Hindawi
Discrete Dynamics in Nature and Society
Volume 2020, Article ID 8634050, 11 pages
https://doi.org/10.1155/2020/8634050

Research Article

Approximation of the Fixed Point of Multivalued Quasi-Nonexpansive Mappings via a Faster Iterative Process with Applications

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Received 24 January 2020; Accepted 24 February 2020; Published 20 March 2020

Academic Editor: Douglas R. Anderson

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In this paper, we approximate the fixed points of multivalued quasi-nonexpansive mappings via a faster iterative process and propose a faster fixed-point iterative method for finding the solution of two-point boundary value problems. We prove analytically and with series of numerical experiments that the Picard–Ishikawa hybrid iterative process has the same rate of convergence as the CR iterative process.

1. Introduction

If the existence of the solution of a fixed-point equation involving an operator T is guaranteed, but an exact solution is not possible, then the requirement of approximating the solution becomes very pertinent. This gives rise to the need of different iterative processes [1–3]. In view of theoretical and practical significance of fixed-point iterative schemes, several authors have constructed and applied different fixed-point iteration schemes in approximating the solution of equations which model certain physical problems (e.g., [2–15]). One of the most important criteria in preferring one fixed-point iteration scheme over the other is the rate of convergence of the iteration scheme. Consequently, a faster fixed-point iteration scheme is always preferred in practice.

In 2018, Bello et al. [16] developed a Mann-type fixed-point iteration scheme for approximating the solution of two-point boundary value problems. It is worth mentioning that the scheme proposed in [16] is a self-correcting, unlike the variational or weighted residual methods of approximation which depend on the selection of suitable coordinate or basis functions (e.g., [16, 17]). They established that the proposed fixed-point iteration method is more suitable to approximate the exact solution than other existing methods. Moreover, a noticeable advantage of this method is that it solves the boundary value problem without constructing Green's function which is always difficult to construct for some problems [16].

The purpose of this paper is to introduce a new faster iterative scheme to approximate the solution of a fixed-point inclusion which involves a multivalued quasi-nonexpansive mapping. A study of a faster fixed-point iterative method for finding the solution of two-point boundary value problems is also carried out. With a series of numerical experiments and an analytical proof, it is established that the Picard–Ishikawa hybrid iterative process, recently introduced by Okeke [3], has the same rate of convergence as the CR iterative process, introduced by Chugh et al. [8]. Our results improve, extend, and generalize several known results in the literature [18–20].

2. Preliminaries

Let X be a Banach space, $x \in X$, and $A \subseteq X$. Define $d(x, A) = \inf\{d(x, y) : y \in A\}$.

Definition 1 (see [20]). Suppose X is a real Banach space. A subset C is called proximinal if for all $x \in X$ there exists an element $c \in C$ such that $d(x, c) = \inf\{\|x - y\| : y \in C\} = d(x, C)$.

It is well known that weakly compact convex subsets of a Banach space X and closed convex subsets of a uniformly convex Banach space are proximinal. Let $N(X)$, $CL(X)$, $CB(X)$, $K(X)$, and 2^X denote the class of all nonvoid subsets, nonvoid closed subsets, nonvoid bounded and closed subsets, nonvoid compact subsets, and subsets of X, respectively. Let $\mathscr{P}(\mathscr{A})$ be the class of all nonvoid proximinal bounded subsets of A. Suppose that H is the generalized Hausdorff metric on $CB(X)$ which is defined as follows:

$$H(A, B) = \max\left\{\sup_{x\in A} d(x, B), \sup_{y\in B} d(y, A)\right\}, \tag{1}$$

for each $A, B \in CB(X)$. We say that a point $x^* \in X$ is a fixed point of $T: X \longrightarrow CL(X)$ *if* $x^* \in Tx^*$. We denote by $F(T) := \{x^* \in X : x^* \in Tx^*\}$ the set of all the fixed points of T and $P_T(x) = \{y \in Tx : \| x - y\| = d(x, Tx)\}$ for a multivalued mapping $T : K \longrightarrow \mathscr{P}(K)$. It is known that multivalued fixed-point theory has applications in economics, differential inclusion, optimization, and control theory [20, 21]. The theory of multivalued mappings is harder than the corresponding theory of single-valued mappings. For further studies of multivalued fixed-point theory, interested reader should see, e.g., [20–25] and the references therein.

The following definitions will be needed in the sequel.

Definition 2. A multivalued mapping $T: A \subseteq X \longrightarrow 2^X$ is called

(i) A contraction if there exists $a \in [0, 1)$ such that

$$H(Tx, Ty) \leq a\|x - y\|, \tag{2}$$

holds for all $x, y \in A$. If $a > 0$, then we say that T is Lipschitzian.

(ii) Nonexpansive if for all $x, y \in A$, we have

$$H(Tx, Ty) \leq \|x - y\|. \tag{3}$$

(iii) Quasi-nonexpansive if for each $x \in A$ and $x^* \in F(T) \neq \emptyset$, we have

$$H(Tx, x^*) \leq \|x - x^*\|. \tag{4}$$

Definition 3 (see [26]). A Banach space X is said to satisfy Opial's condition if for each sequence $\{u_n\}$ in X such that $u_n \rightharpoonup u$ implies that

$$\limsup_{n\longrightarrow\infty}\|u_n - u\| < \limsup_{n\longrightarrow\infty}\|u_n - v\|, \tag{5}$$

for each $v \in X$ with $v \neq u$.

Definition 4. A multivalued mapping $T: C \longrightarrow CB(C)$ is said to be demiclosed at $y \in C$ if for each sequence $\{x_n\}$ in C weakly converges to x and $y_n \in Tx_n$ strongly converges to y, we have $y \in Tx$.

In 2012, Chugh et al. [8] introduced the so-called CR iterative scheme and proved that the iterative scheme converges faster than all of Picard [15], Mann [11], Ishikawa [9], Agarwal et al. [5], Noor [12], and SP [14] iterative schemes. The CR iterative process is given as follows.

Suppose X is a Banach space, T: $X \longrightarrow X$, and $u_0 \in X$. Define the sequence $\{u_n\}_{n=0}^{\infty}$ by

$$\begin{cases} z_n = (1 - \gamma_n)u_n + \gamma_n T u_n, \\ y_n = (1 - \beta_n)Tu_n + \beta_n T z_n, \\ u_{n+1} = (1 - \alpha_n)y_n + \alpha_n T y_n, \\ n \in \mathbb{N}, \end{cases} \tag{6}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences of positive numbers in $[0, 1]$ satisfying $\sum_{n=0}^{\infty}\alpha_n = \infty$. Recently, Okeke [3] introduced the Picard–Ishikawa hybrid iterative process. The author proved that this iterative process converges faster than all of Picard [15], Krasnosel'skii [10], Mann [11], Ishikawa [9], Noor [12], Picard-Mann [27], and Picard–Krasnosel'skii [2] iterative processes in the sense of Berinde [1].

Motivated by the investigation of iterative scheme in [3], we introduce the following multivalued fixed-point iterative process. Suppose X is a Banach space, $T: X \longrightarrow 2^X$, and $x_0 \in X$. Define the sequence $\{x_n\}_{n=0}^{\infty}$ by

$$\begin{cases} z_n = (1 - \beta_n)x_n + \beta_n u_n, \\ y_n = (1 - \alpha_n)x_n + \alpha_n v_n, \\ x_{n+1} = w_n, \\ n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in $[0, 1]$ satisfying appropriate conditions, $u_n \in P_T(x_n)$, $v_n \in P_T(z_n)$, and $w_n \in P_T(y_n)$.

