Research Article

Competitive Coexistence in a Two-Strain Epidemic Model with a Periodic Infection Rate

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In this article, we study the global dynamical behavior of a two-strain SIS model with a periodic infection rate. The positivity and boundedness of solutions are established, and the competitive exclusion conditions are given for the model. The conditions for the global stability of the disease-free equilibrium and persistence of the model are obtained. The conditions of coexistence in this model are also found. Finally, the conditions of uniqueness of the solution are proved.

1. Introduction

Since the first work [1] about the mathematical epidemic model published, there were plenty of results of infectious diseases in modeling and dynamics. These epidemic models usually include two important parameters, the infection rate and recovery rate. Many of the epidemic models focus on the disease eradication and persistence. Among these models, some are applied successfully on infectious disease forecast and control [2–6]. In the analysis of the epidemic model, it is found that infectious diseases often fluctuate over time and exhibit periodic behavior [7–9]. This fluctuation is also observed in the real data [10, 11]. The influenza data from CDC [10] show that the number of influenza cases per week oscillates with a period between two peaks of one year and measles data [11] illustrate the period of two years. The periodic behavior of the incidence of many infectious diseases is caused by the influence of temperature and humidity on virus [12, 13]. In [14, 15], the authors introduced the epidemic models that had periodic coefficients to illustrate the periodical phenomenon. An epidemic SIS model with a periodic infection rate was first considered by Hethcote [15]. Furthermore, Dietz [14] considered SIR and SEIR models with a periodic infection rate.

The causative agents of many diseases are represented by multiple genetically distinct variants. Early autonomous multistrain models suggested that competitive exclusion is the only possible result of the competition of many strains [16]. However, these models disregard many mechanisms such as coinfection [17], mutation [18], cross immunity [19], and periodicity [20], which were proposed as possible mechanisms that could support diversity of causative agents. Castillo-Chavez et al. [21] established a sexually transmitted disease model with two competing strains and found the conditions of coexistence equilibrium and its global stability. The interesting work finished by Nuño et al. [19] considered a two-strain influenza model with cross immunity, and their simulation results showed that there might be up to four coexistence equilibria for their model. Another result about multiple-strain model was finished by Martcheva [20]. In this work, the author introduced a nonautonomous multistrain epidemic model without susceptible individuals and found the coexistence and persistence conditions of this model. All of these results also show that the analysis of multiple-strain model can extend our knowledge about the mechanism of the multiple-strain coexistence and competitive exclusion.

In this paper, we investigate a two-strain SIS model with a periodic infection rate

$$
\begin{cases}
\dfrac{dS}{dt} = \lambda - \dfrac{\beta_1(t)SI_1}{1+S} - \dfrac{\beta_2(t)SI_2}{1+S} + \gamma_1 I_1 + \gamma_2 I_2 - dS, \\[3mm]
\dfrac{dI_1}{dt} = \left(\dfrac{\beta_1(t)S}{1+S} - k_1 \right) I_1, \\[3mm]
\dfrac{dI_2}{dt} = \left(\dfrac{\beta_2(t)S}{1+S} - k_2 \right) I_2,
\end{cases}
\tag{1}
$$

where S is the number of susceptible individuals and I_i ($i = 1, 2$) is the number of individuals that are infected by the disease strain i. $0 < \beta_i(t) < M$ ($i = 1, 2$) is the infection rate of strain i, and it is a periodic function with the period T. "M" here is the upper bound of $\beta_i(t)$ ($i = 1, 2$). $d > 0$ is the death rate of susceptible individuals. $k_i = \gamma_i + d + \alpha_i > 0$, where $\gamma_i > 0$ is the recovery rate of the infected individuals of strain i, $i = 1, 2$. $d_i > d > 0$ is the death rate of the infected individuals of strain i, and $\alpha_i > 0$ is the disease introduced death rate.

One of the most important concepts about the dynamics of epidemic models is the basic reproductive number R_0, which represents the expected number of secondary cases produced by a typical infected individual in a fully susceptible population [22, 23]. According to the definition, it is easy to conclude that an epidemic will never occur when $R_0 < 1$, and instead it will occur if $R_0 > 1$. In this manuscript, we introduce two basic reproductive numbers R_0^1 and R_0^2 for our two-strain model and use these two values as the thresholds to analyze the dynamic behaviors of model (1).

This paper is organized as follows: In Section 2, we analyze the positivity and boundedness of model (1), as well as the global stability of the disease-free equilibrium. The competitive exclusion conditions are also gained in this section. In Section 3, we study the coexistence and stability of T-periodic positive solution of this model. In Section 4, the conditions of uniqueness of the solution are found. Finally, the numerical simulations are done to illustrate our results.

2. The Global Stability of the Disease-Free Equilibrium and the Competitive Exclusion

We first give the positivity and boundedness of model (1).

Theorem 1. *Solutions of model (1) with positive initial conditions are positive and bounded.*

Proof. Assume that the initial condition is $(S(0), I_1(0), I_2(0))$ with $S(0) \geq 0$, $I_1(0) \geq 0$, $I_2(0) \geq 0$, then we have

$$
I_1(t) = \exp\left(\int_0^t \left(\frac{\beta_1(\tau)S(\tau)}{1+S(\tau)} - k_1 \right) d\tau \right) I_1(0) \geq 0,
$$

$$
I_2(t) = \exp\left(\int_0^t \left(\frac{\beta_2(\tau)S(\tau)}{1+S(\tau)} - k_2 \right) d\tau \right) I_2(0) \geq 0.
\tag{2}
$$

The solutions $I_1(t)$ and $I_2(t)$ are nonnegative, and they are positive when $I_1(0) \neq 0$ and $I_2(0) \neq 0$.

