Research Article

The Bondage Number of Generalized Petersen Graphs $P(n, 2)$

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The domination number $\gamma(G)$ of a nonempty graph G is the minimum cardinality among all subsets $D \subseteq V(G)$ such that $N_G[D] = V(G)$. The bondage number $b(G)$ of a graph G is the smallest number of edges whose removal from G results in a graph with larger domination number. The exact value of $b(P(n, 2))$ for $n = 0, 3, 4 \pmod{5}$ and the bounds of $b(P(n, 2))$ for $n = 1, 2 \pmod{5}$ are determined.

1. Introduction

For terminology and notation on graph theory not given here, we refer the reader to [1]. Let $G = (V, E)$ be a finite, undirected graph without loops, multiple edges, and isolated vertices, where $V = V(G)$ is the vertex set and $E = E(G)$ is the edge set. A graph G is *nonempty* if $E(G) \neq \emptyset$. For a vertex v , let $N_G(v) = \{u: uv \in E(G)\}$ be the *open neighborhood* and $N_G[v] = N_G(v) \cup \{v\}$ be the *closed neighborhood*. For a set $D \subseteq V(G)$, the *open neighborhood* $N_G(D)$ is defined to be $\bigcup_{u \in D} (N_G(u))$ and the *closed neighborhood* is $N_G[D] = N_G(D) \cup D$. We use $G[D]$ to denote the subgraph of G induced by D .

A subset $D \subseteq V(G)$ is a *dominating set* of G if $N_G[D] = V(G)$. The *domination number* $\gamma(G)$ is the minimum cardinality of a dominating set of G . A dominating set is called a γ -*set* if its cardinality is $\gamma(G)$. Studies on domination number have long attracted graph theorists for their applications and theoretical interest. The two outstanding domination books by Haynes et al. [2, 3] have a comprehensive research of domination.

The *bondage number* of a nonempty graph G is the smallest number of edges whose removal from G results in a graph with domination number greater than the domination number of G . That is,

$$b(G) = \min\{|B|: B \subseteq E(G), \gamma(G - B) > \gamma(G)\}. \quad (1)$$

The bondage number of a nonempty graph is well defined since the domination number of every spanning subgraph of a nonempty graph G is at least as great as $\gamma(G)$. The bondage number is proposed by Fink et al. [4] in 1990. However, it is quite difficult to compute the exact value of the bondage number for general graphs. Hu and Xu [5] showed that the problem of determining bondage number for arbitrarily given graph is NP-hard. Therefore, it is important to determine bondage numbers for some graphs or networks with simple structure. The exact value of the bondage number has been worked out for only a few classes of graphs, such as complete graphs, paths, cycles, complete t -partite graphs [4], trees [6–8], complete t -partite digraphs [9], de Bruijn and Kautz digraphs [10], the Cartesian product of two cycles [11–13], the Cartesian product of two paths [14], the strong product of two paths [15], and Mycielski Graphs [16]. For a further study on the advances of bondage number, the readers are suggested to refer to [17, 18].

For positive integer numbers n and k , where $n > 2k$, the *generalized Petersen graph* $P(n, k)$ is a graph on $2n$ ($n \geq 3$) vertices with

$$\begin{aligned} V(P(n, k)) &= \{v_i, u_i: 0 \leq i \leq n-1\}; \\ E(P(n, k)) &= \{v_i v_{i+1}, v_i u_i, u_i u_{i+k}: 0 \leq i \leq n-1, \text{subscripts modulo } n\}. \end{aligned} \quad (2)$$

Although the generalized Petersen graph $P(n, k)$ has a special construction, the complexity and difficulty are ubiquitous throughout the study on it for each $k \geq 1$. Even so, many researchers still insist on $P(n, k)$ and obtain many graph parameter values other than the bondage number of $P(n, k)$ [19–25]. Yet, the studies on the domination number and its variations of $P(n, k)$ are attractive. Recall that the problems of determining the domination number and bondage number for arbitrarily given graph are NP-complete [26] and NP-hard, respectively. Therefore, researchers tend to study the domination problem of generalized Petersen graph $P(n, k)$ with a given small value for k or a

given relationship between n and k [27–33]. It is worth mentioning that Fu et al. [28] showed the exact domination number of generalized Petersen graph $P(n, 2)$. Hence, it is meaningful to study the bondage number of generalized Petersen graph $P(n, 2)$ in this paper. The rest of the paper is organized as follows. Section 2 shows the lower bound of $b(P(n, 2))$. In Section 3, we present the upper bounds of $b(P(n, 2))$ for $n = 0, 3, 4 \pmod{5}$ and $n = 1, 2 \pmod{5}$, respectively. In particular, the upper bound of $b(P(n, 2))$ for $n = 0, 3, 4 \pmod{5}$ is just the exact value.

2. Lower Bound on Bondage Number of $P(n, 2)$

Let $n = 5m + t$, that is, $t \equiv n \pmod{5}$ and $m = \lfloor n/5 \rfloor$. Fu et al. [28] proved that $\gamma(P(n, 2)) = n - \lfloor n/5 \rfloor - \lfloor (n+2)/5 \rfloor$ and showed a minimum dominating set D of graph $P(n, 2)$, where

$$D = \begin{cases} \{v_{5i+1}, u_{5i+2}, v_{5i+3}: 0 \leq i \leq m-1\}, & \text{if } t = 0, \\ \{v_{5i+1}, u_{5i+2}, v_{5i+3}: 0 \leq i \leq m-1\} \cup \{v_{n-1}\}, & \text{if } t = 1, \\ \{v_{5i+1}, u_{5i+2}, v_{5i+3}: 0 \leq i \leq m-1\} \cup \{v_{n-2}, u_{n-1}\}, & \text{if } t = 2, \\ \{v_{5i+1}, u_{5i+2}, v_{5i+3}: 0 \leq i \leq m-1\} \cup \{v_{n-2}, u_{n-3}\}, & \text{if } t = 3, \\ \{v_{5i+1}, u_{5i+2}, v_{5i+3}: 0 \leq i \leq m-1\} \cup \{v_{n-3}, u_{n-2}, v_{n-1}\}, & \text{if } P(n, 2)t = 4. \end{cases} \quad (3)$$

In order to determine the lower bound on bondage number of $P(n, 2)$, we need to first prove the following lemma by the definition of bondage number.

Lemma 1. $\gamma(P(n, 2) - \{e_1, e_2\}) \leq \gamma(P(n, 2))$, where e_1 and e_2 are any two edges of G .

Proof. By the definition of graph $P(n, 2)$, we need to consider the following six cases: $e_1 = v_i v_{i+1}$, $e_2 = v_j v_{j+1}$; $e_1 = v_i u_i$, $e_2 = v_j u_j$; $e_1 = u_i u_{i+2}$, $e_2 = u_j u_{j+2}$; $e_1 = v_i v_{i+1}$, $e_2 = v_j u_j$; $e_1 = v_i v_{i+1}$, $e_2 = u_j u_{j+2}$; $e_1 = v_i u_i$, $e_2 = u_j u_{j+2}$, $0 \leq i, j \leq n-1$. Actually, we only show the proof for $e_1 = v_i v_{i+1}$, $e_2 = v_j v_{j+1}$ and $e_1 = v_i u_i$, $e_2 = v_j u_j$, since the proofs for the other four cases are similar (for reference, see Appendix).

