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Research Article

Least Squares Estimation for Discretely Observed Stochastic Lotka–Volterra Model Driven by Small α-Stable Noises

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Stochastic Lotka–Volterra model driven by small α-stable noises is used to describe population dynamics perturbed by random environment. However, parameters in the model are always unknown. The contrast function is given to obtain least squares estimators. The consistency and the rate of convergence of the least squares estimators are proved, and the asymptotic distribution of the estimators are derived by Markov inequality, Cauchy–Schwarz inequality, and Gronwall's inequality. Some numerical examples are provided to verify the effectiveness of the estimators.

1. Introduction

Stochastic differential equations are basic tools for modeling random phenomena in the financial field. Due to the widespread application of the stochastic differential equations in the field of financial economics, it has attracted a large number of scholars to devote themselves to research in this field [1–3]. However, the parameters in stochastic model are always unknown. In the past few decades, the parameter estimation problem for economical models has been studied by many authors. For example, Yu and Phillips [4] utilized the Gaussian method to estimate the parameters of continuous time short-term interest rate models. Faff and Gray [5] discussed the estimation and comparison of short-rate models by the generalised method of moments. Rossi [6] applied particle filters and maximum likelihood estimation to solve the parameter estimation for Cox–Ingersoll–Ross model. Wei et al. [7] utilized the Gaussian estimation method to investigate the parameter estimation for discretely observed Cox–Ingersoll–Ross model. However, it is well known that many financial processes exhibit discontinuous sample paths and heavy tailed properties (e.g., certain moments are infinite). These features cannot be captured by Brownian motion [8, 9]. Therefore, it is natural to replace the driving Brownian motion by Lévy noises. In recent years, with the development of Lévy process theory and its application in the fields of engineering systems, economic systems, and management systems, it has attracted great attention from scholars. Therefore, some authors considered parameter estimation for stochastic differential equations driven by Lévy noises. For example, Li and Ma [10] discussed the asymptotic properties of estimators in a stable Cox–Ingersoll–Ross model. Long [11] analyzed the least squares estimator for discretely observed 2Ornstein–Uhlenbeck processes with small Lévy noises. Then, Long et al. [8] tackled the least squares estimators for discretely observed stochastic processes driven by small Lévy noises. Singh et al. [12] utilized a randomized response model under Poisson distribution to estimate a rare sensitive attribute in two-stage sampling. Wei [13] used least squares estimation to discuss the discretely observed CoxCIngersollCRoss model driven by small symmetrical stable noises and studied the consistency and asymptotic distribution of the estimators. There have been many applications of small noise asymptotics to mathematical finance [14, 15]. From a practical point of view in parametric inference, it is more realistic and interesting to consider asymptotic estimation for diffusion processes with small noise based on discrete observations.

Lotka–Volterra model is often used to model population growth of a single species. However, systems are more or less influenced by random factors. Thus, stochastic

Lotka–Volterra equation, being a reasonable and popular approach to model population dynamics perturbed by random environment, has recently been studied by many authors both from a mathematical perspective and in the context of real biological dynamics [16–19] to explain the change of biodiversity also over time [20–23]. For example, Mao et al. [24] investigated a multidimensional stochastic Lotka–Volterra system driven by one-dimensional standard Brownian motion. They revealed that the environmental noise could suppress population explosion. Later, Mao [25] discussed a finite second moment of the stationary distribution under Brownian noise, which is very important in application. Bao et al. [26] and Bao and Yuan [27] considered a competitive Lotka–Volterra population model with Lévy jumps. Zhao et al. [28] studied the parameter estimation for stochastic Lotka–Volterra model by using the maximum likelihood method from continuous time observations. However, due to the limitation of instrument precision, it is impossible to observe the system from continuous time. Moreover, few literatures considered the consistency and asymptotic distribution of parameter estimators for stochastic Lotka–Volterra driven by α-stable noises. We consider the parameter estimation problem for discretely observed stochastic Lotka–Volterra model with small α-stable noises. The contrast function is given to obtain the least squares estimators. The consistency and asymptotic distribution of the estimators are discussed by Markov inequality, Cauchy–Schwarz inequality, and Gronwall's inequality.

The structure is that the stochastic Lotka–Volterra model driven by small α-stable noises is introduced and the contrast function is given to obtain the least squares estimators in Section 2. In Section 3, the consistency of the estimators is proved and the asymptotic distribution of the estimators is studied. In Section 4, some simulations are made. The conclusion is given in Section 5.

2. Problem Formulation and Preliminaries

$(\Omega, \mathscr{F}, \mathbb{P})$ is a basic probability space equipped with a right continuous and increasing family of σ-algebras $(\mathscr{F}_t)_{t\geq 0}$ and $Z = (Z_t, t \geq 0)$ is a strictly symmetric α-stable Lévy motion.

A random variable η is said to have a stable distribution with index of stability $\alpha \in (0, 2]$, scale parameter $\sigma \in (0, \infty)$, skewness parameter $\beta \in [-1, 1]$, and location parameter $\mu \in (-\infty, \infty)$ if it has the following characteristic function:

$$\phi_\eta(u) = \mathbb{E}\exp\{iu\eta\} = \begin{cases} \exp\left\{-\sigma^\alpha |u|^\alpha \left(1 - i\beta \operatorname{sgn}(u) \tan\frac{\alpha\pi}{2}\right) + i\mu u\right\}, & \text{if } \alpha \neq 1, \\ \exp\left\{-\sigma |u| \left(1 + i\beta \frac{2}{\pi} \operatorname{sgn}(u) \log|u|\right) + i\mu u\right\}, & \text{if } \alpha = 1. \end{cases} \tag{1}$$

We denote $\eta \sim S_\alpha(\sigma, \beta, \mu)$. When $\mu = 0$, we say η is strictly α-stable; if in addition $\beta = 0$, we call η symmetrical α-stable. Throughout this article, α-stable motion is strictly symmetrical and $\alpha \in (1, 2)$.