Remark 1. We remark that iteration (7) is the multivalued version of the Picard–Ishikawa hybrid iterative process, recently introduced by Okeke [3].

The multivalued version of the CR iterative process (6) is given as follows:

$$\begin{cases} z_n = (1 - \gamma_n)u_n + \gamma_n g_n, \\ y_n = (1 - \beta_n)g_n + \beta_n h_n, \\ u_{n+1} = (1 - \alpha_n)y_n + \alpha_n s_n, \\ n \in \mathbb{N}, \end{cases} \tag{8}$$

where $g_n \in P_T(u_n)$, $h_n \in P_T(z_n)$, and $s_n \in P_T(y_n)$.

The aim of this paper is to approximate the fixed point of multivalued quasi-nonexpansive mappings via the newly introduced fixed-point iteration scheme (7). We prove that the Picard–Ishikawa hybrid iterative process (7) has the same rate of convergence as the CR iterative process (6) for a certain class of quasi-nonexpansive and contraction mappings.

Suppose that $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$ are two fixed-point iteration processes that converge to a certain fixed point $x^* F(T)$ of a mapping T; we say that $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ ([28]) if

$$\left\|u_n - x^*\right\| \leq \left\|v_n - x^*\right\|, \quad \forall n \in \mathbb{N}. \tag{9}$$

The following definitions are due to Berinde [1].

Definition 5 (see [1]). Let $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ be two sequences of positive numbers that converge to a and b, respectively. Assume there exists

$$l = \lim_{n \longrightarrow \infty} \frac{\left|a_n - a\right|}{\left|b_n - b\right|}. \tag{10}$$

(i) If $l = 0$, then it is said that the sequence $\{a_n\}_{n=0}^{\infty}$ converges to a faster than the sequence $\{b_n\}_{n=0}^{\infty}$ to b

(ii) If $0 < l < \infty$, then we say that the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ have the same rate of convergence.

Definition 6 (see [1]). Suppose that, for two fixed-point iterative processes $\{u_n\}_{n=0}^{\infty}$ and $\{v_n\}_{n=0}^{\infty}$, both converging to the same fixed point x^*, the error estimates

$$\begin{aligned} \left\|u_n - x^*\right\| &\leq a_n, \quad \text{for all } n \in \mathbb{N}, \\ \left\|v_n - x^*\right\| &\leq b_n, \quad \text{for all } n \in \mathbb{N}, \end{aligned} \tag{11}$$

are available, where $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are two sequences of positive numbers converging to zero. If $\{a_n\}_{n=0}^{\infty}$ converges faster than $\{b_n\}_{n=0}^{\infty}$, then $\{u_n\}_{n=0}^{\infty}$ converges faster than $\{v_n\}_{n=0}^{\infty}$ to x^*.

For the rest of this paper, whenever we make reference to the rate of convergence of iterative processes, we mean the rate of convergence in the sense of Berinde [1] as in Definition 5.

Definition 7 (see [29]). A multivalued nonexpansive mapping $T: C \longrightarrow CB(C)$ is said to satisfy Condition (I) if there exists a continuous nondecreasing function $f: [0, \infty) \longrightarrow [0, \infty)$ with $f(0) = 0$ and $f(r) > 0$ for all $r \in (0, \infty)$ such that $d(x, Tx) \geq f(d(x, F(T)))$ for each $x \in C$.

The following lemma will be needed in this study.

Lemma 1 (see [30]). *Suppose that X is a uniformly convex Banach space and $0 < p \leq t_n \leq q < 1$ for each $n \in \mathbb{N}$. Let $\{x_n\}$ and $\{y_n\}$ be two sequences of X such that $\limsup_{n\longrightarrow\infty}\|x_n\| \leq r$, $\limsup_{n\longrightarrow\infty}\|y_n\| \leq r$, and $\lim_{n\longrightarrow\infty}\|t_n x_n + (1 - t_n)y_n\| = r$ hold for some $r \geq 0$. Then, $\lim_{n\longrightarrow\infty}\|x_n - y_n\| = 0$.*

Lemma 2 (see [31]). *If $T: C \longrightarrow \mathcal{P}(\mathcal{C})$ and $P_T(x) = \{y \in Tx: \|x - y\| = d(x, Tx)\}$. Then, the following are equivalent:*

(i) $x \in F(T)$

(ii) $P_T(x) = \{x\}$

(iii) $x \in F(P_T)$

Moreover, $F(T) = F(P_T)$.

3. Convergence Analysis of the Multivalued Picard–Ishikawa Hybrid Iterative Process

We begin with the following convergence results.

Lemma 3. *Suppose that X is a normed linear space and C is a nonvoid closed convex subset of X. Let $T: C \longrightarrow \mathcal{P}(\mathcal{C})$ such that $F(T) \neq \emptyset$ and P_T is a quasi-nonexpansive mapping. Suppose that $\{x_n\}$ is a sequence defined by the iteration scheme (7). Then, $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists for each $x^* \in F(T)$ and $\lim_{n\longrightarrow\infty} d(x_n, P_T(x_n)) = 0$.*

Proof. To prove that $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists for each $x^* \in F(T) \neq \emptyset$, we proceed as follows. Using (4) and (7), we obtain the following estimates:

$$\begin{aligned} \left\|x_{n+1} - x^*\right\| &= \left\|w_n - x^*\right\| \\ &\leq H\left(P_T\left(y_n\right), P_T\left(x^*\right)\right) \\ &\leq \left\|y_n - x^*\right\|. \end{aligned} \tag{12}$$

Next, by (4) and (7), we have the following estimates:

$$\begin{aligned} \left\|y_n - x^*\right\| &= \left\|(1 - \alpha_n)x_n + \alpha_n v_n - x^*\right\| \\ &\leq (1 - \alpha_n)\left\|x_n - x^*\right\| + \alpha_n\left\|v_n - x^*\right\| \\ &\leq (1 - \alpha_n)\left\|x_n - x^*\right\| + \alpha_n H\left(P_T\left(z_n\right), P_T\left(x^*\right)\right) \\ &\leq (1 - \alpha_n)\left\|x_n - x^*\right\| + \alpha_n\left\|z_n - x^*\right\|. \end{aligned} \tag{13}$$

Next, we obtain that

$$\begin{aligned} \left\|z_n - x^*\right\| &= \left\|(1 - \beta_n)x_n + \beta_n u_n - x^*\right\| \\ &\leq (1 - \beta_n)\left\|x_n - x^*\right\| + \beta_n\left\|u_n - x^*\right\| \\ &\leq (1 - \beta_n)\left\|x_n - x^*\right\| + \beta_n H\left(P_T\left(x_n\right), P_T\left(x^*\right)\right) \\ &\leq (1 - \beta_n)\left\|x_n - x^*\right\| + \beta_n\left\|x_n - x^*\right\| \\ &= \left\|x_n - x^*\right\|. \end{aligned} \tag{14}$$

Using (14) in (13), we have

$$\begin{aligned} \left\|y_n - x^*\right\| &\leq (1 - \alpha_n)\left\|x_n - x^*\right\| + \alpha_n\left\|x_n - x^*\right\| \\ &= \left\|x_n - x^*\right\|. \end{aligned} \tag{15}$$

Using (14) in (12), we have

$$\left\|x_{n+1} - x^*\right\| \leq \left\|x_n - x^*\right\|. \tag{16}$$

This implies that $\{\|x_n - x^*\|\}$ is decreasing. Therefore, $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists for each $x^* \in F(T)$.

