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Research Article

Green's Function and Positive Solutions of a Third-Order Equation with Periodic Boundary Conditions

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We apply the fixed point index to obtain positive solutions of a nonresonant periodic boundary value problem for a third-order differential equation $u''' + \rho^3 u = \lambda f(u)$.

1. Introduction

Third-order differential equations arise in many areas of physics and engineering [1] and describe, for example, deflection of a curved beam having a constant or varying cross section, a three-layered beam, and electromagnetic waves. Boundary value problems for third-order differential equations have been studied by many authors, for example, [2–9] just to name a few. In this paper, we consider a well-known [10, 11] boundary value problem:

$$u'''(t) + \rho^3 u(t) = \lambda f(u(t)), \quad t \in (0, 2\pi), \tag{1}$$

$$u^{(i)}(0) = u^{(i)}(2\pi), \quad i = 0, 1, 2. \tag{2}$$

We improve the results of [10, 11] and obtain positive solutions using the fixed point index. The solutions to (1) and (2) will be sought in the Banach space $\mathscr{B} = C[0, 2\pi]$ endowed with the max-norm. In order to obtain positive solutions, we apply the fixed point theorem of Guo and Lakshmikantham [12] stated in Section 2.

Green's function of $u'(t) + \rho u(t) = 0$ with $u(0) = u(2\pi)$ is

$$G_1(t,s) = \frac{1}{e^{2\pi\rho} - 1} \begin{cases} e^{\rho(2\pi+s-t)}, & 0 \le s \le t \le 2\pi, \\ e^{\rho(s-t)}, & 0 \le t \le s \le 2\pi. \end{cases} \tag{3}$$

Green's function of $u''(t) - \rho u'(t) + \rho^2 u(t) = 0$ with $u^{(i)}(0) = u^{(i)}(2\pi)$, $i = 0, 1$, is

$$G_2(t,s) = \frac{2}{\sqrt{3}\rho\left(e^{\pi\rho} + e^{-\pi\rho} - 2\cos\sqrt{3}\pi\rho\right)} \cdot \begin{cases} g_1(t-s), & 0 \le s \le t \le 2\pi, \\ g_2(s-t), & 0 \le t \le s \le 2\pi, \end{cases} \tag{4}$$

where

$$g_1(x) = e^{(1/2)\rho x}\left(\left(e^{-\pi\rho} - \cos\sqrt{3}\pi\rho\right)\sin\frac{\sqrt{3}}{2}\rho x + \sin\sqrt{3}\pi\rho\cos\frac{\sqrt{3}}{2}\rho x\right), \tag{5}$$

$$g_2(x) = e^{-(1/2)\rho x}\left(\left(e^{\pi\rho} - \cos\sqrt{3}\pi\rho\right)\sin\frac{\sqrt{3}}{2}\rho x + \sin\sqrt{3}\pi\rho\cos\frac{\sqrt{3}}{2}\rho x\right), \tag{6}$$

where $x \in [0, 2\pi]$. To ensure that $G_2(t,s) \ge 0$, we need $\rho \in (0, 1/\sqrt{3})$.

We maximize g_1 and g_2. Introduce, for convenience,

$$\begin{aligned} A(\rho) &= e^{-\pi\rho} - \cos\sqrt{3}\pi\rho, \\ B(\rho) &= \sin\sqrt{3}\pi\rho, \\ C(\rho) &= e^{\pi\rho} - \cos\sqrt{3}\pi\rho. \end{aligned} \tag{7}$$

We formally find that $g_1'(x_1) = 0$ if

$$x_1(\rho) = \frac{2}{\sqrt{3}\rho}\cot^{-1}\frac{\sqrt{3}B(\rho) - A(\rho)}{\sqrt{3}A(\rho) + B(\rho)}, \quad (8)$$

provided $x_1 \in (0, 2\pi)$.