We study the parametric estimation problem for stochastic Lotka–Volterra model driven by small α-stable noises described by the following stochastic differential equation:

$$\begin{aligned} \mathrm{d}X_t &= X_t(\theta - \beta X_t)\mathrm{d}t + \varepsilon X_t \mathrm{d}Z_t, \quad t \in [0, 1], \\ X_0 &= x_0, \end{aligned} \tag{2}$$

where θ and β are unknown parameters, $\varepsilon \in (0, 1]$.

Since the stochastic Lotka–Volterra model is driven by small α-stable noises and due to the complexity of the α-stable noises, it is difficult to obtain the likelihood function. Thus, the maximum likelihood estimation cannot be used. Therefore, the contrast function is given to obtain least squares estimators.

Consider the following contrast function:

$$\rho_{n,\varepsilon}(\theta, \beta) = \sum_{i=1}^{n} \frac{\left|X_{t_i} - X_{t_{i-1}} - X_{t_{i-1}}\left(\alpha - \beta X_{t_{i-1}}\right)\Delta t_{i-1}\right|^2}{\varepsilon^2 X_{t_{i-1}}^2 \Delta t_{i-1}}, \tag{3}$$

where $\Delta t_{i-1} = t_i - t_{i-1} = 1/n$.

We obtain the estimators:

$$\begin{aligned} \widehat{\theta}_{n,\varepsilon} &= \frac{n\sum_{i=1}^n \left(X_{t_i} - X_{t_{i-1}}\right) \sum_{i=1}^n X_{t_{i-1}} + n^2 \sum_{i=1}^n X_{t_{i-1}}^2 - n\sum_{i=1}^n X_{t_i}/X_{t_{i-1}} \sum_{i=1}^n X_{t_{i-1}}^2}{\left(\sum_{i=1}^n X_{t_{i-1}}\right)^2 - n\sum_{i=1}^n X_{t_{i-1}}^2}, \\ \widehat{\beta}_{n,\varepsilon} &= \frac{n^2 \sum_{i=1}^n \left(X_{t_i} - X_{t_{i-1}}\right) + n^2 \sum_{i=1}^n X_{t_{i-1}} - n\sum_{i=1}^n X_{t_i}/X_{t_{i-1}} \sum_{i=1}^n X_{t_{i-1}}}{\left(\sum_{i=1}^n X_{t_{i-1}}\right)^2 - n\sum_{i=1}^n X_{t_{i-1}}^2}. \end{aligned} \tag{4}$$

The work will be carried out under the following assumptions.

Assumption 1. θ and β are positive true values of the parameters.

Assumption 2. $\sup_{0\le t\le 1}\mathbb{E}|X_t|^2<\infty$.

3. Main Results and Proofs

$X^0=(X_t^0,t\ge 0)$ is the solution to the underlying ordinary differential equation under the true value of the parameters:

$$\mathrm{d}X_t^0=X_t^0\left(\theta-\beta X_t^0\right)\mathrm{d}t,\quad X_0^0=x_0. \tag{5}$$

Discretizing equation (2), we obtain

$$X_{t_i}-X_{t_{i-1}}=\theta\int_{t_{i-1}}^{t_i}X_s\mathrm{d}s-\beta\int_{t_{i-1}}^{t_i}X_s^2\mathrm{d}s+\varepsilon\int_{t_{i-1}}^{t_i}X_s\mathrm{d}Z_s. \tag{6}$$

Then, a more explicit decomposition for $\widehat{\theta}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ can be given

$$\begin{aligned}\widehat{\theta}_{n,\varepsilon}&=\frac{\theta\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s\mathrm{d}s1/n\sum_{i=1}^nX_{t_{i-1}}-\theta\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s/X_{t_{i-1}}\mathrm{d}s1/n\sum_{i=1}^nX_{t_{i-1}}^2}{\left(1/n\sum_{i=1}^nX_{t_{i-1}}\right)^2-1/n\sum_{i=1}^nX_{t_{i-1}}^2}\\&+\frac{\beta\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s^2/X_{t_{i-1}}\mathrm{d}s1/n\sum_{i=1}^nX_{t_{i-1}}^2-\beta\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s^2\mathrm{d}s1/n\sum_{i=1}^nX_{t_{i-1}}}{\left(1/n\sum_{i=1}^nX_{t_{i-1}}\right)^2-1/n\sum_{i=1}^nX_{t_{i-1}}^2}\\&+\frac{\varepsilon\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s\mathrm{d}Z_s1/n\sum_{i=1}^nX_{t_{i-1}}-\varepsilon\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s/X_{t_{i-1}}\mathrm{d}Z_s1/n\sum_{i=1}^nX_{t_{i-1}}^2}{\left(1/n\sum_{i=1}^nX_{t_{i-1}}\right)^2-1/n\sum_{i=1}^nX_{t_{i-1}}^2},\\\widehat{\beta}_{n,\varepsilon}&=\frac{\theta\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s\mathrm{d}s-\theta\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s/X_{t_{i-1}}\mathrm{d}s1/n\sum_{i=1}^nX_{t_{i-1}}}{\left(1/n\sum_{i=1}^nX_{t_{i-1}}\right)^2-1/n\sum_{i=1}^nX_{t_{i-1}}^2}\\&+\frac{\beta\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s^2/X_{t_{i-1}}\mathrm{d}s1/n\sum_{i=1}^nX_{t_{i-1}}-\beta\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s^2\mathrm{d}s}{\left(1/n\sum_{i=1}^nX_{t_{i-1}}\right)^2-1/n\sum_{i=1}^nX_{t_{i-1}}^2}\\&+\frac{\varepsilon\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s\mathrm{d}Z_s-\varepsilon\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s/X_{t_{i-1}}\mathrm{d}Z_s1/n\sum_{i=1}^nX_{t_{i-1}}}{\left(1/n\sum_{i=1}^nX_{t_{i-1}}\right)^2-1/n\sum_{i=1}^nX_{t_{i-1}}^2}.\end{aligned} \tag{7}$$

Before giving the theorems, we need to establish some preliminary results.

