Research Article

Half-Logistic Xgamma Distribution: Properties and Estimation under Censored Samples

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This paper proposed a new probability distribution, namely, the half-logistic xgamma (HLXG) distribution. Various statistical properties, such as, moments, incomplete moments, mean residual life, and stochastic ordering of the proposed distribution, are discussed. Parameter estimation of the half-logistic xgamma distribution is approached by the maximum likelihood method based on complete and censored samples. Asymptotic confidence intervals of model parameters are provided. A simulation study is conducted to illustrate the theoretical results. Moreover, the model parameters of the HLXG distribution are estimated by using the maximum likelihood, least square, maximum product spacing, percentile, and Cramer-von Mises (CVM) methods. Superiority of the new model over some existing distributions is illustrated through three real data sets.

1. Introduction

In the recent years, generalization and extension of existing distributions, to enhance flexibility in modeling a variety of data, have received attention of many researchers. Marshall and Olkin [1] proposed a general method of adding a parameter to a family of distributions which is called the Marshall-Olkin-G (MO-G) class. The new parameter gives more flexibility and extends several well-known distributions. For any continuous baseline with the cumulative distribution function (cdf) say $G(\cdot)$, and the probability density function (pdf), say $g(\cdot)$, the MO-G family is defined as follows:

$$f_{\text{MO-G}}(x; \delta) = \frac{\delta g(x)}{[1 - \bar{\delta}(\bar{G}(x))]^2}, \quad (1)$$

where $\delta > 0$, $\bar{\delta} = 1 - \delta$, and $\bar{G}(x) = 1 - G(x)$ is the survival function (sf). Lately, some of the notable new families of distributions include beta-G [2, 3], gamma-G [4], Kumaraswamy-G [5], transformed-transformer [6], exponentiated generalized [7], Weibull-G [8], exponentiated half-logistic-G [9], Kumaraswamy-Weibull-G [10], exponentiated Weibull-G

[11], additive Weibull-G [12], type I half-logistic-G [13], type II half-logistic-G (TIIHL-G) [10], generalized additive Weibull-G [14], odd exponentiated half-logistic-G [15], generalized odd log-logistic-G [16], inverse Weibull-G [17], power Lindley-G [18], and type II generalized Topp-Leone-G [19].

Hassan et al. [20] proposed a class of probability distribution generated by a half-logistic random variable with the following cdf:

$$F_{\text{TIIHL-G}}(x; \lambda, \zeta) = \frac{2[G(x; \zeta)]^\lambda}{1 + [G(x; \zeta)]^\lambda}, \quad \lambda > 0, x \in R, \quad (2)$$

where λ is the shape parameter and $G(x; \zeta)$ is a baseline cdf, which relies on a parameter vector ζ . The pdf corresponding to (2) is given by

$$f_{\text{TIIHL-G}}(x; \lambda, \zeta) = \frac{2\lambda g(x; \zeta)[G(x; \zeta)]^{\lambda-1}}{[1 + [G(x; \zeta)]^\lambda]^2}, \quad \lambda > 0. \quad (3)$$

For $\lambda = 1$ in (3) and $\delta = 0.5$ in (1), pdf (3) provides the MO-G family. The pdf (3) provides a number of known

distributions as particular cases with more flexibility in their skewness and kurtosis.

Recently, Sen et al. [21] proposed the xgamma (XG) distribution, following the idea of Lindley distribution. The XG model is a special finite mixture from exponential with a scale parameter (θ) and gamma with a scale parameter (θ) and shape parameter (3). The pdf and cdf of the XG distribution are, respectively, given by

$$G(x; \theta) = 1 - \frac{(1 + \theta + \theta x + (\theta^2 x^2/2))}{(1 + \theta)} e^{-\theta x}, \quad x > 0, \theta > 0, \quad (4)$$

and

$$g(x; \theta) = \frac{\theta^2}{1 + \theta} \left(1 + \frac{\theta}{2} x^2\right) e^{-\theta x}, \quad x > 0, \theta > 0. \quad (5)$$

Sen et al. [21] investigated the structural properties of the XG distribution, and they have found that in many cases that the XG distribution has more flexibility than the exponential distribution. The extensions of XG distribution were studied in [22, 23] and [24], among others.

Our objective here is two folds: First, based on the TIHLG class, we propose a new distribution related to xgamma distribution. We call the new model as the half-logistic xgamma (HLXG) distribution, and we study its several statistical properties. Second, the maximum likelihood (ML) method is employed to estimate the model parameters of the HLXG based on complete and censored samples. Further, $(1 - \nu)\%$ asymptotic confidence intervals (CIs) of the model parameters are constructed. Simulation and application issues are considered. The rest of the paper is organized as follows. Sections 2 and 3 provide the pdf, cdf, hazard rate function (hrf), and structure properties of the HLXG model. In Section 4, point and approximate CIs of model parameters are derived under complete, type I censoring (TIC) and type II censoring (TIIC). Also, in the same section, the behavior of the estimates is studied via a simulation study. Real data sets are analyzed to demonstrate the flexibility of the HLXG distribution over some known distributions in Section 5. The article ends with some concluding remarks.

2. The Half-Logistic Xgamma Distribution

In this section, we provide a more flexible model by adding one extra shape parameter to the XG model for improving its goodness-of-fit to real data. The motivations of the HLXG distribution are (i) to obtain a more flexible pdf with right skewed, unimodal, and reversed J -shape; (ii) to be capable of modeling decreasing, increasing, and semibathhtub hazard rate shapes; and (iii) to provide more flexibility to model the various types of data.

Definition 1. A random variable X is said to have a HLXG distribution, if its cdf is given by

$$F(x; \theta, \lambda) = 2[1 - A(\theta, x)]^\lambda (1 + [1 - A(\theta, x)]^\lambda)^{-1}, \quad x > 0, \quad (6)$$

where $A(\theta, x) = (1 + \theta + \theta x + (\theta^2 x^2)/2)/(1 + \theta)e^{-\theta x}$.

The corresponding pdf is given as follows:

$$f(x; \lambda, \theta) = \frac{2\lambda\theta^2(1 + \theta x^2/2)e^{-\theta x}[1 - A(\theta, x)]^{\lambda-1}}{(1 + \theta)[1 + [1 - A(\theta, x)]^\lambda]^2}, \quad x, \theta, \lambda > 0. \quad (7)$$

A random variable X with pdf (7) will be denoted by $HLXG \sim (\lambda, \theta)$. For $\lambda = 1$ in (7), we obtain the MO extended xgamma distribution with $\delta = 0.5$ as a special new model. Further, the sf and hrf of the HLXG are given, respectively, by $\bar{F}(x; \lambda, \theta) = 1 - [1 - A(x, \theta)]^\lambda / [1 + [1 - A(x, \theta)]^\lambda]$ and $h(x; \lambda, \theta) = 2\lambda\theta^2(1 + \theta x^2/2)e^{-\theta x}[1 - A(\theta, x)]^{\lambda-1} / (1 + \theta)[1 - [1 - A(\theta, x)]^{2\lambda}]$.

The pdf and hrf plots for the HLXG are displayed in Figure 1 for some given values of parameters.

Figure 1 reveals that the pdf of X is quite flexible and can take asymmetric forms, among others. Also, the hrf can be increasing, decreasing, and semibathhtub shapes. In general, they reinforce the importance of the HLXG model to fit real lifetime data.

3. Main Properties

In this section, we give some important properties of the HLXG distribution such as ordinary and incomplete moments, stochastic ordering, mean residual life, and mean waiting time, among others.

3.1. Expansion. This section provides a useful expansion of the HLXG pdf due its complicated form to obtain its structure properties. Since the generalized binomial series is

$$(1 + z)^{-c} = \sum_{i=0}^{\infty} (-1)^i \binom{c + i - 1}{i} z^i, \quad |z| < 1, c > 0, \quad (8)$$

then, by applying (8) in (7), we have

$$f(x; \lambda, \theta) = \sum_{i=0}^{\infty} \frac{(-1)^i (i + 1) 2\lambda\theta^2}{(1 + \theta)} \left(1 + \frac{\theta x^2}{2}\right) e^{-\theta x} [1 - A(\theta, x)]^{\lambda(i+1)-1}. \quad (9)$$

Employing the binomial expansion in the last term of (9), we have $f(x; \lambda, \theta) = \sum_{i,j=0}^{\infty} (-1)^{i+j} (i + 1) 2\lambda\theta^2 / (1 + \theta) \binom{\lambda(i+1)-1}{j} (1 + \theta x^2/2) e^{-\theta(j+1)x} (A(\theta, x))^j$.