Next, we prove the positivity of $S(t)$ with positive initial values. Assuming the contrary and letting $t_0 > 0$ be the first time such that $S(t_0) = 0$, by the first equation of model (1), we have

$$
\left. \frac{dS(t)}{dt} \right|_{t=t_0} = \lambda + \gamma_1 I_1(t_0) + \gamma_2 I_2(t_0) > 0,
\tag{3}
$$

which implies $S(t) < 0$ for $t \in (t_0 - \varepsilon, t_0)$ and sufficient small $\varepsilon > 0$. This contradicts that t_0 is the first time such that $S(t_0) = 0$. It follows that $S(t) > 0$ for $t > 0$.

Next, we prove the boundedness. Let $N(t) = S(t) + I_1(t) + I_2(t)$ and α, we have

$$
\frac{dN(t)}{dt} = \lambda - dN(t) - \alpha_1 I_1(t) - \alpha_2 I_2(t) \leq \lambda - dN(t).
\tag{4}
$$

If the initial condition $0 \leq N(0) \leq (\lambda/d)$ holds, then by the comparison theorem

$$
N(t) \leq \frac{\lambda}{d} - \left(\frac{\lambda}{d} - N(0) \right) e^{-dt} \leq \frac{\lambda}{d}.
\tag{5}
$$

This inequality shows that the solution of model (1) is bounded.

This completes the proof. □

Remark 1. In the rest of this paper, we assume that $(S(0), I_1(0), I_2(0)) \in D$, where $D = \{(S, I_1, I_2) \mid S \geq 0, I_1 \geq 0, I_2 \geq 0, 0 \leq S + I_1 + I_2 \leq (\lambda/d)\}$.

Model (1) has a disease-free equilibrium $E_0 = ((\lambda/d), 0, 0) = (S_E, 0, 0)$. So, we define the basic reproductive number of strain i as $R_0^i = \mu_i(S_E)/k_i$, where $\mu_i(S_E) = 1/T \int_0^T (\beta_i(t)S_E/1 + S_E dt)$, ($i = 1, 2$). Then, we have the following theorem.

Theorem 2. *For model (1), if $R_0^1 < 1$ and $R_0^2 < 1$, then the disease-free equilibrium E_0 is globally asymptotically stable.*

Proof. The solution of model (1) satisfies

$$
I_i(t) = I_i(0) \exp\left(\int_0^t \left(\frac{\beta_i(\tau)S(\tau)}{1+S(\tau)} - k_i \right) d\tau \right), \quad i = 1, 2.
\tag{6}
$$

For any given $t \geq 0$, there exists an integer number n and a real number s such that $t = nT + s$, where $0 \leq s < T$ and $n \geq 0$. The inequality $S(t) \leq N(t) \leq (\lambda/d)$ implies that

$$I_i(t) \leq I_i(0) \exp\left(\int_0^{nT}\left(\frac{\beta_i(\tau)S_E}{1+S_E} - k_i\right)d\tau\right)\exp\left(\int_{nT}^{nT+s}\left(\frac{\beta_i(\tau)S_E}{1+S_E} - k_i\right)d\tau\right)$$

$$= I_i(0) \exp\left(k_i\left(R_0^i - 1\right)n\right)\exp\left(\frac{S_E}{1+S_E}\int_0^s \beta_i(\tau)d\tau - k_i\right), \tag{7}$$

which implies that $\lim_{t\rightarrow\infty}I_i(t) = 0$; that is, for any $\varepsilon > 0$, there exists a $T_i > 0$, such that $I_i(t) < \varepsilon$. Let $T_0 = \max(T_1, T_2)$ and $\beta_0 = \max_{0 \leq t \leq T}\{\beta_1(t) + \beta_2(t)\}$. If $t > T_0$, the first equation of (1) satisfies

$$\frac{dS}{dt} \geq \lambda - \beta_0\varepsilon - dS. \tag{8}$$

The comparison theorem implies that

$$S(t) \geq \frac{\lambda - \beta_0\varepsilon}{d} - \left(\frac{\lambda - \beta_0\varepsilon}{d} - S(0)\right)e^{-dt}. \tag{9}$$

Since $S(t) \leq N(t) \leq \lambda/d$, we have $\lim_{t\rightarrow\infty}S(t) = (\lambda/d)$. The limits $\lim_{t\rightarrow\infty}S(t) = (\lambda/d)$ and $\lim_{t\rightarrow\infty}I_i(t) = 0\,(i = 1, 2)$ show that E_0 is globally attractive.

The Jacobin matrix at the disease-free equilibrium $E_0 = (S_E, 0, 0)$ is

$$J = \begin{pmatrix} -d & \gamma_1 - \dfrac{\beta_1(t)S_E}{1+S_E} & \gamma_2 - \dfrac{\beta_2(t)S_E}{1+S_E} \\[3mm] 0 & \dfrac{\beta_1(t)S_E}{1+S_E} - k_1 & 0 \\[3mm] 0 & 0 & \dfrac{\beta_2(t)S_E}{1+S_E} - k_2 \end{pmatrix}. \tag{10}$$

The three characteristic multipliers are

$$e^{-Td},$$
$$e^{\int_0^T\left((\beta_1(t)S_E/1+S_E) - k_1\right)dt}, \tag{11}$$
$$e^{\int_0^T\left((\beta_2(t)S_E/1+S_E) - k_2\right)dt}.$$

The condition $R_0^i < 1\,(i = 1, 2)$ implies that if $e^{\int_0^T\left((\beta_i(t)S_E/1+S_E) - k_i\right)dt} < 1\,(i = 1, 2)$, then we claim that the disease-free equilibrium is globally asymptotically stable. □

We give the competitive exclusion conditions of model (1).