Note that

$$\lim_{\rho\longrightarrow 0^+}\frac{A(\rho)}{B(\rho)} = -\frac{1}{\sqrt{3}}, \qquad \lim_{\rho\longrightarrow (1/\sqrt{3})^-}\frac{A(\rho)}{B(\rho)} = \infty. \quad (9)$$

Also

$$\left(\frac{A}{B}\right)'(\rho) = \pi\frac{\sqrt{3} - e^{-\pi\rho}(\sin\sqrt{3}\pi\rho + \sqrt{3}\cos\sqrt{3}\pi\rho)}{\sin^2\sqrt{3}\pi\rho}. \quad (10)$$

Denoting

$$\alpha(\rho) = \sqrt{3} - e^{-\pi\rho}(\sin\sqrt{3}\pi\rho + \sqrt{3}\cos\sqrt{3}\pi\rho), \quad (11)$$

we have $\alpha'(\rho) = 4\pi e^{-\pi\rho}\sin\sqrt{3}\pi\rho > 0$, $\rho \in (0, 1/\sqrt{3})$, and $\alpha(0) = 0$ and we have $(A/B)'(\rho) > 0$. Hence,

$$\phi(\rho) = \frac{\sqrt{3}B(\rho) - A(\rho)}{\sqrt{3}A(\rho) + B(\rho)}, \quad (12)$$

is a decreasing function on $(0, 1/\sqrt{3})$, $\lim_{\rho\longrightarrow 0^+}\phi(\rho) = \infty$, and $\lim_{\rho\longrightarrow (1/\sqrt{3})^-}\phi(\rho) = -1/\sqrt{3}$. We have

$$\lim_{\rho\longrightarrow (1/\sqrt{3})^-} x_1(\rho) = \lim_{\rho\longrightarrow (1/\sqrt{3})^-}\frac{2}{\sqrt{3}\rho}\cot^{-1}\frac{\sqrt{3}B(\rho) - A(\rho)}{\sqrt{3}A(\rho) + B(\rho)} = \frac{4\pi}{3},$$

$$\begin{aligned}\lim_{\rho\longrightarrow 0^+} x_1(\rho) &= \lim_{\rho\longrightarrow 0^+}\frac{2}{\sqrt{3}\rho}\cot^{-1}\phi(\rho) \\ &= -\frac{2}{\sqrt{3}}\lim_{\rho\longrightarrow 0^+}\frac{\phi'(\rho)}{1 + \phi^2(\rho)} \\ &= \frac{2}{\sqrt{3}}\lim_{\rho\longrightarrow 0^+}\frac{A'(\rho)B(\rho) - B'(\rho)A(\rho)}{A^2(\rho) + B^2(\rho)} \\ &= \frac{2}{\sqrt{3}}\lim_{\rho\longrightarrow 0^+}\frac{\sqrt{3} - e^{-\pi\rho}(\sin\sqrt{3}\pi\rho + \sqrt{3}\cos\sqrt{3}\pi\rho)}{e^{-2\pi\rho} + 1 - 2e^{-\pi\rho}\cos\sqrt{3}\pi\rho} \\ &= \frac{2\pi}{\sqrt{3}}\lim_{\rho\longrightarrow 0^+}\frac{2\sin\sqrt{3}\pi\rho}{\cos\sqrt{3}\pi\rho + \sqrt{3}\sin\sqrt{3}\pi\rho - e^{-\pi\rho}} \\ &= \frac{2\pi}{\sqrt{3}}\lim_{\rho\longrightarrow 0^+}\frac{2\sqrt{3}\cos\sqrt{3}\pi\rho}{3\cos\sqrt{3}\pi\rho - \sqrt{3}\sin\sqrt{3}\pi\rho + e^{-\pi\rho}} \\ &= \pi.\end{aligned} \quad (13)$$

It is a lengthy exercise to verify that $x_1(\rho)$ is increasing on $(0, 1/\sqrt{3})$. We infer

$$\pi < x_1(\rho) < \frac{4\pi}{3}. \quad (14)$$

It follows from (8) that

$$\cos\frac{\sqrt{3}}{2}\rho x_1 = \frac{\sqrt{3}B - A}{2\sqrt{A^2 + B^2}}, \qquad \sin\frac{\sqrt{3}}{2}\rho x_1 = \frac{\sqrt{3}A + B}{2\sqrt{A^2 + B^2}}. \quad (15)$$