Again, we use the binomial expansion more than one time; then, we have

$$f(x; \lambda, \theta) = \sum_{i,j=0}^{\infty} \eta_{i,j,m,k} x^{m+k} e^{-\theta(j+1)x} + \psi_{i,j,m,k} x^{m+k+2} e^{-\theta(j+1)x}, \quad (10)$$

where $\eta_{i,j,m,k} = \sum_{m=0}^j \sum_{k=0}^m (-1)^{i+j} (i + 1) 2^{1-k} \lambda \theta^{2+m+k} / (1 + \theta)^{m+1} \binom{\lambda(i+1)-1}{j} \binom{j}{m} \binom{m}{k}$ and $\psi_{i,j,m,k} = \sum_{m=0}^j \sum_{k=0}^m (-1)^{i+j} (i + 1) 2^{-k} \lambda \theta^{3+m+k} / (1 + \theta)^{m+1} \binom{\lambda(i+1)-1}{j} \binom{j}{m} \binom{m}{k}$.

FIGURE 1: The pdf and hrf plots for different values of parameters.

3.2. Moments and Related Statistics. In this section, we obtain moments and some related measures of the HLXG distribution with pdf defined in (10). The s^{th} moment about the origin of the random variable X has the HLXG distribution obtained as follows:

$$\mu'_s = \sum_{i,j=0}^{\infty} \int_0^{\infty} x^s (\eta_{i,j,m,k} x^{m+k} e^{-\theta(j+1)x} + \psi_{i,j,m,k} x^{m+k+2} e^{-\theta(j+1)x}) dx. \quad (11)$$

After simplification, the s^{th} moment of the HLXG distribution is given by

$$\mu'_s = \sum_{i,j=0}^{\infty} \frac{\eta_{i,j,m,k} \Gamma(m+k+s+1)}{(\theta(j+1))^{m+k+s+1}} + \frac{\psi_{i,j,m,k} \Gamma(m+k+s+3)}{(\theta(j+1))^{m+k+s+3}}, \quad (12)$$

where $\Gamma(\cdot)$ is the gamma function. Furthermore, the s^{th} central moment of a given random variable X is defined by

$$\mu_s = E(X - \mu'_1)^s = \sum_{i=0}^s (-1)^i \binom{s}{i} (\mu'_1)^i \mu'_{s-i}. \quad (13)$$

Table 1 gives the numerical values for certain parameter values of mean (μ'_1), variance (σ^2), coefficient of skewness (CS), and coefficient of kurtosis (CK) of the HLXG

distribution. It is evident from Table 1 that skewness and kurtosis heavily depend on the value of the parameters of the new distribution.

Additionally, the s^{th} lower incomplete moment, say $\pi_s(t)$, of the HLXG distribution is given by

$$\pi_s(t) = \sum_{i,j=0}^{\infty} \int_0^t x^s (\eta_{i,j,m,k} x^{m+k} e^{-\theta(j+1)x} + \psi_{i,j,m,k} x^{m+k+2} e^{-\theta(j+1)x}) dx, \quad (14)$$

which leads to

$$\pi_s = \sum_{i,j=0}^{\infty} \frac{\eta_{i,j,m,k} \gamma(m+k+s+1, \theta(j+1)t)}{(\theta(j+1))^{m+k+s+1}} + \frac{\psi_{i,j,m,k} \gamma(m+k+s+3, \theta(j+1)t)}{(\theta(j+1))^{m+k+s+3}}, \quad (15)$$

where $\gamma(\cdot, t)$ is the lower incomplete gamma function. The first incomplete moment, say π_1 , for $s=1$ in (15) is obtained. The Lorenz and Bonferroni curves are the essential applications of the first incomplete moment. The Lorenz curve, say $LO(t)$, and the Bonferroni curve, say $BO(t)$, of the HLXG are obtained, respectively, as follows:

$$LO(t) = \frac{\pi_1(t)}{E(T)} = \frac{\sum_{i,j=0}^{\infty} (\eta_{i,j,m,k} \gamma(m+k+2, \theta(j+1)t) / (\theta(j+1))^{m+k+2}) + \psi_{i,j,m,k} \gamma(m+k+4, \theta(j+1)t) / (\theta(j+1))^{m+k+4}}{\sum_{i,j=0}^{\infty} (\eta_{i,j,m,k} \Gamma(m+k+2) / (\theta(j+1))^{m+k+2}) + \psi_{i,j,m,k} \Gamma(m+k+4) / (\theta(j+1))^{m+k+4}}, \quad (16)$$

TABLE 1: Some moments of the HLXG distribution.

μ'_s	$(\lambda = 0.5, \theta = 2)$	$(\lambda = 1, \theta = 2)$	$(\lambda = 2, \theta = 2)$	$(\lambda = 3, \theta = 2)$	$(\lambda = 0.5, \theta = 0.5)$	$(\lambda = 1, \theta = 0.5)$
μ'_1	0.315	0.559	0.897	1.124	2.192	4.014
σ^2	0.296	0.451	0.606	0.691	7.405	8.322
CS	3.119	2.22	1.59	1.275	2.118	1.775
CK	15.912	9.586	6.537	5.495	9.05	7.273

TABLE 2: ML estimates, biases, MSEs, LBs, UBs, and ALs of the HLXG model under TIIC for $(\lambda = 0.5$ and $\theta = 0.5)$.

n	r	Parameters	ML	Bias	MSE	90%		AL	95%	
						LB	UB		LB	UB
30	21	λ	0.5510	0.0510	0.0208	0.3525	0.7495	0.3970	0.3145	0.7875
		θ	0.5794	0.0794	0.0524	0.2489	0.9099	0.6610	0.1856	0.9732
	27	λ	0.5393	0.0393	0.0169	0.3671	0.7114	0.3443	0.3342	0.7444
		θ	0.5460	0.0460	0.0335	0.3104	0.7816	0.4712	0.2653	0.8267
	30	λ	0.5279	0.0279	0.0121	0.3666	0.6893	0.3226	0.3357	0.7201
		θ	0.5217	0.0217	0.0179	0.3207	0.7227	0.4020	0.2822	0.7612
50	35	λ	0.5198	0.0198	0.0092	0.3758	0.6638	0.2880	0.3482	0.6913
		θ	0.5645	0.0645	0.0275	0.3030	0.8261	0.5232	0.2529	0.8762
	45	λ	0.5147	0.0147	0.0077	0.3883	0.6410	0.2528	0.3641	0.6652
		θ	0.5459	0.0459	0.0138	0.3575	0.7344	0.3768	0.3214	0.7704
	50	λ	0.5137	0.0137	0.0075	0.3928	0.6347	0.2419	0.3696	0.6578
		θ	0.5419	0.0419	0.0103	0.3772	0.7067	0.3295	0.3456	0.7382
100	70	λ	0.5105	0.0105	0.0039	0.4111	0.6099	0.1988	0.3920	0.6289
		θ	0.5155	0.0155	0.0109	0.3486	0.6823	0.3337	0.3167	0.7143
	90	λ	0.5050	0.0050	0.0029	0.4177	0.5922	0.1745	0.4010	0.6089
		θ	0.5017	0.0017	0.0061	0.3802	0.6231	0.2430	0.3569	0.6464
	100	λ	0.5073	0.0073	0.0030	0.4232	0.5915	0.1683	0.4071	0.6076
		θ	0.5049	0.0049	0.0040	0.3971	0.6126	0.2155	0.3765	0.6333

TABLE 3: ML estimates, biases, MSEs, LBs, UBs, and ALs of the HLXG model under complete and TIIC for $(\lambda = 0.5$ and $\theta = 1.2)$.

