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Research Article

Transportation Inequalities for Coupled Fractional Stochastic Evolution Equations Driven by Fractional Brownian Motion

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In this paper, we consider the existence and uniqueness of the mild solution for a class of coupled fractional stochastic evolution equations driven by the fractional Brownian motion with the Hurst parameter $H \in (1/4, 1/2)$. Our approach is based on Perov's fixed-point theorem. Furthermore, we establish the transportation inequalities, with respect to the uniform distance, for the law of the mild solution.

1. Introduction

In the research of various fields of science and engineering, fractional stochastic differential equations (SDEs) play a significant role in the modeling of many complex phenomena in diverse areas. The intensive development in both theory and applications of fractional SDEs was investigated in [1–4]. In addition, many scholars have also developed interest in systems with memory or after effect (i.e., systems with finite delays in the state equation). Therefore, it is necessary to study stochastic evolution equations with finite delays. The equation is widely used in network flow analysis, mathematical finance, astrophysics, hydrology, image processing, and other directions [5, 6]. For the nature of the existence and uniqueness of the mild solutions, Sakthivel et al. [7] considered the nonlinear-type fractional SDE, Li [8] established stochastic delay fractional evolution equations driven by fractional Brownian motion in a Hilbert space, and Mophou [9] was concerned about impulsive fractional semilinear differential equations.

Moreover, under the research of many scholars, transportation inequalities are developed greatly in various SDEs and stochastic systems with respect to the different measure conditions. Among others, the Girsanov transformation argument introduced in [10] has been efficiently applied, e.g., Wu and Zhang [11] considered infinite-dimensional dynamical systems with respect to L^2 metric; Üstünel [12] studied the multivalued SDE and singular SDE under uniform distance; besides, Bao et al. [13] investigated the neutral functional SDE with respect to both the uniform distance and the L^2 distance; Saussereau [14] researched the SDE driven by a fractional Brownian motion; futhermore, Li and Luo [15] took it into account that stochastic delay evolution equations driven by fractional Brownian motion with the hurst parameter $H > 1/2$ under the L^2 metric and the uniform metric; and Boufoussi and Hajji [16] established the transportation inequalities, with respect to the uniform distance, for the law of the mild solution for a neutral stochastic differential equation with finite delay, driven by a fractional Brownian motion with the Hurst parameter lesser than 1/2 in a Hilbert space.

In connection with the aforementioned works, in this paper, we investigate the existence, uniqueness, and property $T_2(C)$ under the uniform distance for the law of mild solution of the coupled fractional stochastic delay evolution equations with finite delay driven by a fractional Brownian motion with the Hurst parameter $0 < H < 1/2$:

$$\begin{cases} {}^cD^{\alpha_1}(x(t)) = (A_1 x(t) + f_1(t, x_t, y_t))\mathrm{d}t + \sigma_1(t)\mathrm{d}B_1^H(t), & t \in [0,T], \\ {}^cD^{\alpha_2}(y(t)) = (A_2 y(t) + f_2(t, x_t, y_t))\mathrm{d}t + \sigma_2(t)\mathrm{d}B_2^H(t), & t \in [0,T], \\ x(t) = \phi_1(t), & t \in [-r,0], \\ y(t) = \phi_2(t), & t \in [-r,0], \end{cases} \tag{1}$$

where ${}^cD^{\alpha_i}$ is the Caputo fractional derivative of order $\alpha_i \in (1/2, 1]$, for each $i = 1, 2$, as for the state $x(\cdot)$, $y(\cdot)$ has values in a real and separable Hilbert X with an inner product $(\cdot,\cdot)_X$ and norm $\|\cdot\|_X$, where $\{A_i, i = 1, 2\}$ are the infinitesimal generators of analytic semigroups of bounded linear operators $\{T_i(t), t \geq 0\}$, B_i^H is the fractional Brownian motion on a real and separable Hilbert space Y, with the Hurst parameter $H \in (0, 1/2)$, and let $r > 0$ denote the constant. As for y_t, we mean the segment solution which is defined in the usual way, that is, if $y(\cdot,\cdot)\colon [-r, T] \times \Omega \longrightarrow X$, then for any $t > 0$

$$y_t(\theta, \omega) = y(t + \theta, \omega), \quad \theta \in [-r, 0], \omega \in \Omega. \tag{2}$$

Before describing the properties fulfilled by operators f_i, σ_i, we need to introduce some nontations and describe some spaces. Let $\mathscr{D}_0$ denote the space of all continuous functions $\varphi\colon [-r, 0] \times \Omega \longrightarrow X$ such that $\varphi(\theta, \cdot)$ is $\mathscr{F}_0$-measurable for each $\theta \in [-r, 0]$ and $\int_{-r}^0 \mathbb{E}\|\varphi\|_X^2 \mathrm{d}t < \infty$. In the space $\mathscr{D}_0$, we endow with the following norm:

$$\|\varphi\|_{D_0}^2 = \int_{-r}^0 \mathbb{E}\|\varphi(t)\|_X^2 \mathrm{d}t. \tag{3}$$

Next, we denote by $C(a, b; L^2(\Omega; X)) = C(a, b; L^2(\Omega, \mathscr{F}, \mathbb{P}; X))$ the Banach space of all continuous functions from $[a, b]$ into $L^2(\Omega; X)$. Now, fixing $T > 0$, we define

$$\mathscr{D}_T = \left\{ y\colon\ y \in C\left(-r, T; L^2(\Omega; X)\right), \sup_{t\in[0,T]} \mathbb{E}\left(\|y(t)\|_X^2\right) < \infty, \int_{-r}^0 \mathbb{E}\|y(t)\|_X^2 \mathrm{d}t < \infty \right\}, \tag{4}$$

endowing with the following norm:

$$\|y\|_{\mathscr{D}_T} = \sup_{t\in[0,T]} \sqrt{\mathbb{E}\left(\|y(t)\|_X^2\right)} + \|y(t)\|_{\mathscr{D}_0}. \tag{5}$$

We give initial data $\varphi_1, \varphi_2 \in \mathscr{D}_0$, and Y is another real and separable Hilbert space, and $B_{Q_i}^H = B_i^H$ is a Y-valued fractional Brownian motion with increment covariance given by a nonnegative trace class operator Q_i, and $L(Y, X)$ represents the space of all bounded, continuous, and linear operators from Y into X.

We denote $f_i\colon J \times D_0 \times D_0 \longrightarrow X$ and $\sigma_i\colon J \longrightarrow L_{Q_i}^0(Y, X)$. Here, let us denote $L_{Q_i}^0(Y, X)$ by the space of all Q_i-Hilbert–Schmidt operators from Y into X for each $i = 1, 2$ which will also be introduced in the next section.

Now, let us present the relevant knowledge of transportation inequalities. To connect the measure distances with the probability measures, we consider the transportation distance, also called as Wasserstein distance. Let (E, d) be a metric space provided with the σ-field $\mathscr{B}$, such that $\mathrm{d}(\cdot,\cdot)$ is $\mathscr{B} \times \mathscr{B}$-measurable. Fixing $p \geq 1$ and for any probability measures μ and ν on E, we define the Wasserstein distance of order p between μ and ν as

$$W_p^d(\mu, \nu) = \inf_{\pi \in \prod(\mu,\nu)} \left(\int_{\mathrm{E}\times\mathrm{E}} \mathrm{d}(x, y)^p \mathrm{d}\pi(x, y) \right)^{(1/p)}, \tag{6}$$

where $\Pi(\mu, \nu)$ denotes the totality of probability measures on $E \times E$ with the marginal μ and ν. The relative entropy of ν with respect to μ is defined as

$$H(\nu, \mu) = \begin{cases} \int \log \dfrac{\mathrm{d}\nu}{\mathrm{d}\mu} \mathrm{d}\nu, & \nu \ll \mu, \\ +\infty, & \text{otherwise}. \end{cases} \tag{7}$$

The probability measure μ satisfies L^p-transportation inequality on (E, d) if there exists a constant $C \geq 0$ such that for any probability measure ν,

$$W_p^d(\mu, \nu) \leq \sqrt{2CH(\nu \mid \mu)}. \tag{8}$$

As usual, we write $\mu \in T_p(C)$ for this relation. The property $T_2(C)$ is of particular interest. We will investigate the property $T_2(C)$ for the law of the mild solution of stochastic delay evolution equations driven by fractional Brownian motion with the Hurst parameter $1/4 < H < 1/2$ under the uniform distance.

This paper is organized as follows. In Section 2, we introduce some preliminaries used in this paper such as stochastic calculus, some properties of generalized Banach spaces, and fractional calculus. In Section 3, we state and prove the existence and uniqueness of the mild solution by using Perov's fixed-point type in generalized Banach spaces. In Section 4, we investigate the property $T_2(C)$ for the law of the solution of fractional stochastic delay evolution equations driven by fractional Brownian motion with the Hurst parameter $1/4 < H < 1/2$ under the uniform metric. In Section 5, we present an example to illustrate the efficiency of the obtained result.

2. Preliminaries

In this section, we introduce some notations and recall definitions and preliminary results which are used throughout this paper.