Theorem 3. *(i) If $(\beta_1(t)/\beta_2(t)) < (k_1/k_2)$ hold, then all solutions of model (1) with $(S(0), I_1(0), I_2(0)) \in D$ satisfies $\lim_{t\rightarrow\infty}I_1(t) = 0$. (ii) If $(\beta_1(t)/\beta_2(t)) > (k_1/k_2)$ hold, then all solutions of model (1) with $(S(0), I_1(0), I_2(0)) \in D$ satisfies $\lim_{t\rightarrow\infty}I_2(t) = 0$.*

Proof. To prove statement (i), we let $m = (k_1/k_2)$ and consider the function

$$V(S, I_1, I_2) = I_1 I_2^{-m}. \tag{12}$$

We have

$$\frac{dV(S, I_1, I_2)}{dt} = (\beta_1(t) - m\beta_2(t))\frac{S}{1+S}V(S, I_1, I_2). \tag{13}$$

The condition $(\beta_1(t)/\beta_2(t)) < m$ implies that

$$(\beta_1(t) - m\beta_2(t))\frac{S(t)}{1+S(t)} < 0, \tag{14}$$

which follows

$$I_1(t)I_2(t)^{-m} = I_1(t_0)I_2(t_0)^{-m}\exp\left(\int_0^t\left((\beta_1(\tau) - m\beta_2(\tau))\frac{S(\tau)}{1+S(\tau)}\right)d\tau\right). \tag{15}$$

Let

$$D_0 = \left\{(S, I_1, I_2)\,\Big|\,\frac{dV(S, I_1, I_2)}{dt} = 0,\, (S, I_1, I_2) \in D\right\}. \tag{16}$$

Then $D_0 = \{(S, I_1, I_2)\,|\,I_1 = 0\}$ is an invariant set of model (1). By Lasalle's principle, we know that all solutions of model (1) in D satisfy $\lim_{t\rightarrow\infty}I_1(t) = 0$.

Using similar argument, we can obtain statement (ii). □

These two conditions are usually called the principle of competitive exclusion [24, 25].

3. Coexistence

This section discusses the persistence of model (1).

Theorem 4. *For model (1), if $R_0^1 > 1$ and $R_0^2 > 1$, then there exists a positive constant $\delta^* > 0$, such that for at least one strain $I(t)$ satisfies $\limsup_{t\rightarrow\infty}I(t) \geq \delta^*$.*

Proof. As $S_E = \lambda/d$ is a global attractor for $(dS/dt) = \lambda - dS$, we can choose small enough $\delta > 0$ that for the system,

$$\frac{dS}{dt} = \lambda + \left(\gamma_1 + \gamma_2 - \frac{\beta_1(t)S}{1+S} - \frac{\beta_2(t)S}{1+S}\right)\eta - dS. \tag{17}$$

If $0 < \eta < \delta$, then $\forall \varepsilon > 0$. There exists a time T that when $t > T$, $|S(t) - S_E| < \varepsilon$. Without loss of generality, from the conditions $R_0^1 = (1/k_1 T)\int_0^T (\beta_1(t)S_E/1 + S_E) > 1$ and $R_0^2 = (1/k_2 T)\int_0^T (\beta_2(t)S_E/1 + S_E) > 1$, we can choose $\delta^* > 0$ and small ε that, for any $t > T$, $S(t) > S_E - \varepsilon$, $(1/k_1 T)\int_0^T (\beta_1(t)(S_E - \varepsilon)/1 + (S_E - \varepsilon)) > 1$, and $(1/k_2 T)\int_0^T \beta_2(t)(S_E - \varepsilon)/1 + (S_E - \varepsilon) > 1$.

We prove by contradiction. Assume both of these two strains I_1 and I_2 satisfy $\limsup_{t\rightarrow\infty}I_1(t) < \delta^*$ and

$\limsup_{t\longrightarrow\infty}I_2(t)<\delta^*$. Without loss of generality, we assume $I_1(t)<\delta^*$ and $I_2(t)<\delta^*$.

As a consequence, for all $t\geq T$, there hold

$$I_1'(t)\geq\left(\frac{\beta_1(t)(S_E-\varepsilon)}{1+(S_E-\varepsilon)}-k_1\right)I_1,$$

$$I_2'(t)\geq\left(\frac{\beta_2(t)(S_E-\varepsilon)}{1+(S_E-\varepsilon)}-k_2\right)I_2,$$

(18)

and it is easy to see that $\lim_{t\longrightarrow\infty}I_1(t)=+\infty$ and $\lim_{t\longrightarrow\infty}I_2(t)=+\infty$, which lead to a contradiction. Hence, the conclusion holds. □

Moreover, if $(\beta_2(t)/\beta_1(t))=(k_2/k_1)=m$ hold for all $t\geq0$, then it is easy to get that

$$\frac{dI_2}{dI_1}=m\frac{I_2}{I_1}.$$

(19)

Therefore,

$$I_2=QI_1^m,$$

(20)

where Q is a constant which is determined by the initial values $I_1(0),I_2(0)$. Then, the dynamical behavior of model (1) is equivalent to that of the following model:

$$\begin{cases}\dfrac{dS}{dt}=\lambda+\left(\gamma_1-\dfrac{\beta_1(t)S}{1+S}\right)I_1+Q\left(\gamma_2-m\dfrac{\beta_1(t)S}{1+S}\right)I_1^m-dS,\\[2mm]\dfrac{dI_1}{dt}=\left(\dfrac{\beta_1(t)S}{1+S}-k_1\right)I_1.\end{cases}$$

(21)

We can get the following theorem about the above model.

Theorem 5. *If $R_0^1>1$, then there exists a positive constant $\delta>0$; then (1) all solutions of model (21) with positive initial conditions will satisfy $\liminf_{t\longrightarrow\infty}I_1(t)\geq\delta$, and model (21) will admit at least one positive periodic solution.*

Proof. Since $R_0^1=\mu_1(S_E)/k_1>1$, there exists a small enough $\eta>0$ such that $\mu_1(S_E-\eta)>k_1$. Let us consider the following equation:

$$\frac{dx}{dt}=\lambda-\beta_1(t)\delta-mQ\beta_1(t)\delta^m-dx,$$

(22)

and the solution of this equation (22) is

$$x(t)=S_E+C_0e^{-dt}-\left(\delta+mQ\delta^m\right)\int_0^t\beta_1(\tau)e^{d(\tau-t)}d\tau,$$

(23)

where C_0 is a constant. As $0<\beta_1(t)<M$, the following inequalities hold:

$$S_E-\left(\delta+mQ\delta^m\right)\frac{M}{d}\leq\lim_{t\longrightarrow\infty}x(t)\leq S_E.$$

(24)

Thus, we can fix a small enough number $\delta=\delta^*>0$ such that $\lim_{t\longrightarrow\infty}x(t)>S_E-\eta$.