n	r	Parameters	ML	Bias	MSE	90%		AL	95%	
						LB	UB		LB	UB
30	21	λ	0.5416	0.0416	0.0229	0.3455	0.7377	0.3922	0.3080	0.7752
		θ	1.5025	0.3025	0.5847	0.4767	2.5283	2.0515	0.2803	2.7247
	27	λ	0.5344	0.0344	0.0144	0.3641	0.7046	0.3405	0.3315	0.7372
		θ	1.4224	0.2224	0.1901	0.7016	2.1433	1.4417	0.5636	2.2813
	30	λ	0.5311	0.0311	0.0141	0.3704	0.6918	0.3215	0.3396	0.7226
		θ	1.3807	0.1807	0.1364	0.7849	1.9765	1.1916	0.6708	2.0906
50	35	λ	0.5318	0.0318	0.0097	0.3834	0.6801	0.2967	0.3550	0.7085
		θ	1.4902	0.2902	0.3362	0.6973	2.2832	1.5859	0.5454	2.4351
	45	λ	0.5152	0.0152	0.0048	0.3892	0.6412	0.2521	0.3650	0.6654
		θ	1.3701	0.1701	0.1509	0.8284	1.9118	1.0834	0.7247	2.0156
	50	λ	0.5117	0.0117	0.0047	0.3929	0.6306	0.2377	0.3701	0.6533
		θ	1.3409	0.1409	0.1264	0.8884	1.7933	0.9048	0.8018	1.8799
100	70	λ	0.5261	0.0261	0.0052	0.4225	0.6297	0.2073	0.4026	0.6496
		θ	1.3005	0.1005	0.1041	0.8214	1.7797	0.9583	0.7296	1.8715
	90	λ	0.5223	0.0223	0.0036	0.4317	0.6128	0.1810	0.4144	0.6301
		θ	1.2696	0.0696	0.0451	0.9241	1.6150	0.6908	0.8580	1.6811
	100	λ	0.5223	0.0223	0.0033	0.4363	0.6083	0.1721	0.4198	0.6248
		θ	1.2722	0.0722	0.0419	0.9750	1.5693	0.5943	0.9181	1.6262

TABLE 4: ML estimates, biases, MSEs, LBs, UBs, and ALs of the HLXG model under complete and TIIC for ($\lambda = 1.7$ and $\theta = 0.9$).

n	r	Parameters	ML	Bias	MSE	90%		AL	95%	
						LB	UB		LB	UB
30	21	λ	1.9508	0.2508	0.4006	1.0597	2.8420	1.7823	0.8890	3.0126
		θ	0.9795	0.0795	0.0445	0.6481	1.3110	0.6630	0.5846	1.3745
	27	λ	1.8748	0.1748	0.2306	1.1078	2.6419	1.5341	0.9609	2.7887
		θ	0.9540	0.0540	0.0332	0.6768	1.2311	0.5543	0.6237	1.2842
	30	λ	1.8608	0.1608	0.1895	1.1304	2.5912	1.4608	0.9905	2.7311
		θ	0.9501	0.0501	0.0278	0.6911	1.2091	0.5180	0.6415	1.2587
50	35	λ	1.8334	0.1334	0.1636	1.1994	2.4675	1.2681	1.0780	2.5889
		θ	0.9632	0.0632	0.0217	0.7077	1.2187	0.5110	0.6588	1.2676
	45	λ	1.8020	0.1020	0.1246	1.2391	2.3649	1.1257	1.1313	2.4727
		θ	0.9494	0.0494	0.0158	0.7344	1.1645	0.4301	0.6932	1.2057
	50	λ	1.7859	0.0859	0.1112	1.2500	2.3219	1.0718	1.1474	2.4245
		θ	0.9416	0.0416	0.0139	0.7415	1.1417	0.4002	0.7032	1.1800
100	70	λ	1.8003	0.1003	0.0931	1.3617	2.2389	0.8772	1.2777	2.3229
		θ	0.9242	0.0242	0.0113	0.7506	1.0977	0.3471	0.7174	1.1309
	90	λ	1.8112	0.1112	0.0777	1.4112	2.2112	0.8000	1.3346	2.2878
		θ	0.9324	0.0324	0.0095	0.7838	1.0811	0.2973	0.7553	1.1096
	100	λ	1.8208	0.1208	0.0743	1.4336	2.2079	0.7743	1.3595	2.2821
		θ	0.9377	0.0377	0.0087	0.7978	1.0777	0.2799	0.7710	1.1045

TABLE 5: ML estimates, biases, MSEs, LBs, UBs, and ALs of the HLXG model under complete and TIC for ($\lambda = 0.5$ and $\theta = 0.5$).

n	T	Parameters	ML	Bias	MSE	90%		AL	95%	
						LB	UB		LB	UB
30	2.5	λ	0.5234	0.0234	0.0128	0.3389	0.7079	0.3690	0.3036	0.7432
		θ	0.5466	0.0466	0.0415	0.2456	0.8475	0.6020	0.1880	0.9052
	4	λ	0.5223	0.0223	0.0116	0.3502	0.6944	0.3442	0.3172	0.7273
		θ	0.5441	0.0441	0.0300	0.2863	0.8019	0.5155	0.2369	0.8512
50	2.5	λ	0.5069	0.0069	0.0058	0.3692	0.6446	0.2754	0.3429	0.6710
		θ	0.5151	0.0151	0.0158	0.2899	0.7403	0.4504	0.2467	0.7834
	4	λ	0.5093	0.0093	0.0048	0.3802	0.6384	0.2582	0.3555	0.6631
		θ	0.5221	0.0221	0.0122	0.3282	0.7161	0.3879	0.2910	0.7532
100	2.5	λ	0.4993	-0.0007	0.0045	0.4040	0.5946	0.1906	0.3858	0.6128
		θ	0.5067	0.0067	0.0089	0.3493	0.6641	0.3148	0.3192	0.6943
	4	λ	0.4990	-0.0010	0.0040	0.4098	0.5882	0.1784	0.3927	0.6053
		θ	0.5052	0.0052	0.0064	0.3706	0.6397	0.2691	0.3449	0.6655

TABLE 6: ML estimates, biases, MSEs, LBs, UBs, and ALs of the HLXG under complete and TIC for ($\lambda = 0.5$ and $\theta = 1.2$).

n	T	Parameters	ML	Bias	MSE	90%		AL	95%	
						LB	UB		LB	UB
30	2.5	λ	0.5270	0.0270	0.0121	0.3630	0.6910	0.3280	0.3316	0.7224
		θ	1.3309	0.1309	0.1940	0.7125	1.9493	1.2367	0.5941	2.0677
	4	λ	0.5299	0.0299	0.0114	0.3687	0.6911	0.3224	0.3379	0.7220
		θ	1.3492	0.1492	0.1885	0.7610	1.9374	1.1764	0.6483	2.0500
50	2.5	λ	0.5220	0.0220	0.0074	0.3963	0.6476	0.2513	0.3723	0.6717
		θ	1.2486	0.0486	0.1130	0.7989	1.6983	0.8994	0.7128	1.7844
	4	λ	0.5219	0.0219	0.0072	0.3989	0.6449	0.2460	0.3754	0.6685
		θ	1.2468	0.0468	0.1042	0.8242	1.6693	0.8451	0.7433	1.7502
100	2.5	λ	0.5001	0.0001	0.0024	0.4157	0.5844	0.1687	0.3996	0.6006
		θ	1.2348	0.0348	0.0406	0.9149	1.5547	0.6398	0.8536	1.6159
	4	λ	0.4995	-0.0005	0.0021	0.4170	0.5821	0.1651	0.4012	0.5979
		θ	1.2310	0.0310	0.0328	0.9313	1.5308	0.5995	0.8739	1.5882

TABLE 7: ML estimates, biases, MSEs, LBs, UBs, and ALs of the HLXG under complete and TIC for $(\lambda = 1.7$ and $\theta = 0.9)$.