Assume that D is the minimum dominating set of graph $P(n, 2)$ constructed in [28]. We prove $\gamma(P(n, 2) - \{v_i v_{i+1}, v_j v_{j+1}\}) \leq \gamma(P(n, 2))$ for $0 \leq i, j \leq n-1$ in the following. Without loss of generality, we assume that $i < j$ and $j = i + k$, where $1 \leq k \leq n$. If $\lfloor n/2 \rfloor + 1 \leq k \leq n$, then $P(n, 2) - \{v_i v_{i+1}, v_j v_{j+1}\} \cong P(n, 2) - \{v_i v_{i+1}, v_{n-j} v_{n-j+1}\}$. In view of $\gamma(P(n, 2) - \{v_i v_{i+1}, v_j v_{j+1}\}) = \gamma(P(n, 2) - \{v_i v_{i+1}, v_{n-j} v_{n-j+1}\})$ and $n - j - i \leq \lfloor n/2 \rfloor$, we only need to prove $\gamma(P(n, 2) - \{v_i v_{i+1}, v_j v_{j+1}\}) \leq \gamma(P(n, 2))$ for $1 \leq k \leq \lfloor n/2 \rfloor$. Let

$$\begin{aligned} E_0 &= \{v_{5i+1} v_{5i+2}, v_{5i+2} v_{5i+3}, v_{5i+4} v_{5i+5} \mid 0 \leq i \leq m-1\}, \\ E_1 &= E_0 \cup \{v_{n-1} v_0\}, \\ E_2 &= E_0 \cup \{v_{n-2} v_{n-1}, v_{n-1} v_0\}, \\ E_3 &= E_0 \cup \{v_{n-3} v_{n-2}, v_{n-1} v_0\}, \\ E_4 &= E_0 \cup \{v_{n-3} v_{n-2}, v_{n-2} v_{n-1}, v_{n-1} v_0\}. \end{aligned} \quad (4)$$

We claim that for each $k \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$, there exists an $i_0 \in \{0, 1, \dots, n-1\}$ such that $v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1} \in E_t$ where $j_0 = i_0 + k$. We consider $t = 0, 1, 2, 3, 4$, respectively.

For $t = 0$, if $k = 0, 1, 3 \pmod{5}$, let $i_0 = 1$; then, $j_0 = i_0 + k = 1, 2, 4 \pmod{5}$ such that $v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1} \in E_0$. If $k = 2, 4 \pmod{5}$, let $i_0 = 2$; then, $j_0 = 4, 1 \pmod{5}$ such that $v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1} \in E_0$.

For $t = 1$, if $k = 1, 3, 4 \pmod{5}$, let $i_0 = n-1-k$. Then, $i_0 = \begin{cases} n-1-k, & \text{if } k = 0, 2, 4 \pmod{5}, \\ n-2-k, & \text{if } k = 1, 3 \pmod{5}. \end{cases}$ and $i_0 = 4, 2, 1 \pmod{5}$ such that $v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1} \in E_1$. If $k = 0, 2 \pmod{5}$, let $i_0 = n-2-k$. Then, $j_0 = n-2 = 4 \pmod{5}$ and $i_0 = 4, 2 \pmod{5}$ such that $v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1} \in E_1$.

For $t = 2$, let $i_0 = n-1-k$ when $k = 0, 2, 4 \pmod{5}$; then, $i_0 = 1, 4, 2 \pmod{5}$ and $j_0 = n-1$ such that $v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1} \in E_2$. Let $i_0 = n-2-k$ when $k = 1, 3 \pmod{5}$; then, $i_0 = 4, 2 \pmod{5}$ and $j_0 = n-2$ such that $v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1} \in E_2$.

For $t = 3$, let $i_0 = n - 1 - k$ when $k = 0, 3 \pmod{5}$, $i_0 = n - 3 - k$ when $k = 1, 4 \pmod{5}$, and $i_0 = n - 4 - k$ when $k = 2 \pmod{5}$. Thus, $v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1} \in E_3$.

For $t = 4$, let $i_0 = n - 1 - k$ if $k = 1, 2, 4 \pmod{5}$ and $i_0 = n - 2 - k$ if $k = 0, 3 \pmod{5}$. Then, $v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1} \in E_4$.

It is easy to verify that D is a dominating set of graph $P(n, 2) - E_t$ for $t = 0, 1, 2, 3, 4$. Hence, D is a dominating set of graph $P(n, 2) - \{v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1}\}$. In consideration of $P(n, 2) - \{v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1}\} \cong P(n, 2) - \{v_i v_{i+1}, v_j v_{j+1}\}$, we get that $\gamma(P(n, 2) - \{v_i v_{i+1}, v_j v_{j+1}\}) = \gamma(P(n, 2) - \{v_{i_0} v_{i_0+1}, v_{j_0} v_{j_0+1}\}) \leq \gamma(P(n, 2))$ for any $0 \leq i < j \leq n - 1$, $j = i + k$ and $k \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$. The result follows.

Next, we prove $\gamma(P(n, 2) - \{v_i u_i, v_j u_j\}) \leq \gamma(P(n, 2))$ for $0 \leq i, j \leq n - 1$. Let $j = i + k$, $k \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$. We claim that D is a dominating set of $P(n, 2) - E_t$ for $t = 0, 1, 2, 3, 4$, where

$$\begin{aligned} E_0 &= \{v_{5i+2} u_{5i+2}, v_{5i+4} u_{5i+4}, v_{5i+5} u_{5i+5} \mid 0 \leq i \leq m - 1\}, \\ E_1 &= (E_0 \setminus \{v_{5m} u_{5m}\}) \cup \{v_0 u_0\}, \\ E_2 &= (E_0 \setminus \{v_{5m} u_{5m}\}) \cup \{v_{n-1} u_{n-1}, v_0 u_0\}, \\ E_3 &= E_0 \cup \{v_{n-1} u_{n-1}, v_0 u_0\}, \\ E_4 &= E_0 \cup \{v_{n-2} u_{n-2}, v_0 u_0\}. \end{aligned} \quad (5)$$

For $t = 0$, let

$$i_0 = \begin{cases} n - k, & \text{if } k = 0, 1, 3 \pmod{5}, \\ n - 1 - k, & \text{if } k = 2, 4 \pmod{5}. \end{cases} \quad (6)$$

For $t = 1$ and $n = 6$, $D' = \{u_0, u_1, v_0, v_3\}$ is a γ -set of $P(6, 2)$, and it is a dominating set of $P(n, 2) - \{v_i u_i \mid 0 \leq i \leq 5\}$. Thus, $\gamma(P(6, 2) - \{v_i u_i, v_j u_j\}) \leq |D'| = \gamma(P(6, 2))$. For $t = 1$ and $n > 6$, let

$$i_0 = \begin{cases} 4, & \text{if } k = 1, \\ n - k, & \text{if } k = 1, 2, 4 \pmod{5} (k \neq 1), \\ n - 2 - k, & \text{if } k = 0 \pmod{5}, \\ n - 4 - k, & \text{if } k = 3 \pmod{5}. \end{cases} \quad (7)$$

For $t = 2$, let

$$i_0 = \begin{cases} 4, & \text{if } k = 1, \\ n - k, & \text{if } k = 0, 3 \pmod{5}, \\ n - 1 - k, & \text{if } k = 1, 2, 4 \pmod{5} (k \neq 1). \end{cases} \quad (8)$$

For $t = 3$, let

$$i_0 = \begin{cases} n - k, & \text{if } k = 1, 3, 4 \pmod{5}, \\ n - 1 - k, & \text{if } k = 0, 2 \pmod{5}. \end{cases} \quad (9)$$

For $t = 4$, let

$$i_0 = \begin{cases} n - k, & \text{if } k = 0, 2, 4 \pmod{5}, \\ n - 2 - k, & \text{if } k = 3 \pmod{5}, \\ n - 4, & \text{if } k = 1 \pmod{5}. \end{cases} \quad (10)$$

The integers i_0 and $j_0 = i_0 + k$ are obtained such that $v_{i_0} u_{i_0}, v_{j_0} u_{j_0} \in E_t$ for $t = 0, 1, 2, 3, 4$ ($n \neq 6$), respectively. Thus,

D is a dominating set of graph $G = P(n, 2) - \{v_{i_0} u_{i_0}, v_{j_0} u_{j_0}\}$, ($n \neq 6$). As a consequence, $\gamma(P(n, 2) - \{v_i v_{i+1}, v_j v_{j+1}\}) = \gamma(G) \leq \gamma(P(n, 2))$.