In fact, $g_1''(x_1) < 0$; thus,

$$\begin{aligned}g_1(x) \le g_1(x_1) &= e^{(1/2)\rho x_1}\left(A\sin\frac{\sqrt{3}}{2}\rho x_1 + B\cos\frac{\sqrt{3}}{2}\rho x_1\right) \\ &= \frac{\sqrt{3}}{2}e^{(1/2)\rho x_1}\sqrt{A^2 + B^2}.\end{aligned} \quad (16)$$

By a similar argument, one can show that $g_2(x) \le g_2(x_2)$, where

$$x_2(\rho) = \frac{2}{\sqrt{3}\rho}\tan^{-1}\frac{\sqrt{3}C(\rho) - B(\rho)}{\sqrt{3}B(\rho) + C(\rho)} \in \left(\frac{2\pi}{3}, \pi\right), \quad (17)$$

so that

$$\begin{aligned}g_2(x) \le g_2(x_2) &= e^{-(1/2)\rho x_2}\left(C\sin\frac{\sqrt{3}}{2}\rho x_2 + B\cos\frac{\sqrt{3}}{2}\rho x_2\right) \\ &= \frac{\sqrt{3}}{2}e^{-(1/2)\rho x_2}\sqrt{C^2 + B^2}.\end{aligned} \quad (18)$$

We are in position to state and prove our first lemma.

Lemma 1. *Green's functions $G_1(t,s)$ and $G_2(t,s)$ satisfy*

$$M_1 = \max_{t,s\in[0,2\pi]} G_1(t,s) = \frac{e^{2\pi\rho}}{e^{2\pi\rho} - 1}, \quad (19)$$

$$m_1 = \max_{t,s\in[0,2\pi]} G_1(t,s) = \frac{1}{e^{2\pi\rho} - 1}, \quad (20)$$

$$M_2 = \max_{t,s\in[0,2\pi]} G_2(t,s) = \frac{1}{\sqrt{e^{\pi\rho} + e^{-\pi\rho} - 2\cos\sqrt{3}\pi\rho}} \cdot \max\left\{e^{(1/2)\rho(x_1 - \pi)}, e^{-(1/2)\rho(\pi - x_2)}\right\}, \quad (21)$$

$$m_2 = \max_{t,s\in[0,2\pi]} G_2(t,s) = \frac{2\sin\sqrt{3}\pi\rho}{\sqrt{3}\rho\left(e^{\pi\rho} + e^{-\pi\rho} - 2\cos\sqrt{3}\pi\rho\right)}. \quad (22)$$

Proof. Identities (19) and (20) are easy to check.

The following identities are useful:

$$\begin{aligned}A^2 + B^2 &= e^{-2\pi\rho} - 2e^{-\pi\rho}\cos\sqrt{3}\pi\rho + 1, \\ C^2 + B^2 &= e^{2\pi\rho} - 2e^{\pi\rho}\cos\sqrt{3}\pi\rho + 1, \\ A^2 + B^2 &= e^{-2\pi\rho}\left(C^2 + B^2\right), \\ e^{\pi\rho} + e^{-\pi\rho} - 2\cos\sqrt{3}\pi\rho &= e^{-\pi\rho}\left(C^2 + B^2\right).\end{aligned} \quad (23)$$