Suppose, by contradiction, that $\limsup_{t\longrightarrow\infty}I_1(t)<\delta^*$. Without loss of generality, we can assume that $I_1(t)<\delta^*$, it follows $0\leq I_1(t)<\delta^*$. By the first equation of model (21), we have the following inequality:

$$S'(t)\geq\lambda-\beta_1(t)\delta^*-mQ\beta_1(t)\delta^{*m}-dS.$$

(25)

Thus, there exists a time T^*; for all $t>T^*$, we have $S(t)>S_E-\eta$. As a consequence, for all $t\geq T$, there holds

$$I_1'(t)\geq\left(\frac{\beta(t)(S_E-\eta)}{1+(S_E-\eta)}-k\right)I_1.$$

(26)

Since $\mu(S_E-\eta)>k$, it is easy to see that $\lim_{t\longrightarrow\infty}I_1(t)=+\infty$, which leads to a contradiction. Hence, $\limsup_{t\longrightarrow\infty}I_1(t)\geq\delta^*$.

Define

$$\begin{aligned}X&=R_+^2,\\X_0&=R_+\times\mathrm{int}R_+,\\\partial X_0&=X/X_0,\end{aligned}$$

(27)

where $R_+=\{x\mid x\geq0\}$ and $\mathrm{int}R_+=\{x\mid x>0\}$. Then, assume $u(t,(S^0,I_1^0))$ is the solution of model (21) and $p((S^0,I_1^0))=u(T,(S^0,I_1^0))$ is the Poincare map.

E_0 is globally attractive in ∂X_0 for p, and E_0 is isolated invariant sets in X with $W^s(E_0)\cap X_0=\varnothing$. Clearly, every orbit in ∂X_0 converges to E_0. By Theorem 1.3.1 in [26], for a stronger repelling property of ∂X_0, we conclude that p is uniformly persistent with respect to $(X_0,\partial X_0)$. Thus, Theorem 3.1.1 in [26] implies the uniform persistence of the solutions of model (21) with respect to $(X_0,\partial X_0)$. By Theorem 1.3.6 in [26], p has a fixed point $(\widehat{S}(0),\widehat{I}_1(0))\in X_0$. Then $\widehat{S}(0)\in R^+$ and $\widehat{I}_1(0)\in\mathrm{int}R^+$. We further claim that $\widehat{S}(0)\in R_+/\{0\}$. Suppose $\widehat{S}(0)=0$, then we have $\lambda+\gamma_1\widehat{I}_1(0)+\gamma_2Q\widehat{I}_1^m(0)=0$, which is a contradiction. So from model (21), we can get $u(t,(\widehat{S}(0),\widehat{I}_1(0)))\in\mathrm{int}(R_+^2)$, $\forall t>0$. Then $(\widehat{S}(0),\widehat{I}_1(0))$ is a component-wise positive fixed point of p. Thus, $(\widehat{S}(t),\widehat{I}_1(t))$ is a positive T-periodic solution of model (21). □

Remark 2. It is easy to conclude that when the condition $\forall t,(\beta_2(t)/\beta_1(t))=(k_2/k_1)=m$ and $R_0^1>1$ hold, there exist at least one persistent solution $(S(t),I_1(t),\text{and }I_2(t))$ with respect to $I_2(t)=QI_1(t)^m$ and $Q\geq0$. Moreover, when $Q=0$, model (21) becomes the plane system that $I_2(t)=0$.

Then, we consider a more special case that the infection rate of model (21) satisfies $\beta_1(t)=b_0+\varepsilon\cos(t)$, where $\varepsilon>0$ is a small number. When $R_0^1>1$ and $\varepsilon=0$, model (21) becomes an autonomous model

$$\begin{cases}\dfrac{dS}{dt}=\lambda+\left(\gamma_1-\dfrac{b_0S}{1+S}\right)I_1+Q\left(\gamma_2-m\dfrac{b_0S}{1+S}\right)I_1^m-dS,\\[2mm]\dfrac{dI_1}{dt}=\left(\dfrac{b_0S}{1+S}-k_1\right)I_1.\end{cases}$$

(28)

The positive equilibrium $E_1=(S^*,I^*)$ for model (28) satisfies

$$\frac{b_0 S^*}{1 + S^*} = k_1,$$

$$\left(\gamma - \frac{b_0 S^*}{1 + S^*}\right) I^* + Q\left(\gamma_2 - m\frac{b_0 S^*}{1 + S^*}\right) I^{*\,m} = dS^* - \lambda. \quad (29)$$

As $(b_0 S^*/1 + S^*) = k_1 > \gamma_1$, $m(b_0 S^*/1 + S^*) = k_2 > \gamma_2$ and $R_0^1 > 1$ implies $dS^* - \lambda > 0$, so there exists unique positive equilibrium $E_1 = (S^*, I^*)$ for model (28). It is easy to check the Jacobin matrix at equilibrium E_1 is

$$\begin{pmatrix} -\dfrac{b_0 I^*}{(1 + S^*)^2} - \dfrac{mQb_0 I^{*\,m}}{(1 + S^*)^2} - d & \gamma_1 - \dfrac{b_0 S^*}{1 + S^*} + mQ\left(\gamma_2 - m\dfrac{b_0 S^*}{1 + S^*}\right) I_1^{*\,(m-1)} \\ \\ \dfrac{b_0 I^*}{(1 + S^*)^2} & 0 \end{pmatrix}, \quad (30)$$

and its two eigenvalues λ_1, λ_2 satisfy

$$\lambda_1 + \lambda_2 = -\frac{b_0 I^*}{(1 + S^*)^2} - \frac{mQb_0 I^{*m}}{(1 + S^*)^2} - d < 0,$$

$$\lambda_1 \lambda_2 = \frac{b_0 I^*}{(1 + S^*)^2}\left(\gamma_1 - \frac{b_0 S^*}{1 + S^*}\right. \quad (31)$$

$$\left. + mQ\left(\gamma_2 - m\frac{b_0 S^*}{1 + S^*}\right) I_1^{*\,(m-1)}\right) > 0,$$

when $R_0^1 > 1$, which implies that the real parts of λ_1 and λ_2 are negative. So the unique equilibrium E_1 is asymptotically stable.