Let $(\Omega, \mathscr{F}, \mathbb{P}, \{\mathscr{F}_t\}_{t\geq 0})$ be a complete probability space furnished with a normal filtration $\{\mathscr{F}_t\}_{t\geq 0}$. We postulate that the operator A_i is self-adjoint and there exists the eigenvectors e_k corresponding to eigenvalues γ_k such that

$$A_i e_k = \gamma_k e_k, e_k = \sqrt{2}\sin(k\pi), \quad \gamma_k = \pi^2 k^2, k \in \mathbb{N}^+. \tag{9}$$

For each $\sigma > 0, A_i^{(\sigma/2)} e_k = \gamma_k^{(\sigma/2)}, k = 1, 2, 3, \ldots$, and let $\dot{H}^\sigma$ be the domain of $\{A_i^{(\sigma/2)}\}$ defined by

$$\dot{H}^\sigma = D\left(A_i^{(\sigma/2)}\right) = \left\{ \nu \in L^2(X), \quad \text{s.t.} \quad \|\nu\|_{\dot{H}^\sigma}^2 = \sum_{k=0}^{\infty} \gamma_k^{(\sigma/2)} \nu_k^2 < \infty \right\}, \tag{10}$$

where the vector $\nu_k = (\nu, e_k)$ and the norm $\|\nu\|_{\dot{H}^\sigma} = \|A_i^{(\sigma/2)} \nu\|$. Let $L^2(X)$ be an X-valued Hilbert space with the inner product $\mathbb{E}(\cdot,\cdot)$ and norm $\mathbb{E}\|\cdot\|$, and it is given by

$$L^2(X) = \left\{ \chi \colon \mathbb{E}\|\chi\|_X^2 = \int_\Omega \|\chi(\omega)\|_X^2 \mathrm{d}\mathbb{P}(\omega) < \infty, \quad \omega \in \Omega \right\}. \tag{11}$$

Definition 1. For $H \in (0,1)$, a continuous centered Gaussian process $\{B^H(t)\}_{t\in[0,\infty)}$ with covariance function

$$\begin{aligned} R_H(t,s) &= \mathbb{E}\left[B^H(t) B^H(s)\right] \\ &= \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right), \quad t, s \in [0,\infty), \end{aligned} \tag{12}$$

is called a one-dimensional fractional Brownian motion (fBm), and H is the Hurst parameter. In particular, when $H = 1/2$, $B^H(t)$ represents a standard Brownian motion.

Now, let us aim at the Wiener integral with respect to the fBm. To begin with, $B^H(t)$ has following integral expression (see [17]):

$$B^H(t) = \int_0^t K_H(t,s) \mathrm{d}B(s), \tag{13}$$

where $B = \{B(t)\colon t \in [0,T]\}$ is a Wiener process, $K_H(t,s)$ is a square integrable kernel, for $0 < H < 1/2$, and $t > s$; the formula is as follows (see [17]):

$$K_H(t,s) = c_H\left[\left(\frac{t}{s}\right)^{H-(1/2)} (t-s)^{H-(1/2)} - \left(H - \frac{1}{2}\right) s^{(1/2)-H} \int_s^t (u-s)^{H-(1/2)} u^{H-(3/2)} \mathrm{d}u\right], \tag{14}$$

where $c_H = \sqrt{H/((1-H)\beta(1-2H, H+1/2))}$ and $\beta(\cdot,\cdot)$ denotes the Beta function. $K_H(t,s) = 0, t \le s$. Since $0 < H < 1/2$, from (14), we can infer that

$$\begin{aligned} \left|K_H(t,s)\right| &\le c_H\left[(t-s)^{H-(1/2)} + \left(\frac{1}{2} - H\right) s^{(1/2)-H} \int_s^t (u-s)^{H-(1/2)} u^{-H-(1/2)} u^{2H-1} \mathrm{d}u\right] \\ &\le c_H\left[(t-s)^{H-(1/2)} + \left(\frac{1}{2} - H\right) s^{(1/2)-H} \int_s^t (u-s)^{H-(1/2)} u^{-H-(1/2)} \mathrm{d}u\right]. \end{aligned} \tag{15}$$

Then, we obtain

$$\left|K_H(t,s)\right| \le 2c_H\left((t-s)^{H-(1/2)} + s^{H-(1/2)}\right). \tag{16}$$

Taking the derivative of (14) with respect to t, we can have

$$\frac{\partial K_H}{\partial t}(t,s) = c_H(H - (1/2))\left(\frac{t}{s}\right)^{H-(1/2)} (t-s)^{H-(3/2)}. \tag{17}$$

Apparently, we can obtain the following inequality:

$$\left|\frac{\partial K_H}{\partial t}(t,s)\right| = c_H\left(\frac{1}{2} - H\right)(t-s)^{H-(3/2)}. \tag{18}$$

Let $\mathscr{H}$ be the Hilbert space defined as the closure of the vector space spanned by the set of step functions $\{\mathbf{I}_{[0,t]}, t \in [0,T]\}$ with respect to the scalar product:

$$\langle \mathbf{I}_{[0,t]}, \mathbf{I}_{[0,s]} \rangle = R_H(t,s), \quad \forall t, s \in [0,T]. \tag{19}$$

Now, we consider the operator $K_{H,T}^*$ from $\mathscr{H}$ to $L^2([0,T])$ defined by

$$\left(K_{H,T}^* \varphi\right)(s) = K_H(T,s)\varphi(s) + \int_s^T (\varphi(r) - \varphi(s)) \frac{\partial K_H}{\partial r}(r,s) \mathrm{d}r. \tag{20}$$

Furthermore, $K_{H,T}^*$ is an isometry between $\mathscr{H}$ and $L^2([0,T])$(see [18]). Taking account for $B = \{B(t), t \in [0,T]\}$ defined by

$$B(t) = B^H\left(\left(K_H^*\right)^{-1} \mathrm{I}_{[0,t]}\right), \tag{21}$$

it turns out that B is a Wiener process. Moreover, for any $\varphi \in \mathscr{H}$, with (13), we have

$$\int_0^T \varphi(S) \mathrm{d}B^H(s) := B^H(\varphi) = \int_0^T \left(K_{H,T^\varphi}^*\right)(t) \mathrm{d}B(t). \tag{22}$$

For any $0 \le t \le T$, we can also deduce

$$\begin{aligned} \int_0^t \varphi(s) \mathrm{d}B^H(s) &:= \int_0^T \left(K_{H,T}^* \varphi \mathrm{I}_{[0,t)}\right)(s) \mathrm{d}B(s) \\ &= \int_0^t \left(K_{H,t}^* \varphi\right)(s) \mathrm{d}B(s), \end{aligned} \tag{23}$$

where $K_{H,t}^{*}$ is defined in the same way as in (20) with t instead of T. Next, we will use the notation $K_{H,t}^{*}$ without specifying the parameter $t \in [0, T]$.

Let $(X, \|\cdot\|_X, \langle\cdot,\cdot\rangle_X)$ and $(Y, \|\cdot\|_Y, \langle\cdot,\cdot\rangle_Y)$ be two real, separable Hilbert spaces and let $\mathscr{L}(Y, X)$ denote the space of all bounded linear operators from Y to X. Let $Q \in \mathscr{L}(Y, X)$ be a nonnegative self-adjoint operator i.e., $Qe_n = \lambda_n e_n$ with trace $\operatorname{tr} Q = \sum_{n=1}^{\infty} \lambda_n < \infty$, where $\lambda_n \in \mathbb{R}^+$ and $\{e_n\}_{n\geq 1}$ is a complete orthonormal basis in Y. We define the infinite-dimensional fBm on Y with covariance Q by the following formula:

$$B_Q^H(t) = \sum_{n=1}^{\infty} \sqrt{\lambda_n} e_n B_n^H(t), \quad t \geq 0, \tag{24}$$

where $\{B_n^H(t)\}_{n\in\mathbb{N}}$ be a sequence of one-dimensional mutually independent standard fractional Brownian motions on $(\Omega, \mathscr{F}, \mathbb{P})$. $B_n^H(t)$ is a Y-valued Gaussian process, starting from 0, and has zero mean and covariance:

$$\begin{aligned} \mathbb{E}\langle B_Q^H(t), x\rangle\langle B_Q^H(s), y\rangle = R(s,t)\langle Q(x), y\rangle, \\ \text{for all } x, y \in Y \text{ and } t, s \in [0, T]. \end{aligned} \tag{25}$$

Let $\mathscr{L}_2^0(Y, X)$ be the space of all $\xi \in \mathscr{L}(Y, X)$ such that $\xi Q^{(1/2)}$ is a Hilbert–Schmidt operator. The norm is given by

$$\|\xi\|_{L_2^0(Y,X)}^2 := \sum_{n=1}^{\infty} \left\| \sqrt{\lambda_n} \xi(s) e_n \right\|^2 = \operatorname{tr}(\xi Q \xi^*) < \infty. \tag{26}$$

Then, ξ is called a Q-Hilbert–Schmidt operator from Y to X.