Next, we show that

$$\lim_{n\longrightarrow\infty} d\left(x_n, P_T\left(x_n\right)\right) = 0. \tag{17}$$

Let

$$\lim_{n\longrightarrow\infty}\left\|x_n - x^*\right\| = k, \tag{18}$$

for some constant $k \geq 0$. If $k = 0$, then (17) holds trivially. Suppose that $k > 0$; because $d(x_n, P_T(x_n)) \leq \|x_n - u_n\|$, it suffices to establish that $\lim_{n\longrightarrow\infty}\|x_n - u_n\| = 0$. Clearly, we have

$$\begin{aligned}\left\|u_n - x^*\right\| &\leq H\left(P_T\left(x_n\right), P_T\left(x^*\right)\right)\\ &\leq \left\|x_n - x^*\right\|.\end{aligned} \tag{19}$$

This implies that

$$\limsup_{n\longrightarrow\infty}\left\|u_n - x^*\right\| \leq k. \tag{20}$$

Using (14) and (15), we obtain

$$\begin{aligned}&\limsup_{n\longrightarrow\infty}\left\|z_n - x^*\right\| \leq k,\\ &\limsup_{n\longrightarrow\infty}\left\|y_n - x^*\right\| \leq k.\end{aligned} \tag{21}$$

Observe that

$$\begin{aligned}\left\|v_n - x^*\right\| &\leq H\left(P_T\left(z_n\right), P_T\left(x^*\right)\right)\\ &\leq \left\|z_n - x^*\right\|\\ &\leq \left\|x_n - x^*\right\|.\end{aligned} \tag{22}$$

Hence, we have

$$\limsup_{n\longrightarrow\infty}\left\|v_n - x^*\right\| \leq k. \tag{23}$$

Similarly, we have

$$\begin{aligned}\left\|w_n - x^*\right\| &\leq H\left(P_T\left(y_n\right), P_T\left(x^*\right)\right)\\ &\leq \left\|y_n - x^*\right\|\\ &\leq \left\|x_n - x^*\right\|.\end{aligned} \tag{24}$$

Hence,

$$\limsup_{n\longrightarrow\infty}\left\|w_n - x^*\right\| \leq k. \tag{25}$$

Clearly,

$$\lim_{n\longrightarrow\infty}\left\|x_{n+1} - x^*\right\| = \lim_{n\longrightarrow\infty}\left\|w_n - x^*\right\| = k. \tag{26}$$

Hence, using (23), (25), (26), and Lemma 1, we obtain

$$\lim_{n\longrightarrow\infty}\left\|v_n - w_n\right\| = 0. \tag{27}$$

From (25), we have

$$\liminf_{n\longrightarrow\infty}\left\|x_{n+1} - x^*\right\| \leq \liminf_{n\longrightarrow\infty}\left(\left\|w_n - v_n\right\| + \left\|v_n - x^*\right\|\right). \tag{28}$$

Hence, we have

$$k \leq \liminf_{n\longrightarrow\infty}\left\|v_n - x^*\right\|. \tag{29}$$

Using (23) and (29), we have

$$\lim_{n\longrightarrow\infty}\left\|v_n - x^*\right\| = k. \tag{30}$$

Similarly, we can show that

$$\lim_{n\longrightarrow\infty}\left\|z_n - x^*\right\| = k. \tag{31}$$

This means that

$$\begin{aligned}\lim_{n\longrightarrow\infty}\left\|z_n - x^*\right\| &= \lim_{n\longrightarrow\infty}\left\|(1-\beta_n)x_n + \beta_n u_n - x^*\right\|\\ &= \lim_{n\longrightarrow\infty}\left\|(1-\beta_n)\left(x_n - x^*\right)\right.\\ &\quad\left. + \beta_n\left(u_n - x^*\right)\right\|\\ &= k.\end{aligned} \tag{32}$$

It follows from (20), (32), and Lemma 1 that

$$\lim_{n\longrightarrow\infty}\left\|x_n - u_n\right\| = 0. \tag{33}$$

Therefore, we have

$$\lim_{n\longrightarrow\infty} d\left(x_n, P_T\left(x_n\right)\right) = 0. \tag{34}$$

The proof of Lemma 3 is completed.

Next, we prove the following strong convergence theorem. □

Theorem 1. *Suppose that C is a nonvoid compact convex subset of a real Banach space X, $T: C \longrightarrow \mathscr{P}(\mathscr{C})$ with $F(T) \neq \emptyset$, and P_T is a quasi-nonexpansive mapping. Let $\{x_n\}$ be the iterative sequence defined in (7). Then, the sequence $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. We have already established in Lemma 3 that $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists for each $x^* \in F(T) \neq \emptyset$. Because C is compact, it follows that there exists a subsequence $\left\{x_{n_j}\right\}$ of $\{x_n\}$ such that $\lim_{n\longrightarrow\infty}\|x_{n_j} - y^*\| = 0$ for some $y^* \in C$. Hence, by Lemma 3 and the triangle inequality, we have

$$\begin{aligned}d\left(y^*, P_T\left(y^*\right)\right) &\leq d\left(x_{n_j}, y^*\right) + d\left(x_{n_j}, P_T\left(x_{n_j}\right)\right)\\ &\quad + H\left(P_T\left(x_{n_j}\right), P_T\left(y^*\right)\right)\\ &\leq \left\|x_{n_j} - y^*\right\| + \left\|x_{n_j} - u_{n_j}\right\|\\ &\quad + \left\|x_{n_j} - y^*\right\| \longrightarrow 0, \quad \text{as } n \longrightarrow \infty.\end{aligned} \tag{35}$$

This means that

$$d\left(y^*, P_T\left(y^*\right)\right) = 0. \tag{36}$$

Therefore, y^* is a fixed point of P_T. By Lemma 2, the set of fixed points of P_T coincides with the set of fixed points of T, so the sequence $\{x_n\}$ converges strongly to a fixed point $y^* \in F(T)$. The proof of Theorem 1 is completed.

We next prove the following results for multivalued mappings satisfying Condition (I) of Senter and Dotson [29]. □

Theorem 2. *Suppose that C is a nonvoid closed and convex subset of a real Banach space X. Let $T: C \longrightarrow \mathscr{P}(\mathscr{C})$ satisfies Condition (I) such that $F(T) \neq \emptyset$ and P_T is a quasi-nonexpansive mapping. Then, the sequence $\{x_n\}$ defined by (7) converges strongly to a fixed point x^* of T.*

Proof. By Lemma 3, $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists for each $x^* \in F(T) = F(P_T)$(Lemma 2). If $\lim_{n\longrightarrow\infty}\|x_n - x^*\| = 0$, then the result trivially holds. We assume that $\lim_{n\longrightarrow\infty}\|x_n - x^*\| = k > 0$. Using relation (16), we have $\|x_{n+1} - x^*\| \leq \| x_n - x^*\|$. Therefore, we have

$$d(x_{n+1}, F(T)) \leq d(x_n, F(T)). \tag{37}$$

This implies that $\lim_{n\longrightarrow\infty}\mathrm{d}(x_{n+1}, F(T))$ exists. Next, we show that $\lim_{n\longrightarrow\infty}\mathrm{d}(x_{n+1}, F(T)) = 0$. Assume on contrary that $\lim_{n\longrightarrow\infty}d(x_{n+1}, F(T)) = \mu > 0$. Then, for each $n \in \mathbb{N}$, we have

$$\begin{aligned} p_n &= \frac{u_n - x^*}{\|x_n - x^*\|}, \\ q_n &= \frac{x_n - x^*}{\|x_n - x^*\|}. \end{aligned} \tag{38}$$

