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Research Article

Synchronizability of Multilayer Star and Star-Ring Networks

Yang Deng, Zhen Jia, and Feimei Yang

College of Science, Guilin University of Technology, Guilin 541004, China

Correspondence should be addressed to Zhen Jia; jjjzzz0@163.com

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Synchronization of multilayer complex networks is one of the important frontier issues in network science. In this paper, we strictly derived the analytic expressions of the eigenvalue spectrum of multilayer star and star-ring networks and analyzed the synchronizability of these two networks by using the master stability function (MSF) theory. In particular, we investigated the synchronizability of the networks under different interlayer coupling strength, and the relationship between the synchronizability and structural parameters of the networks (i.e., the number of nodes, intralayer and interlayer coupling strengths, and the number of layers) is discussed. Finally, numerical simulations demonstrated the validity of the theoretical results.

1. Introduction

Network science is an interdisciplinary subject which abstracts physical, biological, economic and social systems into networks composed of nodes and edges and studies their structural characteristics, dynamic evolution and dynamic characteristics. The network synchronization as an important emerging phenomenon of a population of dynamically interacting units in various fields of science has attracted much attention. Synchronization of complex networks has been widely studied, and much research works have been done over the past few decades and achieved fruitful research results [1–24]. However, most of these works are focusing on isolated networks; in reality, most systems are not isolated but interrelated such as the combination of aviation and railway transportation networks in the transportation system; the interdependence of the server and the terminal system in computer networks; in power infrastructure, the interactive control between the power station and the computer central control system; and in the social network, the compound overlap between the real interpersonal and online interpersonal networks, which constitute more complex networks, called multilayer networks. Therefore, in recent years, the focus of complex network research has gradually shifted from the single layer network to the multilayer network. The research of multilayer network has become an important research direction and attracts tensive attention from scholars.

Although the research on multilayer networks is still in its infancy, a series of influential research results have emerged. In 2013, Gómez et al. proposed a diffusion dynamics model based on multilayer networks and the supra-Laplacian matrix of the networks [25]. Granell et al. analyzed the correlation between the two processes of epidemic transmission and epidemic and proposed the information awareness of preventing its infection on the multilayer network [26]. In 2014, Aguirre et al. discussed the eigenvalue spectrum of two completely identical star networks coupled by an edge between layers [27] and considered synchronizability with different coupling modes between layers. In 2016, Xu et al. studied the eigenvalue spectrum and synchronizability of the two-layer star network and theoretically provided the analytic value of the eigenvalue spectrum of the two-layer star network with full connection between layers [28, 29]. In 2017, Li et al. investigated some rules and properties about synchronizability of duplex networks composed of two networks interconnected by two links, for a specific duplex network composed of two star networks, analytical expressions containing the largest and smallest nonzero eigenvalues of the (weighted) Laplacian matrix, and the interlink weight, as well as the network size, which are given for three different interlayer connection patterns [30]. Sun et al. studied the synchronizability of multilayer unidirectional coupling star networks and strictly derived the eigenvalues of such networks in the case of unidirectional coupling between layers [31]. Later, Wei et al. further studied the synchronizability of a double-layer regular network based on the master stability function (MSF) theory by using numerical simulation [32]. In 2019, Tang et al. extended the master stability function to multilayer networks with different intralayer and interlayer coupling functions and derived three master stability equations that determined complete synchronization, intralayer synchronization and interlayer synchronization regions [33]. Deng et al. studied the problem of synchronization of two kinds of multiplex chain networks under different coupling modes between layers and derived the eigenvalue spectrum of the supra-Laplacian matrices of those networks [34]. However, due to the structural complexity of multilayer networks, there is almost no strict theoretical derivation on the eigenvalues of the multilayer network; most of the studies are based on the results of numerical simulation on synchronizability of that. To provide more useful foundations for getting insight into understanding synchronizability of multilayer networks and explore the main influencing factors of synchronizability, in our paper, two kinds of typical multilayer networks (i.e., multilayer star and star-ring networks) are considered on the basis of the literature [28] that studied the synchronizability of the two-layer star network; we not only strictly derived the eigenvalue spectrum of the supra-Laplacian matrices of multilayer star and star-ring networks (not limited to two layers) but also studied the relationships between the synchronizability and structural parameters of that.