Definition 2. Let φ: $[0, T] \longrightarrow \mathscr{L}_2^0(Y, X)$. Then, the Wiener integral of φ with respect to the fBm B_Q^H is defined as follows:

$$\begin{aligned} \int_0^t \varphi(s) \mathrm{d}B_Q^H(s) &:= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \varphi(s) e_n \mathrm{d}B_n^H(s) \\ &= \sum_{n=1}^{\infty} \int_0^t \sqrt{\lambda_n} \left(K_H^*(\varphi e_n)\right)(s) \mathrm{d}B_n(s), \end{aligned} \tag{27}$$

where B_n is the standard Brownian motion used to represent B_n^H as in (13), and the sum above is finite when $\sum_{n=1}^{\infty} \lambda_n \|K_H^*(\varphi e_n)\| < \infty$.

The classical Banach contraction principle was extended for contractive maps on spaces endowed with a vector-valued metric space by Perov [19] in 1964 and Precup [20, 21]. Now, we recall some useful definitions and results.

Definition 3. Let Z be a nonempty set. We denote by a vector-valued metric on Z defined as a mapping d: $Z \times Z \longrightarrow \mathbb{R}^n$ with the following properties:

(1) $\mathrm{d}(u, v) \geq 0$ for all $u, v \in Z$; $\mathrm{d}(u, v) = 0$, only if $u = v$.
(2) $\mathrm{d}(u, v) = \mathrm{d}(u, v)$ for all $u, v \in Z$.
(3) $\mathrm{d}(u, v) \leq \mathrm{d}(u, w) + \mathrm{d}(w, v)$ for all $u, v, w \in Z$.

Now, we consider a generalized metric space (Z, d). For $r = (r_1, \ldots, r_n) \in \mathbb{R}_+^n$, we will define the open ball centered in x_0 with radius r:

$$B(x_0, r) = \{x \in Z: \mathrm{d}(x_0, x) < r\}, \tag{28}$$

and the closed ball centered in x_0 with radius r:

$$\overline{B(x_0, r)} = \{x \in Z: \mathrm{d}(x_0, x) \leq r\}. \tag{29}$$

We state that for a generalized metric space, the notation of open and closed sets, convergence, Cauchy sequence, and completeness in a generalized metric space are similar to those in usual metric spaces. If $x, y \in \mathbb{R}^n$, $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n)$, by $x \leq y$ then we mean $x_i \leq y_i$, $i = 1, \ldots, n$. Also, $|x| = (|x_1|, \ldots, |x_n|)$ and $\max(x, y) = \max(\max(x_1, y_1), \ldots, \max(x_n, y_n))$. If $c \in \mathbb{R}$, then $x \leq c$ means $x_i \leq c$ for each $i = 1, \ldots, n$.

Definition 4. A generalized metric space (Z, d) where $\mathrm{d}(x, y) := \begin{pmatrix} \mathrm{d}_1(x, y) \\ \vdots \\ \mathrm{d}_n(x, y) \end{pmatrix}$ is complete, if for every $i = 1, \ldots, n$, (Z, d_i) is a complete metric space.

Definition 5. We denote that a real square matrix M is convergent to zero if and only if its spectral radius $\rho(M)$ is strictly less than 1. In other words, it means that all the eigenvalues of M are in the open unit disc (i.e., $|\lambda| < 1$, for every $\lambda \in \mathbb{C}$ with $\det(M - \lambda I) = 0$, where I denotes the unit matrix of $\mathscr{M}_{n\times n}(\mathbb{R})$).

Definition 6. We denote that a nonsingular matrix $A = (a_{ij})_{1\leq i,j\leq n} \in \mathscr{M}_{n\times n}(\mathbb{R})$ has the absolute value property if

$$A^{-1}|A| \leq I, \tag{30}$$

where

$$|A| = \left(\left|a_{ij}\right|\right)_{1\leq i,j\leq n} \in \mathscr{M}_{n\times n}(\mathbb{R}). \tag{31}$$

Now, we need to use the following fixed-point theorem to prove the existence and uniqueness of mild solution for (1).

Theorem 1 (see [19]). *Let (Z, d) be a complete generalized metric space with $d: Z \times Z \longrightarrow \mathbb{R}^n$ and let operator $N: Z \longrightarrow Z$ be such that*

$$\mathrm{d}(N(x), N(y)) \leq M \mathrm{d}(x, y), \tag{32}$$

for all $x, y \in Z$ and some nonnegative square matrix M. If the matrix M is convergent to 0, that is, $M^k \longrightarrow 0$ as $k \longrightarrow 0$, then operator N has a unique fixed point $x_* \in Z$:

$$\mathrm{d}\left(N^k(x_0, x_*)\right) \leq M^k (I - M)^{-1} \mathrm{d}(N(x_0, x_0)), \tag{33}$$

for every $x_0 \in Z$ and $k \geq 1$.

Definition 7. The fractional integral of index α with the lower limit 0 for a function f can be written as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} \mathrm{d}s, \quad t > 0, \alpha > 0, \tag{34}$$

provided the right-hand side is pointwise defined on $[0, +\infty)$, where Γ is the gamma function, which is defined by $\Gamma(y) := \int_0^{\infty} t^{y-1} e^{-t} \mathrm{d}t$.

Definition 8. The Caputo derivative of index α for a function $f \in C^n([0, \infty))$ is defined as

$$^{C}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}} \mathrm{d}s = I^{n-\alpha} f^{(n)}(t), \quad t > 0, n-1 < \alpha < n. \tag{35}$$

3. Existence and Uniqueness

In this section, we investigate the existence and uniqueness of a mild solution for (1). First of all, we will give some hypotheses which will be used to prove our main result; for this question, we assume that the following conditions hold.

($\mathscr{H}$.1) There exists constants $a_{f_i}, b_{f_i} \in \mathbb{R}^+$ for each $i = 1, 2, \ldots$ such that

$$\int_0^t \left\| f_i(s, x_s, y_s) - f_i(s, \overline{x}_s, \overline{y}_s) \right\|_X^2 \leq a_{f_i} \int_{-r}^t \| x(s) - \overline{x}(s) \|_X^2 \mathrm{d}s + b_{f_i} \int_{-r}^t \| y(s) - \overline{y}(s) \|_X^2 \mathrm{d}s, \quad \text{for all } x, y, \overline{x}, \overline{y} \in C([-r, T]; X). \tag{36}$$

($\mathscr{H}$.2) The function $\sigma: [0, T] \longrightarrow L_Q^0(Y, X)$ satisfies the following Hölder continuous conditions, that is, there exists a constant $C_{\sigma} > 0$ such that for all $t, s \in [0, T]$,

$$\| \sigma(t) - \sigma(s) \|_{\mathscr{L}_Q^0} \leq C_{\sigma} |t-s|^{\gamma}, \tag{37}$$

where $\gamma > 1 - 2H$.

Now, we state the following definition of mild solution for our problem.

Definition 9. A $\mathscr{H}$-valued process $u(t) = (x(t), y(t))$ is called a mild solution of (1) with respect to the probability space $(\Omega, \mathscr{F}, \mathbb{P})$, if $x, y \in C(-r, T; L^2(\Omega; X))$, $(x(t), y(t)) = (\phi_1(t), \phi_2(t))$ for $t \in [-r, 0]$, and for each $t \in [0, T] = J$, $u(t)$ it satisfies the following integral equation:

$$\begin{cases} x(t) = T_{\alpha_1} \phi_1(0) + \displaystyle\int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1}(t-s) f_1(s, x_s, y_s) \mathrm{d}s + \int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1}(t-s) \sigma_1(s) \mathrm{d}B_1^H(s), & \mathbb{P}-\text{a.s.}, t \in J, \\ y(t) = T_{\alpha_2} \phi_2(0) + \displaystyle\int_0^t (t-s)^{\alpha_2 - 1} E_{\alpha_2}(t-s) f_2(s, x_s, y_s) \mathrm{d}s + \int_0^t (t-s)^{\alpha_2 - 1} E_{\alpha_2}(t-s) \sigma_2(s) \mathrm{d}B_2^H(s), & \mathbb{P}-\text{a.s.}, t \in J, \end{cases} \tag{38}$$

where

$$\begin{aligned} T_{\alpha_i}(t) &= \int_0^{\infty} \eta_{\alpha_i}(\theta) T_i(t^{\alpha_i} \theta) \theta, \quad i = 1, 2, \\ E_{\alpha_i}(t) &= \alpha_i \int_0^{\infty} \theta \eta_{\alpha_i}(\theta) T_i(t^{\alpha_i} \theta) \theta, \quad i = 1, 2, \end{aligned} \tag{39}$$

in which $T(t) = e^{-tA_i}, t \geq 0$ is an analytic semigroup generated by the operator $-A_i$, and the Mainardi's Wright-type function with $\alpha_i \in (0, 1)$ is given by

$$\eta_{\alpha_i}(z) = \sum_0^{+\infty} \frac{(-z)^n}{n! \Gamma(-\alpha_i) n + 1 - \alpha_i}, \quad \alpha_i \in (0, 1), z \in \mathbb{C}. \tag{40}$$