Then, using (16) and (18), we have

$$\max_{t,s\in[0,2\pi]} G_2(t,s) = \frac{2}{\sqrt{3}\rho\left(e^{\pi\rho}+e^{-\pi\rho}-2\cos\sqrt{3}\pi\rho\right)} \cdot \max\{g_1(x_1), g_2(x_2)\}$$
$$= \frac{2e^{\rho\pi\rho}}{\sqrt{3}(C^2+B^2)}\max\{g_1(x_1), g_2(x_2)\}$$
$$= \frac{e^{\pi\rho}}{\rho(C^2+B^2)}\max\left\{e^{(1/2)\rho x_1}\sqrt{A^2+B^2}, e^{-(1/2)\rho x_2}\sqrt{C^2+B^2}\right\}$$
$$= \frac{e^{\pi\rho}}{\rho\sqrt{C^2+B^2}}\max\left\{e^{(1/2)\rho x_1-\rho\pi}, e^{-(1/2)\rho x_2}\right\}$$
$$= \frac{e^{(1/2)\pi\rho}}{\rho\sqrt{e^{\pi\rho}+e^{-\pi\rho}-2\cos\sqrt{3}\pi\rho}} \cdot \max\left\{e^{(1/2)\rho x_1-\rho\pi}, e^{-(1/2)\rho x_2}\right\}$$
$$= \frac{1}{\rho\sqrt{e^{\pi\rho}+e^{-\pi\rho}-2\cos\sqrt{3}\pi\rho}} \cdot \max\left\{e^{(1/2)\rho(x_1-\pi)}, e^{(1/2)\rho(\pi-x_2)}\right\}, \tag{24}$$

where x_1 and x_2 are given by (8) and (17), respectively. In addition,

$$\min_{x\in[0,2\pi]} g_1(x) = \min\{g_1(0), g_1(2\pi)\} = \sin\sqrt{3}\pi\rho = \min\{g_2(0), g_2(2\pi)\} = \min_{x\in[0,2\pi]} g_2(x). \tag{25}$$

Hence,

$$\min_{t,s\in[0,2\pi]} G_2(t,s) = \frac{2\sin\sqrt{3}\pi\rho}{\sqrt{3}\rho\left(e^{\pi\rho}+e^{-\pi\rho}-2\cos\sqrt{3}\pi\rho\right)}. \tag{26}$$

Observe that (21) and (22) are exact and improve the corresponding estimates stated in Lemma 3 of [10], namely,

$$\frac{2\sin\sqrt{3}\pi\rho}{\sqrt{3}\rho\left(e^{\pi\rho}+1\right)^2} \le G_2(t,s) \le \frac{2}{\sqrt{3}\rho\sin\sqrt{3}\pi\rho}. \tag{27}$$

Indeed, $(e^{\pi\rho}+1)^2 > e^{\pi\rho}+e^{-\pi\rho}-2\cos\sqrt{3}\pi\rho$, so (22) improves the first inequality. From (14) and (17), it is clear that

$$\max\left\{e^{(1/2)\rho(x_1-\pi)}, e^{(1/2)\rho(\pi-x_2)}\right\} < e^{(1/2)\pi\rho},$$
$$\frac{\max\left\{e^{(1/2)\rho(x_1-\pi)}, e^{(1/2)\rho(\pi-x_2)}\right\}}{\sqrt{e^{\pi\rho}+e^{-\pi\rho}-2\cos\sqrt{3}\pi\rho}} < \frac{1}{\sqrt{1+e^{-2\pi\rho}-2e^{-\pi\rho}\cos\sqrt{3}\pi\rho}}$$
$$= \frac{1}{\sqrt{\left(e^{-\pi\rho}-\cos\sqrt{3}\pi\rho\right)^2+\sin^2\sqrt{3}\pi\rho}} < \frac{2}{\sqrt{3}\sin\sqrt{3}\pi\rho}, \tag{28}$$

that is, (21) is preferred to the constant in the second inequality.