Hence, we have $\|p_n\| = 1$ and $\|q_n\| \leq 1$ because $\|u_n - x^*\| \leq H(P_T(x_n), P_T(x^*)) \leq \|x_n - x^*\|$. Using the fact that f satisfies Condition (I), we have

$$\begin{aligned} \|q_n - p_n\| &= \left\| \frac{x_n - x^*}{\|x_n - x^*\|} - \frac{u_n - x^*}{\|x_n - x^*\|} \right\| \\ &= \frac{\|x_n - u_n\|}{\|x_n - x^*\|} \\ &\geq \frac{d(x_n, Tx_n)}{\|x_n - x^*\|} \\ &\geq \frac{f(d(x_n, F(T))}{\|x_n - x^*\|}. \end{aligned} \tag{39}$$

Because f is continuous, we obtain that $\liminf_n\|q_n - p_n\| \geq (f(\mu)/k) > 0$ for each $n \in \mathbb{N}$. It follows from (7), (18), and (31) that

$$\begin{aligned} \lim_{n\longrightarrow\infty}\|(1-\beta_n)q_n + \beta_n p_n\| &= \lim_{n\longrightarrow\infty}\left\|(1-\beta_n)\frac{x_n - x^*}{\|x_n - x^*\|} + \beta_n\frac{u_n - x^*}{\|x_n - x^*\|}\right\| \\ &= \lim_{n\longrightarrow\infty}\left\|\frac{(1-\beta_n)x_n + \beta_n u_n - x^*}{\|x_n - x^*\|}\right\| \\ &= \lim_{n\longrightarrow\infty}\left\{\frac{\|z_n - x^*\|}{\|x_n - x^*\|}\right\} \\ &= \frac{k}{k} \\ &= 1. \end{aligned} \tag{40}$$

This implies that

$$\lim_{n\longrightarrow\infty}\|(1-\beta_n)q_n + \beta_n p_n\| = 1. \tag{41}$$

Therefore, by Lemma 1, we have $\lim_{n\longrightarrow\infty}\|q_n - p_n\| = 0$, a contradiction. Therefore, we have $\lim_{n\longrightarrow\infty}d(x_{n+1}, F(T)) = 0$, and hence,

$$\lim_{n\longrightarrow\infty}\|x_n - x^*\| = 0. \tag{42}$$

Thus, the sequence $\{x_n\}$ converges strongly to a fixed point $x^* \in F(T) \neq \emptyset$. The proof of Theorem 2 is completed.

Next, we prove the following weak convergence result. □

Theorem 3. *Suppose X is a uniformly convex Banach space satisfying Opial's condition and C is a nonvoid closed convex subset of X. Let $T: C \longrightarrow \mathscr{P}(\mathscr{C})$ be a multivalued mapping such that $F(T) \neq \emptyset$ and P_T is a quasi-nonexpansive mapping. Suppose that $\{x_n\}$ is the sequence defined in (7). If $I - P_T$ is demiclosed at zero. Then, the sequence $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Suppose that $x^* \in F(T) = F(P_T)$. It follows from Lemma 3 that $\lim_{n\longrightarrow\infty}\|x_n - x^*\|$ exists for each x^*. Next, we show that the sequence $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. We proceed as follows: let y_1^* and y_2^* be weak limits of the subsequences $\{x_{n_k}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$, respectively. By (33), we have that $\lim_{n\longrightarrow\infty}\|x_n - u_n\| = 0$. Using the fact that the mapping $I - P_T$ is demiclosed at zero, we have $y_1^* \in F(P_T) = F(T)$. Similarly, we can show that $y_2^* \in F(T)$. Next, we prove that the weak limit is unique. Suppose that $y_1^* \neq y_2^*$. Because X satisfies Opial's condition, we have

$$\begin{aligned}\lim_{n\longrightarrow\infty}\left\|x_n-y_1^*\right\| &= \lim_{n_k\longrightarrow\infty}\left\|x_{n_k}-y_1^*\right\| \\ &< \lim_{n_k\longrightarrow\infty}\left\|x_{n_k}-y_2^*\right\| \\ &= \lim_{n\longrightarrow\infty}\left\|x_n-y_2^*\right\| \\ &= \lim_{n_j\longrightarrow\infty}\left\|x_{n_j}-y_2^*\right\| \\ &< \lim_{n_j\longrightarrow\infty}\left\|x_{n_j}-y_1^*\right\| \\ &= \lim_{n\longrightarrow\infty}\left\|x_n-y_1^*\right\|,\end{aligned} \tag{43}$$

a contradiction. Hence, the sequence $\{x_n\}$ converges weakly to a point of $F(T)$. The proof of Theorem 3 is completed.

Next, we prove that the Picard–Ishikawa hybrid iterative process (7) and the CR iterative process (8) have the same rate of convergence. □

Proposition 1. *Suppose that X is a normed linear space and C is a nonvoid closed convex subset of X. Let $T: C \longrightarrow \mathscr{P}(\mathscr{C})$ such that $F(T) \neq \emptyset$ and P_T is a quasi-nonexpansive mapping. Suppose $x_0 = u_0 \in C$, $\{x_n\}$ is a sequence defined by the Picard–Ishikawa hybrid iterative process (7), and $\{u_n\}$ is the CR iterative process (8) converging to the same fixed point $x^* \in F(T)$, where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\gamma_n\}$ are sequences in [0, 1]. Then, the sequences $\{x_n\}$ and $\{u_n\}$ have the same rate of convergence.*

Proof. By (16), we have

$$\begin{aligned}\left\|x_{n+1}-x^*\right\| &\leq \left\|x_n-x^*\right\| \\ &\leq \left\|x_{n-1}-x^*\right\| \\ &\leq \left\|x_{n-2}-x^*\right\| \\ &\vdots \\ &\leq \left\|x_0-x^*\right\|.\end{aligned} \tag{44}$$

Let

$$\left|a_n-x^*\right| = \left\|x_0-x^*\right\|. \tag{45}$$

Using (4) and (8), we have

$$\begin{aligned}\left\|u_{n+1}-x^*\right\| &= \left\|(1-\alpha_n)y_n+\alpha_n s_n-x^*\right\| \\ &\leq (1-\alpha_n)\left\|y_n-x^*\right\|+\alpha_n\left\|s_n-x^*\right\| \\ &\leq (1-\alpha_n)\left\|y_n-x^*\right\|+\alpha_n H\left(P_T(y_n),P_T(x^*)\right) \\ &\leq (1-\alpha_n)\left\|y_n-x^*\right\|+\alpha_n\left\|y_n-x^*\right\| \\ &= \left\|y_n-x^*\right\|.\end{aligned} \tag{46}$$

Next, we have the following estimate:

$$\begin{aligned}\left\|y_n-x^*\right\| &= \left\|(1-\beta_n)g_n+\beta_n h_n-x^*\right\| \\ &\leq (1-\beta_n)\left\|g_n-x^*\right\|+\beta_n\left\|h_n-x^*\right\| \\ &\leq (1-\beta_n)H\left(P_T(u_n),P_T(x^*)\right) \\ &\quad +\beta_n H\left(P_T(z_n),P_T(x^*)\right) \\ &\leq (1-\beta_n)\left\|u_n-x^*\right\|+\beta_n\left\|z_n-x^*\right\|,\end{aligned} \tag{47}$$

$$\begin{aligned}\left\|z_n-x^*\right\| &= \left\|(1-\gamma_n)u_n+\gamma_n g_n-x^*\right\| \\ &\leq (1-\gamma_n)\left\|u_n-x^*\right\|+\gamma_n\left\|g_n-x^*\right\| \\ &\leq (1-\gamma_n)\left\|u_n-x^*\right\|+\gamma_n H\left(P_T(u_n),P_T(x^*)\right) \\ &\leq (1-\gamma_n)\left\|u_n-x^*\right\|+\gamma_n\left\|u_n-x^*\right\| \\ &= \left\|u_n-x^*\right\|.\end{aligned} \tag{48}$$