The paper is structured as follows. Some preliminaries are introduced in Section 2. Section 3 studies the eigenvalue spectrum and synchronizability of the multilayer star networks. In Section 4, the eigenvalue spectrum and synchronizability of the multilayer star-ring networks are studied. Section 5 explores the relationship between the synchronizability and structural parameters of two kinds of multilayer networks. Finally, Section 6 gives the conclusion of this paper.

2. Preliminaries

2.1. Dynamics Model of the Multilayer Network. The dynamic equation of the i-th node in the multilayer network, with M layers, is [33, 34]

$$\dot{x}_i^K = f\left(x_i^K\right) + a\sum_{j=1}^{N}\omega_{ij}^K H\left(x_j^K\right) + d\sum_{L=1}^{M} d_i^{KL}\Gamma\left(x_i^L\right), \quad i = 1, 2, \ldots, N; K = 1, 2, \ldots, M, \tag{1}$$

where $\dot{x}_i^K = f(x_i^K)$ $(i = 1, 2, \ldots, N; K = 1, 2, \ldots, M)$ describes the isolated dynamics for the i-th node in the K-th layer, $f(*)\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a well-defined vector function, $H(*)\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and a are the inner coupling function and coupling strength for nodes within each layer, respectively, and $\Gamma(*)\colon \mathbb{R}^n \longrightarrow \mathbb{R}^n$ and d are the inner coupling function and coupling strength for nodes across layers, respectively. Here, $W^K = (\omega_{ij}^K) \in \mathbb{R}^{N\times N}$ is the coupling weight configuration matrix of the K-th layer. Explicitly, if the i-th node is connected with the j-th $(j \neq i)$ node within the K-th layer, $\omega_{ij}^K = 1$; otherwise, $\omega_{ij}^K = 0$, and there is

$$\omega_{ii}^K = -\sum_{\substack{j=1\\ j\neq i}}^{N}\omega_{ij}^K, \quad i, j = 1, 2, \ldots, N, K = 1, 2, \ldots, M. \tag{2}$$

Let $L^{(K)} = -aW^K$, which is a Laplacian matrix. If the i-th node in the K-th layer is connected with its replica in the L-th $(L \neq K)$ layer, $d_i^{KL} = 1$; otherwise, $d_i^{KL} = 0$, and there is

$$d_i^{KK} = -\sum_{\substack{L=1\\ L\neq K}}^{M} d_i^{KL}, \quad L, K = 1, 2, \ldots, M. \tag{3}$$

It is obvious that $D = (d_i^{KL}) \in \mathbb{R}^{M\times M}$ is also a negative Laplacian matrix.

Let $\mathscr{L}$ be the supra-Laplacian matrix of equation (1), $\mathscr{L}_{\mathrm{I}}$ be the supra-Laplacian matrix representing the interlayer topology, and $\mathscr{L}_L$ be the supra-Laplacian matrix describing the intralayer topology. Then, $\mathscr{L}$ can be written as

$$\mathscr{L} = \mathscr{L}_I + \mathscr{L}_L. \tag{4}$$

Taking L_{I} to be the Laplacian matrix of the interlayer networks, we have

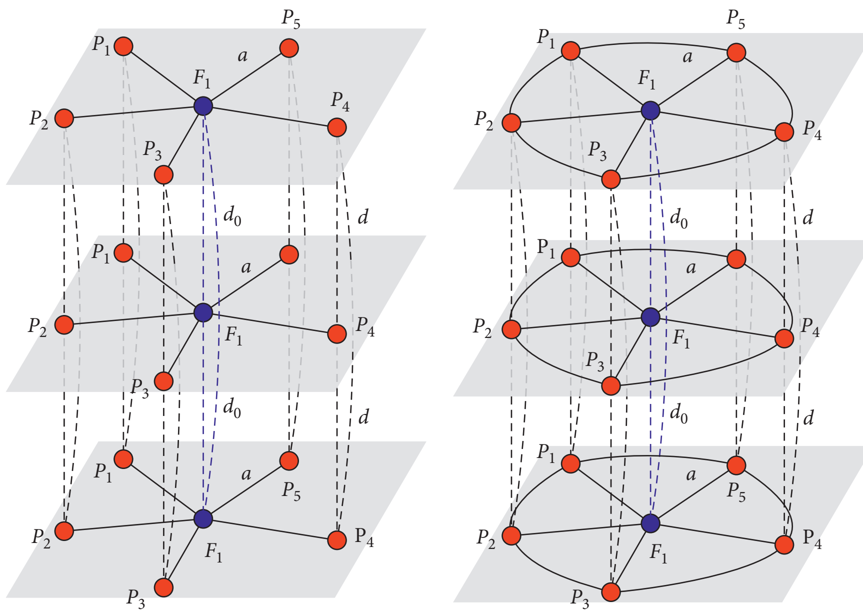
$$\mathscr{L}_I = L_I \otimes I_N, \tag{5}$$