Now, Green's function of $u'''(t)+\rho^3u(t)=0$, $u^{(i)}(0)=u^{(i)}(2\pi)$, $i=0,1,2$, is determined from (3)–(6) by

$$G(t,s) = \int_0^{2\pi} G_1(t,\tau)G_2(\tau,s)\,d\tau = L_1H_1(t,s) + L_2H_2(t,s), \tag{29}$$

where

$$H_1(t,s) = \begin{cases} e^{(1/2)\rho(t-s)}\left(\sin\left(\frac{\sqrt{3}}{2}\rho(t-s)-\frac{\pi}{6}\right) - e^{\pi\rho}\sin\left(\frac{\sqrt{3}}{2}\rho(t-s-2\pi)-\frac{\pi}{6}\right)\right), & s\le t, \\ e^{(1/2)\rho(t-s+2\pi)}\left(\sin\left(\frac{\sqrt{3}}{2}\rho(t-s+2\pi)-\frac{\pi}{6}\right) - e^{\pi\rho}\sin\left(\frac{\sqrt{3}}{2}\rho(t-s)-\frac{\pi}{6}\right)\right), & t\le s, \end{cases}$$
$$H_2(t,s) = \begin{cases} e^{\rho(s-t)}, & s\le t, \\ e^{\rho(s-t-2\pi)}, & t\le s, \end{cases}$$
$$L_1 = \frac{2}{3\rho^2\left(1+e^{2\pi\rho}-2e^{\pi\rho}\cos\sqrt{3}\pi\rho\right)},$$
$$L_2 = \frac{1}{3\rho^2\left(1-e^{-2\pi\rho}\right)}. \tag{30}$$

One can find (29) in [11]. The structure of (29) is too complex to be analyzed the same way as $G_2(t,s)$. Instead, by Lemma 1,

$$G(t,s) = \int_0^{2\pi} G_1(t,\tau)G_2(\tau,s)d\tau \le M_2 \int_0^{2\pi} G_1(t,\tau)d\tau = \frac{M_2}{\rho}, \tag{31}$$

and, similarly,

$$G(t,s) \ge \frac{m_2}{\rho}. \tag{32}$$

Also,

$$G(t,s) = \int_0^{2\pi} G_1(t,\tau)G_2(\tau,s)d\tau \le M_1 \int_0^{2\pi} G_2(t,\tau)d\tau = \frac{M_1}{\rho^2},$$
$$G(t,s) \ge \frac{m_1}{\rho^2}. \tag{33}$$

Thus,

$$\frac{1}{\rho^2}\max\{m_1, \rho m_2\} = m \le G(t,s) \le M = \frac{1}{\rho^2}\min\{M_1, \rho M_2\}. \tag{34}$$

Comparing m_1 to ρm_2 and M_1 to ρM_2, respectively, we find that the constants "compete" and we cannot simplify (34). Finally, note that Theorem 2.6 in [11] contains similar estimates only for $0 < \rho < (2/3\sqrt{3})$ while (34) is fulfilled for $0 < \rho < (1/\sqrt{3})$.

In the next section, as an application of (34), we obtain a positive solution of (1) and (2). □

2. Main Results

Assume that (H_1) f: $[0,\infty) \longrightarrow [0,\infty)$ is continuous and $\lambda > 0$.

It is clear that the map T: $\mathscr{B} \longrightarrow \mathscr{B}$ defined by

$$Tu(t) = \lambda \int_0^{2\pi} G(t,s) f(u(s))ds, \tag{35}$$

is completely continuous. Define

$$\mathscr{C} = \left\{u \in \mathscr{B}\colon u(t) \ge \frac{m}{M}\|u\|\right\}. \tag{36}$$

Obviously, $\mathscr{C}$ is a cone in $\mathscr{B}$. By (34), T: $\mathscr{C} \longrightarrow \mathscr{C}$ and $u \in \mathscr{B}$ is a solution of (1) and (2) if $u \in \mathscr{B}$ is a fixed point of T.

Let

$$\mathscr{C}_R = \{u \in \mathscr{C}\colon \|u\| < R\},$$
$$\partial\mathscr{C}_R = \{u \in \mathscr{C}\colon \|u\| = R\}. \tag{37}$$

We apply the following theorem [12].

Theorem 1. *Let $\mathscr{B}$ be a Banach space. Assume that T: $\overline{\mathscr{C}}_r \longrightarrow \mathscr{C}$ is completely continuous such that $Tu \ne u$ for $u \in \partial\mathscr{C}_r$.*

(A_1) If $\|Tu\| \ge \|u\|$ for $u \in \partial\mathscr{C}_r$, then $i(T, \mathscr{C}_r, \mathscr{C}) = 0$.