Using (48) in (47), we obtain

$$\begin{aligned}\left\|y_n-x^*\right\| &\leq (1-\beta_n)\left\|u_n-x^*\right\|+\beta_n\left\|u_n-x^*\right\| \\ &= \left\|u_n-x^*\right\|.\end{aligned} \tag{49}$$

Using (49) in (46), we have

$$\begin{aligned}\left\|u_{n+1}-x^*\right\| &\leq \left\|u_n-x^*\right\| \\ &\leq \left\|u_{n-1}-x^*\right\| \\ &\leq \left\|u_{n-2}-x^*\right\| \\ &\vdots \\ &\leq \left\|u_0-x^*\right\|.\end{aligned} \tag{50}$$

Let

$$\left|b_n-x^*\right| = \left\|u_0-x^*\right\|. \tag{51}$$

Hence, using (45), (51), and the condition that $x_0 = u_0 \in C$, we obtain

$$\lim_{n\longrightarrow\infty}\frac{\left|a_n-x^*\right|}{\left|b_n-x^*\right|} = \lim_{n\longrightarrow\infty}\frac{\left\|x_0-x^*\right\|}{\left\|u_0-x^*\right\|} = \frac{\left\|x_0-x^*\right\|}{\left\|u_0-x^*\right\|} = \frac{\left\|x_0-x^*\right\|}{\left\|x_0-x^*\right\|} = 1. \tag{52}$$

Because $0 < l = 1 < \infty$, it follows that the sequences $\{x_n\}$ and $\{u_n\}$ have the same rate of convergence. □

4. Applications to Solution of Two-point Boundary Value Problems

In this section, we apply the Picard–Ishikawa hybrid iterative process recently introduced by Okeke [3] for finding the solution of two-point boundary value problems for a second-order differential equation. Let X be a real Banach space and $x_0 \in X$; the Picard–Ishikawa hybrid iterative process $\{x_n\}$ is defined as follows:

$$\begin{cases} z_n = (1-\beta_n)x_n + \beta_n T x_n, \\ y_n = (1-\alpha_n)x_n + \alpha_n T z_n, \\ x_{n+1} = T y_n, \\ n \in \mathbb{N}, \end{cases} \tag{53}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are appropriate sequences in [0, 1]. It is known (see [3]) that this fixed-point iterative process converges faster than all of Picard [15], Krasnosel'skii [10], Mann [11], Ishikawa [9], Noor [12], Picard-Mann [27], and Picard–Krasnosel'skii [2] iterations.

In this section, we propose a Picard–Ishikawa hybrid-type algorithm for finding the solution of two-point boundary value problem.

Green's function is given as follows. Suppose $h(t)$ is continuous on $[a, b]$ for each $t \in [a, b]$. Functions satisfying $x'' = h(t)$ are of the following form:

$$x = K_0 + K_1 t + \int_a^b (t-s)h(s)\mathrm{d}s. \tag{54}$$

Therefore, the unique solution of the following boundary value problem

$$x'' = h(t), \tag{55}$$

$$\begin{cases} x(a) = 0, \\ x(b) = 0, \end{cases} \tag{56}$$

could be expressed in form of equation (54) with appropriate values for constants K_0 and K_1. Once the values of the constants K_0 and K_1 are determined, one can easily prove that the solution $x(t)$ of equations (55) and (56) can be expressed in form of $x(t) = \int_a^b G(t,s)h(s)\mathrm{d}s$, where $G(t, s)$ is called Green's function of the boundary value problem $x'' = 0$, $x(a) = 0$, and $x(b) = 0$ and it is defined as follows:

$$G(t,s) = \begin{cases} \dfrac{(t-a)(s-b)}{b-a}, & \text{for } a \le t \le s, \\[2ex] \dfrac{(t-b)(s-a)}{b-a}, & \text{for } s \le t \le b. \end{cases} \tag{57}$$

Similarly, the solution of the following boundary value problem

$$\begin{cases} x'' = h(t), \\ x(a) = \xi_0, \\ x(b) = \xi_1, \end{cases} \tag{58}$$

can be expressed in the following form:

$$x(t) = \int_a^b G(t,s)h(s)\mathrm{d}s + w(t), \tag{59}$$

where $w(t)$ is the solution of the equation $x'' = 0$ satisfying $w(a) = \xi_0$ and $w(b) = \xi_1$.

Assume that the function $f(t, x, x')$ is continuous on $[a,b] \times \mathbb{R}^2$. It follows from (59) that if the function $w(t)$ is the solution of the equation $x'' = 0$ with $w(a) = \xi_0$ and $w(b) = \xi_1$, then the function $x(t) \in C^{(2)}[a, b]$ is a solution of the following boundary value problem:

$$\begin{cases} x'' = f(t, x, x'), \\ x(a) = \xi_0, \\ x(b) = \xi_1, \end{cases} \tag{60}$$

if and only if $x(t)$ belongs to $C^{(1)}[a, b]$ and it is a solution of the following integral equation:

$$x(t) = \int_a^b G(t,s) f(s, x(s), x'(s))\mathrm{d}s + w(t)), \quad \text{on } [a,b]. \tag{61}$$

Hence, suppose $T: C^{(1)}[a, b] \longrightarrow C^{(1)}[a, b]$ is defined by

$$T[x(t)] = \int_a^b G(t,s) f(s, x(s), x'(s))\mathrm{d}s + w(t), \tag{62}$$

for each $x \in C^{(1)}[a, b]$ with $a \le t \le b$, then a fixed point of the mapping T is a solution of equation (61). Hence, it is also a solution of the boundary value problem (60).

Bello et al. [16] considered the following two-point boundary value problem:

$$x'' = f(t, x, x'), \quad a \le t \le b, \tag{63}$$

$$\begin{cases} \lambda_0 x(a) + \nu_0 x'(a) = \xi_0, \\ \lambda_1 x(b) + \nu_1 x'(b) = \xi_1, \end{cases} \tag{64}$$

where λ_i and ν_i are real constants such that $\lambda_i^2 + \nu_i^2 > 0$, $i = 0, 1$.

In order to solve (63) and (64) using our proposed Picard–Ishikawa hybrid iterative process (53), we first transform (63) and (64) to the following relation:

$$\begin{cases} z_n'' = (1-\beta_n)x_n'' + \beta_n x_n'' = (1-\beta_n)x_n'' + \beta_n f(t, x_n, x_n'), \\ y_n'' = (1-\alpha_n)x_n'' + \alpha_n z_n'' = (1-\alpha_n)x_n'' + \alpha_n f(t, z_n, z_n'), \\ x_{n+1}'' = y_{n+1}'' = f(t, y_n, y_n'), \\ \lambda_0 x_{n+1}(a) + \nu_0 x_{n+1}'(a) = \xi_0, \\ \lambda_1 x_{n+1}(b) + \nu_1 x_{n+1}'(b) = \xi_1, \end{cases} \tag{65}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0, 1] satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$. One can prove that $x(t)$ is a solution of (63) and (64) if and only if $x(t)$ is a solution of the following equivalent integral equation:

$$x(t) = \int_a^b G(t,s) f(s, x(s), x'(s))\mathrm{d}s + w(t), \quad \text{on } [a,b], \tag{66}$$

where $G(t, s)$ is Green's function of the following associated boundary value problem:

$$\begin{cases} x'' = 0, \\ \lambda_0 x(a) + \nu_0 x'(a) = \xi_0, \\ \lambda_1 x'(b) + \nu_1 x'(b) = \xi_1, \end{cases} \tag{67}$$

and $w(t)$ is the solution of (67).