Lemma 1 (see [29]). *Z is a strictly α-stable Lévy process and $\phi\in L_{a.s.}^\alpha$. Then,*

$$\int_0^t\phi(s)\mathrm{d}Z_s=Z'\circ\int_0^t\phi_+^\alpha(s)\mathrm{d}s-Z''\circ\int_0^t\phi_-^\alpha(s)\mathrm{d}s,\ \text{a.s.} \tag{8}$$

If Z is symmetric, that is, $\beta=0$, then, there exists some α-stable Lévy process $Z'\overset{d}{=}Z$, such that

$$\int_0^t\phi(s)\mathrm{d}Z_s=Z'\circ\int_0^t|\phi(s)|^\alpha\mathrm{d}s,\ \text{a.s.} \tag{9}$$

Lemma 2. *When $\varepsilon\longrightarrow 0$ and $n\longrightarrow\infty$, we have*

$$\sup_{0\le t\le 1}\left|X_t-X_t^0\right|\overset{P}{\longrightarrow}0. \tag{10}$$

Proof. Integrating on both sides of equation (2) and (8), we have

$$\begin{aligned}X_t-X_t^0&=\theta\int_0^t\left(X_s-X_s^0\right)\mathrm{d}s-\beta\int_0^t\left(X_s^2-\left(X_s^0\right)^2\right)\mathrm{d}s\\&+\varepsilon\int_0^tX_s\mathrm{d}Z_s.\end{aligned} \tag{11}$$

By using the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}
\left|X_t - X_t^0\right|^2 &\leq 3\theta^2\left|\int_0^t \left(X_s - X_s^0\right)\mathrm{d}s\right|^2 + 3\beta^2\left|\int_0^t \left(X_s^2 - \left(X_s^0\right)^2\right)\mathrm{d}s\right|^2 + 3\varepsilon^2\left|\int_0^t X_s \mathrm{d}Z_s\right|^2 \\
&\leq 3t\theta^2 \int_0^t \left|X_s - X_s^0\right|^2 \mathrm{d}s + 12t\beta^2K^2 \int_0^t \left|X_s - X_s^0\right|^2 \mathrm{d}s + 3\varepsilon^2\left|\int_0^t X_s \mathrm{d}Z_s\right|^2 \\
&= 3t\left(\theta^2 + 4\beta^2K^2\right)\int_0^t \left|X_s - X_s^0\right|^2 \mathrm{d}s + 3\varepsilon^2\left|\int_0^t X_s \mathrm{d}Z_s\right|^2 \\
&\leq 3t\left(\theta^2 + 4\beta^2K^2\right)\int_0^t \left|X_s - X_s^0\right|^2 \mathrm{d}s + 3\varepsilon^2 \sup_{0\leq t\leq 1}\left|\int_0^t X_s \mathrm{d}Z_s{}^2\right|,
\end{aligned} \tag{12}$$

where K is the upper bound of X_t.

According to Gronwall's inequality, we have

$$\left|X_t - X_t^0\right|^2 \leq 3\varepsilon^2 e^{3t^2\left(\theta^2+4\beta^2K^2\right)} \sup_{0\leq t\leq 1}\left|\int_0^t X_s \mathrm{d}Z_s\right|^2. \tag{13}$$

Squaring on both sides of equation (17),

$$\sup_{0\leq t\leq 1}\left|X_t - X_t^0\right| \leq \sqrt{3}\varepsilon e^{3/2\left(\theta^2+4\beta^2K^2\right)} \sup_{0\leq t\leq 1}\left|\int_0^t X_s \mathrm{d}Z_s\right|. \tag{14}$$

By the Markov inequality, for any given $\delta > 0$, when $\varepsilon \longrightarrow 0$, we have

$$\begin{aligned}
&\mathbb{P}\left(\left|\sqrt{3}\,\varepsilon e^{3/2\left(\theta^2+4\beta^2K^2\right)} \sup_{0\leq t\leq 1}\right|\int_0^t X_s \mathrm{d}Z_s\| > \delta\right) \\
&\quad\leq \delta^{-1}\sqrt{3}\varepsilon e^{3/2\left(\theta^2+4\beta^2K^2\right)}\mathbb{E}\left[\sup_{0\leq t\leq 1}\left|\int_0^t X_s \mathrm{d}Z_s\right|\right] \\
&\quad\leq C\delta^{-1}\sqrt{3}\varepsilon e^{3/2\left(\theta^2+4\beta^2K^2\right)}\mathbb{E}\left[\left(\int_0^1 X_s^\alpha \mathrm{d}s\right)^{1/\alpha}\right] \\
&\quad\leq C\delta^{-1}\sqrt{3}\varepsilon e^{3/2\left(\theta^2+4\beta^2K^2\right)}K \longrightarrow 0,
\end{aligned} \tag{15}$$

namely,

$$\sup_{0\leq t\leq 1}\left|X_t - X_t^0\right| \xrightarrow{P} 0. \tag{16}$$

The proof is complete. □

Lemma 3. *When $\varepsilon \longrightarrow 0$ and $n \longrightarrow \infty$, we have*

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 \left(X_t^0\right)^2 \mathrm{d}t. \tag{17}$$

Proof. As

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 = \frac{1}{n}\sum_{i=1}^n \left(X_{t_{i-1}}^0\right)^2 + \frac{1}{n}\sum_{i=1}^n\left(X_{t_{i-1}}^2 - \left(X_{t_{i-1}}^0\right)^2\right), \tag{18}$$

we obtain

$$\frac{1}{n}\sum_{i=1}^n \left(X_{t_{i-1}}^0\right)^2 \xrightarrow{P} \int_0^1 \left(X_t^0\right)^2 \mathrm{d}t. \tag{19}$$