The operators $\{T_{\alpha_i}(t)\}_{t \geq 0}$ and $\{E_{\alpha_i}(t)\}_{t \geq 0}$ in (38) have the following properties [22]:

Lemma 1 (see [23]). *For any $t > 0$ and $\chi \in X$, $\{T_{\alpha_i}(t)\}_{t \geq 0}$ and $\{E_{\alpha_i}(t)\}_{t \geq 0}$ are linear and bounded operators. Moreover, for $0 < \alpha_i < 1$ and $0 \leq v < 2$, there exists a constant $C > 0$ such that*

$$\begin{aligned} \left\| T_{\alpha_i}(t) \chi \right\|_{\dot{H}^v} &\leq C t^{-(\alpha_i v/2)} \| \chi \|_X, \\ \left\| E_{\alpha_i}(t) \chi \right\|_{\dot{H}^v} &\leq C t^{-(\alpha_i v/2)} \| \chi \|_X. \end{aligned} \tag{41}$$

Lemma 2 (see [23]). *For any $T > 0$ and $\chi \in X$, the operator $E_{\alpha_i}(t)$ is strongly continuous. Moreover, for $0 < \alpha_i < 1$ and $0 \leq v < 2$ and $0 \leq t_1 < t_2 \leq T$, there exists a constant $C > 0$ such that*

$$\left\| \left(E_{\alpha_i}(t_2) - E_{\alpha_i}(t_1) \right) \chi \right\|_{\dot{H}^v} \leq C (t_2 - t_1)^{(\alpha_i v/2)} \| \chi \|_X. \tag{42}$$

Lemma 3 (see [23]). *Let $S_{\alpha_i}(t) = t^{\alpha_i - 1} E_{\alpha_i}(t)$, for $\forall \chi \in X$, $0 \leq v < 2$, and $0 < \alpha_i < 1$, there exists a constant $C > 0$ such that*

$$\begin{aligned} \left\| S_{\alpha_i}(t) \chi \right\|_{\dot{H}^v} &\leq C t^{((2-v)\alpha_i - 2/2)} \| \chi \|_X, \\ \left\| \left[S_{\alpha_i}(t_2) - S_{\alpha_i}(t_1) \right] \chi \right\|_{\dot{H}^v} &\leq C (t_2 - t_1)^{(2-(2-v)\alpha_i/2)} \| \chi \|_X. \end{aligned} \tag{43}$$

The following lemma proves that the stochastic integral in (38) is well defined.

Lemma 4. *Under the assumptions on A, $E_{\alpha_i}(t)$, and $\sigma(t)$, for $0 \leq v < 2$, $0 < \alpha_i < 1$, and $1/4 < H < 1/2$, the stochastic integral in (38) is well defined and satisfies the following:*

$$\mathbb{E}\left\|\int_0^t (t-s)^{\alpha_i-1}E_{\alpha_i}(t-s)\sigma_2(s)\right\|_{\dot{H}^{\nu}}^2 \leq Ct^{\delta} < \infty, \quad (44)$$

where the index should satisfy

$$\delta = \min\{(2-\nu)\alpha_i + 4H - 3, (2-\nu)\alpha_i + 2H + \gamma - 3, 2H - (2-\nu)\alpha_i + 1\} > 0. \quad (45)$$

Proof. Using the Wiener integral with respect to fBm and noticing the expression of K_t^* and the properties of the Itô integral, for $0 < H < (1/2)$, we get

$$\begin{aligned}
&\mathbb{E}\left\|\int_0^t (t-s)^{\alpha_i-1}\mathrm{E}_{\alpha_i}(t-s)\sigma(s)\mathrm{d}B^H(s)\right\|_{\dot{H}^{\nu}}^2 \\
&= \sum_{k=1}^{\infty}\mathbb{E}\left\|\int_0^t \lambda_k^{1/2}\left(K_t^* S_{\alpha_i}(t-s)e_k\sigma\right)(s)\mathrm{d}\beta_k(s)\right\|_{\dot{H}^{\nu}}^2 \\
&= \sum_{k=1}^{\infty}\int_0^t \mathbb{E}\left\|\lambda_k^{1/2}\left(K_t^* S_{\alpha_i}(t-s)e_k\sigma\right)(s)\right\|_{\dot{H}^{\nu}}^2 \mathrm{d}s \\
&= \sum_{k=1}^{\infty}\int_0^t \mathbb{E}\Big\|\lambda_k^{1/2} S_{\alpha_i}(t-s)e_k\sigma(s)K_H(t,s)e_k \\
&\quad + \int_s^t \lambda_k^{1/2}\left[S_{\alpha_i}(t-r)\sigma(r) - S_{\alpha_i}(t-s)\sigma(s)\right]\frac{\partial K_H}{\partial r}(r,s)e_k\mathrm{d}r\Big\|_{\dot{H}^{\nu}}^2 \mathrm{d}s \\
&\leq 2\sum_{k=1}^{\infty}\int_0^t \mathbb{E}\left\|\lambda_k^{1/2} S_{\alpha_i}(t-s)\sigma(s)K_H(t,s)e_k\right\|_{\dot{H}^{\nu}}^2 \mathrm{d}s \\
&\quad + 4\sum_{k=1}^{\infty}\int_0^t \mathbb{E}\left\|\int_s^t \lambda_k^{1/2}\left[S_{\alpha_i}(t-r)(\sigma(r)-\sigma(s))\right]\frac{\partial K_H}{\partial r}(r,s)e_k\mathrm{d}r\right\|_{\dot{H}^{\nu}}^2 \mathrm{d}s \\
&\quad + 4\sum_{k=1}^{\infty}\int_0^t \mathbb{E}\left\|\int_s^t \lambda_k^{1/2}\sigma(s)\left[S_{\alpha_i}(t-r) - S_{\alpha_i}(t-s)\right]\frac{\partial K_H}{\partial r}(r,s)e_k\mathrm{d}r\right\|_{\dot{H}^{\nu}}^2 \mathrm{d}s \\
&:= I_1 + I_2 + I_3.
\end{aligned} \quad (46)$$

With the help of the following inequality (see [24]):

$$K_H(t,s) \leq C(H)(t-s)^{H-(1/2)}s^{H-(1/2)}, \quad (47)$$

furthermore, combining Lemma 3 and Hölder inequality, we obtain

$$\begin{aligned}
I_1 &= 2\sum_{k=1}^{\infty}\int_0^t \mathbb{E}\left\|\lambda_k^{1/2} S_{\alpha i}(t-s)\sigma(s)K_H(t,s)e_k\right\|_{\dot{H}^{\nu}}^2 \mathrm{d}s \\
&\leq 2C(H)\hat{\sigma}\left(\int_0^t (t-s)^{(2-\nu)\alpha_i+2H-3}s^{2H-1}\sum_{k=1}^{\infty}\mathrm{E}\left\|\lambda_k^{1/2}e_k\right\|^2 \mathrm{d}s\right) \\
&\leq 2C(H)\hat{\sigma}Tr(Q)\left(\int_0^t (t-s)^{2\left[(2-\nu)\alpha_i+2H-3\right]}\mathrm{d}s\right)^{(1/2)}\left(\int_0^t s^{2(2H-1)}\mathrm{d}s\right)^{(1/2)} \\
&\leq C(H,Q)\hat{\sigma}t^{(2-\nu)\alpha_i+4H-3},
\end{aligned} \quad (48)$$

where $\widehat{\sigma}:=\sup_{0\le s\le T}\|\sigma(s)\|_{\mathscr{L}_Q^0}<+\infty$. On the contrary, utilizing ($\mathscr{H}$..2), expression (17), and Hölder inequality, we get

$$\begin{aligned} I_2 &= 4\sum_{k=1}^{\infty}\int_0^t \mathbb{E}\left\|\int_s^t \lambda_k^{1/2}\left[S_{\alpha_i}(t-r)(\sigma(r)-\sigma(s))\right]\frac{\partial K_H}{\partial r}(r,s)e_k \mathrm{d}r\right\|_{\dot{H}^\nu}^2 \mathrm{d}s \\ &\le 4c_H^2C_\sigma^2 2-\nu Tr(Q)\left(H-\frac{1}{2}\right)^2\int_0^t s^{1-2H}\int_s^t\left\|\left[(t-r)^{\frac{(2-\nu)\alpha_i-2}{2}}(r-s)^{\gamma}\right](r-s)^{H-(3/2)}\gamma^{(1/2)-H}\right\|_{\dot{H}^\nu}^2 \mathrm{d}r\mathrm{d}s \\ &\le C(H,Q)\int_0^t s^{1-2H}\left(\int_s^t (t-r)^{2(2-\nu)\alpha_i-4}(r-s)^{2(2H-3+\gamma)}\mathrm{d}r\right)^{(1/2)}\left(\int_s^t r^{4H-2}\mathrm{d}r\right)^{(1/2)}\mathrm{d}s \\ &\le C(H,Q)\beta\left(2(2H-3+\gamma)+1,2(2-\nu)\alpha_i-3\right)^{(1/2)}\times\beta\left(2-2H,(2-\nu)\alpha_i+4H+\gamma-4\right)t^{(2-\nu)\alpha_i+2H+\gamma-3}, \end{aligned} \tag{49}$$

where $\beta(p,q)=\int_0^1 x^{p-1}(1-x)^{q-1}\mathrm{d}x$ is the standard Beta function, and we have used $t_2^\omega-t_1^\omega\le C(t_2-t_1)^\omega$ for $0\le\omega\le 1$, in the above derivation.