(A_2) If $\|Tu\| \le \|u\|$ for $u \in \partial\mathscr{C}_r$, then $i(T, \mathscr{C}_r, \mathscr{C}) = 1$.

We will restrict our attention to the case.

(H_2) $f_0, f_\infty \ne 0$, where $f_0 = \lim_{x\longrightarrow 0} f(x)/x$ and $f_\infty = \lim_{x\longrightarrow\infty} f(x)/x$.

Recall that

$$\int_0^{2\pi} G(t,s)ds = \frac{1}{\rho^3}. \tag{38}$$

If there exists $\delta > 0$ such that $f(x) \ge \delta x$, then for $u \in \mathscr{C}$,

$$\begin{aligned} Tu(t) &= \lambda \int_0^{2\pi} G(t,s) f(u(s))ds \\ &\ge \lambda\delta \int_0^{2\pi} G(t,s)u(s)ds \\ &\ge \lambda\delta \frac{m}{M} \int_0^{2\pi} G(t,s)ds\|u\| \\ &= \lambda \frac{\delta m}{M\rho^3}\|u\|. \end{aligned} \tag{39}$$

Similarly, if there exists $\delta > 0$ such that $f(x) \le \gamma x$, then, for $u \in \mathscr{C}$,

$$\begin{aligned} Tu(t) &= \lambda \int_0^{2\pi} G(t,s) f(u(s))ds \\ &\le \lambda\gamma \int_0^{2\pi} G(t,s)u(s)ds \\ &\le \lambda\gamma \int_0^{2\pi} G(t,s)ds\|u\| \\ &= \lambda \frac{\gamma}{\rho^3}\|u\|. \end{aligned} \tag{40}$$

From these inequalities, by Theorem 1, we have the following result.

Theorem 2. *If (H_1) and (H_2) are satisfied, then (1) and (2) have at least one positive solution provided:*

$$\lambda \in \left(\frac{M\rho^3}{m\max\{f_\infty, f_0\}}, \frac{\rho^3}{\min\{f_\infty, f_0\}}\right). \tag{41}$$

Proof. Consider $f_\infty > f_0$. Let $f_\infty > \epsilon > 0$ be such that

$$\lambda \in \left(\frac{M\rho^3}{m(f_\infty - \epsilon)}, \frac{\rho^3}{f_0 + \epsilon}\right). \tag{42}$$

There exists $R_1 < 0$ such that $f(x) \le (f_0 + \epsilon)x$, $x \in [0, R_1]$, that is, $f(u(t)) \le (f_0 + \epsilon)u(t)$, $u \in \partial\mathscr{C}_{R_1}$. Hence,

$$\|Tu\| \le \lambda(f_0 + \epsilon)\frac{1}{\rho^3}\|u\| < \|u\|, \quad u \in \partial\mathscr{C}_{R_1}. \tag{43}$$

There exists $R > R_1$ such that $f(x) \ge (f_\infty - \epsilon)x$, $x \ge R$. Let $R_2 = \max\{2R_1, MR/m\}$. Then, $u(t) \ge (m/M)\|u\| \ge R$, $u \in \partial\mathscr{C}_{R_2}$. As above, for $u \in \partial\mathscr{C}_{R_2}$,

$$\|Tu\| \geq \lambda \left(f_{\infty} - \epsilon \right) \frac{m}{M\rho^{3}} \|u\| > \|u\|. \tag{44}$$

It follows that $i(T, \mathscr{C}_{R_1}, \mathscr{C}) = 1$ and $i(T, \mathscr{C}_{R_2}, \mathscr{C}) = 0$ so that $i(T, \mathscr{C}_{R_2}/\overline{\mathscr{C}}_{R_1}, \mathscr{C}) = -1$, that is, T has a fixed point in $\mathscr{C}_{R_2}/\overline{\mathscr{C}}_{R_1}$.

The case $f_{\infty} < f_0$ is handled similarly. □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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