According to Lemma 2, when $\varepsilon \longrightarrow 0$ and $n \longrightarrow \infty$, we have

$$\begin{aligned}
\left|\frac{1}{n}\sum_{i=1}^n\left(X_{t_{i-1}}^2 - \left(X_{t_{i-1}}^0\right)^2\right)\right| &\leq \frac{1}{n}\sum_{i=1}^n\left|X_{t_{i-1}} + X_{t_{i-1}}^0\right|\left|X_{t_{i-1}}t - nX_{t_{i-1}}^0\right| \\
&\leq 2K \sup_{0\leq t\leq 1}\left|X_t - X_t^0\right| \xrightarrow{P} 0.
\end{aligned} \tag{20}$$

According to equations (23) and (24), we obtain

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \int_0^1 \left(X_t^0\right)^2 \mathrm{d}t. \tag{21}$$

The proof is complete.

Applying the same methods used in Lemma 3, we have

$$\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} \int_0^1 X_t^0 \mathrm{d}t. \tag{22}$$

In the following theorem, the consistency of the least squares estimators is proved. □

Theorem 1. *When $\varepsilon \longrightarrow 0$, $n \longrightarrow \infty$, and $\varepsilon n^{1-1/\alpha} \longrightarrow 0$, the least squares estimators $\widehat{\theta}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ are consistent, namely,*

$$\widehat{\theta}_{n,\varepsilon} \xrightarrow{P} \theta, \widehat{\beta}_{n,\varepsilon} \xrightarrow{P} \beta. \tag{23}$$

Proof. According to Lemma 3, we have

$$\left(\frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}\right)^2 - \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \left(\int_0^1 X_t^0 \mathrm{d}t\right)^2 - \int_0^1 \left(X_t^0\right)^2 \mathrm{d}t. \tag{24}$$

When $\varepsilon \longrightarrow 0$ and $n \longrightarrow \infty$, we obtain

$$\theta\sum_{i=1}^n \int_{t_{i-1}}^{t_i} X_s \mathrm{d}s \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}} \xrightarrow{P} \theta\int_0^1 X_t \mathrm{d}t \int_0^1 X_t^0 \mathrm{d}t, \theta\sum_{i=1}^n \int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} \mathrm{d}s \frac{1}{n}\sum_{i=1}^n X_{t_{i-1}}^2 \xrightarrow{P} \theta\int_0^1 \frac{X_t}{X_t^0} \mathrm{d}t \int_0^1 \left(X_t^0\right)^2 \mathrm{d}t. \tag{25}$$

According to Lemma 2, we have

$$\theta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s \mathrm{d}s \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} - \theta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} \mathrm{d}s \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2 \xrightarrow{P} \theta\left(\left(\int_0^1 X_t^0 \mathrm{d}t\right)^2 - \int_0^1 \left(X_t^0\right)^2 \mathrm{d}t\right). \tag{26}$$

From equations (28) and (29), we obtain

$$\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} \frac{X_s^2}{X_{t_{i-1}}} \mathrm{d}s \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}}^2 - \beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s^2 \mathrm{d}s \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \xrightarrow{P} 0. \tag{27}$$

By the Markov inequality, we have

$$\begin{aligned} P\left(\left|\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s \mathrm{d}Z_s\right| > \delta\right) &\le \delta^{-1}\varepsilon\sum_{i=1}^{n}\mathbb{E}\left|\int_{t_{i-1}}^{t_i} X_s \mathrm{d}Z_s\right| \le C\delta^{-1}\varepsilon\sum_{i=1}^{n}\mathbb{E}\left(\int_{t_{i-1}}^{t_i} X_s^\alpha \mathrm{d}s\right)^{1/\alpha} \\ &\le CK\delta^{-1}\varepsilon n^{1-1/\alpha} \longrightarrow 0, \end{aligned} \tag{28}$$

where C is constant.

As $\varepsilon \longrightarrow 0$, $n \longrightarrow \infty$, and $\varepsilon n^{1-1/\alpha} \longrightarrow 0$, we obtain

$$\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s \mathrm{d}Z_s \xrightarrow{P} 0. \tag{29}$$

According to Lemma 2, we obtain

$$\begin{aligned} \left|\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} \mathrm{d}Z_s\right| &\le \varepsilon\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}}\right|\left|\int_{t_{i-1}}^{t_i} \sqrt{X_s}\, \mathrm{d}Z_s\right| \le \varepsilon\sum_{i=1}^{n}\left(\left|\frac{1}{X_{t_{i-1}}^0}\right| + \left|\frac{1}{X_{t_{i-1}}} - \frac{1}{X_{t_{i-1}}^0}\right|\right)\left|\int_{t_{i-1}}^{t_i} X_s \mathrm{d}Z_s\right| \\ &\le \varepsilon\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}^0}\right|\left|\int_{t_{i-1}}^{t_i} X_s \mathrm{d}Z_s\right| + \varepsilon\sup_{0\le t\le 1}\left|\frac{1}{X_t} - \frac{1}{X_t^0}\right|\left|\int_{t_{i-1}}^{t_i} X_s \mathrm{d}Z_s\right|. \end{aligned} \tag{30}$$

By the Markov inequality, we have

$$\begin{aligned} P\left(|\varepsilon\sum_{i=1}^{n}|\frac{1}{X_{t_{i-1}}^0}\left|\left|\int_{t_{i-1}}^{t_i} X_s \mathrm{d}Z_s\right|\right| > \delta\right) &\le \delta^{-1}\varepsilon\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}^0}\right|\mathbb{E}\left|\int_{t_{i-1}}^{t_i} X_s \mathrm{d}Z_s\right| \\ &\le C\delta^{-1}\varepsilon\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}^0}\right|\mathbb{E}\left(\int_{t_{i-1}}^{t_i} X_s^\alpha \mathrm{d}s\right)^{1/\alpha} \le CK\delta^{-1}\varepsilon n^{1-1/\alpha}\frac{1}{n}\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}^0}\right| \xrightarrow{P} 0. \end{aligned} \tag{31}$$

