9\cdot\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{y_{21}(n-2)/(n-2-ny_{11})} + r_{12}\int_\Omega\left|\nabla v^{(\sigma+l-1)/2}\right|^2\mathrm{d}x, \tag{25}$$

where $r_{11} = (la_2 n y_{10}/\rho_0) + (la_2(\sigma+l+\beta-2)d/2\varepsilon_4\rho_0)$ and $r_{12} = (la_2(\sigma+l+\beta-2)d/\rho_0)\cdot(2\varepsilon_4/(\sigma+l-1)^2) - (4l(\sigma-1)/(\sigma+l-1)^2)$.

For the third term on the right side of (11), using Hölder inequality and Young's inequality, we have

$$\sigma k_1^{\delta+1}(t)\int_\Omega u^{\sigma+p-1}v^q\mathrm{d}x \le \sigma k_1^{\delta+1}(t)\left(\int_\Omega u^{(\sigma+p-1)\sigma/(\sigma-q)}\mathrm{d}x\right)^{(\sigma-q)/\sigma}\cdot\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{q/\sigma}$$
$$\le (\sigma-q)k_1^{\delta+1}(t)\int_\Omega u^{(\sigma+p-1)\sigma/(\sigma-q)}\mathrm{d}x + qk_1^{\delta+1}(t)\int_\Omega v^\sigma\mathrm{d}x. \tag{26}$$

For the first term on the right side of (26), using (4), (5), and (7) and taking care of the given condition $\delta = ((\sigma-q)(mn-n+2\sigma))/(2\sigma(p+q-1))$, we have

$$
\begin{aligned}
k_1^{\delta+1}(t)\int_\Omega u^{(\sigma+p-1)\sigma/(\sigma-q)}\mathrm{d}x &\le k_1^{\delta+1}(t)\left(\int_\Omega u^{(\sigma+m-1)n/(n-2)}\mathrm{d}x\right)^{x_{13}}\left(\int_\Omega u^\sigma \mathrm{d}x\right)^{x_{23}}\\
&\le k_1^{\delta+1}(t)\left(C^{2n/(n-2)}2^{(n/(n-2))-1}\right)^{x_{13}}\left[\left(\int_\Omega u^{\sigma+m-1}\mathrm{d}x\right)^{x_{13}n/(n-2)}\right.\\
&\quad\left.+\left(\int_\Omega\left|\nabla u^{(\sigma+m-1)/2}\right|^2\mathrm{d}x\right)^{x_{13}n/(n-2)}\right]\left(\int_\Omega u^\sigma \mathrm{d}x\right)^{x_{23}}\\
&=\lambda_1 k_1^{\delta+1}(t)\left(\int_\Omega u^{\sigma+m-1}\mathrm{d}x\right)^{x_{13}n/(n-2)}\left(\int_\Omega u^\sigma \mathrm{d}x\right)^{x_{23}}\\
&\quad+\lambda_1 k_1^{\delta+1}(t)\left(\int_\Omega\left|\nabla u^{(\sigma+m-1)/2}\right|^2\mathrm{d}x\right)^{x_{13}n/(n-2)}\left(\int_\Omega u^\sigma \mathrm{d}x\right)^{x_{23}}\\
&\le\frac{\lambda_1 x_{13}n}{n-2}k_1^{\delta+1}(t)\int_\Omega u^{\sigma+m-1}\mathrm{d}x+\frac{\lambda_1(n-2-nx_{13})}{n-2}\left(k_1^{1-(2x_{13}\delta/(n-2-nx_{13}))}+\varepsilon_7^{-(nx_{13}\delta/(n-2-nx_{13}))}\right)\cdot\\
&\quad\cdot\left(k_1^\delta(t)\int_\Omega u^\sigma\mathrm{d}x\right)^{x_{23}(n-2)/(n-2-nx_{13})}+\frac{\lambda_1 x_{13}n\varepsilon_7}{n-2}k_1^\delta(t)\int_\Omega\left|\nabla u^{(\sigma+m-1)/2}\right|^2\mathrm{d}x,
\end{aligned}
\tag{27}
$$

where $\lambda_1=(C^{2n/(n-2)}2^{(n/(n-2))-1})^{x_{13}}$, $x_{13}=(\sigma(p+q-1)(n-2))/((\sigma-q)(mn-n+2\sigma))$, and $x_{23}=((\sigma-q)(mn-n+2\sigma)-\sigma(p+q-1)(n-2))/((\sigma-q)(mn-n+2\sigma))$.