The right-hand side of model (21) $f(S, I, \mu)$ can be illustrated as

$$f(S, I, b_1) = \begin{pmatrix} \lambda + \left(\gamma_1 - \dfrac{b_0 S}{1 + S}\right) I_1 + Q\left(\gamma_2 - m\dfrac{b_0 S}{1 + S}\right) I_1^m - dS \\ \\ \left(\dfrac{b_0 S}{1 + S} - k_1\right) I_1 \end{pmatrix}$$

$$+ \varepsilon \begin{pmatrix} -\dfrac{\cos(t) S I_1}{1 + S} - Qm\dfrac{\cos(t) S I_1^m}{1 + S} \\ \\ \dfrac{\cos(t) S I_1}{1 + S} \end{pmatrix}. \quad (32)$$

It is easy to see $f(S, I, \varepsilon)$ is analytic with $S, I \in R^+$ and $\varepsilon \in R$, and when $\epsilon = 0$, model (21) becomes (28). Then by Theorem 1.1 in Chapter 14 in [27], there exists a T-periodic solution $(S(t), I(t))$ near equilibrium E_1 for small ε, and by Theorem 1.2 in Chapter 14 in [27], this solution $(S(t), I(t))$ is asymptotically stable.

Thus, we have the following theorem.

Theorem 6. *For model (21), if $R_0^1 > 1$ and $\beta_1(t) = b_0 + \varepsilon \cos(t)$, where $\varepsilon > 0$ and small, then there exists a asymptotically stable positive 2π-periodic solution $(S(t), I(t))$ for model (21).*

4. Uniqueness of the Solution on the Surface

In this section, we mainly talk about the uniqueness of the solution of model (21), and the solution is on the surface $I_2 = QI_1^m$. We consider the model

$$\begin{cases} \dfrac{dS}{dt} = \lambda + \left(\gamma_1 - \dfrac{\beta(t)S}{1 + S}\right) I + Q\left(\gamma_2 - \dfrac{m\beta(t)S}{1 + S}\right) I^m - dS, \\ \\ \dfrac{dI}{dt} = \left(\dfrac{\beta(t)S}{1 + S} - k\right) I, \end{cases} \quad (33)$$

where $\beta(t) = b_0 + \varepsilon \cos(t)$ and $b_0 = 8$, $\lambda = 1$, $\gamma_1 = \gamma_2 = 1$, $d = 1$, $k = 2$, $Q = 2$, and $m = 2$.

Theorem 7. *When ε is small enough, model (5) exhibits unique 2π-periodic solution.*

Proof. Assume system (33) has a solution

$$S_p(t) = S_0(t) + \varepsilon S_1(t) + \varepsilon^2 S_2(t) + \varepsilon^3 S_3(t) + \ldots,$$

$$I_p(t) = I_0(t) + \varepsilon I_1(t) + \varepsilon^2 I_2(t) + \varepsilon^3 I_3(t) + \ldots. \quad (34)$$

Using (34) in model (35), and equating the powers of ε, we get the following equations:

$$\begin{cases} \dfrac{dS_0(t)}{dt} = \lambda + \left(\gamma_1 - \dfrac{b_0 S_0(t)}{1 + S_0(t)}\right) I_0(t) \\ \\ \quad + Q\left(\gamma_2 - \dfrac{mb_0 S_0(t)}{1 + S_0(t)}\right) I_0(t)^m - dS_0(t), \\ \\ \dfrac{dI_0(t)}{dt} = \left(\dfrac{b_0 S_0(t)}{1 + S_0(t)} - k\right) I_0(t), \end{cases} \quad (35)$$

$$\begin{pmatrix} \dfrac{dS_j(t)}{dt} \\ \\ \dfrac{dI_j(t)}{dt} \end{pmatrix} = J\begin{pmatrix} S_j(t) \\ \\ I_j(t) \end{pmatrix} + \begin{pmatrix} F_j(t) \\ \\ G_j(t) \end{pmatrix},$$

where F_j and G_j are periodic functions of the period 2π, and they are only relevant to $S_0(t), I_0(t), \ldots, S_{j-1}(t), I_{j-1}(t)$, and

$$J = \begin{pmatrix} -2\dfrac{b_0 Q I_0(t)^2}{(1+S_0(t))^2} - \dfrac{b_0 I_0(t)}{(1+S_0(t))^2} - d & \gamma_1 - \dfrac{b_0 S_0(t)}{1+S_0(t)} + 2Q\left(\gamma_2 - 2\dfrac{b_0 S_0(t)}{1+S_0(t)}\right) I_0(t) \\[3mm] \dfrac{b_0 I_0(t)}{(1+S_0(t))^2} & \dfrac{b_0 S_0(t)}{1+S_0(t)} - k \end{pmatrix}. \tag{36}$$