It is well known that $\gamma(P(n, 2) - \{e_1, e_2\}) \geq \gamma(P(n, 2))$ for any two edges e_1, e_2 . Combining Lemma 1, we obtain that $\gamma(P(n, 2) - \{e_1, e_2\}) = \gamma(P(n, 2))$, which implies the following theorem. $\square$

Theorem 1. $b(P(n, 2)) \geq 3$.

3. Upper Bound on Bondage Number of $P(n, 2)$

Let $n = 5m + t$, where $t \equiv n \pmod{5}$ and $m = \lfloor n/5 \rfloor$. The main results we get in this section are $b(P(n, 2)) \leq 3$ for $t = 0, 3, 4$ and $b(P(n, 2)) \leq 5$ for $t = 1, 2$. We prove the above results mainly by mathematical induction on m in Lemma 3. As a recursive condition in mathematical induction, the inequation shown in Lemma 2 is important and necessary. In addition, due to the complexity inherent in the study of domination number and generalized Petersen graph, the proof for Lemma 2 is inevitably tedious and difficult. Let $G(n) = P(n, 2) - v_2$; we first present an algorithm which constructs from $G(n)$ a smaller graph $G(n - 5) = P(n - 5, 2) - v_2$.

Figure 1 gives an illustration for Algorithm 1.

Observation 1. For each $n \geq 10$, the graph G'' returned by Algorithm 1 is isomorphic to $G(n - 5) = P(n - 5, 2) - v_2$.

Lemma 2. For each $n \geq 10$, $\gamma(G(n)) \geq \gamma(G(n - 5)) + 3$.

Proof. We assume that $n \geq 10$, and show $\gamma(G(n - 5)) \leq \gamma(G(n)) - 3$. Let S be a γ -set of $G(n)$ and G' and G'' be the graphs returned by Algorithm 1. Then, $G'' \cong G(n - 5)$. We will identify $V(G(n - 5))$ with $V(G'')$ and $E(G(n - 5)) = E(G'')$. Then, $V(G(n - 5)) = V(G(n)) - T$, where $T = \{u_i, v_i \mid 5 \leq i \leq 9\}$. Let $S' = S \cap V(G(n - 5))$. Then, we get that $|S'| = |S| - |S \cap T| = \gamma(G(n)) - |S \cap T|$ and the set S' can dominate all vertices in $V(G(n - 5))$, except possibly vertices in $R = \{u_3, u_4, v_4, v_{10}, u_{10}, u_{11}\}$. We consider the following cases. $\square$

Case 1. $|S \cap T| \geq 5$.

Since $\{u_4, u_{10}\}$ dominates the vertices in R and $S' = S \cap V(G')$ dominates the vertices in $V(G(n - 5)) \setminus R$, we have that $S' \cup \{u_4, u_{10}\}$ forms a dominating set of $G(n - 5)$. Hence, $\gamma(G(n - 5)) \leq |S' \cup \{u_4, u_{10}\}| = |S| - |S \cap T| + 2 \leq \gamma(G(n)) - 3$.

Case 2. $|S \cap T| = 4$. Then, $|S'| = |S| - 4 = \gamma(G(n)) - 4$.

The vertices in $S \cap T$ can dominate at most three vertices in R ; otherwise, all the four vertices in $S \cap T$ belong to $Q = \{v_5, u_5, u_6, u_8, u_9, v_9\}$. Thus, v_7 cannot be dominated by S since $N_{G(n)}[v_7] \cap Q = \emptyset$. Therefore, $|S \cap Q| \leq 3$.

Subcase 1. $|S \cap Q| = 0$. Since $|S \cap T| = 4$, we get that $|S \cap T| = N_{G(n)}[v_7]$. Thus, $S - \{v_7\}$ is a dominating set of

FIGURE 1: Algorithm 1.

$G(n)$, a contradiction with the minimality of S . Hence, this case does not exist.

Subcase 2. $|S \cap Q| = 1$. In this case, there is only one vertex $x \in R$ dominated by $S \cap T$; then, $S' \cup \{x\}$ is a dominating set of $G(n-5)$. Thus, $\gamma(G(n-5)) \leq |S' \cup \{x\}| \leq \gamma(G(n)) - 3$.

Subcase 3. $|S \cap Q| = 2$.

Suppose that $x, y \in R$ are dominated by $S \cap T$; thus, x and y may not be dominated by S' . If $z \in N_{G(n-5)}[x] \cap N_{G(n-5)}[y]$ for some vertex $z \in V(G(n-5))$, then $S' \cup \{z\}$ is a dominating set of $G(n-5)$ and the result follows. If $N_{G(n-5)}[x] \cap N_{G(n-5)}[y] = \emptyset$, then there exist the following three possible cases (see Figure 2). To make the following proof easier to understand, we attach the corresponding graph for each case. Furthermore, in each figure a vertex colored dark indicates a vertex in S , a vertex $\oplus$ indicates a possible vertex in S , and a vertex $\odot$ indicates a vertex not dominated by $S \cap Q$ and S' . Since $|S \cap Q| = 2$, the vertices $\odot$ should be dominated by the two vertices in $S \cap (T - Q) = S \cap N_{G(n)}[v_7]$.

- (1) $S \cap Q = \{u_5, u_6\}$ (see Figure 2(a)), that is, $u_3, u_4 \in R$ are dominated by $S \cap T$.

It is obvious that $S' \cup \{v_3, v_4\}$ or $S' \cup \{v_3, u_4\}$ is a dominating set of $G(n-5)$ in this case. For dominating vertex v_4 in graph $G(n)$, at least one vertex in $\gamma-$ belongs to S . Hence,

$$\gamma(G(n-5)) \leq |S' \cup \{v_3, v_4\}| \leq |S'| + 1 = \gamma(G(n)) - 3 \quad (11)$$

or

$$\gamma(G(n-5)) \leq |S' \cup \{v_3, u_4\}| \leq |S'| + 1 = \gamma(G(n)) - 3. \quad (12)$$

- (2) $S \cap Q = \{u_6, u_8\}$ (see Figure 2(b)), that is, $u_4, u_{10} \in R$ are dominated by $S \cap T$.

In this case, $S' \cup \{v_4, v_{10}\}$ can be a dominating set of $G(n-5)$. If $\{v_4, v_{10}\} \cap S = \emptyset$, then $S \cap T = (S \cap Q) \cup \{v_6, v_8\}$ for dominating vertices v_5 and v_9 in graph $G(n)$. It implies that vertex u_7 cannot be dominated by $S = S' \cup (S \cap T)$, a contradiction. Therefore, $\{v_4, v_{10}\} \cap S \neq \emptyset$. It follows that

$$\gamma(G(n-5)) \leq |S' \cup \{v_4, v_{10}\}| \leq |S'| + 1 = \gamma(G(n)) - 3. \quad (13)$$

- (3) $S \cap Q = \{u_5, u_9\}$ (see Figure 2(c)), that is, $u_3, u_{11} \in R$ are dominated by $S \cap T$.