n	T	Parameters	ML	Bias	MSE	90%			95%		
						LB	UB	AL	LB	UB	AL
30	2.5	λ	1.9974	0.2974	0.5593	1.0347	2.9601	1.9254	0.8503	3.1445	2.2942
		θ	0.9956	0.0956	0.0612	0.6387	1.3525	0.7137	0.5704	1.4208	0.8504
	4	λ	1.9080	0.2080	0.3347	1.0963	2.7196	1.6233	0.9409	2.8750	1.9341
		θ	0.9626	0.0626	0.0389	0.6704	1.2547	0.5842	0.6145	1.3106	0.6961
50	2.5	λ	1.7271	0.0271	0.1449	-0.8747	4.3290	5.2037	-1.3729	4.8272	6.2001
		θ	0.8697	-0.0303	0.0675	0.3054	1.4340	1.1286	0.1973	1.5420	1.3447
	4	λ	1.7204	0.0204	0.0910	1.1726	2.2683	1.0957	1.0677	2.3732	1.3055
		θ	0.8970	-0.0030	0.0113	0.6827	1.1112	0.4284	0.6417	1.1522	0.5105
100	2.5	λ	1.7088	0.0088	0.0741	1.2775	2.1400	0.8624	1.1950	2.2226	1.0276
		θ	0.8887	-0.0113	0.0307	0.7046	1.0727	0.3681	0.6694	1.1079	0.4386
	4	λ	1.7168	0.0168	0.0592	1.3324	2.1012	0.7688	1.2588	2.1748	0.9160
		θ	0.9046	0.0046	0.0095	0.7523	1.0569	0.3047	0.7231	1.0861	0.3630

and

$$BO(t) = \frac{LO(t)}{F(t; \lambda, \theta)} = \frac{\sum_{i,j=0}^{\infty} (\eta_{i,j,m,k} \gamma(m+k+2, \theta(j+1)t) / (\theta(j+1))^{m+k+2}) + \psi_{i,j,m,k} \gamma(m+k+4, \theta(j+1)t) / (\theta(j+1))^{m+k+4}}{F(t; \lambda, \theta) \sum_{i,j=0}^{\infty} (\eta_{i,j,m,k} \Gamma(m+k+2) / (\theta(j+1))^{m+k+2}) + \psi_{i,j,m,k} \Gamma(m+k+4) / (\theta(j+1))^{m+k+4}}. \quad (17)$$

3.3. Mean Residual Life and Mean Waiting Time. The mean residual life (MRL) function at age t measures the expected remaining lifetime of an individual of age t . The MRL of X is given by

$$M(t) = \frac{1}{\bar{F}(t)} [E(T) - \pi_1(T)] - t. \quad (18)$$

Hence, the MRL of X can be obtained as follows:

$$M(t) = \frac{1}{\bar{F}(t; \lambda, \theta)} \left\{ \left[\sum_{i,j=0}^{\infty} \frac{\eta_{i,j,m,k} \Gamma(m+k+2)}{(\theta(j+1))^{m+k+2}} + \frac{\psi_{i,j,m,k} \Gamma(m+k+4)}{(\theta(j+1))^{m+k+4}} \right] - t \right\}. \quad (19)$$

The mean waiting time (MWT) of an item failed in an interval $[0, t]$ is defined as

$$\bar{M}(t) = t - \frac{\pi_1(t)}{F(t)}. \quad (20)$$

Hence, the MWT of the HLXG distribution is determined as follows:

$$\bar{M}(t) = t - \frac{1}{F(t; \lambda, \alpha, \beta)} \left[\sum_{i,j=0}^{\infty} \frac{\eta_{i,j,m,k} \gamma(m+k+2, \theta(j+1)t)}{(\theta(j+1))^{m+k+2}} + \frac{\psi_{i,j,m,k} \gamma(m+k+4, \theta(j+1)t)}{(\theta(j+1))^{m+k+4}} \right]. \quad (21)$$

3.4. Stochastic Ordering. Shaked and Shanthikumar [25] stated that for independent random variables X and Y with cdfs, F_X and F_Y , respectively, X is said to be smaller than Y in the following ordering, if the following is available:

- (i) Stochastic order ($X \leq_{sr} Y$) if $F_X(x) \geq F_Y(x)$ for all x
- (ii) Likelihood ratio order ($X \leq_{lr} Y$) if $f_X(x)/f_Y(x)$ is decreasing in x
- (iii) Hazard rate order ($X \leq_{hr} Y$) if $h_X(x) \geq h_Y(x)$ for all x

TABLE 8: The sf and hrf estimates at $t_0 \in \{3, 5, 10\}$ of the HLXG distribution under TIIC.

Parameters λ	θ	n	r	Exact values	$\hat{F}(3)$	$\hat{F}(5)$	$\hat{F}(10)$	$\hat{h}(3)$	$\hat{h}(5)$	$\hat{h}(10)$
0.5	0.5	30	21		0.209338	0.099149	0.013145	0.366873	0.381720	0.426493
			27	$\bar{F}(3) = 0.233346$	0.223111	0.110608	0.016766	0.345210	0.357746	0.397168
			30	$\bar{F}(5) = 0.122910$	0.232564	0.119128	0.019874	0.329717	0.340573	0.376164
			35	$\bar{F}(10) = 0.022299$	0.205363	0.099156	0.013952	0.358267	0.371267	0.413400
			45		0.213266	0.105614	0.016006	0.346226	0.357975	0.397136
		50	50	$h(3) = 0.316873$	0.215099	0.107107	0.016494	0.343618	0.355109	0.393633
			70	$h(5) = 0.325682$	0.228756	0.118063	0.020173	0.326463	0.336412	0.370866
			90	$h(10) = 0.357710$	0.234576	0.123315	0.022232	0.317714	0.326775	0.359109
			100		0.233687	0.122333	0.021784	0.319687	0.328987	0.361823
			21		0.021299	0.002269	0.000004	1.061410	1.172880	1.317870
0.5	1.2	30	27	$\bar{F}(3) = 0.0413473$	0.025608	0.003150	0.000008	0.994555	1.097540	1.238630
			30	$\bar{F}(5) = 0.0074909$	0.028204	0.003736	0.000013	0.960171	1.058560	1.197370
			35	$\bar{F}(10) = 0.0000601$	0.021555	0.002348	0.000004	1.051110	1.161310	1.305730
			45		0.028081	0.003789	0.000013	0.951551	1.048730	1.186930
			50	$h(3) = 0.8152270$	0.029972	0.004258	0.000017	0.927689	1.021570	1.158020
		100	70	$h(5) = 0.8926640$	0.034017	0.005185	0.000026	0.895045	0.984345	1.118220
			90	$h(10) = 1.0191500$	0.036430	0.005859	0.000034	0.870220	0.955919	1.087670
			100		0.036200	0.005796	0.000033	0.872305	0.958308	1.090240
			21		0.269852	0.071870	0.001641	0.617601	0.696475	0.803238
			27	$\bar{F}(3) = 0.2849660$	0.275798	0.076556	0.001969	0.598531	0.674527	0.778366
1.7	0.9	30	30	$\bar{F}(5) = 0.0861552$	0.276290	0.077161	0.002020	0.595790	0.671264	0.774649
			35	$\bar{F}(10) = 0.0028425$	0.264273	0.072155	0.001778	0.607585	0.682591	0.787315
			45	$h(3) = 0.5597790$	0.268432	0.074945	0.001969	0.597151	0.670820	0.773960
		50	50		0.271044	0.076638	0.002087	0.591197	0.664155	0.766387
			70	$h(5) = 0.6288880$	0.284424	0.082838	0.002446	0.576170	0.649138	0.749466
			90	$h(10) = 0.7261120$	0.280601	0.080623	0.002291	0.582698	0.656203	0.757493
			100		0.278566	0.079352	0.002201	0.586778	0.660712	0.762618

TABLE 9: The sf and hrf estimates at $t_0 \in \{3, 5, 10\}$ of the HLXG distribution under complete and TIC.