For the first term on the right side of (27), using (4) and Young's inequality, we get

$$\int_\Omega u^{\sigma+m-1}\mathrm{d}x\le x_{14}\varepsilon_7\int_\Omega u^{(\sigma+p-1)\sigma/(\sigma-q)}+x_{24}\varepsilon_7^{-(x_{14}/x_{24})}\int_\Omega u^\sigma\mathrm{d}x, \tag{28}$$

where $x_{14}=((m-1)(\sigma-q))/(\sigma(\sigma+q-1))$, $x_{24}=(\sigma(\sigma+q-1)-(m-1)(\sigma-q))/(\sigma(\sigma+q-1))$, and ε_7 is a positive constant which will be defined later.

Combining (27) and (28), if we choose suitable ε_7 such that $(\lambda_1 nx_{13}x_{14}\varepsilon_7)/(n-2)=1/2$, we have

$$
\begin{aligned}
(\sigma-q)k_1^{\delta+1}(t)\int_\Omega u^{(\sigma+p-1)\sigma/(\sigma-q)}\mathrm{d}x&\le K_1(t)k_1^\delta(t)\int_\Omega u^\sigma\mathrm{d}x\\
&\quad+K_2(t)\left(k_1^\delta(t)\int_\Omega u^\sigma\mathrm{d}x\right)^{x_{23}(n-2)/(n-2-nx_{13})}\\
&\quad+\lambda_2 k_1^\delta(t)\int_\Omega\left|\nabla u^{(\sigma+m-1)/2}\right|^2\mathrm{d}x,
\end{aligned}
\tag{29}
$$

where $K_1(t)=(2(\sigma-q)\lambda_1 nx_{13}x_{24}/(n-2))\varepsilon_7^{-(x_{14}/x_{24})}k_1(t)$, $K_2(t)=(2(\sigma-q)\lambda_1(n-2-nx_{13})/(n-2))(k_1^{1-(2x_{13}\delta/(n-2-nx_{13}))}(t)+\varepsilon_7^{-(nx_{13}/(n-2-nx_{13}))})$, and $\lambda_2=2(\sigma-q)\lambda_1 nx_{13}\varepsilon_7/(n-2)$.

Combining (26) and (29), we obtain

$$
\begin{aligned}
\sigma k_1^{\delta+1}(t)\int_\Omega u^{\sigma+p-1}v^q\mathrm{d}x&\le K_1(t)k_1^\delta(t)\int_\Omega u^\sigma\mathrm{d}x\\
&\quad+K_2(t)\left(k_1^\delta(t)\int_\Omega u^\sigma\mathrm{d}x\right)^{x_{23}(n-2)/(n-2-nx_{13})}\\
&\quad+\lambda_2 k_1^\delta(t)\int_\Omega\left|\nabla u^{(\sigma+m-1)/2}\right|^2\mathrm{d}x\\
&\quad+qk_1^{\delta+1}(t)\int_\Omega v^\sigma\mathrm{d}x.
\end{aligned}
\tag{30}
$$

By the same way, for the fifth term on the right side of (11), using (4), we have

$$
\begin{aligned}
\sigma k_2^{\chi+1}(t)\int_\Omega v^{\sigma+r-1}u^s\mathrm{d}x&\le(\sigma-s)k_2^{\chi+1}(t)\int_\Omega v^{(\sigma+r-1)\sigma/(\sigma-s)}\mathrm{d}x\\
&\quad+sk_2^{\chi+1}(t)\int_\Omega u^\sigma\mathrm{d}x.
\end{aligned}
\tag{31}
$$