Finally, for I_3, applying Lemma 3 and expression (17), we have

$$\begin{aligned} I_3 &= 4\sum_{k=1}^{\infty}\int_0^t \mathbb{E}\left\|\int_s^t \lambda_k^{1/2}\sigma(s)\left[S_{\alpha_i}(t-r)-S_{\alpha_i}(t-s)\right]\frac{\partial K_H}{\partial r}(r,s)e_k\mathrm{d}r\right\|_{\dot{H}^\nu}^2\mathrm{d}s \\ &\le 4c_H^2\left(H-\frac{1}{2}\right)^2 Tr(Q)\widehat{\sigma}^2\int_0^t s^{1-2H}\int_0^t\left\|(r-s)^{2H-(2-\nu)\alpha_i}\gamma^{2H-1}(r-s)^{2H-3}\right\|_{\dot{H}^\nu}\mathrm{d}r\mathrm{d}s \\ &\le C(H,Q)\widehat{\sigma}^2\int_0^t s^{1-2H}(t-s)^{4H-(2-\nu)\alpha_i-1}\mathrm{d}s \\ &\le C(H,Q)\widehat{\sigma}^2\beta\left(2-2H,4H-(2-\nu)\alpha_i\right)t^{2H-(2-\nu)\alpha_i+1}. \end{aligned} \tag{50}$$

Then, when $\delta=\min\{(2-\nu)\alpha_i+4H-3,\ (2-\nu)\alpha_i+2H+\gamma-3,2H-(2-\nu)\alpha_i+1\}>0$ and $(1/4)<H<(1/2)$ and combining the above estimation inequalities of I_1, I_2, and I_3, we can obtain

$$\mathbb{E}\left\|\int_0^t (t-s)^{\alpha_i-1}E_{\alpha_i}(t-s)\sigma(s)\mathrm{d}B^H(s)\right\|_{\dot{H}^\nu}^2\le Ct^\delta<\infty, \tag{51}$$

where $C>0$ is a constant depending only on H,Q,ν,γ,α_i, and function $\sigma(s)$. □

Theorem 2. *Assume that* ($\mathscr{H}$.1) – ($\mathscr{H}$.2) *are satisfied and the matrix*

$$M_{\text{trice}}=\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4\end{pmatrix},\quad B_j\ge 0,\ j=1,2,3,4, \tag{52}$$

where

$$\begin{aligned} B_1 &= \sqrt{\frac{T^{(2-\nu)\alpha_1-1}a_{f_1}}{(2-\nu)\alpha_1-1}}, \\ B_2 &= \sqrt{\frac{T^{(2-\nu)\alpha_1-1}b_{f_1}}{(2-\nu)\alpha_1-1}}, \\ B_3 &= \sqrt{\frac{T^{(2-\nu)\alpha_2-1}a_{f_2}}{(2-\nu)\alpha_2-1}}, \\ B_4 &= \sqrt{\frac{T^{(2-\nu)\alpha_2-1}b_{f_2}}{(2-\nu)\alpha_2-1}}. \end{aligned} \tag{53}$$

If M converges to zero, then problem (1) has a unique solution.

Proof. We consider the operator $N:\mathscr{D}_T\times\mathscr{D}_T\longrightarrow\mathscr{D}_T\times\mathscr{D}_T$ defined by

$$N(x,y)=\left(N_1(x,y),N_2(x,y)\right),\ (x,y)\in\mathscr{D}_T\times\mathscr{D}_T, \tag{54}$$

where

$$N_1(x,y)=\begin{cases}\phi_1(t), & t\in[-r,0],\\ T_{\alpha_1}(t)\phi_1(0)+\int_0^t(t-s)^{\alpha_1-1}E_{\alpha_1}(t-s)f_1(s,x_s,y_s)\mathrm{d}s+\int_0^t(t-s)^{\alpha_1-1}E_{\alpha_1}(t-s)\sigma_1(s)\mathrm{d}B_1^H(s), & \mathbb{P}-\text{a.s}, t\in J,\end{cases}$$
$$N_2(x,y)=\begin{cases}\phi_2(t), & t\in[-r,0],\\ T_{\alpha_2}(t)\phi_2(0)+\int_0^t(t-s)^{\alpha_2-1}E_{\alpha_2}(t-s)f_2(s,x_s,y_s)\mathrm{d}s+\int_0^t(t-s)^{\alpha_2-1}E_{\alpha_2}(t-s)\sigma_2(s)\mathrm{d}B_2^H(s), & \mathbb{P}-\text{a.s}, \quad t\in J.\end{cases} \tag{55}$$

Now, we prove that $N(x,y)$ has a fixed point by Theorem 1. Indeed, let $(x,y),(\overline{x},\overline{y})\in\mathscr{D}_T\times\mathscr{D}_T$, and by using Lemma 1 and Hölder inequality, we obtain that

$$\begin{aligned}&\mathbb{E}\left\|N_1(x(t),y(t))-N_1(\overline{x}(t),\overline{y}(t))\right\|_{\dot{H}^\nu}^2\\&\le\int_0^t(t-s)^{2\alpha_1-2}\left\|S_{\alpha_1}(t-s)\right\|_{\dot{H}^\nu}^2\mathrm{d}s\mathbb{E}\int_0^t\left\|\left[f_1(s,x_s,y_s)-f_1(s,\overline{x}_s,\overline{y}_s)\right]\right\|_X^2\mathrm{d}s\\&\le\frac{t^{(2-\nu)\alpha_1-1}a_{f_1}}{(2-\nu)\alpha_1-1}\int_0^t\mathbb{E}\|x(s)-\overline{x}(s)\|_X^2\mathrm{d}s+\frac{t^{(2-\nu)\alpha_1-1}b_{f_1}}{(2-\nu)\alpha_1-1}\int_0^t\mathbb{E}\|y(s)-\overline{y}(s)\|_X^2\mathrm{d}s\\&\le\frac{t^{(2-\nu)\alpha_1-1}a_{f_1}}{(2-\nu)\alpha_1-1}\int_0^t\sup_{\tau\in J}\mathbb{E}\|x(\tau)-\overline{x}(\tau)\|_X^2\mathrm{d}s\\&\quad+\frac{t^{(2-\nu)\alpha_1-1}b_{f_1}}{(2-\nu)\alpha_1-1}\int_0^t\sup_{\tau\in J}\mathbb{E}\|y(\tau)-\overline{y}(\tau)\|_X^2\mathrm{d}s.\end{aligned} \tag{56}$$

Therefore, since $(x,y)=(\overline{x},\overline{y})$ over the interval $[-r,0]$, by taking supremum in the above inequality, we have

$$\left\|N_1(x,y)-N_1(\overline{x},\overline{y})\right\|_{\mathscr{D}_T}^2\le B_1^2\|x-\overline{x}\|_{\mathscr{D}_T}^2+B_2^2\|y-\overline{y}\|_{\mathscr{D}_T}^2, \tag{57}$$

where

$$B_1=\sqrt{\frac{T^{(2-\nu)\alpha_1-1}a_{f_1}}{(2-\nu)\alpha_1-1}},\qquad B_2=\sqrt{\frac{T^{(2-\nu)\alpha_1-1}b_{f_1}}{(2-\nu)\alpha_1-1}}. \tag{58}$$

Repeating the above process, we can also obtain

$$\begin{aligned}&\mathbb{E}\left\|N_2(x(t),y(t))-N_2(\overline{x}(t),\overline{y}(t))\right\|_{\dot{H}^\nu}^2\\&\le\frac{t^{(2-\nu)\alpha_2-1}a_{f_2}}{(2-\nu)\alpha_2-1}\int_0^t\sup_{\tau\in J}\mathbb{E}\|x(\tau)-\overline{x}(\tau)\|_X^2\mathrm{d}s\\&\quad+\frac{t^{(2-\nu)\alpha_2-1}b_{f_2}}{(2-\nu)\alpha_2-1}\int_0^t\sup_{\tau\in J}\mathbb{E}\|y(\tau)-\overline{y}(\tau)\|_X^2\mathrm{d}s.\end{aligned} \tag{59}$$

Thus,

$$\left\|N_2(x,y)-N_2(\overline{x},\overline{y})\right\|_{\mathscr{D}_T}^2\le B_3^2\|x-\overline{x}\|_{\mathscr{D}_T}^2+B_4^2\|y-\overline{y}\|_{\mathscr{D}_T}^2, \tag{60}$$

where

$$B_3=\sqrt{\frac{T^{(2-\nu)\alpha_2-1}a_{f_2}}{(2-\nu)\alpha_2-1}},\qquad B_2=\sqrt{\frac{T^{(2-\nu)\alpha_2-1}b_{f_2}}{(2-\nu)\alpha_2-1}}. \tag{61}$$