As $\varepsilon \longrightarrow 0$, $n \longrightarrow \infty$, and $\varepsilon n^{1-1/\alpha} \longrightarrow 0$, we obtain

$$\varepsilon\sum_{i=1}^{n}\left|\frac{1}{X_{t_{i-1}}^0}\right|\left|\int_{t_{i-1}}^{t_i} \sqrt{X_s}\, \mathrm{d}Z_s\right| \xrightarrow{P} 0. \tag{32}$$

According to Lemma 2, when $\varepsilon \longrightarrow 0$ and $n \longrightarrow \infty$,

$$\varepsilon\sup_{0\le t\le 1}\left|\frac{1}{X_t} - \frac{1}{X_t^0}\right|\left|\int_{t_{i-1}}^{t_i} X_s \mathrm{d}Z_s\right| \xrightarrow{P} 0. \tag{33}$$

From equations (37) and (38), we have

$$\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} \frac{X_s}{X_{t_{i-1}}} \mathrm{d}Z_s \xrightarrow{P} 0. \tag{34}$$

According to equations (27), (30), (31), (33), and (39), when $\varepsilon \longrightarrow 0$, $n \longrightarrow \infty$, and $\varepsilon n^{1-1/\alpha} \longrightarrow 0$,

$$\widehat{\theta}_{n,\varepsilon} \xrightarrow{P} \theta. \tag{35}$$

Using the same methods in Theorem 1, we obtain

$$\theta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s\mathrm{d}s-\theta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}\mathrm{d}s\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\xrightarrow{P}0, \tag{36}$$

$$\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s^2}{X_{t_{i-1}}}\mathrm{d}s\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}-\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s^2\mathrm{d}s\xrightarrow{P}\beta\left(\left(\int_0^1 X_t^0\mathrm{d}t\right)^2-\int_0^1\left(X_t^0\right)^2\mathrm{d}t\right). \tag{37}$$

Together with the results that

$$\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}X_s\mathrm{d}Z_s\xrightarrow{P}0, \tag{38}$$

$$\varepsilon\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\frac{X_s}{X_{t_{i-1}}}\mathrm{d}Z_s\frac{1}{n}\sum_{i=1}^{n}X_{t_{i-1}}\xrightarrow{P}0, \tag{39}$$

when $\varepsilon\longrightarrow 0$, $n\longrightarrow\infty$, and $\varepsilon n^{1-1/\alpha}\longrightarrow 0$, we have

$$\widehat{\beta}_{n,\varepsilon}\xrightarrow{P}\beta. \tag{40}$$

The proof is complete. □

Theorem 2. *When $\varepsilon\longrightarrow 0$, $n\longrightarrow\infty$, and $n\varepsilon\longrightarrow\infty$,*

$$\varepsilon^{-1}\left(\widehat{\theta}_{n,\varepsilon}-\theta\right)\xrightarrow{d}\frac{\left(\left(\int_0^1\left(X_t^0\right)^{\alpha}\mathrm{d}t\right)^{1/\alpha}\int_0^1 X_t^0\mathrm{d}t-\int_0^1\left(X_t^0\right)^2\mathrm{d}t\right)}{\left(\int_0^1 X_t^0\mathrm{d}t\right)^2-\int_0^1\left(X_t^0\right)^2\mathrm{d}t}S_\alpha(1,0,0),\ \varepsilon^{-1}\left(\widehat{\beta}_{n,\varepsilon}-\beta\right)\xrightarrow{d}\frac{\left(\left(\int_0^1\left(X_t^0\right)^{\alpha}\mathrm{d}t\right)^{1/\alpha}-\int_0^1 X_t^0\mathrm{d}t\right)}{\left(\int_0^1 X_t^0\mathrm{d}t\right)^2-\int_0^1\left(X_t^0\right)^2\mathrm{d}t}S_\alpha(1,0,0). \tag{41}$$

Proof. According to the explicit decomposition for $\widehat{\theta}_{n,\varepsilon}$, we have

$$\begin{aligned}\varepsilon^{-1}\left(\widehat{\theta}_{n,\varepsilon}-\theta\right)&=\frac{\varepsilon^{-1}\theta\left(\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s\mathrm{d}s-1/n\sum_{i=1}^nX_{t_{i-1}}\right)1/n\sum_{i=1}^nX_{t_{i-1}}}{\left(1/n\sum_{i=1}^nX_{t_{i-1}}\right)^2-1/n\sum_{i=1}^nX_{t_{i-1}}^2}\\&+\frac{\varepsilon^{-1}\theta\left(1-\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s/X_{t_{i-1}}\mathrm{d}s\right)1/n\sum_{i=1}^nX_{t_{i-1}}^2}{\left(1/n\sum_{i=1}^nX_{t_{i-1}}\right)^2-1/n\sum_{i=1}^nX_{t_{i-1}}^2}\\&+\frac{\varepsilon^{-1}\beta\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s^2/X_{t_{i-1}}\mathrm{d}s1/n\sum_{i=1}^nX_{t_{i-1}}^2-\varepsilon^{-1}\beta\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s^2\mathrm{d}s1/n\sum_{i=1}^nX_{t_{i-1}}}{\left(1/n\sum_{i=1}^nX_{t_{i-1}}\right)^2-1/n\sum_{i=1}^nX_{t_{i-1}}^2}\\&+\frac{\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s\mathrm{d}Z_s1/n\sum_{i=1}^nX_{t_{i-1}}-\sum_{i=1}^n\int_{t_{i-1}}^{t_i}X_s/X_{t_{i-1}}\mathrm{d}Z_s1/n\sum_{i=1}^nX_{t_{i-1}}^2}{\left(1/n\sum_{i=1}^nX_{t_{i-1}}{}^2-1/n\sum_{i=1}^nX_{t_{i-1}}^2\right.}.\end{aligned} \tag{42}$$