For the first term on the right side of (31), using (4), (6), and (7) and taking care of the given condition $\chi=((\sigma-s)(ln-n+2\sigma))/(2\sigma(r+s-1))$, we have

$$
\begin{aligned}
k_2^{\chi+1}(t)\int_\Omega v^{(\sigma+r-1)\sigma/(\sigma-s)}\mathrm{d}x&\le k_2^{\chi+1}(t)\left(\int_\Omega v^{(\sigma+l-1)n/(n-2)}\mathrm{d}x\right)^{y_{13}}\left(\int_\Omega v^\sigma\mathrm{d}x\right)^{y_{23}}\\
&\le\frac{\lambda_3 y_{13}n}{n-2}k_2^{\chi+1}(t)\int_\Omega v^{\sigma+l-1}\mathrm{d}x+\frac{\lambda_3(n-2-ny_{13})}{n-2}\left(k_2^{1-(2y_{13}\chi/(n-2-ny_{13}))}(t)+\varepsilon_8^{-(ny_{13}\chi/n-2-ny_{13})}\right)\cdot\\
&\quad\cdot\left(k_2^\chi(t)\int_\Omega v^\sigma\mathrm{d}x\right)^{y_{23}(n-2)/(n-2-ny_{13})}+\frac{\lambda_3 y_{13}n\varepsilon_8}{n-2}k_2^\chi(t)\int_\Omega\left|\nabla v^{(\sigma+l-1)/2}\right|^2\mathrm{d}x,
\end{aligned}
\tag{32}
$$

where $\lambda_3 = (C^{2n/(n-2)}2^{(n/(n-2))-1})^{y_{13}}$, $y_{13} = (\sigma(r+s-1)(n-2))/((\sigma-s)(ln-n+2\sigma))$, $y_{23} = ((\sigma-s)(ln-n+2\sigma)-\sigma(r+s-1)(n-2))/((\sigma-s)(ln-n+2\sigma))$, and ε_8 is a positive constant which will be defined later.

For the first term on the right side of (32), using (4), we get

$$\int_\Omega v^{\sigma+l-1}\mathrm{d}x \le y_{14}\varepsilon_9\int_\Omega v^{(\sigma+r-1)\sigma/(\sigma-s)} + y_{24}\varepsilon_9^{-(y_{14}/y_{24})}\int_\Omega v^\sigma \mathrm{d}x, \tag{33}$$

where $y_{14} = ((l-1)(\sigma-s))/(\sigma(\sigma+s-1))$, $y_{24} = (\sigma(\sigma+s-1)-(l-1)(\sigma-s))/(\sigma(\sigma+s-1))$, and ε_9 is a positive constant which will be defined later.

Combining (32) and (33), if we choose suitable ε_9 such that $\lambda_2 n y_{13} y_{14}\varepsilon_9/(n-2) = 1/2$, we have

$$\begin{aligned}(\sigma-s)k_2^{\chi+1}(t)\int_\Omega v^{(\sigma+r-1)\sigma/(\sigma-s)}\mathrm{d}x &\le K_3(t)k_2^\chi(t)\int_\Omega v^\sigma \mathrm{d}x \\ &+ K_4(t)\left(k_2^\chi(t)\int_\Omega v^\sigma \mathrm{d}x\right)^{y_{23}(n-2)/(n-2-ny_{13})} \\ &+ \lambda_4 k_2^\chi(t)\int_\Omega \left|\nabla v^{(\sigma+l-1)/2}\right|^2 \mathrm{d}x,\end{aligned} \tag{34}$$

where $K_3(t) = ((2(\sigma-s)\lambda_3 n y_{13} y_{24})/(n-2))\varepsilon_9^{-(y_{14}/y_{24})} k_2(t)$, $K_4(t) = ((2(\sigma-s)\lambda_3(n-2-ny_{13}))/(n-2))(k_2^{1-(2y_{13}\delta/(n-2-ny_{13}))}(t) + \varepsilon_9^{-(ny_{13}/(n-2-ny_{13}))})$, and $\lambda_4 = (2(\sigma-s)\lambda_3 n y_{13}\varepsilon_9)/(n-2)$.

Combining (31) and (34), we obtain

$$\begin{aligned}\sigma k_2^{\chi+1}(t)\int_\Omega v^{\sigma+r-1}u^s \mathrm{d}x &\le K_3(t)k_2^\chi(t)\int_\Omega v^\sigma \mathrm{d}x + K_4(t) \\ &\cdot\left(k_2^\chi(t)\int_\Omega v^\sigma \mathrm{d}x\right)^{y_{23}(n-2)/(n-2-ny_{13})} \\ &+ \lambda_4 k_2^\chi(t)\int_\Omega\left|\nabla v^{(\sigma+l-1)/2}\right|^2\mathrm{d}x + s k_2^{\chi+1}(t)\int_\Omega u^\sigma \mathrm{d}x.\end{aligned} \tag{35}$$