We set $b_0 = 8$, $\lambda = 1$, $\gamma_1 = \gamma_2 = 1$, $d = 1$, $k = 2$, and $Q = 2$ and obtain

$$S_0(t) = \frac{1}{3},$$
$$\tag{37}$$
$$I_0(t) = \frac{1}{3},$$

The fundamental matrix $\Phi(t)$ of the linear homogeneous differential equation system of system (35) is

$$\Phi(t) = \begin{pmatrix} \phi_{11}(t), \phi_{12}(t) \\ \phi_{21}(t), \phi_{22}(t) \end{pmatrix}, \tag{38}$$

where

$$\phi_{11}(t) = e^{-7/4t}\cos(1/4t\sqrt{23}) - \frac{7\sqrt{23}e^{-7/4t}\sin(1/4t\sqrt{23})}{23},$$

$$\phi_{12}(t) = -\frac{12\sqrt{23}e^{-7/4t}\sin(1/4t\sqrt{23})}{23},$$

$$\phi_{21}(t) = \frac{6\sqrt{23}e^{-7/4t}\sin(1/4t\sqrt{23})}{23},$$

$$\phi_{22}(t) = e^{-7/4t}\cos(1/4t\sqrt{23}) + \frac{7\sqrt{23}e^{-7/4t}\sin(1/4t\sqrt{23})}{23}.$$
$$\tag{39}$$

The inverse matrix of $\Phi(t)$ is $\Psi(t)$:

$$\Psi(t) = \begin{pmatrix} \psi_{11}(t), \psi_{12}(t) \\ \psi_{21}(t), \psi_{22}(t) \end{pmatrix}, \tag{40}$$

where

$$\psi_{11}(t) = \frac{1}{23}\frac{7\sqrt{23}\sin(1/4t\sqrt{23}) + 23\cos(1/4t\sqrt{23})}{e^{-7/4t}\left((\cos(1/4t\sqrt{23}))^2 + (\sin(1/4t\sqrt{23}))^2\right)},$$

$$\psi_{12}(t) = \frac{12\sqrt{23}\sin(1/4t\sqrt{23})}{23e^{-7/4t}\left((\cos(1/4t\sqrt{23}))^2 + (\sin(1/4t\sqrt{23}))^2\right)},$$

$$\psi_{21}(t) = -\frac{6\sqrt{23}\sin(1/4t\sqrt{23})}{23e^{-7/4t}\left((\cos(1/4t\sqrt{23}))^2 + (\sin(1/4t\sqrt{23}))^2\right)},$$

$$\psi_{22}(t) = \frac{1}{23}\frac{-7\sqrt{23}\sin(1/4t\sqrt{23}) + 23\cos(1/4t\sqrt{23})}{e^{-7/4t}\left((\cos(1/4t\sqrt{23}))^2 + (\sin(1/4t\sqrt{23}))^2\right)}.$$
$$\tag{41}$$

We obtain the solution of system (35)

$$\begin{pmatrix} S_j(t) \\ I_j(t) \end{pmatrix} = \Phi(t)\begin{pmatrix} c_{1j} \\ c_{2j} \end{pmatrix}\Psi(t)\int_0^t \Psi(\tau)\begin{pmatrix} F_j(\tau) \\ G_j(\tau) \end{pmatrix}d\tau. \tag{42}$$

By periodicity, we obtain

$$\begin{pmatrix} c_{1j} \\ c_{2j} \end{pmatrix} = (\Phi(0) - \Phi(2\pi))^{-1}\Phi(2\pi)\int_0^{2\pi}\Psi(\tau)\begin{pmatrix} F_j(\tau) \\ G_j(\tau) \end{pmatrix}d\tau. \tag{43}$$

After simple calculation, we get $\det(\Phi(0) - \Phi(2\pi))^{-1} \approx 1 \neq 0$. Therefore, we obtain the unique existence of the periodic solution of period 2π of model (33). □

5. Numerical Simulation

Numerical simulations are performed to illustrate the dynamic behaviors of model (1) and (21).

Figures 1(a) and 1(b) show the solution of (21) under the conditions $R_0^1 = 1.5$ and $R_0^1 = 2$. From these two figures, we can see that there exists one stable periodic solution of model (21) under different values of R_0, and when the value of R_0 becomes larger, the mean value of $I(t)$ becomes larger.

Figures 2(a) and 2(b) show the results of Theorem 3. When the competitive exclusion condition holds, one strain will drive another into extinction.

Figures 3(a) and 3(b) show the coexistence of model (1) under the condition $\beta_1(t)/k_1 = \beta_2(t)/k_2$. Figure 3(a) is under the initial condition $(S(0), I_1(0), I_2(0)) = (2, 1, 2)$ and Figure 3(b) is $((S(0), I_1(0), I_2(0)) = (2, 2, 1)$.

Figure 4(a) is the solution of model (1) with $\beta_1 = 3 + \sin(t)$, $\beta_2 = 3 + \cos(t)$, $\lambda = 1$, $d = 1$, $k_1 = 1$, $\gamma_1 = 0.1$, $k_2 = 1$, $\gamma_2 = 0.1$, and $R_0^1 = R_0^2$. Figure 4(b) is the solution of model (1) with $\beta_1 = 3 + \sin(t)$, $\beta_2 = 3.5 + \sin(t)$, $\lambda = 1$, $d = 1$, $k_1 = 1.2$, $\gamma_1 = 0.1$, $k_2 = 1.401$, $\gamma_2 = 0.1$, $R_0^1 = 1.25$, $R_0^2 = 1.2491$, and $R_0^1 \neq R_0^2$. These two figures illustrate that the basic reproductive numbers of different strains cannot determine the coexistence of model (1). One strain can drive another into extinction even with the same reproductive number, and the two strains can coexist with different basic reproductive numbers.