Next, suppose $T: C^{(1)}[a, b] \longrightarrow C^{(1)}[a, b]$ is defined as follows:

$$\begin{cases} T[x(t)] = \int_a^b G((t,s)f(s,x(s),x'(s))\mathrm{d}s + w(t), \\ T[z(t)] = \int_a^b G(t,s)f(s,z(s),z'(s))\mathrm{d}s + w(t), \\ T[y(t)] = \int_a^b G(t,s)f(s,y(s),y'(s))\mathrm{d}s + w(t), \end{cases} \quad (68)$$

where T is a mapping such that any solution $x(t)$ of (63) and (64) is a fixed point of T [32].

We next embark on the derivation of our fixed-point iterative method for solving the proposed boundary value problem. Suppose D is a nonvoid convex subset of a real Banach space X and $T: D \longrightarrow \mathscr{P}(\mathscr{D})$ is a multivalued mapping. For arbitrary $x_0 \in D$, let $\{x_n\}$ be the iterative sequence generated by (53), with sequences $\{\alpha_n\}$ and $\{\beta_n\}$ in [0, 1] satisfying $\sum_{n=0}^{\infty} \alpha_n = \infty$.

Next, we compare our method (65) with (53) to establish their equivalence. We obtain this by first differentiating (68) as follows:

$$\begin{cases} (Tx_n)' = \int_a^b \frac{\partial}{\partial t} G(t,s)f(s,x_n(s),x_n'(s))\mathrm{d}s + w'(t), \\ (Tz_n)' = \int_a^b \frac{\partial}{\partial t} G(t,s)f(s,z_n(s),z_n'(s))\mathrm{d}s + w'(t), \\ (Ty_n)' = \int_a^b \frac{\partial}{\partial t} G(t,s)f(s,y_n(s),y_n'(s))\mathrm{d}s + w'(t). \end{cases} \quad (69)$$

Secondly, differentiating (53), we have

$$\begin{cases} z_n'' = (1-\beta_n)x_n'' + \beta_n (Tx_n)'', \\ y_n'' = (1-\alpha_n)x_n'' + \alpha_n (Tz_n)'', \\ x_{n+1}'' = (Ty_n)'', \\ n \in \mathbb{N}. \end{cases} \quad (70)$$

Our third step is to differentiate (69) as follows:

$$\begin{cases} (Tx_n)'' = \int_a^b \frac{\partial}{\partial t} G(t,x_n(s),x_n'(s))\mathrm{d}s + w'(t), \\ (Tz_n)'' = \int_a^b \frac{\partial}{\partial t} G(t,z_n(s),z_n'(s))\mathrm{d}s + w'(t), \\ (Ty_n)'' = \int_a^b \frac{\partial}{\partial t} G(t,y_n(s),y_n'(s))\mathrm{d}s + w'(t). \end{cases} \quad (71)$$

Lastly, substituting (71) in (70), we have

$$\begin{cases} z_n'' = (1-\beta_n)x_n'' + \beta_n\left(\int_a^b \frac{\partial}{\partial t} G(t,x_n(s),x_n'(s))\mathrm{d}s + w'(t)\right), \\ y_n'' = (1-\alpha_n)x_n'' + \alpha_n\left(\int_a^b \frac{\partial}{\partial t} G(t,z_n(s),z_n'(s))\mathrm{d}s + w'(t)\right), \\ x_{n+1}'' = \int_a^b \frac{\partial}{\partial t} G(t,y_n(s),y_n'(s))\mathrm{d}s + w'(t), \\ n \in \mathbb{N}. \end{cases} \quad (72)$$

Hence, (72) can be written as follows, which is our proposed fixed-point iteration method:

$$\begin{cases} z_n'' = (1-\beta_n)x_n'' + \beta_n x_n'' = (1-\beta_n)x_n'' + \beta_n f(t,x_n,x_n'), \\ y_n'' = (1-\alpha_n)x_n'' + \alpha_n z_n'' = (1-\alpha_n)x_n'' + \alpha_n f(t,z_n,z_n'), \\ x_{n+1}'' = y_n'' = f(t,y_n,y_n'), \\ n \in \mathbb{N}. \end{cases} \quad (73)$$

Next, we give our main results for this section as follows.

Theorem 4. *Let $T: C^{(1)}[a, b] \longrightarrow C^{(1)}[a, b]$ be a contraction mapping defined in (68) with contractive constant $a \in (0, 1)$. Suppose $x_0 \in C^{(1)}[a, b]$ is an affine function. Construct the sequence $\{x_n\}$ generated by the Picard–Ishikawa hybrid iterative process (53) satisfying $x'' = 0$ and the boundary condition defined in (64) with $\{\alpha_n\}$ and $\{\beta_n\}$ sequences in [0, 1] such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then, $\{x_n\}$ converges to a unique solution $x^* \in C^{(1)}[a, b]$ of (63) and (64).*

Proof. The existence and uniqueness of $x^* \in C^{(1)}[a, b]$ is guaranteed by the famous Banach contraction mapping principle. We now show that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. Using (53), we have

$$\begin{aligned} \|x_{n+1} - x^*\| &= \|Ty_n - x^*\| \\ &\le a\|y_n - x^*\|. \end{aligned} \quad (74)$$

Next, we have the following estimates:

$$\begin{aligned} \|y_n - x^*\| &= \|(1-\alpha_n)x_n + \alpha_n Tz_n - x^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + \alpha_n\|Tz_n - x^*\| \\ &\le (1-\alpha_n)\|x_n - x^*\| + a\alpha_n\|z_n - x^*\|. \end{aligned} \quad (75)$$

Next, we have

$$\begin{aligned} \|z_n - x^*\| &= \|(1-\beta_n)x_n + \beta_n Tx_n - x^*\| \\ &\le (1-\beta_n)\|x_n - x^*\| + \beta_n\|Tx_n - x^*\| \\ &\le (1-\beta_n)\|x_n - x^*\| + a\beta_n\|x_n - x^*\| \\ &= (1-\beta_n(1-a))\|x_n - x^*\|. \end{aligned} \quad (76)$$

Using (76) in (75), we have

$$\|y_n - x^*\| \le (1-\alpha_n)\|x_n - x^*\| + a\alpha_n(1-\beta_n(1-a))\|x_n - x^*\| = (1-\alpha_n(1-a(1-\beta_n(1-a))))\|x_n - x^*\|. \tag{77}$$

Using (77) in (74), we obtain

$$\|x_{n+1} - x^*\| \le a[1-\alpha_n(1-a(1-\beta_n(1-a)))]\|x_n - x^*\|. \tag{78}$$

Continuing this process, we obtain the following inequalities:

$$\begin{cases} \|x_{n+1} - x^*\| \le a[1-\alpha_n(1-a(1-\beta_n(1-a)))]\|x_n - x^*\| \\ \le a[1-\alpha_{n-1}(1-a(1-\beta_{n-1}(1-a)))]\|x_{n-1} - x^*\| \\ \le a[1-\alpha_{n-2}(1-a(1-\beta_{n-2}(1-a)))]\|x_{n-2} - x^*\| \\ \vdots \\ \le a[1-\alpha_0(1-a(1-\beta_0(1-a)))]\|x_0 - x^*\|. \end{cases} \tag{79}$$