Hence,

$$\begin{aligned}\|N(x,y)-N(\overline{x},\overline{y})\|_{\mathscr{D}_T}&=\begin{pmatrix}\left\|N_1(x,y)-N_1(\overline{x},\overline{y})\right\|_{\mathscr{D}_T}\\\left\|N_2(x,y)-N_2(\overline{x},\overline{y})\right\|_{\mathscr{D}_T}\end{pmatrix}\\&\le\begin{pmatrix}B_1&B_2\\B_3&B_4\end{pmatrix}\begin{pmatrix}\|x-\overline{x}\|_{\mathscr{D}_T}\\\|y-\overline{y}\|_{\mathscr{D}_T}\end{pmatrix}.\end{aligned} \tag{62}$$

Therefore,

$$\|N(x,y)-N(\overline{x},\overline{y})\|_{\mathscr{D}_T} \le M_{\text{trice}} \begin{pmatrix} \|x-\overline{x}\|_{\mathscr{D}_T} \\ \|y-\overline{y}\|_{\mathscr{D}_T} \end{pmatrix}, \quad \text{for all, } (x,y), (\overline{x},\overline{y}) \in \mathscr{D}_T \times \mathscr{D}_T. \tag{63}$$

From Theorem 1, the mapping N has a unique fixed $(x,y) \in \mathscr{D}_T \times \mathscr{D}_T$ which is a unique solution of equation (1). □

Remark 1. Noticing that $B_1, B_2, B_3, B_4 \in \mathbb{R}^+$, if $B_1B_4 - B_2B_3 - (B_1 + B_4) > 0$, then

$$\begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}, \tag{64}$$

is convergent to zero.

4. Transportation Inequalities

In this section, we consider the property $T_2(C)$, for the law of the mild solution of equation (1), on the space $\mathbb{C} = C([0,T],X)$ endowed with the uniform metric d_∞. Precisely, we have the following theorem.

Theorem 3. *Assume that* $(H.1)$ *and* $(\mathscr{H}.2)$ *holds, and let* $\mathbb{P}_{\phi_1}, \mathbb{P}_{\phi_2}$ *be the law of* $x(\phi_1,\cdot), y(\phi_2,\cdot)$*, the solution process of equation (1). Using the metric*

$$\mathrm{d}_\infty(x,y) = \sup_{t\in J} \|x-y\|_X, \quad x,y \in C([0,T],X), \tag{65}$$

the probability measure $\mathbb{P}_{\phi_1}, \mathbb{P}_{\phi_2}$ *satisfies "*$T_2(C)$*" in the sense that*

$$\left[W_2^{d_\infty}\left(\mathbb{Q}_1, \mathbb{P}_{\phi_1}\right)\right]^2 + \left[W_2^{d_\infty}\left(\mathbb{Q}_1, \mathbb{P}_{\phi_2}\right)\right]^2 \le 2C\left[H_1\left(\mathbb{Q}_2 \mid \mathbb{P}_{\phi_1}\right) + H_1\left(\mathbb{Q}_2 \mid \mathbb{P}_{\phi_2}\right)\right], \tag{66}$$

on the metric space $C([0,T],X)$ *with the metric* d_∞.

Proof. Let $\mathbb{P}_{\phi_1}, \mathbb{P}_{\phi_2}$ be the law of $x(t,\phi_1), y(t,\phi_2), t \in [0,T]$ on $\mathscr{C} = C[(0,T),X]$ and $\mathbb{Q}_i$ be any probability measure on $\mathbb{C}$ such that $\mathbb{Q}_i \ll \mathbb{P}_{\phi_i}$. Define

$$\widetilde{\mathbb{Q}}_i := \frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}_{\phi_i}}(x(\cdot,\phi_i))\mathbb{P}, \tag{67}$$

which is a probability measure on $(\Omega, \mathscr{F})$. Recalling the definition of entropy and adopting a measure-transformation argument,

$$\begin{aligned} H(\widetilde{\mathbb{Q}} \mid \mathbb{P}) &= \begin{pmatrix} H_1\left(\widetilde{\mathbb{Q}}_2 \mid \mathbb{P}\right) \\ H_2\left(\widetilde{\mathbb{Q}}_2 \mid \mathbb{P}\right) \end{pmatrix}, \\ H\left(\mathbb{Q} \mid \mathbb{P}_\phi\right) &= \begin{pmatrix} H_1\left(\mathbb{Q}_1 \mid \mathbb{P}_{\phi_1}\right) \\ H_2\left(\mathbb{Q}_2 \mid \mathbb{P}_{\phi_2}\right) \end{pmatrix}, \\ H_i\left(\widetilde{\mathbb{Q}}_i \mid \mathbb{P}\right) &= \int_\Omega \log\left(\frac{\mathrm{d}\widetilde{\mathbb{Q}}_i}{\mathrm{d}\mathbb{P}}\right)\mathrm{d}\widetilde{\mathbb{Q}}_i \\ &= \int_\Omega \log\left(\frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}_\phi}(x(\cdot,\phi))\right)\frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}_{\phi_i}}(x(\cdot,\phi_i))\mathrm{d}\mathbb{P} \\ &= \int_{\mathscr{C}} \log\left(\frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}_\phi}\right)\frac{\mathrm{d}\mathbb{Q}_i}{\mathrm{d}\mathbb{P}_{\phi_i}}\mathrm{d}\mathbb{P}_{\phi_i} \\ &= H\left(\mathbb{Q}_i \mid \mathbb{P}_{\phi_i}\right), \quad i = 1,2. \end{aligned} \tag{68}$$

Following [25], then there exists a predictable process $h_1(t), h_2(t) \in X, \quad t \in J$ with

$$\int_0^{\mathrm{T}} \left\|h_i(s)\right\|_X^2 \mathrm{d}s < +\infty, \quad i = 1,2, \mathbb{P}-\text{a.s}, \tag{69}$$

such that

$$\begin{aligned} &\left(H_1\left(\mathbb{Q}_1 \mid \mathbb{P}_{\phi_1}\right), H_2\left(\mathbb{Q}_2 \mid \mathbb{P}_{\phi_2}\right)\right) \\ &\quad = \left(\frac{1}{2}\mathbb{E}_{\widetilde{\mathbb{Q}}_1}\int_0^{\mathrm{T}} \left\|h_1(s)\right\|_X^2 \mathrm{d}s, \frac{1}{2}\mathbb{E}_{\widetilde{\mathbb{Q}}_1}\int_0^{\mathrm{T}} \left\|h_2(s)\right\|_X^2 \mathrm{d}s\right). \end{aligned} \tag{70}$$

By the Girsanov theorem, the process $\widetilde{B}_1(t)$ and $\widetilde{B}_2(t)$ which are defined by

$$\begin{aligned} \widetilde{B}_1(t) &= B_1(t) - \int_0^t h_1(s)\mathrm{d}s, \\ \widetilde{B}_2(t) &= B_1(t) - \int_0^t h_2(s)\mathrm{d}s, \end{aligned} \tag{71}$$

are two Brownian motions with respect to $\{\mathscr{F}_t\}_{t\ge0}$ on the probability space $(\Omega, \mathscr{F}, \widetilde{\mathbb{Q}}_i)$. Let us consider the $\widetilde{\mathbb{Q}}_i$-fractional Brownian motion $\left\{\widetilde{B}_t^H(t)\right\}_{t\in J}$ defined by

$$\begin{aligned} \widetilde{B}_t^H(t) &= \int_0^t K_H(t,s)\mathrm{d}\widetilde{B}_i(s) \\ &= \int_0^t K_H(t,s)\mathrm{d}B_i(s) - \left(K_H h_i\right)(t), \quad i = 1,2, \end{aligned} \tag{72}$$

where $K_H h_i$ is defined by $(K_H h_i)(t) = \int_0^t K_H(t,s)h_i(s)\mathrm{d}s$. By the Fubini theorem, we obtain

$$
\begin{aligned}
(K_H h_i)(t) &= \int_0^t c_H \left[\left(\frac{t}{s}\right)^{H-(1/2)} (t-s)^{H-(1/2)} - \left(H - \frac{1}{2}\right) s^{(1/2)-H} \int_s^t u^{H-(3/2)} (u-s)^{H-(1/2)} \mathrm{d}u \right] h_i(s)\mathrm{d}s \\
&= \int_0^t c_H \left[\left(\frac{t}{u}\right)^{H-(1/2)} (t-u)^{H-(1/2)} h_i(u) - \left(H - \frac{1}{2}\right) u^{H-(3/2)} \int_0^u u^{(1/2)-H} (u-s)^{H-(1/2)} h_i(s)\mathrm{d}s \right] \mathrm{d}u \\
&:= \int_0^t g_i(u)\mathrm{d}u, \quad i = 1, 2.
\end{aligned} \tag{73}
$$