Since $(N_{G(n)}[v_4] \setminus \{v_5\}) \cap S \neq \emptyset$ and $(N_{G(n)}[v_{10}] \setminus \{v_9\}) \cap S \neq \emptyset$, at least one vertex of $\{v_3, v_4, u_4\}$ and $\{v_{10}, u_{10}, v_{11}\}$ belongs to S . Since $v_3, u_{10} \in N_{G(n-5)}(\{u_3\})$, $S' \cup \{u_{11}\}$ is a dominating set of $G(n-5)$ if one vertex of $\{v_3, u_{10}\}$ belongs to S . Since $u_4, v_{11} \in N_{G(n-5)}(\{u_{11}\})$, $S' \cup \{u_3\}$ is a dominating set of $G(n-5)$ if one vertex of $\{u_4, v_{11}\}$ belongs to S . In the following, we just need to consider $\{v_4, v_{10}\} \subseteq S$. Then, $(S' \setminus \{v_{10}\}) \cup \{v_{11}, u_{10}\}$ is a dominating set of $G(n-5)$. Hence,

$$\gamma(G(n-5)) \leq |S'| + 1 = \gamma(G(n)) - 3. \quad (14)$$

Subcase 4. $|S \cap Q| = 3$; then, $|S \cap N_{G(n)}[v_7]| = 1$ and there exist three vertices in R dominated by $S \cap T$. Suppose that $x, y, z \in R$ are dominated by $S \cap T$. If $\{x, y, z\} \subseteq N_{G(n-5)}[w]$ where $w \in V(G(n-5))$, then $S' \cup \{w\}$ is a dominating set of $G(n-5)$, which means that

$$\gamma(G(n-5)) \leq |S' \cup \{w\}| = \gamma(G(n)) - |S \cap T| + 1 \leq \gamma(G(n)) - 3. \quad (15)$$

Otherwise, we have the following seven possible cases by the symmetry (see Figure 3).

- (1) $S \cap Q = \{u_5, u_6, u_9\}$ (see Figure 3(a)); then, $u_3, u_4, u_{11} \in R$ are dominated by $S \cap T$ and $S' \cup \{v_3, u_4\}$ could be a dominating set of $G(n-5)$.

It is obvious that at least one vertex of $\{v_3, v_4, u_4\}$ belongs to S to dominate v_4 . If $v_3 \in S$ or $u_4 \in S$, then $\gamma(G(n-5)) \leq |S' \cup \{v_3, u_4\}| \leq |S'| + 1 = \gamma(G(n)) - 3$. If $v_4 \in S$, then $(S' \setminus \{v_4\}) \cup \{v_3, u_4\}$ is a dominating set of $G(n-5)$ (in graph $G(n)$, v_{10} is dominated by $E_0 = \{v_{5i+1}v_{5i+2}, v_{5i+2}v_{5i+3}, v_{5i+4}v_{5i+5}, v_{5i+4}u_{5i+4}, v_{5i+5}u_{5i+5} \mid 0 \leq i \leq m-1\}$, $E_1 = (E_0 \setminus \{v_{5m}u_{5m}\}) \cup \{v_{n-1}v_0, v_0u_0\}$, $E_2 = (E_0 \setminus \{v_{5m}u_{5m}\}) \cup \{v_{n-1}v_0, v_{n-1}u_{n-1}, v_0u_0\}$, $E_3 = E_0 \cup \{v_{n-1}v_0, v_{n-2}u_{n-2}, v_0u_0\}$, $E_4 = E_0 \cup \{v_{n-2}v_{n-1}, v_{n-1}v_0, v_{n-2}u_{n-2}, v_0u_0\}$). Thus, $\gamma(G(n-5)) \leq |S'| + 1 = \gamma(G(n)) - 3$.

- (2) $S \cap Q = \{u_5, u_6, v_9\}$ (see Figure 3(b)); then, $u_3, u_4, v_{10} \in R$ are dominated by $S \cap T$, and $S' \cup \{v_3, v_4\}$ or $S' \cup \{u_4, u_{10}\}$ can be a dominating set of $G(n-5)$ in this case.

For dominating vertex v_4 in graph $G(n)$, at least one vertex in $\{v_3, v_4, u_4\}$ belongs to S . Hence, $\gamma(G(n-5)) \leq |S'| + 1 = \gamma(G(n)) - 3$.

- (3) $S \cap Q = \{u_5, v_5, u_9\}$ (see Figure 3(c)), that is, $u_3, v_4, u_{11} \in R$ are dominated by $S \cap Q$ and $u_6, v_7, v_8, u_8 \in T$ are not dominated by $S \cap Q$.

Since the remaining one vertex in $S' - (S \cap Q)$ belongs to $N_{G(n)}[v_7]$, $u_4 \in S$ or $v_6 \in S$ to dominate u_6 . If $v_6 \in S$, then $S \cap T = \{u_5, v_5, v_6, u_9\}$, and thus the vertex v_8 cannot be dominated by S , a contradiction. If $u_4 \in S$, then $S' \cup \{u_3\}$ is a dominating set of $G(n-5)$. So, $\gamma(G(n-5)) \leq |S'| + 1 = \gamma(G(n)) - 3$.

Input: the graph $G(n)$ with $V(G(n)) = V(P(n, 2)) \setminus \{v_2\}$, $E(G(n)) = E(P(n, 2)) \setminus \{v_1v_2, v_2v_3, v_2u_2\}$, $n \geq 10$.

Output: the graph G'' .

Step 1. Delete the subset of vertices $\{v_i, u_i \mid 5 \leq i \leq 9\}$, along with their 18 incident edges. Denote the resulting graph by G' .

Step 2. Define the graph G'' to have vertex set $V(G'') = V(G')$ and edge set $E(G'') = E(G') \cup \{v_4v_{10}, u_3u_{10}, u_4v_{11}\}$. Return G'' .

ALGORITHM 1: The construction for graph $G(n-5)$ from graph $G(n)$.

FIGURE 2: The four possible cases of S in Subcase 3.

FIGURE 3: The seven possible cases of S in Subcase 4.

- (4) $S \cap Q = \{u_5, v_5, v_9\}$ (see Figure 3(d)); then, $u_3, v_4, v_{10} \in R$ are dominated by $S \cap T$ and $u_6, v_7, u_8 \in T$ are not dominated by $S \cap Q$.

Because the only one vertex in $S \cap N_{G(n)}[v_7]$ cannot dominate u_6 and u_8 simultaneously, $u_4 \in S$ or $u_{10} \in S$. Actually, $S' \cup \{u_4, u_{10}\}$ is a dominating set of $G(n-5)$, which follows that $\gamma(G(n-5)) \leq |S' \cup \{u_4, u_{10}\}| \leq |S'| + 1 = \gamma(G(n)) - 3$.

- (5) $S \cap Q = \{u_5, u_6, u_8\}$ (see Figure 3(e)); then, $u_3, u_4, u_{10} \in R$ are dominated by $S \cap T$.

At least one vertex of $\{v_3, v_4, u_4\}$ belongs to S to dominate v_4 . If $u_4 \in S$ or $v_4 \in S$, then $S' \cup \{u_3\}$ is a dominating set of $G(n-5)$. If $v_3 \in S$, then $(S' \setminus \{v_3\}) \cup \{u_3, u_4\}$ is a dominating set of $G(n-5)$. It follows that $\gamma(G(n-5)) \leq |S'| + 1 = \gamma(G(n)) - 3$.