Parameters		n	T	Exact values	$\hat{F}(3)$	$\hat{F}(5)$	$\hat{F}(10)$	$\hat{h}(3)$	$\hat{h}(5)$	$\hat{h}(10)$
λ	θ									
0.5	0.5	30	2.5	$\bar{F}(3) = 0.233346$	0.216432	0.107133	0.016198	0.346273	0.358332	0.397679
			4	$\bar{F}(5) = 0.122910$	0.217353	0.107951	0.016484	0.344680	0.356563	0.395511
		50	2.5	$\bar{F}(10) = 0.022299$	0.227453	0.117434	0.020097	0.326357	0.336167	0.370517
			4	$h(3) = 0.316873$	0.224381	0.114758	0.019105	0.330856	0.341117	0.376559
		100	2.5	$h(5) = 0.325682$	0.229037	0.119549	0.021132	0.321265	0.330391	0.363401
			4	$h(10) = 0.357710$	0.229797	0.120191	0.021372	0.320289	0.329326	0.362107
0.5	1.2	30	2.5	$\bar{F}(3) = 0.0413473$	0.031630	0.004572	0.000020	0.919579	1.012360	1.148190
			4	$\bar{F}(5) = 0.0074909$	0.030409	0.004257	0.000017	0.934418	1.029280	1.166240
		50	2.5	$\bar{F}(10) = 0.0000601$	0.038328	0.006391	0.000041	0.853517	0.936765	1.067000
			4	$h(3) = 0.8152270$	0.038497	0.006439	0.000041	0.852064	0.935098	1.065200
		100	2.5	$h(5) = 0.8926640$	0.037983	0.006485	0.000044	0.842627	0.924181	1.053380
			4	$h(10) = 1.0191500$	0.038291	0.006580	0.000046	0.839676	0.920789	1.049710
1.7	0.9	30	2.5	$\bar{F}(3) = 0.2849660$	0.265858	0.068976	0.001462	0.629726	0.710327	0.818858
			4	$\bar{F}(5) = 0.0861552$	0.274869	0.075262	0.001860	0.604664	0.681868	0.786724
		50	2.5	$\bar{F}(10) = 0.0028425$	0.310250	0.098725	0.003743	0.533615	0.603083	0.696883
			4	$h(3) = 0.5597790$	0.290238	0.088247	0.002953	0.556504	0.626226	0.723173
		100	2.5	$h(5) = 0.6288880$	0.293989	0.090596	0.003149	0.550078	0.619223	0.715160
			4	$h(10) = 0.7261120$	0.284586	0.085431	0.002760	0.562942	0.632723	0.730538

- (iv) Mean residual life order ($X \leq_{\text{mrl}} Y$) if $m_X(x) \geq m_Y(x)$ for all x

We have the following chain of implications among the various partial orderings discussed above:

$$\begin{array}{ccc} & \leq_{\text{hr}} & \\ X \leq_{\text{lr}} Y \implies X & \Downarrow & Y \implies X \leq_{\text{mrl}} Y \\ & X \leq_{\text{sr}} Y & \end{array} \quad (22)$$

Theorem 1. Let $X \sim \text{HLXG}(\lambda_1, \theta_1)$ and $Y \sim \text{HLXG}(\lambda_2, \theta_2)$. If $\lambda_1 \geq \lambda_2$ and $\theta_1 \geq \theta_2$, then $X \leq_{\text{lr}} Y$, $X \leq_{\text{hr}} Y$, $X \leq_{\text{mrl}} Y$, and $X \leq_{\text{sr}} Y$.

Proof. It is sufficient to show that $f_X(x)/f_Y(x)$ is a decreasing function of x ; the likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{\lambda_1 \theta_1^2 (1 + \theta_1 x^2/2) e^{-\theta_1 x} [1 - A(\theta_1, x)]^{\lambda_1 - 1} (1 + \theta_2) [1 + [1 - A(\theta_2, x)]^{\lambda_2}]^2}{(1 + \theta_1) [1 + [1 - A(\theta_1, x)]^{\lambda_1}]^2 \lambda_2 \theta_2^2 (1 + \theta_2 x^2/2) e^{-\theta_2 x} [1 - A(\theta_2, x)]^{\lambda_2 - 1}}, \quad (23)$$

therefore,

$$\begin{aligned} \frac{d}{dx} \log \frac{f_X(x)}{f_Y(x)} &= \frac{\theta_1 x}{(1 + \theta_1 x^2/2)} - \theta_1 - \frac{(\lambda_1 - 1) A'(\theta_1, x)}{1 - A(\theta_1, x)} \\ &\quad - \frac{2\lambda_2 [1 - A(\theta_2, x)]^{\lambda_2 - 1} A'(\theta_2, x)}{1 + [1 - A(\theta_2, x)]^{\lambda_2}} \\ &\quad - \frac{\theta_2 x}{(1 + \theta_2 x^2/2)} + \theta_2 + \frac{(\lambda_2 - 1) A'(\theta_2, x)}{1 - A(\theta_2, x)} \\ &\quad + \frac{2\lambda_1 [1 - A(\theta_1, x)]^{\lambda_1 - 1} A'(\theta_1, x)}{1 + [1 - A(\theta_1, x)]^{\lambda_1}} < 0, \end{aligned} \quad (24)$$

where $A'(\theta_i, x) = -e^{-\theta_i x} [\theta_i^2 + \theta_i^2 x^2/2 + \theta_i]$, $i = 1, 2$. Thus, $f_X(x)/f_Y(x)$ is decreasing in x , and hence $X \leq_{\text{lr}} Y$. Similarly, we can conclude for $X \leq_{\text{hr}} Y$, $X \leq_{\text{mrl}} Y$, and $X \leq_{\text{sr}} Y$. $\square$

3.5. Quantile Function. The HLXG quantile function, say $Q(u) = F^{-1}(u)$, is straightforward to be computed by inverting (6) as follows:

$$(1 + \theta) \left[1 - \left(\frac{u}{2 - u} \right)^{1/\lambda} \right] - \left(1 + \theta + \theta x_{(q)} + \frac{\theta^2 x_{(q)}^2}{2} \right) = 0, \quad (25)$$

where $x_q = Q(u)$. The uniroot function of the R software can be used to solve nonlinear equation (25), numerically, and then the HLXG random variable X can be generated, where u has the uniform distribution on the interval $(0, 1)$.

4. Parameter Estimation under Censored Samples

In survival analysis and the industrial life testing model, it is necessary to minimize the cost and/or the duration of a life-testing experiment, so one may choose to terminate the test early, which results in the so-called censored sampling scheme. TIC and TIIC are the most popular types of censoring. The life testing experiment is ended at a predetermined time, say T , for the TIC, while for the TIIC, the test is ended when a predetermined number of failures, say r , is reached.

In the following sections, the maximum likelihood (ML) estimators of the model parameters of the HLXG model based on complete, TIC, and TIIC are provided. Approximate CIs are constructed. Further, estimators of survival function and hazard rate function for different mission times are given.

4.1. Maximum Likelihood Estimators via TIIC. Let $X_{(1)} < X_{(2)} < \dots < X_{(n)}$ be a TIIC of size r from a life test on n items whose lifetimes have the HLXG model with parameters λ and θ . The log-likelihood function, denoted by $\ln \ell_1$, of r failures and $(n - r)$ censored values is given by

$$\begin{aligned} \ln \ell_1 &= \ln c + r \ln 2\lambda + 2r \ln \theta + \sum_{i=1}^r \ln \left(1 + \frac{\theta x_{(i)}^2}{2} \right) - \sum_{i=1}^r \theta x_{(i)} \\ &\quad + (\lambda - 1) \sum_{i=1}^r \ln [1 - A(\theta, x_{(i)})] - r \ln (1 + \theta) \\ &\quad - 2 \sum_{i=1}^r \ln \left[1 + [1 - A(\theta, x_{(i)})]^\lambda \right] + (n - r) \\ &\quad \times \langle \ln \{ 1 - [1 - A(\theta, x_{(r)})]^\lambda \} - \ln \{ 1 + [1 - A(\theta, x_{(r)})]^\lambda \} \rangle, \end{aligned} \quad (26)$$

where $c = n!/(r-1)!(n-r)!$, $A(\theta, x_{(i)}) = (1 + \theta + \theta x_{(i)} + \theta^2 x_{(i)}^2/2(1 + \theta))e^{-\theta x_{(i)}}$, and $A(\theta, x_{(r)}) = (1 + \theta + \theta x_{(r)} + \theta^2 x_{(r)}^2/2(1 + \theta))e^{-\theta x_{(r)}}$.