From Lemma 2, when $\varepsilon\longrightarrow 0$, $n\longrightarrow\infty$, and $n\varepsilon\longrightarrow\infty$,

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s\mathrm{d}s - \frac{1}{n}\sum_{i=1}^{n} X_{t_{i-1}} \xrightarrow{P} 0,$$

$$\begin{aligned}\left|\varepsilon^{-1}n^{-1}\sum_{i=1}^{n} X_{t_{i-1}}\right| &\leq \varepsilon^{-1}n^{-1}\sum_{i=1}^{n}\left|X_{t_{i-1}}\right| = \varepsilon^{-1}n^{-1}\sum_{i=1}^{n}\left|X_{t_{i-1}} - X_{t_{i-1}}^0 + X_{t_{i-1}}^0\right| \\ &\leq \varepsilon^{-1}n^{-1}\sum_{i=1}^{n}\left(\left|X_{t_{i-1}} - X_{t_{i-1}}^0\right| + \left|X_{t_{i-1}}^0\right|\right) \leq \varepsilon^{-1}n^{-1}\sup_{0\leq t\leq 1}\left(\left|X_t - X_t^0\right| + \left|X_t^0\right|\right) \xrightarrow{P} 0.\end{aligned} \tag{43}$$

From equations (45) and (46), we have

$$\begin{aligned}&\frac{\varepsilon^{-1}\theta\left(\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s\mathrm{d}s - 1/n\sum_{i=1}^{n} X_{t_{i-1}}\right)1/n\sum_{i=1}^{n} X_{t_{i-1}}}{\left(1/n\sum_{i=1}^{n} X_{t_{i-1}}\right)^2 - 1/n\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{P} 0, \\ &\frac{\varepsilon^{-1}\theta\left(1 - \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s/X_{t_{i-1}}\mathrm{d}s\right)1/n\sum_{i=1}^{n} X_{t_{i-1}}^2}{\left(1/n\sum_{i=1}^{n} X_{t_{i-1}}\right)^2 - 1/n\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{P} 0,\end{aligned} \tag{44}$$

$$\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s^2/X_{t_{i-1}}\mathrm{d}s1/n\sum_{i=1}^{n} X_{t_{i-1}}^2 - \varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s^2\mathrm{d}s1/n\sum_{i=1}^{n} X_{t_{i-1}}}{\left(1/n\sum_{i=1}^{n} X_{t_{i-1}}\right)^2 - 1/n\sum_{i=1}^{n} X_{t_{i-1}}^2} \xrightarrow{P} 0. \tag{45}$$

According to Lemma 2, we obtain

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s\mathrm{d}Z_s = \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s^0\mathrm{d}Z_s + \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(X_s - X_s^0\right)\mathrm{d}Z_s. \tag{46}$$

Using the Markov inequality and Holder's inequality, for any given $\delta > 0$, we have

$$\begin{aligned}P\left(\left|\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i}\left(X_s - X_s^0\right)\mathrm{d}Z_s\right| > \delta\right) &\leq \delta^{-1}\sum_{i=1}^{n}\mathbb{E}\left(\left|\int_{t_{i-1}}^{t_i}\left(X_s - X_s^0\right)\mathrm{d}Z_s\right|\right) \\ &\leq C\delta^{-1}\sum_{i=1}^{n}\mathbb{E}\left(\left(\int_{t_{i-1}}^{t_i}\left|X_s - X_s^0\right|^\alpha\mathrm{d}s\right)^{1/\alpha}\right) \leq C\delta^{-1}\sum_{i=1}^{n}\mathbb{E}\left(\sup_{0\leq t\leq 1}\left|X_t - X_t^0\right|n^{-1/\alpha}\right)^{1/2} \\ &\leq C\delta^{-1}n^{1-1/\alpha}\mathbb{E}\left(\sup_{0\leq t\leq 1}\left|X_t - X_t^0\right|\right) \longrightarrow 0,\end{aligned} \tag{47}$$

as $\varepsilon \longrightarrow 0$, $n \longrightarrow \infty$, and $n^{1-1/\alpha} \longrightarrow 0$.

Moreover,

$$\begin{aligned}\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s^0\mathrm{d}Z_s &= \int_0^1\sum_{i=1}^{n} X_s^0 1_{(t_{i-1},t_i]}(s)\mathrm{d}Z_s \\ &= Z'\circ\int_0^1\sum_{i=1}^{n}\left(X_s^0 1_{(t_{i-1},t_i]}(s)\right)^\alpha\mathrm{d}s,\end{aligned} \tag{48}$$

where $Z' \overset{d}{=} Z$.

According to Lemma 1, we have

$$\int_0^1\sum_{i=1}^{n}\left(X_s^0 1_{(t_{i-1},t_i]}(s)\right)^\alpha\mathrm{d}s \longrightarrow \int_0^1\left(X_s^0\right)^\alpha\mathrm{d}s, \tag{49}$$

$$Z'\circ\int_0^1\sum_{i=1}^{n}\left(X_s^0 1_{(t_{i-1},t_i]}(s)\right)^\alpha\mathrm{d}s \xrightarrow{\text{a.s.}} Z'\circ\int_0^1\left(X_s^0\right)^\alpha\mathrm{d}s. \tag{50}$$

From equations (53) and (54),

$$\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s^0 \mathrm{d}Z_s \xrightarrow{d} \left(\int_0^1 \left(X_t^0\right)^{\alpha}\right)^{1/\alpha} S_{\alpha}(1,0,0). \tag{51}$$