Using the inequalities in (79), we have

$$\|x_{n+1} - x^*\| \le a^{(n+1)}\prod_{k=0}^{n}[1-\alpha_k(1-a(1-\beta_k(1-a)))]\|x_0 - x^*\|. \tag{80}$$

Because $a \in (0, 1)$ and $\alpha_n, \beta_n \in [0, 1]$ for each $n \in \mathbb{N}$, $[1-\alpha_n(1-a(1-\beta_n(1-a)))] < 1$. It is well known in analysis that $1 - y \le e^{-y}$ for each $y \in [0, 1]$. Hence, using these facts in (80), we have

$$\|x_{n+1} - x^*\| \le a^{(n+1)}\|x_0 - x^*\| e^{-(1-a(1-\beta_n(1-a)))\sum_{k=0}^{n}\alpha_k} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty. \tag{81}$$

This means that $x_n \longrightarrow x^*$ as $n \longrightarrow \infty$. The proof of Theorem 4 is completed. □

Remark 2. Theorem 4 and the results of this section is an improvement on the results of Bello et al. [16] because it is known (see [3]) that the Picard–Ishikawa hybrid iteration method used in our results converges faster than the Mann-type method used by Bello et al. [16].

5. Numerical Examples

In this section, we provide some numerical examples to validate our analytical results. We compare the speed of convergence between the CR iterative scheme $\{u_n\}$ in (6) and the Picard–Ishikawa hybrid iterative scheme $\{x_n\}$ in (7). In the following figures, we denote the CR iterative process by CR and the Picard–Ishikawa hybrid iterative process by PI. All the codes were written in MATLAB (R2010a) and run on PC with Intel(R) Core(TM) i3-4030U CPU @ 1.90 GHz.

We begin with the following example ([20], Example 2.3).

Example 1. Suppose $(\mathbb{R}, \|\cdot\|)$ is a normed linear space with the usual norm and $C = [0, 1]$. Define $T : C \longrightarrow \mathscr{P}(\mathscr{K})$ by $Tx = [0, (2x + 1/4)]$. Khan et al. [20] proved that T is quasi-nonexpansive with $F(T) = [0, (1/2)]$. We consider the following cases for our numerical experiments:

Case 1: choose $x_0 = u_0 = 0.1$, $\alpha_n = (n/5n + 1)$, $\beta_n = (n/10n + 1)$, and $\gamma_n = (1/7n + 2)$, and the number of iteration for each iterative scheme is $n = 100$. The graph of this case is presented in Figure 1.

Case 2: choose $x_0 = u_0 = 1$ and $\alpha_n = \beta_n = \gamma_n = (1/8n + 7)$, and the number of iteration for each iterative scheme is $n = 100$. The graph of this case is presented in Figure 2.

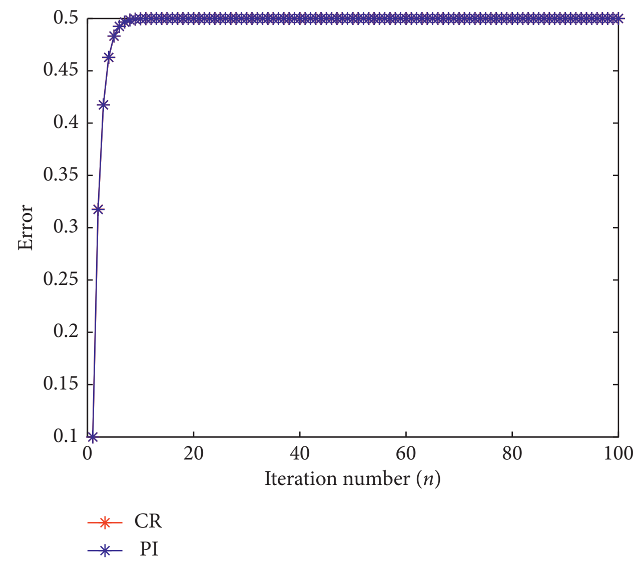

Figure 1: Errors versus iteration numbers (n): Example 1 (Case 1).

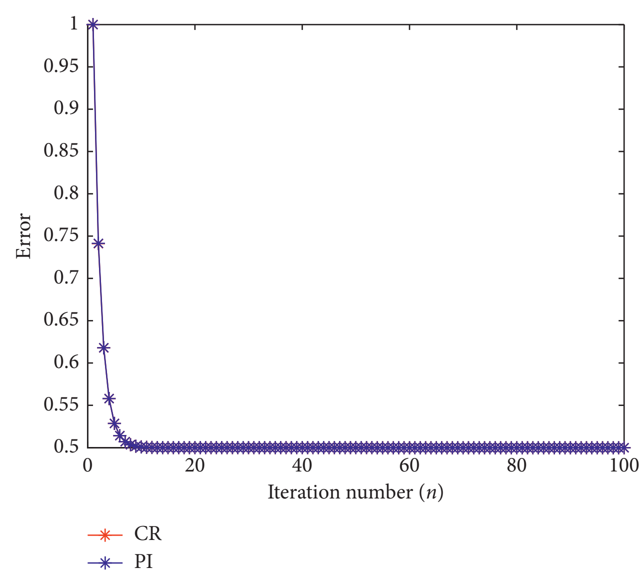

Figure 2: Errors versus iteration numbers (n): Example 1 (Case 2).

Example 2. Let $T : X \longrightarrow X$ be a mapping such that $Tx = (x/3)$, with $a = (1/2)$ and $X = [0, \infty)$. Suppose the first

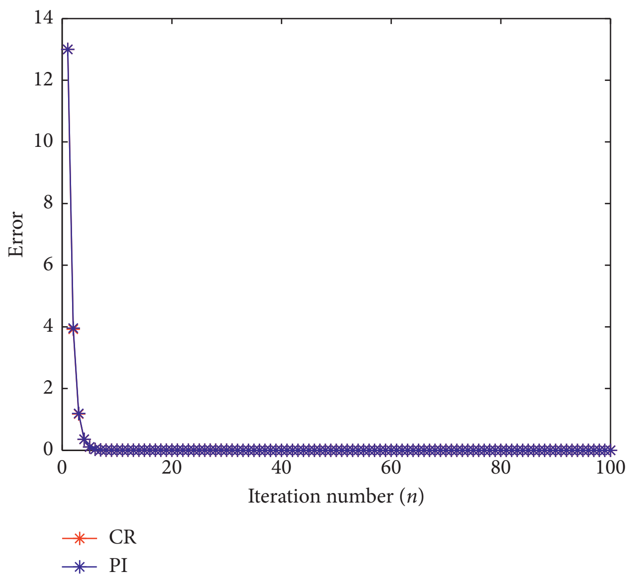

Figure 3: Errors versus iteration numbers (n): Example 2.

iteration $u_0 = x_0 = 13$ and the number of iterations for each iterative scheme is $n = 100$. Choose $\alpha_n = (n/6n + 2)$, $\beta_n = (n/10n + 1)$, and $\gamma_n = (1/3n + 2)$. Then, Figure 3 shows the errors versus iteration numbers for this example.

Next, we present the following graphs of errors versus iteration numbers (n) for each case.