Consequently, under the measure $\widetilde{\mathbb{Q}}_i$, the process $\{u(t,\phi) = (x(t,\phi_1), y(t,\phi_2))\}_{t\in J}$ satisfies that

$$
\begin{aligned}
x(t) &= \begin{cases} \phi_1(t), & t \in [-r, 0], \\ T_{\alpha_1}(t)\phi_1(0) + \int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1}(t-s) f_1(s, x_s, y_s)\mathrm{d}s + \int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1}(t-s)\sigma_1(s)\mathrm{d}\widetilde{B}_1^H(s) & \\ \quad + \int_0^t (t-s)^{\alpha_1 - 1} E_{\alpha_1}(t-s)\sigma_1(s) g_1(s)\mathrm{d}s, & \mathbb{P} - \text{a.s. } t \in J, \end{cases} \\
y(t) &= \begin{cases} \phi_2(t), & t \in [-r, 0], \\ T_{\alpha_2}(t)\phi_2(0) + \int_0^t (t-s)^{\alpha_2 - 1} E_{\alpha_2}(t-s) f_2(s, x_s, y_s)\mathrm{d}s + \int_0^t (t-s)^{\alpha_2 - 1} E_{\alpha_2}(t-s)\sigma_2(s)\mathrm{d}\widetilde{B}_2^H(s) & \\ \quad + \int_0^t (t-s)^{\alpha_2 - 1} E_{\alpha_2}(t-s)\sigma_2(s) g_2(s)\mathrm{d}s, & \mathbb{P} - \text{a.s. } t \in J. \end{cases}
\end{aligned} \tag{74}
$$

We now consider the solution $(\overline{x}, \overline{y})$ (under $(\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2)$) of the following equation:

$$
\begin{cases}
{}^cD^{\alpha_1}(x(t)) = (A_1\overline{x}(t) + f_1(t, \overline{x}_t, \overline{y}_t))\mathrm{d}t + \sigma_1(t)\mathrm{d}\widetilde{B}_1^H(t), & t \in [0, T], \\
{}^cD^{\alpha_2}(y(t)) = (A_2\overline{y}(t) + f_2(t, \overline{x}_t, \overline{y}_t))\mathrm{d}t + \sigma_2(t)\mathrm{d}\widetilde{B}_2^H(t), & t \in [0, T], \\
x(t) = \phi_1(t), & t \in [-r, 0], \\
y(t) = \phi_2(t), & t \in [-r, 0].
\end{cases} \tag{75}
$$

By Theorem 2, under $\widetilde{\mathbb{Q}}_1, \widetilde{\mathbb{Q}}_2$, the law of $\overline{x}(t,\phi_1), \overline{y}(t,\phi_1), t \in J$ is $\mathbb{P}_{\phi_1}, \mathbb{P}_{\phi_2}$; therefore, $(x(t), \overline{x}(t)), (y(t), \overline{y}(t)), t \in [0, T]$ under $\widetilde{\mathbb{Q}}$ is a coupling of $(\mathbb{Q}, \mathbb{P}_\phi)$ and it follows that

$$
\left[W_2^{d_\infty}(\mathbb{Q}, \mathbb{P}_\phi)\right]^2 \le \begin{pmatrix} E_{\widetilde{\mathbb{Q}}_1}(d_\infty(x, \overline{x}))^2 \\ E_{\widetilde{\mathbb{Q}}_2}(d_\infty(y, \overline{y}))^2 \end{pmatrix} = \begin{pmatrix} E_{\widetilde{\mathbb{Q}}_1}\left(\sup_{t\in J}\|x(t) - \overline{x}(t)\|_X^2\right) \\ E_{\widetilde{\mathbb{Q}}_2}\left(\sup_{t\in J}\|y(t) - \overline{y}(t)\|_X^2\right) \end{pmatrix}, \tag{76}
$$

where we also use the Cauchy inequality

$$
(a+b)^2 \le 2a^2 + 2b^2. \tag{77}
$$

Now, we can use the result above to estimate the distance between u and $\overline{u}$ with respect to d_∞:

$$\begin{aligned}\|x(t)-\overline{x}(t)\|_X^2 &= \Big\|\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1}(t-s)\left(f_1(s,x_s,y_s)\right)-\left(f_1(s,\overline{x}_s,\overline{y}_s)\right)ds \\ &\quad+\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1}(t-s)\sigma_1(s)g_1(s)ds\Big\|_X^2 \\ &\le \Big\|\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1}(t-s)\left(f_1(s,x_s,y_s)\right)-\left(f_1(s,\overline{x}_s,\overline{y}_s)\right)ds\Big\|_X^2 \\ &\quad+\Big\|\int_0^t (t-s)^{\alpha_1-1}E_{\alpha_1}(t-s)\sigma_1(s)g_1(s)ds\Big\|_X^2 \\ &:= 2\left(J_1+J_2\right).\end{aligned} \tag{78}$$

By using the Hölder inequality, condition ($\mathscr{H}$.1), and Lemma 3, we obtain

$$\begin{aligned}J_1 &\le \int_0^t \left\|(t-s)^{\alpha_1-1}E_{\alpha_1}(t-s)\right\|_{\dot{H}^\nu}^2 ds\int_0^t \left\|\left(f_1(s,x_s,y_s)-f_1(s,x_s,y_s)\right)\right\|_X^2 ds \\ &\le \frac{t^{(2-\nu)\alpha_1-1}a_{f_1}}{(2-\nu)\alpha_1-1}\int_0^t \|x(s)-\overline{x}(s)\|_X^2 ds+\frac{t^{(2-\nu)\alpha_1-1}b_{f_1}}{(2-\nu)\alpha_1-1}\int_0^t \|y(s)-\overline{y}(s)\|_X^2 ds \\ &\le \frac{t^{(2-\nu)\alpha_1-1}a_{f_1}}{(2-\nu)\alpha_1-1}\int_0^t \sup_{\tau\in[0,s]}\|x(\tau)-\overline{x}(\tau)\|_X^2 ds+\frac{t^{(2-\nu)\alpha_1-1}b_{f_1}}{(2-\nu)\alpha_1-1}\int_0^t \sup_{\tau\in[0,s]}\|y(\tau)-\overline{y}(\tau)\|_X^2 ds.\end{aligned} \tag{79}$$

For the second term, using the Fubini theorem and Hölder inequality, we obtain

$$\begin{aligned}J_2 &\le 2c_H^2\widehat{\sigma}\left\|\int_0^t S_{\alpha_1}(t-s)\left(\frac{t}{s}\right)^{H-(1/2)}(t-s)^{H-(1/2)}h_1(s)ds\right\|^2 \\ &\quad+2\left(H-\frac{1}{2}\right)c_H^2\widehat{\sigma}\left(\left\|\int_0^t S_{\alpha_1}(t-s)s^{H-(3/2)}\int_0^s u^{(1/2)-H}(s-u)^{H-(1/2)}h_1(u)du ds\right\|^2\right) \\ &\le 2c_H^2\widehat{\sigma}\frac{t^{(2-\nu)\alpha_1+2H-2}}{(2-\nu)\alpha_1+2H-2}\int_0^t \left\|h_1(s)\right\|_X^2 ds+4(1-H)\left(H-\frac{1}{2}\right)^2 c_H^2\widehat{\sigma}\beta\left(2H,(2-\nu)\alpha_i-1\right) \\ &\quad\times\beta\left(2-2H,(2-\nu)\alpha_i+4H+4\right)t^{(2-\nu)\alpha_i+4H-3}\int_0^t \left\|h_1(u)\right\|_X^2 du \\ &:= C_1^*\int_0^t \left\|h_1(s)\right\|_X^2 ds.\end{aligned} \tag{80}$$

Combining (78)–(80), we have

$$\begin{aligned}\|x(s)-\overline{x}(s)\|_X^2 &\le \frac{t^{(2-\nu)\alpha_1-1}a_{f_1}}{(2-\nu)\alpha_1-1}\int_0^t \sup_{\tau\in J}\|x(\tau)-\overline{x}(\tau)\|_X^2 ds \\ &\quad+\frac{t^{(2-\nu)\alpha_1-1}b_{f_1}}{(2-\nu)\alpha_1-1}\int_0^t \sup_{\tau\in J}\|y(\tau)-\overline{y}(\tau)\|_X^2 ds \\ &\quad+C_1^*\int_0^t \left\|h_1(s)\right\|_X^2 ds.\end{aligned} \tag{81}$$

Similarly, we have

$$\|y(s)-\overline{y}(s)\|_X^2 \leq \frac{t^{(2-\nu)\alpha_1-1}a_{f_2}}{(2-\nu)\alpha_2-1}\int_0^t \sup_{\tau\in J}\|x(\tau)-\overline{x}(\tau)\|_X^2 ds + \frac{t^{(2-\nu)\alpha_1-1}b_{f_2}}{(2-\nu)\alpha_2-1}\int_0^t \sup_{\tau\in J}\|y(\tau)-\overline{y}(\tau)\|_X^2 ds + C_2^*\int_0^t \left\|h_2(s)\right\|_X^2 ds. \quad (82)$$