The first partial derivatives of the log-likelihood function with respect to λ and θ are obtained as follows:

$$\begin{aligned} \frac{\partial \ln \ell_1}{\partial \lambda} &= \frac{r}{\lambda} + \sum_{i=1}^r \ln[1 - A(\theta, x_{(i)})] \\ &\quad - \sum_{i=1}^r \frac{2 \ln[1 - A(\theta, x_{(i)})]}{[1 - A(\theta, x_{(i)})]^{-\lambda} + 1} \\ &\quad - \frac{2(n-r)[1 - A(\theta, x_{(r)})]^\lambda \ln[1 - A(\theta, x_{(r)})]}{1 - [1 - A(\theta, x_{(r)})]^{2\lambda}}, \end{aligned} \quad (27)$$

$$\begin{aligned} \frac{\partial \ln \ell_1}{\partial \theta} &= \frac{r(2+\theta)}{\theta(1+\theta)} + \sum_{i=1}^r \frac{x_{(i)}^2}{2+\theta x_{(i)}^2} - \sum_{i=1}^r x_i \\ &\quad - \sum_{i=1}^r \frac{(\lambda-1)\partial A(\theta, x_{(i)})/\partial \theta}{[1 - A(\theta, x_{(i)})]} \\ &\quad + \sum_{i=1}^r \frac{2\lambda[1 - A(\theta, x_{(i)})]^{\lambda-1}\partial A(\theta, x_{(i)})/\partial \theta}{1 + [1 - A(\theta, x_{(i)})]} \\ &\quad + \frac{2(n-r)\lambda[1 - A(\theta, x_{(r)})]^{\lambda-1}\partial A(\theta, x_{(r)})/\partial \theta}{1 - [1 - A(\theta, x_{(r)})]^{2\lambda}}, \end{aligned} \quad (28)$$

where

$$\frac{\partial A(\theta, x_{(i)})}{\partial \theta} = \frac{-2(2\theta + \theta^2)x_{(i)} - \theta^2 x_{(i)}^2 - (\theta^2 + \theta^3)x_{(i)}^3}{2(1+\theta)^2} e^{-\theta x_{(i)}}, \quad (29)$$

$$\frac{\partial A(\theta, x_{(r)})}{\partial \theta} = \frac{-2(2\theta + \theta^2)x_{(r)} - \theta^2 x_{(r)}^2 - (\theta^2 + \theta^3)x_{(r)}^3}{2(1+\theta)^2} e^{-\theta x_{(r)}}. \quad (30)$$

Putting $\partial \ln \ell_1 / \partial \lambda$ and $\partial \ln \ell_1 / \partial \theta$ equal to zero and solving these equations numerically provide the ML estimator of λ and θ , respectively. Note that for $r = n$, we obtain the ML of the model parameters in case of a complete sample.

4.2. Maximum Likelihood Estimators via TIC. Suppose that n items, whose lifetimes have the HLXG distribution, are placed on a life test, and the test is terminated at a specified time T before all n items have failed. The number of failures r and all failure times are random variables. The log-likelihood function, based on TIC, is given by

$$\begin{aligned} \ln \ell_2 &= \ln c + r \ln 2\lambda + 2r \ln \theta + \sum_{i=1}^r \ln \left(1 + \frac{\theta x_{(i)}^2}{2} \right) - \sum_{i=1}^r \theta x_{(i)} \\ &\quad + (\lambda - 1) \sum_{i=1}^r \ln[1 - A(\theta, x_{(i)})] - r \ln(1 + \theta) \\ &\quad - 2 \sum_{i=1}^r \ln \left[1 + [1 - A(\theta, x_{(i)})]^\lambda \right] + (n - r) \\ &\quad \langle \ln \{1 - [1 - A(\theta, T)]^\lambda\} - \ln \{1 + [1 - A(\theta, T)]^\lambda\} \rangle, \end{aligned} \quad (31)$$

where $A(\theta, T) = (1 + \theta + \theta T + \theta^2 T^2/2(1 + \theta))e^{-\theta T}$.

The first partial derivatives of the log-likelihood function with respect to λ and θ are obtained as follows:

$$\begin{aligned} \frac{\partial \ln \ell_2}{\partial \lambda} &= \frac{r}{\lambda} + \sum_{i=1}^r \ln[1 - A(\theta, x_{(i)})] - \sum_{i=1}^r \frac{2 \ln[1 - A(\theta, x_{(i)})]}{[1 - A(\theta, x_{(i)})]^{-\lambda} + 1} \\ &\quad - \frac{2(n-r)[1 - A(\theta, T)]^\lambda \ln[1 - A(\theta, T)]}{1 - [1 - A(\theta, T)]^{2\lambda}}, \end{aligned} \quad (32)$$

$$\begin{aligned} \frac{\partial \ln \ell_2}{\partial \theta} &= \frac{r(2+\theta)}{\theta(1+\theta)} + \sum_{i=1}^r \frac{x_{(i)}^2}{2+\theta x_{(i)}^2} - \sum_{i=1}^r x_{(i)} \\ &\quad - \sum_{i=1}^r \frac{(\lambda-1)\partial A(\theta, x_{(i)})/\partial \theta}{[1 - A(\theta, x_{(i)})]} \\ &\quad + \sum_{i=1}^r \frac{2\lambda[1 - A(\theta, x_{(i)})]^{\lambda-1}\partial A(\theta, x_{(i)})/\partial \theta}{1 + [1 - A(\theta, x_{(i)})]} \\ &\quad + \frac{2(n-r)\lambda[1 - A(\theta, T)]^{\lambda-1}\partial A(\theta, T)/\partial \theta}{1 - [1 - A(\theta, T)]^{2\lambda}}, \end{aligned} \quad (33)$$

where

$$\frac{\partial A(\theta, T)}{\partial \theta} = \frac{-2(2\theta + \theta^2)T - \theta^2 T^2 - (\theta^2 + \theta^3)T^3}{2(1+\theta)^2} e^{-\theta T}. \quad (34)$$

Putting $\partial \ln \ell_2 / \partial \lambda$ and $\partial \ln \ell_2 / \partial \theta$ equal to zero and solving these equations numerically provide the ML estimators of λ and θ .

4.3. Approximate Confidence Intervals. In this section, approximate confidence intervals (CIs) of the model parameters for the HLXG distribution are obtained. The asymptotic variances and the covariance matrix of the ML estimates are given by the elements of the inverse of the Fisher information matrix $I_{ij}(\theta) \cong E\{-\partial^2 \ln \ell_1 / \partial \lambda \partial \theta\}$. Unfortunately, the exact mathematical expressions for the above expectation are very difficult to obtain. Therefore, the Fisher information matrix is

given by $I_{ij}(\theta) = \{-\partial^2 \ln \ell / \partial \lambda \partial \theta\}$, which is obtained by dropping the expectation on operation E and replacing λ and θ with $\hat{\lambda}$ and $\hat{\theta}$, respectively (see [26]). The asymptotic variance covariance matrix F for the maximum likelihood estimates can be written as follows:

$$F = I_{ij}(\hat{\lambda}, \hat{\theta}) = -\frac{\partial^2 \ln \ell_1}{\partial \ln \lambda \partial \ln \theta} \Big|_{\lambda=\hat{\lambda}, \theta=\hat{\theta}}. \quad (35)$$

Hence, the approximate 100 $(1 - \nu)\%$ two-sided CIs for λ and θ are, respectively, given by

$$\hat{\lambda} \pm Z_{\nu/2} \sqrt{\text{var}(\hat{\lambda})}, \quad \hat{\theta} \pm Z_{\nu/2} \sqrt{\text{var}(\hat{\theta})}, \quad (36)$$

where $Z_{\nu/2}$ is the upper $\nu/2^{\text{th}}$ percentile of the standard normal distribution. By a similar way, we can find the approximate 100 $(1 - \nu)\%$ two-sided CIs for λ and θ under TIIC and a complete sample.