- (6) $S \cap Q = \{u_6, v_9, u_9\}$ (see Figure 3(f)); then, $u_4, v_{10}, u_{11} \in R$ are dominated by $S \cap T$ and $v_5, u_5, v_7, u_8 \in T$ are not dominated by $S \cap Q$.

The only one vertex in $S \cap N_{G(n)}[v_7]$ cannot dominate v_5 and u_5 simultaneously; hence, $v_4 \in S$ or $u_3 \in S$. For $v_4 \in S$, we get that $S' \cup \{v_{11}\}$ is a dominating set of $G(n-5)$. In view of $(N_{G(n)}[v_4] \setminus \{v_5\}) \cap S \neq \emptyset$, $\{v_3, u_4, v_4\} \cap S \neq \emptyset$. For $u_3 \in S$, $(S' \setminus \{v_3\}) \cup \{v_4, v_{11}\}$ is a γ -set of $G(n)$ if $v_3 \in S$, and $S' \cup \{v_{11}\}$ is a dominating set of $G(n-5)$ if $u_4 \in S$ or $v_4 \in S$. Therefore, $\gamma(G(n-5)) \leq |S'| + 1 = \gamma(G(n)) - 3$.

- (7) $S \cap Q = \{u_5, v_5, u_8\}$ (see Figure 3(g)), that is, $u_3, v_4, u_{10} \in R$ are dominated by $S \cap Q$ and $v_7, v_9, u_9 \in T$ are not dominated by $S \cap Q$.

Clearly, $S' \cup \{u_3, v_4\}$ or $S' \cup \{v_3, u_3\}$ is a dominating set of $G(n-5)$ in this case. For dominating vertex v_3 in graph $G(n)$, we have that $\{u_3, v_3, v_4\} \cap S \neq \emptyset$. Hence, $\gamma(G(n-5)) \leq |S'| + 1 = \gamma(G(n)) - 3$.

Case 3. $|S \cap T| = 3$.

The vertices in $S \cap T$ can dominate at most two vertices in R ; otherwise, $(S \cap T) \subseteq (S \cap Q)$, which means that v_7 cannot be dominated by S . Therefore, $|S \cap Q| \leq 2$.

Subcase 5. $|S \cap Q| = 0$. We have nothing to do, and S' is a dominating set of $G(n-5)$ and $\gamma(G(n-5)) \leq |S'| = \gamma(G(n)) - 3$.

Subcase 6. $|S \cap Q| = 1$. By symmetry, v_3 or not, we have the following four cases (see Figure 4).

- (1) $S \cap Q = \{v_5\}$ (see Figure 4(a)); then, $v_4 \in R$ is dominated by $S \cap Q$.

The two vertices in $S \cap N_{G(n)}[v_7]$ cannot dominate u_6 and v_9 at the same time; otherwise u_7 cannot be dominated by S , a contradiction. Hence, at least one vertex of $\{u_4, v_{10}\}$ belongs to S' . Therefore, S' is a dominating set of $G(n-5)$ and $\gamma(G(n-5)) \leq |S'| = \gamma(G(n)) - 3$.

- (2) $S \cap Q = \{u_5\}$ (see Figure 4(b)); then, $u_3 \in R$ is dominated by $S \cap Q$.

The two vertices in $S \cap N_{G(n)}[v_7]$ cannot dominate u_6 and u_9 at the same time, that is, $S \cap N_{G(n)}[v_7] \neq \{v_6, u_7\}$. Otherwise, v_8 cannot be dominated by S , a contradiction. Hence, at least one vertex of $\{u_4, u_{11}\}$ belongs to S' . In addition, $\{u_3, v_3, v_4\} \cap S \neq \emptyset$ for dominating u_3 in graph $G(n)$. If $v_3 \in S$ or $u_3 \in S$, then S' can be a dominating set of $G(n-5)$. If $v_4 \in S$, in view of at least one vertex of $\{u_4, u_{11}\}$ belonging to S' , then $(S' \setminus \{v_4\}) \cup \{v_3\}$ is a dominating set of $G(n-5)$. Therefore, $\gamma(G(n-5)) \leq |S'| = \gamma(G(n)) - 3$.

- (3) $S \cap Q = \{u_6\}$ (see Figure 4(c)); then, $u_4 \in R$ is dominated by $S \cap Q$.

The two vertices in $S \cap N_{G(n)}[v_7]$ cannot dominate v_5 and u_9 at the same time; otherwise, v_8 cannot be dominated by S , a contradiction. Hence, at least one vertex of $\{v_4, u_{11}\}$ belongs to S' . It implies that S' is a dominating set of $G(n-5)$ and $\gamma(G(n-5)) \leq |S'| = \gamma(G(n)) - 3$.

- (4) $S \cap Q = \{u_9\}$ (see Figure 4(d)).

If $v_6 \notin S$, then $u_4 \in S$ to dominate u_6 . And S' is a dominating set of $G(n-5)$.

If $v_6 \in S$, then $u_7 \notin S$; otherwise, $N_{G(n)}[v_8] \cap S = \emptyset$. Thus, $u_3 \in S$ to dominate u_5 . Since $(N_{G(n)}[v_4] \setminus \{v_5\}) \cap S \neq \emptyset$ and $(S \setminus \{v_3\}) \cup \{v_4\}$ is a dominating set of $G(n)$, at least one vertex of $\{v_4, u_4\}$ belongs to S . If $u_4 \in S$, then S' is a dominating set of $G(n-5)$. If $v_4 \in S$, then $(S' \setminus \{v_4\}) \cup \{u_4\}$ is a dominating set of $G(n-5)$.

Subcase 7. $|S \cap Q| = 2$; then, $|S \cap N_{G(n)}[v_7]| = 1$ and the vertices $\odot$ should be dominated by the only one vertex in $S \cap N_{G(n)}[v_7]$. We consider eleven possible cases.

- (1) $S \cap Q = \{v_5, u_5\}$ (see Figure 5(a)); then, $v_4, u_3 \in R$ is dominated by $S \cap Q$.

We claim that $S \cap N_{G(n)}[v_7] \neq \{v_6\}$; otherwise, v_8 cannot be dominated by S . Hence, $u_4 \in S$ for dominating u_6 . In addition, $\{v_3, u_3, v_4\} \cap S \neq \emptyset$ to dominate v_3 . If $v_3 \in S$ or $u_3 \in S$, then S' is a dominating set of $G(n-5)$. If $v_4 \in S$, then $(S' \setminus \{v_4\}) \cup \{v_3\}$ is a dominating set of $G(n-5)$. Thus, $\gamma(G(n-5)) \leq |S' \setminus \{v_4\} \cup \{v_3\}| = \gamma(G(n)) - 3$.

- (2) $S \cap Q = \{v_9, u_9\}$ (see Figure 5(b)); then, $v_{10}, u_{11} \in R$ is dominated by $S \cap Q$.

If $S \cap N_{G(n)}[v_7] = \{v_8\}$ or $S \cap N_{G(n)}[v_7] = \{u_7\}$, then v_6 cannot be dominated by S , which is a contradiction to the definition of S . If $S \cap N_{G(n)}[v_7] = \{v_7\}$, then $S - \{v_7\} + \{v_6\}$ is a dominating set of $G(n)$. Hence, we assume that $S \cap N_{G(n)}[v_7] = \{v_6\}$. Thus, $S \cup T = \{v_6, v_9, u_9\}$. Moreover, we get that $u_3, u_{10} \in S$ to dominate u_5 and u_8 , respectively. Since $(N_{G(n)}[v_4] \setminus \{v_5\}) \cap S \neq \emptyset$, at least one vertex of $\{v_3, v_4, u_4\}$ belongs to S . If $u_4 \in S$, then S' is a dominating set of $G(n-5)$. If $v_4 \in S$ or $v_3 \in S$, then

$$i_0 = \begin{cases} n-1, & \text{if } k=1, \\ n+4-k, & \text{if } k=1 \pmod{5} (k \neq 1), \\ n-k, & \text{if } k=2, 4 \pmod{5}, \\ n-2-k, & \text{if } k=0, 3 \pmod{5}. \end{cases} \quad \text{or}$$

$(S' \setminus \{v_3\}) \cup \{u_4\}$ is a dominating set of $G(n-5)$. Therefore,

$$\gamma(G(n-5)) \leq |S'| = \gamma(G(n)) - 3. \quad (16)$$

- (3) $S \cap Q = \{v_5, u_9\}$ (see Figure 5(c)).