Remark 3. Clearly, from Figures 1 and 2 of Example 1, we see that the Picard–Ishikawa hybrid iterative process and the CR iterative process have the same rate of convergence for a class of multivalued quasi-nonexpansive mappings. Similarly, from Figure 3 of Example 2, we see that the Picard–Ishikawa hybrid iterative process and the CR iterative process have the same rate of convergence for a class of contraction mappings.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare no conflict of interests.

Authors' Contributions

All authors contributed equally to the writing of this paper.

Acknowledgments

The first author's research was supported by the Abdus Salam School of Mathematical Sciences, Government College University, Lahore, Pakistan, through grant number ASSMS/2018-2019/452. This research was funded by Basque Government (IT1207-19).

References

[1] V. Berinde, *Iterative Approximation of Fixed Points*, Springer, Berlin, Germany, 2007.

[2] G. A. Okeke and M. Abbas, "A solution of delay differential equations via Picard-Krasnoselskii hybrid iterative process," *Arabian Journal of Mathematics*, vol. 6, no. 1, pp. 21–29, 2017.

[3] G. A. Okeke, "Convergence analysis of the Picard-Ishikawa hybrid iterative process with applications," *Afrika Matematika*, vol. 30, no. 5-6, pp. 817–835, 2019.

[4] M. Abbas and T. Nazir, "A new faster iteration process applied to constrained minimization and feasibility problems," *Matematicki Vesnik*, vol. 66, no. 2, pp. 223–234, 2014.

[5] R. P. Agarwal, D. O'Regan, and D. R. Sahu, "Iterative construction of fixed points of nearly asymptotically nonexpansive mappings," *Journal of Nonlinear and Convex Analysis*, vol. 8, no. 1, pp. 61–79, 2007.

[6] H. Akewe, G. A. Okeke, and A. F. Olayiwola, "Strong convergence and stability of Kirk-multistep-type iterative schemes for contractive-type operators," *Fixed Point Theory and Applications*, vol. 2014, no. 1, p. 45, 2014.

[7] H. Akewe and G. A. Okeke, "Convergence and stability theorems for the Picard-Mann hybrid iterative scheme for a general class of contractive-like operators," *Fixed Point Theory and Applications*, vol. 2015, no. 1, p. 66, 2015.

[8] R. Chugh, V. Kumar, and S. Kumar, "Strong convergence of a new three step iterative scheme in Banach spaces," *American Journal of Computational Mathematics*, vol. 2, no. 4, pp. 345–357, 2012.

[9] S. Ishikawa, "Fixed points by a new iteration method," *Proceedings of the American Mathematical Society*, vol. 44, no. 1, p. 147, 1974.

[10] M. A. Krasnosel'skii, "Two observations about the method of successive approximations," *Uspekhi Matematicheskikh Nauk*, vol. 10, pp. 123–127, 1955.

[11] W. R. Mann, "Mean Value methods in iteration," *Proceedings of the American Mathematical Society*, vol. 4, no. 3, p. 506, 1953.

[12] M. A. Noor, "New approximation schemes for general variational inequalities," *Journal of Mathematical Analysis and Applications*, vol. 251, no. 1, pp. 217–229, 2000.

[13] G. A. Okeke, "Iterative approximation of fixed points of contraction mappings in complex valued Banach spaces," *Arab Journal of Mathematical Sciences*, vol. 25, no. 1, pp. 83–105, 2019.

[14] W. Phuengrattana and S. Suantai, "On the rate of convergence of Mann, Ishikawa, Noor and SP iterations for continuous functions on an arbitrary interval," *Journal of Computational and Applied Mathematics*, vol. 235, no. 9, pp. 3006–3014, 2011.

[15] E. Picard, "Memoire sur la theorie des equations aux derivees partielles et la methode des approximations successives," *Journal de Mathématiques Pures et Appliquées*, vol. 6, pp. 145–210, 1890.

[16] N. Bello, A. Jibril Alkali, and A. Roko, "A fixed point iterative method for the solution of two-point boundary value problems for a second order differential equations," *Alexandria Engineering Journal*, vol. 57, no. 4, pp. 2515–2520, 2018.

[17] A. Kilicman, K. R. Adeboye, and M. Wadai, "A variational fixed point iteration technique for the solution of second order differential equations," *Malaysian Journal of Science*, vol. 35, no. 1, pp. 29–36, 2006.

[18] M. Abbas and B. E. Rhoades, "Fixed point theorems for two new classes of multivalued mappings," *Applied Mathematics Letters*, vol. 22, no. 9, pp. 1364–1368, 2009.

[19] M. Abbas, H. Iqbal, and A. Petrusel, "Fixed points for multivalued Suzuki type $(\theta, \mathfrak{R})$-contraction mapping with applications," *Journal of Function Spaces*, vol. 2019, Article ID 9565804, 13 pages, 2019.

[20] S. H. Khan, M. Abbas, and S. Ali, "Fixed points of multivalued quasi-nonexpansive mappings using a faster iterative process," *Acta Mathematica Sinica, English Series*, vol. 30, no. 7, pp. 1231–1241, 2014.

[21] L. Gorniewicz, *Topological Fixed Point Theory of Multivalued Mappings*, Kluwer Academic Pub., Dordrecht, Netherlands, 1999.

[22] W. Cholamjiak, P. Cholamjiak, and S. Suantai, "Convergence of iterative schemes for solving fixed point of multi-valued nonself mappings and equilibrium problems," *Journal of Nonlinear Sciences and Applications*, vol. 8, no. 6, pp. 1245–1256, 2015.

[23] S. Nadler Jr., "Multi-valued contraction mappings," *Pacific Journal of Mathematics*, vol. 30, no. 2, pp. 475–488, 1969.

[24] S. Suantai, W. Cholamjiak, and P. Cholamjiak, "An implicit iteration process for a finite family of multi-valued mappings in Banach spaces," *Applied Mathematics Letters*, vol. 25, no. 11, pp. 1656–1660, 2012.

[25] S. Suantai, W. Cholamjiak, and P. Cholamjiak, "Strong convergence theorems of a finite family of quasi-nonexpansive and Lipschitz multi-valued mappings," *Afrika Matematika*, vol. 26, no. 3-4, pp. 345–355, 2015.

[26] Z. Opial, "Weak convergence of the sequence of successive approximations for nonexpansive mappings," *Bulletin of the American Mathematical Society*, vol. 73, no. 4, pp. 591–598, 1967.

[27] S. H. Khan, "A Picard-Mann hybrid iterative process," *Fixed Point Theory and Applications*, vol. 2013, no. 1, p. 69, 2013.

[28] B. E. Rhoades, "Comments on two fixed point iteration methods," *Journal of Mathematical Analysis and Applications*, vol. 56, no. 3, pp. 741–750, 1976.

[29] H. F. Senter and W. G. Dotson, "Approximating fixed points of nonexpansive mappings," *Proceedings of the American Mathematical Society*, vol. 44, no. 2, p. 375, 1974.

[30] J. Schu, "Weak and strong convergence to fixed points of asymptotically nonexpansive mappings," *Bulletin of the Australian Mathematical Society*, vol. 43, no. 1, pp. 153–159, 1991.

[31] Y. Song and Y.-J. Cho, "Some notes on Ishikawa iteration for multivalued mappings," *Bulletin of the Korean Mathematical Society*, vol. 48, no. 3, pp. 575–584, 2011.

[32] M. Wadai and A. Kilicman, "On the two point boundary value problems using variational fixed point iterative scheme," *Bulletin of the Malaysian Mathematical Sciences Society*, vol. 11, no. 2, pp. 137–160, 2017.