4.4. Simulation Study. This section provides a simulation study to assess the behavior of the estimators in case of complete, TIC, and TIIC. Mean square errors (MSEs), biases, lower bounds (LBs) of the CIs, upper bounds (UBs) of the CIs, and average lengths (ALs) of 90% and 95% are calculated via Mathematica 9. The following algorithm is designed as follows:

- (i) 10000 random samples of size $n = 30, 50$, and 100 are generated from HLXG distribution. The corresponding number of failures is selected as $r = (21, 27, 30), (35, 45, 50)$, and $(70, 90, 100)$.
- (ii) Values of true parameters (λ, θ) are taken as $(0.5, 0.5)$, $(0.5, \theta = 1.2)$, and $(1.7, 0.9)$.
- (iii) The termination time is selected as $T = 2.5$ and 4 in case of TIC.
- (vi) The ML estimates, MSEs, biases, LBs, UBs, and ALs for selected values of parameters are calculated.
- (v) Numerical outcomes are listed in Tables 2–4 based on complete and TIIC, while Tables 5–7 contain simulation results based on TIC.
- (vi) ML estimates of sf, $\bar{F}(t_0)$, hrf, and $h(t_0)$, at different mission time t_0 where $t_0 \in \{3, 5, 10\}$ of the HLXG distribution, are calculated. Observed values under TIIC and complete samples are recorded in Table 8, while Table 9 contains numerical results based on TIC.

From Tables 2–9, we observed the following:

The Biases, MSEs, and average length of λ and θ decrease as n increases

As the censoring level time values, T , increase, the biases, MSEs, and ALs of λ and θ decrease

The biases, MSEs, and ALs of λ and θ decrease as the number of failures r increases

The lengths of the CIs become narrower as n increases

The ALs of the CIs increase as the confidence levels increase from 90% to 95%. Also, as the number of failures r increases, the AL of CIs decreases

As the mission time increases, the estimates of sf decrease while the hrf estimates increase for all sampling schemes

As the sample size increases, the sf and the hrf estimates are increasing

The MSEs, biases, and ALs are increasing as θ increases from 0.5 to 1.2 at $\lambda = 0.5$

5. Data Analysis

In this section, we fit our proposed HLXG model to the following three real data sets. Furthermore, parameter estimation of the three real data sets by using the ML, least square (LS), maximum product spacing (MPS), percentile (PE), and Cramer–von Mises (CVM) methods is provided.

The first data consists of 100 observations of breaking stress of carbon fibres (in Gba) given by Nichols and Padgett [27]. The data are given as follows:

3.7, 3.11, 4.42, 3.28, 3.75, 2.96, 3.39, 3.31, 3.15, 2.81, 1.41, 2.76, 3.19, 1.59, 2.17, 3.51, 1.84, 1.61, 1.57, 1.89, 2.74, 3.27, 2.41, 3.09, 2.43, 2.53, 2.81, 3.31, 2.35, 2.77, 2.68, 4.91, 1.57, .65, 2.12, 1.8, 0.85, 4.38, 2, 1.18, 1.71, 1.17, 2.17, 0.39, 2.79, 1.08, 2.73, 0.98, 1.73, 1.59, 1.92, 2.38, 3.56, 2.55, 3.22, 3.39, 4.9, 1.69, 3.11, 3.6, 2.05, 1.61, 2.03, 2.48, 1.25, 2.48, 1.12, 2.88, 2.87, 3.19, 1.87, 2.95, 2.67, 4.2, 2.85, 2.55, 2.17, 2.97, 3.68, 0.81, 1.22, 5.08, 1.69, 3.68, 4.70, 2.03, 2.82, 2.50, 3.22, 3.15, 1.47, 5.56, 1.84, 1.36, 2.59, 2.83, 2.56, 3.33, 2.93, and 2.97.

The second data set (gauge lengths of 10 mm) is obtained by Kundu and Raqab [28]. This data set consists of 63 observations and are listed as follows:

1.901, 2.132, 2.203, 2.228, 2.257, 2.350, 2.361, 2.396, 2.397, 2.445, 2.454, 2.474, 2.518, 2.522, 2.525, 2.532, 2.575, 2.614, 2.616, 2.618, 2.624, 2.659, 2.675, 2.738, 2.740, 2.856, 2.917, 2.928, 2.937, 2.937, 2.977, 2.996, 3.030, 3.125, 3.139, 3.145, 3.220, 3.223, 3.235, 3.243, 3.264, 3.272, 3.294, 3.332, 3.346, 3.377, 3.408, 3.435, 3.493, 3.501, 3.537, 3.554, 3.562, 3.628, 3.852, 3.871, 3.886, 3.971, 4.024, 4.027, 4.225, 4.395, and 5.020.

The third data set is censored data discussed by Sickel-Santanello et al. [29] and given by Klein and Moeschberger [30]. The data consist of death times (in weeks) of patients with cancer of the tongue with an aneuploid DNA profile. Data under study were based on the effects of ploidy on the prognosis of patients with cancers of the mouth. Patients were selected in whom a paraffin-embedded sample of the cancerous tissue had been taken, and the times to reinfection for patients with sexually transmitted diseases were calculated. Follow-up survival data were obtained on each patient. The tissue samples were examined using a flow cytometer to determine if the tumor had an aneuploid (abnormal) or diploid (normal) DNA profile using a technique discussed by Sickel-Santanello et al. [29].

The observations are 1, 3, 3, 4, 10, 13, 13, 16, 16, 24, 26, 27, 28, 30, 30, 32, 41, 51, 61, 65, 67, 70, 72, 73, 74, 77, 79, 80, 81, 87, 87, 88, 89, 91, 93, 93, 96, 97, 100, 101, 104, 104, 108, 109, 120, 131, 150, 157, 167, 231, 240, and 400.

TABLE 10: Analytical results of the HLXG model and the other competing models for the first data.

Model	ML estimates (SE)	$-2 \ln L$	AIC	CAIC	BIC	HQIC
HLXG	$\hat{\lambda} = 6.023 (1.018)$ $\hat{\theta} = 1.371 (0.099)$	288.519	292.519	292.642	292.519	294.627
GO	$\hat{\alpha} = 0.789 (0.0774)$ $\hat{\lambda} = 0.099 (0.031)$	297.868	301.868	301.992	301.868	303.977
EBXII	$\hat{\alpha} = 2.118 (0.76539)$ $\hat{\beta} = 13.687 (8.346)$ $\hat{\lambda} = 1.358 (0.38)$	316.051	322.051	322.301	322.051	325.214
MOIL	$\hat{\alpha} = 6.501 (0.87256)$ $\hat{\beta} = 3.151 (0.909)$ $\hat{\lambda} = 0.0046 (0.00323)$	304.48	310.48	310.73	310.48	313.643

TABLE 11: Goodness-of-fit measures of the HLXG model and the other competing models for the first data.

Model	A^*	P value	$K-S$	P value
HLXG	1.0953	0.0072	0.101	0.26
GO	1.25295	0.0029	0.1026	0.243
EBXII	2.77675	$5.489 * 10^{-7}$	0.14053	0.039
MOIL	2.30358	$7.821 * 10^{-6}$	0.11066	0.173

TABLE 12: Analytical results of the HLXG model and the other competing models for the second data.

Model	ML estimates (SE)	$-2 \ln L$	AIC	CAIC	BIC	HQIC
HLXG	$\hat{\lambda} = 73.168 (30.67447)$ $\hat{\theta} = 2.193 (0.189)$	113.683	117.683	117.887	117.282	119.369
GO	$\hat{\alpha} = 1.399 (0.12793)$ $\hat{\lambda} = 0.008978 (0.000448)$	139.296	142.296	142.496	141.895	143.982
EBXII	$\hat{\alpha} = 3.926 (2.7811)$ $\hat{\beta} = 1320 (270.9)$ $\hat{\lambda} = 1.646 (0.937)$	116.228	122.228	122.634	121.626	124.756
MOIL	$\hat{\alpha} = 6.943 (0.85665)$ $\hat{\beta} = 3.544 (0.608)$ $\hat{\lambda} = 0.0043 (0.00378)$	162.595	168.595	169.002	167.993	171.124

TABLE 13: Goodness-of-fit measures of the HLXG model and the other competing models for the second data.