According to equations (45)–(51) and (55), we have

$$\varepsilon^{-1}\left(\widehat{\theta}_{n,\varepsilon}-\theta\right) \xrightarrow{d} \frac{\left(\left(\int_0^1 (X_t^0)^{\alpha}\mathrm{d}t\right)^{1/\alpha}\int_0^1 X_t^0\mathrm{d}t - \int_0^1\left(X_t^0\right)^2\mathrm{d}t\right)}{\left(\int_0^1 X_t^0\mathrm{d}t\right)^2 - \int_0^1\left(X_t^0\right)^2\mathrm{d}t} S_{\alpha}(1,0,0). \tag{52}$$

From equation (8), we obtain

$$\begin{aligned}\varepsilon^{-1}\left(\widehat{\beta}_{n,\varepsilon}-\beta\right) &= \frac{\varepsilon^{-1}\theta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s\mathrm{d}s - \varepsilon^{-1}\theta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s/X_{t_{i-1}}\mathrm{d}s1/n\sum_{i=1}^{n}X_{t_{i-1}}}{\left(1/n\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - 1/n\sum_{i=1}^{n}X_{t_{i-1}}^2} \\ &\quad + \frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s^2/X_{t_{i-1}}\mathrm{d}s1/n\sum_{i=1}^{n}X_{t_{i-1}} - \varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s^2\mathrm{d}s}{\left(1/n\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - 1/n\sum_{i=1}^{n}X_{t_{i-1}}^2} \\ &\quad + \frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s\mathrm{d}Z_s - \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s/X_{t_{i-1}}\mathrm{d}Z_s1/n\sum_{i=1}^{n}X_{t_{i-1}}}{\left(1/n\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - 1/n\sum_{i=1}^{n}X_{t_{i-1}}^2} - \varepsilon^{-1}\beta.\end{aligned} \tag{53}$$

According to equations (25)–(27) and (55), we have

$$\frac{\varepsilon^{-1}\theta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s\mathrm{d}s - \varepsilon^{-1}\theta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s/X_{t_{i-1}}\mathrm{d}s1/n\sum_{i=1}^{n}X_{t_{i-1}}}{\left(1/n\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - 1/n\sum_{i=1}^{n}X_{t_{i-1}}^2} \xrightarrow{P} 0, \tag{54}$$

$$\frac{\varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s^2/X_{t_{i-1}}\mathrm{d}s1/n\sum_{i=1}^{n}X_{t_{i-1}} - \varepsilon^{-1}\beta\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s^2\mathrm{d}s}{\left(1/n\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - 1/n\sum_{i=1}^{n}X_{t_{i-1}}^2} - \varepsilon^{-1}\beta \xrightarrow{P} 0, \tag{55}$$

$$\frac{\sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s\mathrm{d}Z_s - \sum_{i=1}^{n}\int_{t_{i-1}}^{t_i} X_s/X_{t_{i-1}}\mathrm{d}Z_s1/n\sum_{i=1}^{n}X_{t_{i-1}}}{\left(1/n\sum_{i=1}^{n}X_{t_{i-1}}\right)^2 - 1/n\sum_{i=1}^{n}X_{t_{i-1}}^2} \xrightarrow{d} \frac{\left(\left(\int_0^1 (X_t^0)^{\alpha}\mathrm{d}t\right)^{1/\alpha} - \int_0^1 X_t^0\mathrm{d}t\right)}{\left(\int_0^1 X_t^0\mathrm{d}t\right)^2 - \int_0^1\left(X_t^0\right)^2\mathrm{d}t} S_{\alpha}(1,0,0). \tag{56}$$

From equations (54)–(56), we have

$$\varepsilon^{-1}\left(\widehat{\beta}_{n,\varepsilon}-\beta\right) \xrightarrow{d} \frac{\left(\left(\int_0^1 (X_t^0)^{\alpha}\mathrm{d}t\right)^{1/\alpha} - \int_0^1 X_t^0\mathrm{d}t\right)}{\left(\int_0^1 X_t^0\mathrm{d}t\right)^2 - \int_0^1\left(X_t^0\right)^2\mathrm{d}t} S_{\alpha}(1,0,0). \tag{57}$$

The proof is complete. □

4. Simulation

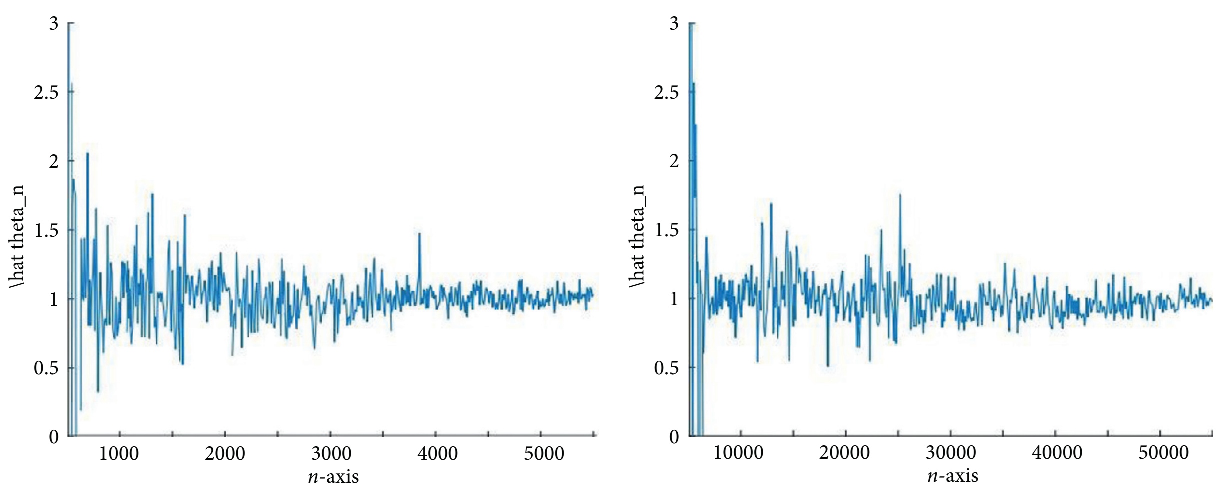
In this experiment, we generate a discrete sample $(X_{t_{i-1}})_{i=1,\ldots,n}$ and compute $\widehat{\theta}_{n,\varepsilon}$ and $\widehat{\beta}_{n,\varepsilon}$ from the sample. Let $x_0 = 0.2$. For every given true value of the parameters- (θ,β),

Table 1: Least squares estimator simulation results of θ and β.