6. Discussion

In this paper, we study the dynamical behaviors of a two-strain SIS model with a periodic infection rate and get its
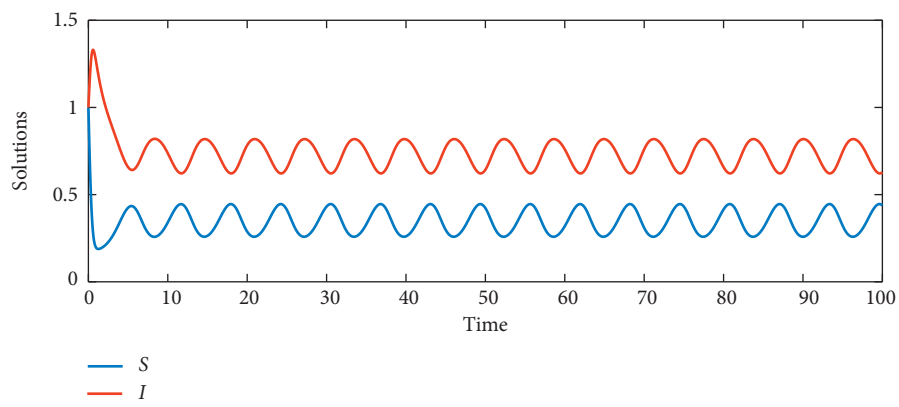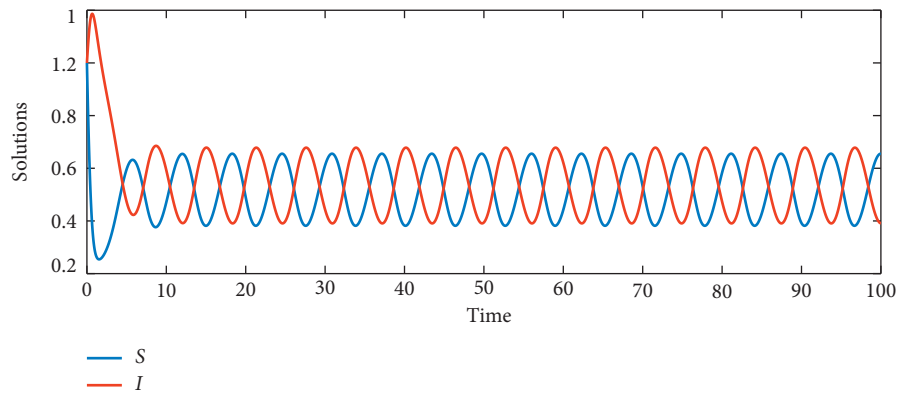
(a)

(b)

Figure 1: The simulated solution of model (21) under the conditions (a) $R_0^1 = 1.5$ and (b) $R_0^1 = 2$.
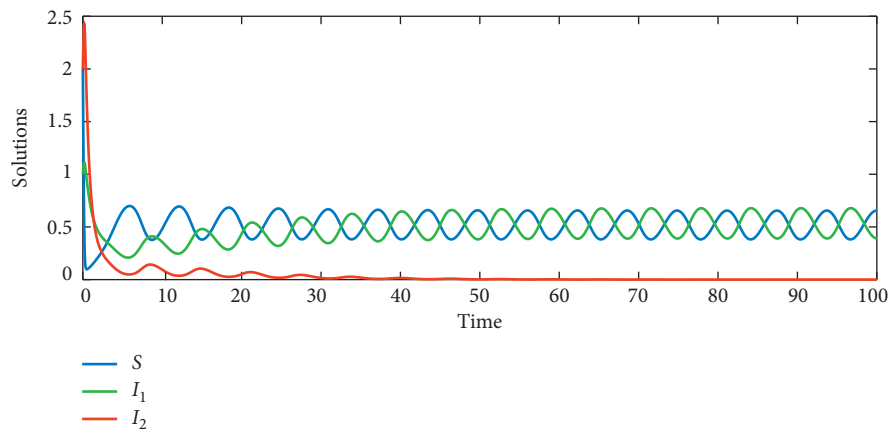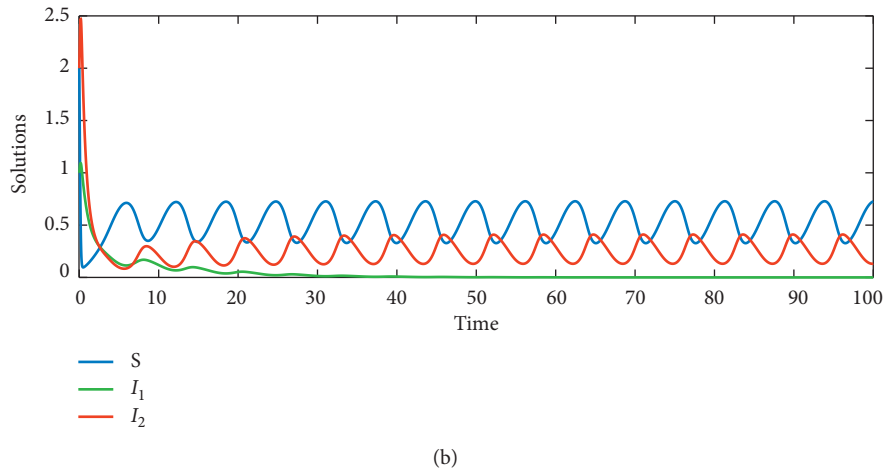

(a)

Figure 2: Continued.

(b)