Combining (11), (18), (25), (30), and (35), we have

$$\begin{aligned}\phi'(t) &\le (L + \widetilde{K}_1 + \widetilde{K}_2(t) + \widetilde{K}_3(t))\phi(t) + \widetilde{K}_4(t)\left(k_1^\delta(t)\int_\Omega u^\sigma \mathrm{d}x\right)^{x_{21}(n-2)/(n-2-nx_{11})} \\ &+ \widetilde{K}_5(t)\left(k_2^\chi(t)\int_\Omega v^\sigma \mathrm{d}x\right)^{y_{21}(n-2)/(n-2-ny_{11})} + (r_6\sigma+\lambda_2)k_1^\delta(t)\int_\Omega\left|\nabla u^{(\sigma+m-1)/2}\right|^2\mathrm{d}x \\ &+ (r_{12}\sigma+\lambda_4)k_2^\chi(t)\int_\Omega\left|\nabla v^{(\sigma+l-1)/2}\right|^2\mathrm{d}x + K_2(t)\left(k_1^\delta(t)\int_\Omega u^\sigma \mathrm{d}x\right)^{x_{23}(n-2)/(n-2-nx_{13})} \\ &+ K_4(t)\left(k_2^\chi(t)\int_\Omega v^\sigma \mathrm{d}x\right)^{y_{23}(n-2)/(n-2-ny_{13})},\end{aligned} \tag{36}$$

where $\widetilde{K}_1 = \max\{\sigma r_5 r_2 + m a_1 n x_{20}\rho_0^{-1}, \sigma r_{11} r_8 + l a_2 n y_{20}\rho_0^{-1}\}$, $\widetilde{K}_2(t) = K_1(t) + K_3(t)$, $\widetilde{K}_3(t) = \max\{q k_1^{\delta+1}(t)k_2^{-\chi}(t), s k_2^{\chi+1}(t)k_1^{-\delta}(t)\}$, $\widetilde{K}_4(t) = r_5 r_3\sigma k_1^{-(2x_{11}\delta/(n-2-nx_{11}))}(t)$, and $\widetilde{K}_5(t) = r_{11}r_9\sigma k_2^{-(2y_{11}\delta/(n-2-ny_{11}))}(t)$.

If we choose suitable ε_1, ε_4, ε_7, and ε_9 such that $r_6\sigma + \lambda_2 \le 0, r_{12}\sigma + \lambda_4 \le 0$, we can rewrite (36) as

$$\begin{aligned}\phi'(t) &\le \widetilde{K}_6(t)\phi(t) + \widetilde{K}_4(t)(\phi(t))^{\xi_1} + \widetilde{K}_5(t)(\phi(t))^{\xi_2} \\ &+ K_2(t)(\phi(t))^{\xi_3} + K_4(t)(\phi(t))^{\xi_4},\end{aligned} \tag{37}$$

where $\xi_1 = 1 + (2x_{11}/(n-2-nx_{11}))$, $\xi_2 = 1 + (2y_{11}/(n-2-ny_{11}))$, $\xi_3 = 1 + (2x_{13}/(n-2-nx_{13}))$, $\xi_4 = 1 + (2y_{13}/(n-2-ny_{13}))$, and $\widetilde{K}_6 = L + \widetilde{K}_1 + \widetilde{K}_2(t) + \widetilde{K}_3(t)$.

Let

$$\Theta(t^*) = \int_0^{t^*} K(\tau)\mathrm{d}\tau, \tag{38}$$

where $K(t) = \widetilde{K}_6(t) + \widetilde{K}_4(t) + \widetilde{K}_5(t) + K_2(t) + K_4(t)$.

Integrating (37) from 0 to t^*, we have

$$\Theta(t^*) \ge \int_{\phi(0)}^\infty \frac{1}{\eta + \eta^{\xi_1} + \eta^{\xi_2} + \eta^{\xi_3} + \eta^{\xi_4}}\mathrm{d}\eta = S. \tag{39}$$

Considering $\xi_i > 1$ $(i = 1, 2, 3, 4)$, the integration of the right side of (39) exists. It is clear that $\Theta(t^*)$ is an increasing function. So, we can get

$$t^* \ge \Theta^{-1}(S), \tag{40}$$

where Θ^{-1} is the inverse function of Θ.

The proof of Theorem 1 is complete.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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