True value (θ, β)	Size n	Average value $\widehat{\theta}_{n,\varepsilon}$	$\widehat{\beta}_{n,\varepsilon}$	Absolute error $\lvert\widehat{\theta}_{n,\varepsilon} - \theta\rvert$	$\lvert\widehat{\beta}_{n,\varepsilon} - \beta\rvert$
(1, 2)	1000	1.1526	2.1368	0.1526	0.1368
	3000	1.0379	2.0151	0.0379	0.0151
	5000	1.0010	2.0007	0.0010	0.0007
(3, 4)	1000	3.1643	4.1538	0.1643	0.1538
	3000	3.0236	4.0320	0.0236	0.0320
	5000	3.0012	4.0014	0.0012	0.0014

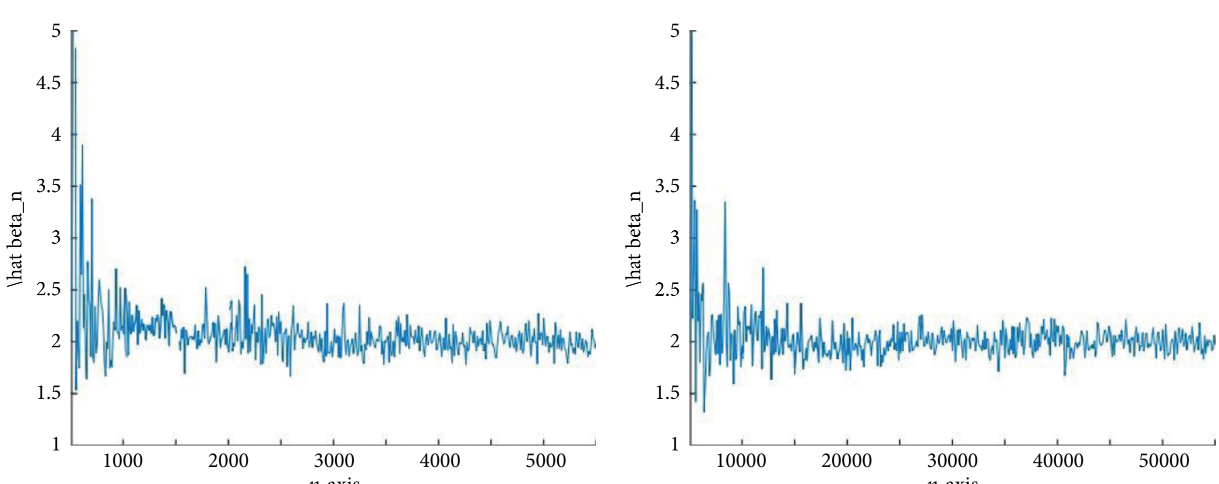
Table 2: Least squares estimator simulation results of θ and β.

True value (θ, β)	Size n	Average value $\widehat{\theta}_{n,\varepsilon}$	$\widehat{\beta}_{n,\varepsilon}$	Absolute error $\lvert\widehat{\theta}_{n,\varepsilon} - \theta\rvert$	$\lvert\widehat{\beta}_{n,\varepsilon} - \beta\rvert$
(1, 2)	10,000	1.1041	2.0925	0.1041	0.0925
	30,000	1.0053	2.0087	0.0053	0.0087
	50,000	1.0005	2.0007	0.0005	0.0007
(3, 4)	10,000	3.1154	4.1025	0.1154	0.1025
	30,000	3.0127	4.0089	0.0127	0.0089
	50,000	3.0009	4.0003	0.0009	0.0003



(a) (b)

Figure 1: The simulation of the estimator $\widehat{\theta}_{n,\varepsilon}$ with $\theta = 1$.



(a) (b)

Figure 2: The simulation of the estimator $\widehat{\beta}_{n,\varepsilon}$ with $\beta = 2$.

the size of the sample is represented as"Size n" and given in the first column of the table. In Table 1, $\varepsilon = 0.1$, the size is increasing from 1000 to 5000. In Table 2, $\varepsilon = 0.01$, the size is increasing from 10,000 to 50,000. The tables list the value of "$\widehat{\theta}_{n,\varepsilon}$", "$\widehat{\beta}_{n,\varepsilon}$," and the absolute errors of least squares estimators.

Two tables illustrate that when n is large enough and ε is small enough, the obtained estimators are very close to the true parameter value. Therefore, the methods used in this paper are effective and the obtained estimators are good.

In Figure 1, $\theta = 1$; when $\varepsilon = 0.1$, the size is increasing from 1000 to 5000; when $\varepsilon = 0.01$, the size is increasing from 10,000 to 50,000. In Figure 2, $\beta = 2$; when $\varepsilon = 0.1$, the size is increasing from 1000 to 5000; when $\varepsilon = 0.01$, the size is increasing from 10,000 to 50,000. Two figures illustrate that when n is large enough and ε is small enough, the obtained estimators are very close to the true parameter value.

5. Conclusion

The parameter estimation problem for discretely observed stochastic Lotka–Volterra model driven by small α-stable noises has been studied. The contrast function has been given to obtain the least squares estimators. The consistency and asymptotic distribution of the least squares estimators have been discussed by using the Markov inequality, Cauchy–Schwarz inequality, and Gronwall's inequality. Further research topics will include parameter and state estimation for partially observed stochastic system driven by α-stable noises.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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