FIGURE 2: The simulated solution of model (1) under the conditions (a) $\beta_1(t)/k_1 = 3 + \sin(t)$ and $\beta_2(t)/k_2 = 2.85 + 0.95\sin(t)$ and (b) $\beta_1(t)/k_1 = 2.72 + 0.91\sin(t)$ and $\beta_2(t)/k_2 = 3 + \sin(t)$.
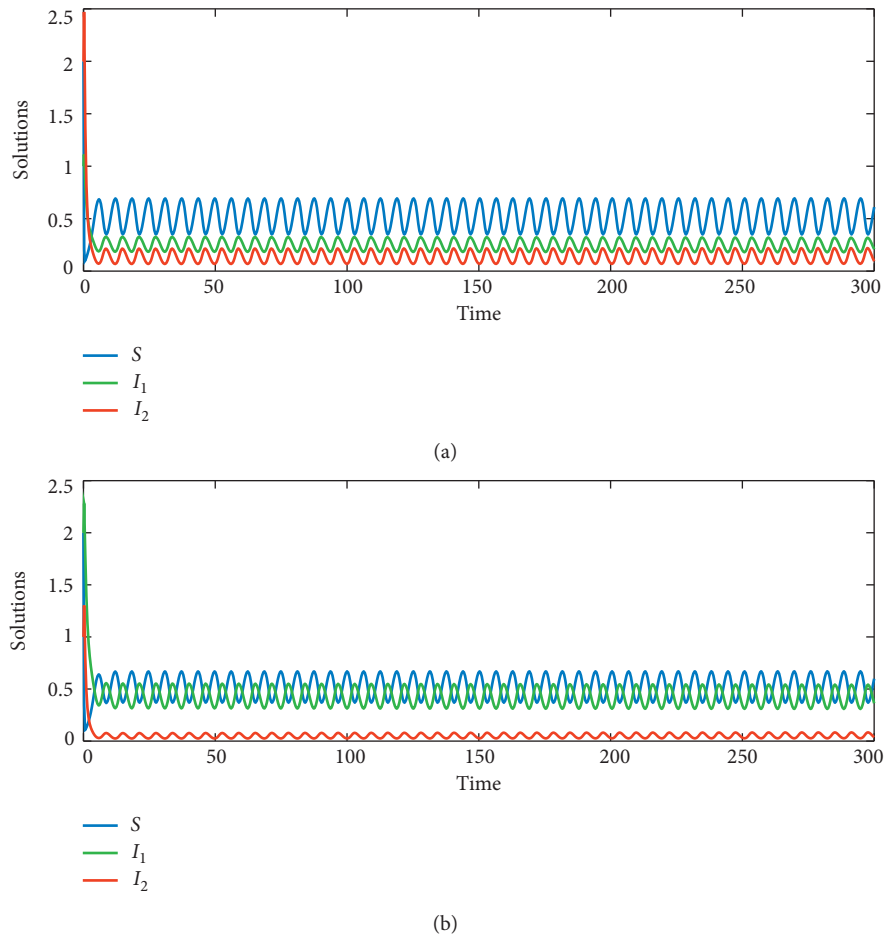

(a)

(b)

FIGURE 3: The simulated solution of model (1) with $\beta_1(t)/k_1 = \beta_2(t)/k_2 = 3 + \sin(t)$ and the initial conditions (a) $(S(0), I_1(0), I_2(0)) = (2, 1, 2)$ and (b) $((S(0), I_1(0), I_2(0)) = (2, 2, 1)$.

persistence, competitive exclusion, and coexistence conditions. We gain the global stability conditions of the disease-free equilibrium E_0 and establish the competitive exclusion condition of the two strains. The coexistence and uniqueness are also discussed and proved. Our results lead to a new insight into the mechanism of two strains interaction and provide a new approach to investigate the inference of the periodic infection rate on the coexistence of two strains. It is
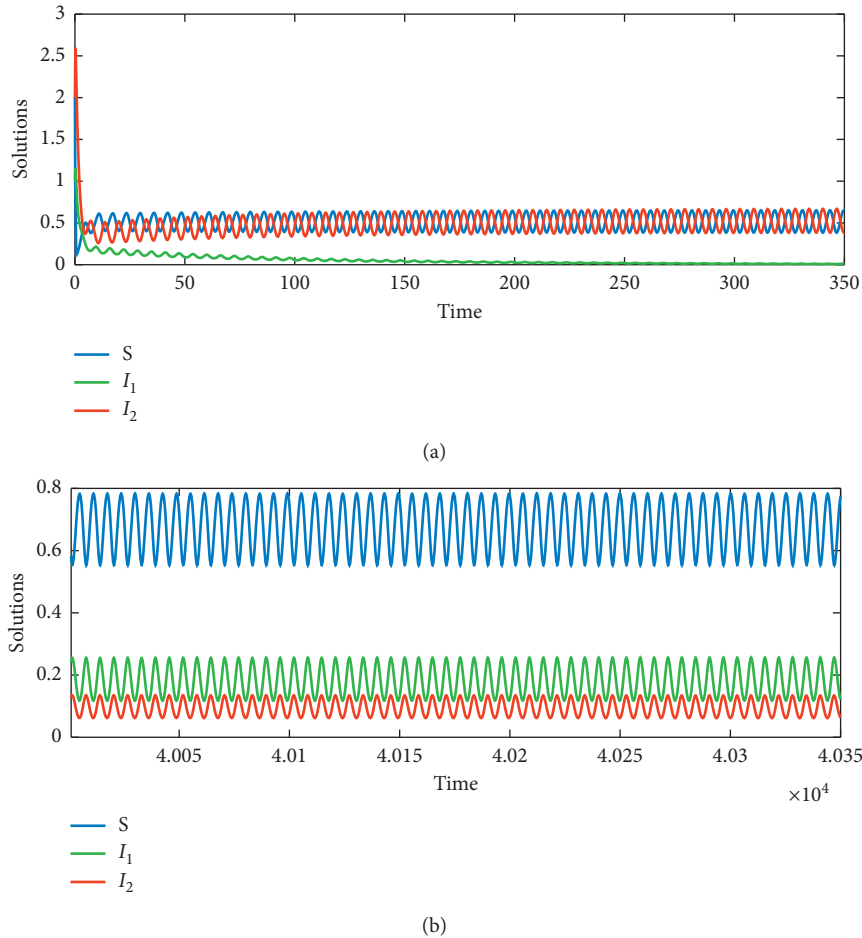
(a)

(b)

Figure 4: (a) The simulated solution of model (1) with $\beta_1 = 3 + \sin(t)$, $\beta_2 = 3 + \cos(t)$, $\lambda = 1$, $d = 1$, $k_1 = 1$, $\gamma_1 = 0.1$, $k_2 = 1$, and $\gamma_2 = 0.1$. (b) The stable simulated solution of model (1) with $\beta_1 = 3 + \sin(t)$, $\beta_2 = 3.5 + \sin(t)$, $\lambda = 1$, $d = 1$, $k_1 = 1.2$, $\gamma_1 = 0.1$, $k_2 = 1.401$, and $\gamma_2 = 0.1$.

worth to mention that the analysis in this article can be applied to the n-strain epidemic model with a periodic infection rate and obtain similar results.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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