Adding (81) and (82), we obtain

$$\sup_{s\in J}\left(\|x(s)-\overline{x}(s)\|_X^2+\|y(s)-\overline{y}(s)\|_X^2\right) \leq \widehat{C}\int_0^t \sup_{\tau\in[0,s]}\left(\|x(\tau)-\overline{x}(\tau)\|_X^2+\|y(\tau)-\overline{y}(\tau)\|_X^2\right)ds + C^*\int_0^t \left\|h_1(s)\right\|_X^2+\left\|h_2(s)\right\|_X^2 ds, \quad (83)$$

where $\widehat{C}=\max\{2(t^{(2-\nu)\alpha_1-1}(a_{f_1}+a_{f_2})/(2-\nu)\alpha_1-1), 2(t^{(2-\nu)\alpha_1-1}(b_{f_1}+b_{f_2})/(2-\nu)\alpha_1-1)\}$ and $C^*=\max\{C_1^*, C_2^*\}$. By using the Gronwall inequality, we have

$$\sup_{s\in[0,t]}\left(\|x(s)-\overline{x}(s)\|_X^2+\|y(s)-\overline{y}(s)\|_X^2\right) \leq C^*\exp(\widehat{C}T)\int_0^T \left\|h_1(s)\right\|_X^2+\left\|h_2(s)\right\|_X^2 ds. \quad (84)$$

Hence, it follows that

$$\left[W_2^{d_\infty}\left(\mathbb{Q}_1,\mathrm{P}_{\phi_1}\right)\right]^2+\left[W_2^{d_\infty}\left(\mathbb{Q}_2,\mathrm{P}_{\phi_2}\right)\right]^2 \leq C^*\exp(\widehat{C}T)E_{\widetilde{\mathbb{Q}}_1}\left(\int_0^T \left\|h_1(s)\right\|_X^2+\left\|h_2(s)\right\|_X^2 ds\right) \leq 2C\left[H_1\left(\mathbb{Q}_1 \,\middle|\, p_{\phi_1}\right)+H_2\left(\mathbb{Q}_2 \,\middle|\, p_{\phi_2}\right)\right], \quad (85)$$

where $C=C^*\exp(\widehat{C}T)$. The proof is complete. □

Remark 2. In [15], by using the Girsanov theorem for fractional Brownian motion, the authors established the transportation inequalities for the law of the mild solution to stochastic evolution equations driven by the fractional Brownian motion with the Hurst parameter $H\in((1/2),1)$. Besides, Boufoussi and Hajji [16] established the transportation inequalities for the law of the mild solution to stochastic evolution equations driven by the fractional Brownian motion with the Hurst parameter $H\in(0,(1/2))$. However, the transportation inequalities for fractional stochastic evolution equations driven by the fractional Brownian motion are more complicated. On the contrary, for a coupled system, we have to consider the transportation inequalities for the law of the random vector $(X(t),Y(t))$, which is more difficult. So, our results generalize and improve the results in [15, 16].

5. An Example

In this section, we present an example to illustrate the usefulness and applicability of our results. We consider the following fractional stochastic partial differential equation with delay effects:

$$\begin{cases} {}^cD^{\alpha_1}(u(t,x)) = \dfrac{\partial^2}{\partial x^2}u(t,x)+\left(1-a_1u(t,x(t-\tau))(\sin t+\sin(\sqrt{2}t))\right) & \\ \quad -b_1v(t,x(t-\tau))(\cos t+\cos(\sqrt{2t}))+e^{-t}\left(dB^H/dt\right), & t\in[0,T], 0\leq x\leq\pi, \\ {}^cD^{\alpha_1}(u(t,x)) = \dfrac{\partial^2}{\partial x^2}u(t,x)+\left(1-a_2u(t,x(t-\tau))(\sin t+\sin(\sqrt{2}t))\right) & \\ \quad -b_2v((t,x(t-\tau))(\cos t+\cos(\sqrt{2t})))+e^{-t}\left(dB^H/dt\right), & t\in[0,T], 0\leq x\leq\pi, \\ u(t,0)=u(t,\pi)=0, & t\in[0,T], \\ v(t,0)=v(t,\pi)=0, & t\in[0,T], \\ u(t,x)=\phi_1(t,x), & t\in[-r,0], 0\leq x\leq\pi, \\ v(t,x)=\phi_2(t,x), & t\in[-r,0], 0\leq x\leq\pi, \end{cases} \quad (86)$$

where $a_i, b_i > 0, \alpha_i \in ((2/3), 1]$ and $\tau > 0, B^H$ denotes a fractional Brownian motion. To rewrite this system into the abstract form (1), we set

$$\begin{aligned} f_1(t, \phi_{1t}, \phi_{2t})(\eta) &= 1 - a_1(\phi_1(\eta_t)(\sin t + \sin(\sqrt{2}t))) \\ &\quad - b_1(\phi_2(\eta_t)(\cos t + \cos(\sqrt{2t}))), \\ f_2(t, \phi_{1t}, \phi_{2t})(\eta) &= 1 - a_2(\phi_1(\eta_t)(\sin t + \sin(\sqrt{2}t))) \\ &\quad - b_2(\phi_1(\eta_t)(\cos t + \cos(\sqrt{2}t))), \\ \sigma_1(t) &= e^{-t}, \\ \sigma_2(t) &= 2e^{-t}, \end{aligned} \tag{87}$$

and $\mathscr{K} = \mathscr{H} = L^2([0, \pi])$. We denote the operator A by $Au = u''$, with domain $D(A) = \{u \in \mathscr{H}, u'' \in \mathscr{H}$ and $u(0) = u(\pi) = 0\}$.

Then, it is easy to obtain

$$Az = -\sum_{n=1}^{\infty} e^{-n^2 t}\langle z, e_n\rangle e_n, \quad z \in \mathscr{H}, \tag{88}$$

and A is the infinitesimal generator of an analytic semigroup $\{S(t)\}_{t\geq 0}$ on $\mathscr{H}$, which has following the following formula:

$$S(t)u = \sum_{n=1}^{\infty} e^{-n^2 t}\langle u, e_n\rangle e_n, \quad u \in \mathscr{H}, \tag{89}$$

where $e_n(u) = (2/\pi)^{(1/2)}\sin(nu), n = 1, 2, \ldots,$ is the orthogonal set of eigenvectors of A. If the analytic semigroup $\{S(t)\}, t \in J$, is compact, then there exists a constant $K \geq 1$ such that $\|S(t)\|^2 \leq K$.

In order to define the operator Q: $\mathscr{K} \longrightarrow \mathscr{K}$, we choose a sequence $\{Qe_n = \sigma_n e_n\}$ and assume that

$$tr(Q) = \sum_{n=1}^{\infty} \sqrt{\sigma_n} < \infty. \tag{90}$$

Define the process $B^H(s)$ by

$$B^H = \sum_{n=1}^{\infty} \sqrt{\sigma_n} B_n^H(t) e_n, \tag{91}$$

where $1/4 < H < 1/2$ and $\{B_n^H\}_{n\in\mathbb{N}}$ is a sequence of two-sided one-dimensional mutually independent fractional Brownian motions. Thus, one has

$$\begin{aligned} \left\|f_1(t, x, y) - f_2(t, \overline{x}, \overline{y})\right\|^2 &\leq 8a_1\|x - \overline{x}\|_{\mathscr{D}_0} + 8b_1\|y - \overline{y}\|_{\mathscr{D}_0}, \\ \left\|f_2(t, x, y) - f_2(t, \overline{x}, \overline{y})\right\|^2 &\leq 8a_2\|x - \overline{x}\|_{\mathscr{D}_0} + 8b_2\|y - \overline{y}\|_{\mathscr{D}_0}. \end{aligned} \tag{92}$$

On account of the conditions, it is straightforward to check that ($\mathscr{H}$.1) and ($\mathscr{H}$.2) hold. Let

$$M = 2\sqrt{2}\begin{pmatrix} \sqrt{\dfrac{T^{(2-\nu)_{\alpha_1}-1}a_1}{(2-\nu)\alpha_1 - 1}} & \sqrt{\dfrac{T^{(2-\nu)_{\alpha_1}-1}b_1}{(2-\nu)\alpha_1 - 1}} \\ \sqrt{\dfrac{T^{(2-\nu)_{\alpha_2}-1}a_2}{(2-\nu)\alpha_2 - 1}} & \sqrt{\dfrac{T^{(2-\nu)_{\alpha_2}-1}b_2}{(2-\nu)\alpha_1 - 1}} \end{pmatrix}. \tag{93}$$

If M converges to zero, then assumptions in Theorem 2 are fulfilled, then we can conclude that the law of the unique mild solution of system (86) on $[0, T]$ satisfies the property $T_2(C)$.

6. Conclusion

In this paper, by Perov's fixed-point theorem, some stochastic analysis technique, and the properties of operator semigroup, we show the existence and uniqueness of the mild solution for a class of coupled fractional stochastic evolution equations driven by the fractional Brownian motion with the Hurst parameter $H \in (1/4, 1/2)$. Furthermore, we establish the transportation inequalities for the law of the mild solution, with respect to the uniform distance. In our next paper, we will explore the existence, uniqueness, and the transportation inequalities of the mild solution for a class of coupled fractional stochastic evolution equations driven by the fractional Brownian motion with the Hurst parameter $H \in (0, 1/4)$.

Data Availability

The data used to support the findings of the study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All authors contributed equally to this work and read and approved the final manuscript.

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