- (4) $S \cap Q = \{v_5, v_9\}$ (see Figure 5(d)).

- (5) $S \cap Q = \{u_5, u_9\}$ (see Figure 5(e)).

- (6) $S \cap Q = \{u_6, u_8\}$ (see Figure 5(f)).

For (3), (4), (5), and (6), it is obvious that S' is a dominating set of $G(n-5)$.

- (7) $S \cap Q = \{u_6, u_9\}$ (see Figure 5(g)). $(S' \setminus \{v_4\}) \cup \{u_4\}$ is a dominating set of $G(n-5)$.

- (8) $S \cap Q = \{u_6, v_9\}$ (see Figure 5(h)). S' is a dominating set of $G(n-5)$.

- (9) $S \cap Q = \{u_8, u_9\}$ (see Figure 5(i)).

We claim that $S \cap N_{G(n)}[v_7] = \{v_6\}$ or $S \cap N_{G(n)}[v_7] = \{v_7\}$. Since $S - \{v_7\} + \{v_6\}$ could be a dominating set of $G(n)$, we assume that $S \cap N_{G(n)}[v_7] = \{v_6\}$. In addition, $u_3 \in S$ for dominating u_5 . Since $(N_{G(n)}[v_4] \setminus \{v_5\}) \cap S \neq \emptyset$ and $(S' \setminus \{v_3\}) \cup \{v_4\}$ is a dominating set of $G(n)$, at least one vertex of $\{v_4, u_4\}$ belongs to S . For $u_4 \in S$, S' is a dominating set of $G(n-5)$. For $v_4 \in S$, $(S' \setminus \{v_4\}) \cup \{u_4\}$ is a dominating set of $G(n-5)$. The result follows.

- (10) $S \cap Q = \{u_5, u_6\}$ (see Figure 5(j)); then, $u_{11} \in S$.

FIGURE 4: The four possible cases of S in Subcase 6. FIGURE 5: The eleven possible cases of S in Subcase 7.

Since $(N_{G(n)}[v_3]) \cap S \neq \emptyset$, at least one vertex of $\{v_3, v_4, u_3\}$ belongs to S . If $v_3 \in S$ or $u_3 \in S$, then S' is a dominating set of $G(n-5)$. If $v_4 \in S$, then $(S' \setminus \{v_4\}) \cup \{v_3\}$ is a dominating set of $G(n-5)$. Thus, $\gamma(G(n-5)) = \gamma(G(n-5)) \leq |S' \setminus \{v_4\}| \cup |v_3| = \gamma(G(n)) - 3$.

- (11) $S \cap Q = \{u_8, v_9\}$ (see Figure 5(k)). S' is a dominating set of $G(n-5)$.

Case 4. $|S \cap T| = 2$.

We say that $v_7 \in S$; otherwise, $|S \cap T| \geq 3$ to dominate the vertices in $N_{G(n)}(v_7)$. Hence, the vertices in $S \cap T$ have the following five cases by symmetry.

- (1) $S \cap T = \{u_5, v_7\}$ (see Figure 6(a)).
- (2) $S \cap T = \{u_6, v_7\}$ (see Figure 6(b)).
- (3) $S \cap T = \{v_5, v_7\}$ (see Figure 6(c)).
- (4) $S \cap T = \{v_6, v_7\}$ (see Figure 6(d)).

In each of the above four circumstances, $S' \setminus \{v_{10}\}$ is a dominating set of $G(n-5)$. Therefore, $\gamma(G(n-5)) \leq |S' \setminus \{v_{10}\}| = \gamma(G(n)) - 3$.

- (5) $S \cap T = \{u_7, v_7\}$ (see Figure 6(e)).

Since $(N_{G(n)}[u_{11}] \setminus \{u_9\}) \cap S \neq \emptyset$, at least one vertex of $\{u_{11}, v_{11}, u_{13}\}$ belongs to S . If $u_{11} \in S$ or $v_{11} \in S$, then $S' \setminus \{v_{10}\}$ is a dominating set of $G(n-5)$. At least one vertex of $\{v_{11}, v_{12}, v_{13}, u_{12}\}$ belongs to S to dominate v_{12} . If $v_{12} \in S$ or $u_{12} \in S$, then $S' \setminus \{u_{10}\}$ is a dominating set of $G(n-5)$.

Hence, we only need to consider $\{v_{13}, u_{13}\} \subseteq S$ to dominate v_{12} and u_{11} . Let $A = \{v_4, u_4, v_{10}, u_{10}, v_{13}, u_{13}\}$ and $B = \{u_3, u_4, v_{11}, u_{13}, u_{14}\}$. Then, we can deduce that $A \subseteq S'$, which is shown in Figure 6(f). Let $S'' = (S' \setminus A) \cup B$, which is shown in Figure 6(g). It is easy to verify that S'' is a dominating set of $G(n-5)$ and $\gamma(G(n-5)) \leq |S''| = |S'| - 1 = \gamma(G(n)) - 3$.

Case 5. $|S \cap T| = 1$; then, $S \cap T = \{v_7\}$ and $R \subseteq S$. Hence, $S' \setminus \{v_4, v_{10}\}$ is a dominating set of $G(n-5)$ and

FIGURE 6: The four possible cases of S in Subcase 6.

$\gamma(G(n-5)) \leq |S' \setminus \{v_4, v_{10}\}| = \gamma(G(n)) - 3$. This completes the proof.

Lemma 3. Let $n = 5m + t$ and $G(n) = P(n, 2) - v_2$. Then, $\gamma(P(n, 2) - \{v_1 v_2, v_2 v_3, v_2 u_2\}) > \gamma(P(n, 2))$ for $i_0 = \{n - k, \text{ if } k = 1, 3 \pmod{5}, n - 1 - k, \text{ if } k = 0, 2, 4 \pmod{5}\}$.

Proof. It is sufficient to show that $\gamma(G(n)) \geq \gamma(P(n, 2))$ for $t = 0, 3, 4$. Let S be a minimum dominating set of $G(n)$. If $N_{P(n, 2)}(v_2) \cap S \neq \emptyset$, then S is a dominating set of $P(n, 2)$. Hence, $\gamma(G(n)) = |S| \geq \gamma(P(n, 2))$.

In the following, we suppose that S satisfies $N_{P(n, 2)}(v_2) \cap S = \emptyset$. We prove Lemma 3 by using mathematical induction on m . There are at least three vertices in S to dominate the vertices in $N_{P(n, 2)}(v_2) = \{v_1, u_2, v_3\}$, at least one vertex of $\{v_0, u_1\}$, $\{u_0, u_4\}$, and $\{u_3, v_4\}$, respectively. Clearly, there are eight possible cases of these three vertices in S . By the symmetry of graph $G(n)$, we only need to consider the following four cases: $\{v_0, u_0, u_3\} \subseteq S$, $\{v_0, u_0, v_4\} \subseteq S$, $\{u_0, u_1, u_3\} \subseteq S$, and $\{u_0, u_1, v_4\} \subseteq S$. Let $m = 1$ and $t = 0$. Then, $\gamma(G(n)) = |S| \geq 3 = \gamma(P(n, 2))$. Let $m = 1$ and $t = 3, 4$. It is easy to verify that $\gamma(G(n)) = |S| \geq \gamma(P(n, 2))$ by analyzing the above four cases, respectively, and thus omitted here.