Model	A^*	P value	$K-S$	P value
HLXG	0.4434	0.286	0.0941	0.632
GO	1.6283	$3.52 * 10^{-4}$	0.139	0.175
EBXII	0.6169	0.108	0.096	0.607
MOIL	22.3882	0	0.2076	0.0088

TABLE 14: Analytical results of the HLXG model and the other competing models for the third data.

Model	ML estimates (SE)	$-2 \ln L$	AIC	CAIC	BIC	HQIC
HLXG	$\hat{\lambda} = 0.619 (0.09717)$ $\hat{\theta} = 0.021 (0.00326)$	559.128	563.128	563.252	562.56	564.624
GO	$\hat{\alpha} = 0.00194 (0.00169)$ $\hat{\lambda} = 5.489 (5.569)$	559.789	563.789	563.912	563.221	565.285
EBXII	$\hat{\alpha} = 2.997 (1.60698)$ $\hat{\beta} = 62.12 (76.415)$ $\hat{\lambda} = 0.324 (0.136)$	591.406	597.406	597.656	596.554	599.65
MOIL	$\hat{\alpha} = 1.703 (0.24084)$ $\hat{\beta} = 393.894 (284.417)$ $\hat{\lambda} = 0.032 (0.038)$	573.349	579.349	579.599	578.497	581.593

TABLE 15: Goodness-of-fit measures of the HLXG model and the other competing models for the third data.

Model	A^*	P value	$K-S$	P value
HLXG	1.0383	0.00991	0.1386	0.27
GO	1.0829	0.0077	0.1572	0.153
EBXII	3.8372	$1.468 * 10^{-9}$	0.2181	0.014
MOIL	2.4612	$3.225 * 10^{-6}$	0.171	0.095

FIGURE 2: Log-likelihood for the first data.

FIGURE 3: Log-likelihood for the second data.

FIGURE 4: Log-likelihood for the third data.

FIGURE 5: The empirical pdf, cdf, sf, and pp plots for the first data.

First, we want to compare the fits of the HLXG model with the Gompertz (GO), exponentiated Burr XII (EBXII), and MO inverse Lomax (MOIL) distributions. We estimate the model parameters by using the ML method. We compare the goodness-of-fit of the models using $-2\ln L$ where L denotes the log-likelihood function evaluated at the ML estimates, Akaike information criterion (AIC), corrected Akaike information criterion (CAIC), Hannan–Quinn information criterion (HQIC), and Bayesian information criterion (BIC). We also reported the Kolmogorov–Smirnov ($K-S$) and Anderson–Darling (A^*) statistics and their corresponding P values. The pdfs of the IL, GIL, EIL, MOIL, and IW models are, respectively, given by

$$f_{GO}(x; \alpha, \lambda) = \alpha \lambda e^{\lambda} e^{\alpha x} e^{-\lambda e^{\alpha x}}, \quad x, \alpha, \lambda > 0,$$

$$f_{EBXII}(x; \alpha, \beta, \lambda) = \alpha \beta \lambda x^{\lambda-1} (1+x^{\lambda})^{-\alpha-1} \cdot \left[1 - (1+x^{\lambda})^{-\alpha} \right]^{\beta-1}, \quad x, \alpha, \beta, \lambda > 0, \quad (37)$$

$$f_{MOIL}(x; \alpha, \beta, \lambda) = \frac{\alpha \beta \lambda (1 + \beta/x)^{-\alpha-1}}{x^2 [1 - (1-\lambda)[1 - (1 + \beta/x)^{-\alpha}]]^2}, \quad x, \alpha, \beta, \lambda > 0. \quad (38)$$

FIGURE 6: The empirical pdf, cdf, sf, and pp plots for the second data.

The ML estimates (and the corresponding standard errors in parentheses) of parameters and values of $-2 \ln L$, AIC, BIC, CAIC, and HQIC and the values of $K-S$ and A^* statistics and their corresponding P values for the first data set are displayed in Tables 10 and 11, respectively, for the second data set in Tables 12 and 13, respectively, and for the third data set, in Tables 14 and 15, respectively. The log-likelihood for the first, second, and third data sets are given in Figures 2–4, respectively.

It is observed from Tables 10–15 that the HLXG distribution has the smallest values of $-2 \ln L$, AIC, CAIC, BIC, HQIC, $K-S$, and A^* but the largest P value among all other

competitive models. Hence, it is evidenced that our proposed model performed the best for all three data sets. The relative histogram and the fitted densities and the plots of the fitted empirical cdf and the empirical sf and $p-p$ plots of the three data sets are displayed in Figures 5–7, respectively. These figures supported the above conclusions to some extent.

Tables 16–18 provide the ML, LS, MPS, PC, and CV estimates of the unknown parameters of the HLXG distribution based on four methods of estimation. Also, values of $-2 \ln L$, AIC, BIC, HQIC, and CAIC are listed. From these tables, we conclude that all estimation methods performed well.

FIGURE 7: The empirical pdf, cdf, sf, and pp plots for the third data.

TABLE 16: The parameter estimates under various methods and the goodness-of-fit statistics for (Nicholas) data.

Method	$\hat{\theta}$	$\hat{\lambda}$	$-2 \ln L$	AIC	BIC	HQIC	CAIC
ML	1.371	6.023	288.519	292.519	292.519	294.627	292.642
LS	1.375	6.477	289.157	293.157	293.157	295.265	293.28
MPS	0.015	0.118	774.822	778.822	778.822	780.931	778.946
PC	1.063	3.7	298.925	302.925	302.925	305.034	303.049
CV	1.392	6.723	289.437	293.437	293.437	295.545	293.56

TABLE 17: The parameter estimates under various methods and the goodness-of-fit statistics for the second (Kundu) data.

Method	$\hat{\theta}$	$\hat{\lambda}$	$-2 \ln L$	AIC	BIC	HQIC	CAIC
ML	2.193	73.168	113.683	117.683	117.282	119.369	117.887
LS	1.996	46.235	114.934	118.934	118.533	120.62	119.134
MPS	0.027	0.143	489.716	493.716	493.315	495.402	493.916
PC	2.006	47.873	114.748	118.748	118.347	120.434	118.948
CV	2.031	50.444	114.496	118.496	118.095	120.182	118.696

TABLE 18: The parameter estimates under various methods and the goodness-of-fit statistics for the third (Sickle) data.

Method	$\hat{\theta}$	$\hat{\lambda}$	$-2\ln L$	AIC	BIC	HQIC	CAIC
ML	0.021	0.619	559.128	563.128	562.56	564.624	563.252
LS	0.02	0.62	559.22	563.22	562.652	564.716	563.465
MPS	0.0019	0.122	659.689	663.689	663.121	665.185	663.933
PC	0.018	0.53	560.487	564.487	563.919	565.983	564.732
CV	0.021	0.651	559.283	563.283	562.715	564.779	563.528

6. Some Concluding Remarks

A new probability distribution related to xgamma, called the half-logistic xgamma distribution, is proposed in this paper. Some important statistical properties, including moments, incomplete moments, mean residual life, mean waiting time, quantile function, and stochastic ordering of the suggested model, are derived. Parameter estimation of model parameters is discussed via the maximum likelihood method from complete and censored samples. Further, survival and hazard rate function estimates are obtained for different sample sizes. Also, asymptotic confidence intervals of parameters are constructed. Good performance and accuracy of the maximum likelihood estimators of the parameters are examined through a simulation study. Based on maximum likelihood, least square, maximum product spacing, percentile, and Cramer–von Mises (CV) methods, we estimate the model parameters of the HLXG distribution using three real data sets. Furthermore, three real data sets are analyzed to demonstrate the superiority of the proposed half-logistic xgamma distribution compared to some existing distributions.

Data Availability

In order to obtain the numerical data set used to carry out the analysis reported in the manuscript, please contact Mahmoud Elsehetry at ma_sehetry@hotmail.com.

Conflicts of Interest

The authors declare no conflicts of interest.

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