Assume that Lemma 3 is true for $m = k$; we get that

$$\begin{aligned} \gamma(G(5(k+1)+t)) &\geq \gamma(G(5k+t)) + 3 \\ &\geq 5k+t - \left\lfloor \frac{5k+t}{5} \right\rfloor - \left\lfloor \frac{5k+t+2}{5} \right\rfloor + 3 \\ &= 5(k+1)+t - \left\lfloor \frac{5(k+1)+t}{5} \right\rfloor - \left\lfloor \frac{5(k+1)+t+2}{5} \right\rfloor \\ &= \gamma(P(5(k+1)+t, 2)), \end{aligned} \quad (17)$$

by Lemma 2. Therefore, The assertion holds for $m = k + 1$. This completes the proof. $\square$

Lemma 4 (see [34]). $b(G) \leq d(x) + d(y) - 1$ for any two adjacent vertices x and y in a graph G ; that is,

$$b(G) \leq \min_{xy \in E(G)} \{d_G(x) + d_G(y) - 1\}. \quad (18)$$

Theorem 2. If $t = 0, 3, 4$, then $b(P(n, 2)) = 3$; if $t = 1, 2$, then $3 \leq b(P(n, 2)) \leq 5$.

Proof. By Lemma 3, we have that $\gamma(P(n, 2) - \{v_1 v_2, v_2 v_3, v_2 u_2\}) > \gamma(P(n, 2))$ for $t = 0, 3, 4$; thus, $b(P(n, 2)) \leq 3$. By Lemma 4, $b(P(n, 2)) \leq 5$ for $t = 1, 2$. Combining Theorem 1 gives that $b(P(n, 2)) = 3$ when $t = 0, 3, 4$ and $3 \leq b(P(n, 2)) \leq 5$ when $t = 1, 2$. This completes the proof. $\square$

Appendix

Proof for $\gamma(P(n, 2) - \{u_i u_{i+2}, u_j u_{j+2}\}) \leq \gamma(P(n, 2))$, $0 \leq i, j \leq n - 1$.

Assume that $j = i + k$; we only need to consider $k \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$. It is clear that D is a dominating set of $P(n, 2) - E_t$ for $t = 0, 1, 2, 3, 4$, where

$$\begin{aligned} E_0 &= \{u_{5i+1} u_{5i+3}, u_{5i+3} u_{5i+5}, u_{5i+4} u_{5i+6} \mid 0 \leq i \leq m - 1\}, \\ E_1 &= E_0 \cup \{u_{n-1} u_1\}, \\ E_2 &= E_0 \cup \{u_{n-2} u_0, u_{n-1} u_1\}, \\ E_3 &= E_0 \cup \{u_{n-2} u_0, u_{n-1} u_1\}, \\ E_4 &= E_0 \cup \{u_{n-3} u_{n-1}, u_{n-2} u_0, u_{n-1} u_1\}. \end{aligned} \quad (A.1)$$

When $t = 0$, let

$$i_0 = \begin{cases} n - 1 - k, & \text{if } k = 0, 1, 3 \pmod{5}, \\ n - 2 - k, & \text{if } k = 2, 4 \pmod{5}. \end{cases} \quad (A.2)$$

When $t = 1$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 1, 2, 4 \pmod{5}, \\ n-2-k, & \text{if } k = 0, 3 \pmod{5}. \end{cases} \quad (\text{A.3})$$

When $t = 2$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 0, 2, 3 \pmod{5}, \\ n-2-k, & \text{if } k = 1, 4 \pmod{5}. \end{cases} \quad (\text{A.4})$$

When $t = 3$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 1, 3, 4 \pmod{5}, \\ n-2-k, & \text{if } k = 0, 2 \pmod{5}. \end{cases} \quad (\text{A.5})$$

When $t = 4$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 0, 2, 4 \pmod{5}, \\ n-2-k, & \text{if } k = 1, 3 \pmod{5}. \end{cases} \quad (\text{A.6})$$

Then, i_0 and $j_0 = i_0 + k$ such that $u_{i_0}u_{i_0+2}, u_{j_0}u_{j_0+2} \in E_t$, $t = 0, 1, 2, 3, 4$. Therefore, for each $k \in \{1, 2, \dots, \lfloor n/2 \rfloor\}$, D is a dominating set of graph $G = P(n, 2) - \{u_{i_0}u_{i_0+2}, u_{j_0}u_{j_0+2}\}$. Hence, $\gamma(G) \leq \gamma(P(n, 2))$.

It implies that $\gamma(P(n, 2) - \{u_{i_0}u_{i_0+2}, u_{j_0}u_{j_0+2}\}) = \gamma(G) \leq \gamma(P(n, 2))$.

Proof for $\gamma(P(n, 2) - \{v_i v_{i+1}, v_j u_j\}) \leq \gamma(P(n, 2))$, $0 \leq i, j \leq n-1$.

Assume that $j = i + k$; we just need to prove the lemma is true for $k \in \{0, 1, \dots, \lfloor n/2 \rfloor\}$. The γ -set D of $P(n, 2)$ is a dominating set of $P(n, 2) - E_t$ for $t = 0, 1, 2, 3, 4$, where

$$\begin{aligned} E_0 &= \{v_{5i+1}v_{5i+2}, v_{5i+2}v_{5i+3}, v_{5i+4}v_{5i+5}, v_{5i+4}u_{5i+4}, v_{5i+5}u_{5i+5} \mid 0 \leq i \leq m-1\}, \\ E_1 &= (E_0 \setminus \{v_{5m}u_{5m}\}) \cup \{v_{n-1}v_0, v_0u_0\}, \\ E_2 &= (E_0 \setminus \{v_{5m}u_{5m}\}) \cup \{v_{n-1}v_0, v_{n-1}u_{n-1}, v_0u_0\}, \\ E_3 &= E_0 \cup \{v_{n-1}v_0, v_{n-1}u_{n-1}, v_0u_0\}, \\ E_4 &= E_0 \cup \{v_{n-2}v_{n-1}, v_{n-1}v_0, v_{n-2}u_{n-2}, v_0u_0\}. \end{aligned} \quad (\text{A.7})$$

When $t = 0$, let

$$i_0 = \begin{cases} n-k, & \text{if } k = 1, 3, 4 \pmod{5}, \\ n-1-k, & \text{if } k = 0, 2 \pmod{5}. \end{cases} \quad (\text{A.8})$$

When $t = 1$, let

$$i_0 = \begin{cases} n-1, & \text{if } k = 1, \\ n+4-k, & \text{if } k = 1 \pmod{5} (k \neq 1), \\ n-k, & \text{if } k = 2, 4 \pmod{5}, \\ n-2-k, & \text{if } k = 0, 3 \pmod{5}. \end{cases} \quad (\text{A.9})$$

When $t = 2$, let

$$i_0 = \begin{cases} n-k, & \text{if } k = 1, 3 \pmod{5}, \\ n-1-k, & \text{if } k = 0, 2, 4 \pmod{5}. \end{cases} \quad (\text{A.10})$$

When $t = 3$, let

$$i_0 = \begin{cases} n-k, & \text{if } k = 1, 4 \pmod{5}, \\ n-1-k, & \text{if } k = 0, 3 \pmod{5}, \\ 2, & \text{if } k = 2. \end{cases} \quad (\text{A.11})$$

When $t = 4$, let

$$i_0 = \begin{cases} n-k, & \text{if } k = 2 \pmod{5}, \\ n-2-k, & \text{if } k = 0, 3 \pmod{5} (k \neq 1), \\ n-4-k, & \text{if } k = 1, 4 \pmod{5}. \end{cases} \quad (\text{A.12})$$

It follows that $v_{i_0}v_{i_0+1}, v_{j_0}u_{j_0} (j_0 = i_0 + k) \in E_t$, $t = 0, 1, 2, 3, 4$. Hence, D is a dominating set of graph $G = P(n, 2) - \{v_{i_0}v_{i_0+1}, v_{j_0}u_{j_0}\}$, which means that $\gamma(P(n, 2) - \{v_{i_0}v_{i_0+1}, v_{j_0}u_{j_0}\}) = \gamma(G) \leq \gamma(P(n, 2))$.

Proof for $\gamma(P(n, 2) - \{v_i v_{i+1}, u_j u_{j+2}\}) \leq \gamma(P(n, 2))$, $0 \leq i, j \leq n-1$.

Let $j = i + k$; in the following we show that for each $k \in \{0, 1, \dots, \lfloor n/2 \rfloor\}$, $c(P(n, 2) - v_i v_{i+1}, u_j u_{j+2}) \leq c(P(n, 2))$, $0 \leq i, j \leq n-1$ is true. D is a dominating set of $P(n, 2) - E_t$ ($t = 0, 1, 2, 3, 4$), where

$$\begin{aligned} E_0 &= \{v_{5i+1}v_{5i+2}, v_{5i+2}v_{5i+3}, v_{5i+4}v_{5i+5}, u_{5i+1}u_{5i+3}, u_{5i+3}u_{5i+5}, \\ &\quad u_{5i+4}u_{5i+6} \mid 0 \leq i \leq m-1\}, \\ E_1 &= E_0 \cup \{v_{n-1}v_0, u_{n-1}u_1\}, \\ E_2 &= E_0 \cup \{v_{n-2}v_{n-1}, v_{n-1}v_0, u_{n-2}u_0, u_{n-1}u_1\}, \\ E_3 &= E_0 \cup \{v_{n-3}v_{n-2}, v_{n-1}v_0, u_{n-2}u_0, u_{n-1}u_1\}, \\ E_4 &= E_0 \cup \{v_{n-3}v_{n-2}, v_{n-2}v_{n-1}, v_{n-1}v_0, u_{n-3}u_{n-1}, u_{n-2}u_0, u_{n-1}u_1\}. \end{aligned} \quad (\text{A.13})$$

For $t = 0$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 0, 2, 3 \pmod{5}, \\ n-2-k, & \text{if } k = 1, 4 \pmod{5}. \end{cases} \quad (\text{A.14})$$

For $t = 1$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 1, 3, 4 \pmod{5}, \\ n-2-k, & \text{if } k = 0, 2 \pmod{5}. \end{cases} \quad (\text{A.15})$$

For $t = 2$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 0, 2, 4 \pmod{5}, \\ n-2-k, & \text{if } k = 1, 3 \pmod{5}. \end{cases} \quad (\text{A.16})$$

For $t = 3$, let

$$i_0 = \begin{cases} n-3, & \text{if } k = 1, \\ n-1-k, & \text{if } k = 0, 1, 3 \pmod{5} (k \neq 1), \\ n-2-k, & \text{if } k = 2, 4 \pmod{5}. \end{cases} \quad (\text{A.17})$$

For $t = 4$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 1, 2, 4 \pmod{5}, \\ n-2-k, & \text{if } k = 0, 3 \pmod{5}. \end{cases} \quad (\text{A.18})$$

Clearly, $v_{i_0}v_{i_0+1}, u_{j_0}u_{j_0+2} (j_0 = i_0 + k) \in E_t$. So, for each $k \in \{0, 1, \dots, \lfloor n/2 \rfloor\}$, D is a dominating set of graph

$G = P(n, 2) - \{v_{i_0} u_{i_0+1}, u_{j_0} u_{j_0+2}\}$. Hence, $\gamma(P(n, 2) - \{v_i u_{i+1}, u_j u_{j+2}\}) = \gamma(G) \leq \gamma(P(n, 2))$.

Proof for $\gamma(P(n, 2) - \{v_i u_i, u_j u_{j+2}\}) \leq \gamma(P(n, 2))$, $0 \leq i, j \leq n-1$.

Assume that $j = i + k$, $k \in \{0, 2, \dots, \lfloor n/2 \rfloor\}$. D is a dominating set of $P(n, 2) - E_t$ ($t = 0, 1, 2, 3, 4$), where

$$\begin{aligned} E_0 &= \{v_{5i+2} u_{5i+2}, v_{5i+4} u_{5i+4}, v_{5i+5} u_{5i+5}, u_{5i+1} u_{5i+3}, u_{5i+3} u_{5i+5}, \\ &\quad u_{5i+4} u_{5i+6} \mid 0 \leq i \leq m-1\}, \\ E_1 &= E_0 \setminus \{v_{5m} u_{5m}\} \cup \{v_0 u_0, u_{n-1} u_1\}, \\ E_2 &= E_0 \setminus \{v_{5m} u_{5m}\} \cup \{v_{n-1} u_{n-1}, v_0 u_0, u_{n-2} u_0, u_{n-1} u_1\}, \\ E_3 &= E_0 \cup \{v_{n-1} u_{n-1}, v_0 u_0, u_{n-2} u_0, u_{n-1} u_1\}, \\ E_4 &= E_0 \cup \{v_{n-2} u_{n-2}, v_0 u_0, u_{n-3} u_{n-1}, u_{n-2} u_0, u_{n-1} u_1\}. \end{aligned} \quad (\text{A.19})$$

When $t = 0$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 0, 2, 4 \pmod{5}, \\ n-2-k, & \text{if } k = 1, 3 \pmod{5}. \end{cases} \quad (\text{A.20})$$

When $t = 1$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 1, 3 \pmod{5}, \\ n-2-k, & \text{if } k = 0, 2, 4 \pmod{5}. \end{cases} \quad (\text{A.21})$$

When $t = 2$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 2, 4 \pmod{5}, \\ n-2-k, & \text{if } k = 1, 3 \pmod{5}, \\ n-3-k, & \text{if } k = 0 \pmod{5}. \end{cases} \quad (\text{A.22})$$

When $t = 3$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 0, 2, 3 \pmod{5}, \\ n-2-k, & \text{if } k = 1, 4 \pmod{5}. \end{cases} \quad (\text{A.23})$$

When $t = 4$, let

$$i_0 = \begin{cases} n-1-k, & \text{if } k = 1, 3, 4 \pmod{5}, \\ n-2-k, & \text{if } k = 0, 2 \pmod{5}. \end{cases} \quad (\text{A.24})$$

Since $v_{i_0} u_{i_0}, u_{j_0} u_{j_0+2} \in E_t$ where $j_0 = i_0 + k$, D is a dominating set of graph $P(n, 2) - \{v_{i_0} u_{i_0}, u_{j_0} u_{j_0+2}\}$ for each $k \in \{0, 2, \dots, \lfloor n/2 \rfloor\}$. Then, $\gamma(P(n, 2) - \{v_i u_i, u_j u_{j+2}\}) = \gamma(G) \leq \gamma(P(n, 2))$.

Data Availability

All data generated or analyzed during this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

Acknowledgments

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