where $\otimes$ is the Kronecker product, $L_{\mathrm{I}} = -dD$, and I_N is the $N \times N$ identity matrix. As for $\mathscr{L}_L$, it can be represented by the direct sum of the Laplacian matrix $L^{(K)}$ within each layer, namely,

$$\mathscr{L}_L = \begin{pmatrix} L^{(1)} & 0 & \cdots & 0\\ 0 & L^{(2)} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & L^{(M)} \end{pmatrix} = \bigoplus_{K=1}^{M} L^{(K)}. \tag{6}$$

The eigenvalues of the supra-Laplacian matrix of the networks are recorded as $0 = \lambda_1 < \lambda_2 \leq \lambda_3 \leq \cdots \leq \lambda_{\max}$. According to the MSF theory, the synchronizability of network (1) is determined by the minimum nonzero eigenvalue or the ratio $R = \lambda_{\max}/\lambda_2$ of the maximum eigenvalue to the minimum nonzero eigenvalue of the supra-Laplacian matrix $\mathscr{L}$. Generally, when the network synchronous region is unbounded, the greater the λ_2, the stronger the synchronizability is; when the synchronous region is bounded, the smaller the $R = \lambda_{\max}/\lambda_2$, the stronger the synchronizability is.

2.2. Two Types of Multilayer Networks. This paper focuses on two types of multilayer networks, that is, the multilayer star and star-ring networks. For a multilayer star network, each layer is made up of identical star subnets, as shown in Figure 1(a), a three-layer star network. For a multilayer



(a) (b)

Figure 1: Schematic diagram of the network structure. (a) A three-layer star network structure. (b) A three-layer star-ring network structure.

star-ring network, each layer is made up of identical star-ring subnets, as shown in Figure 1(b), a three-layer star-ring network. The nodes in each layers carry the identical dynamics, and each node in one layer is connected to its replicas in other layers (the hub node F_1 is connected to its replicas in other layers, and the leaf nodes P_i are connected to their replicas in other layers). Suppose each multilayer network has M layers and $N * M$ nodes (each layer contains N nodes), the intralayer coupling strength of each layer is a, and the interlayer coupling strength between the hub nodes is d_0 and between the leaf nodes is d.

In order to analyze the synchronizability of these two types of multilayer networks, we will separately calculate the eigenvalue spectrum of the networks and analyze the relationship between the synchronizability and structural parameters in the following sections.

3. The Eigenvalue Spectrum and Synchronizability of Multilayer Star Networks

For the derivation of the network eigenvalues, let us introduce the following lemma.

Lemma 1 (see [31]). *Let A, B be $N \times N$ matrices and M be an integer, then*

$$\begin{vmatrix} A & B & \cdots & B \\ B & A & \cdots & B \\ \vdots & \vdots & \ddots & \vdots \\ B & B & \cdots & A \end{vmatrix}_{M\times M} = |A + (M-1)B| \cdot |A - B|^{(M-1)}. \tag{7}$$

Considering the star networks composed of M layers (each layer consisting of N nodes), the node dynamics equation of the networks is as shown in equation (1). In the following, we derive the eigenvalue spectrum and synchronizability of the multilayer star networks under two different conditions, namely, the interlayer coupling strength between the hub nodes d_0 and between the leaf nodes d is different $(d_0 \neq d)$ or the same $(d_0 = d)$.

3.1. *The Eigenvalue Spectrum and Synchronizability of Multilayer Star Networks with $d_0 \neq d$.*

When $d_0 \neq d$, the supra-Laplacian matrix of the multilayer star networks is

$$\mathscr{L}=\begin{bmatrix}
(N-1)a+(M-1)d_0 & -a & \cdots & -a & -d_0 & 0 & \cdots & 0 & \cdots\cdots\cdots\cdots & -d_0 & 0 & \cdots & 0\\
-a & a+(M-1)d & \cdots & 0 & 0 & -d & \cdots & 0 & \cdots\cdots\cdots\cdots & 0 & -d & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \cdots\cdots\cdots\cdots & \vdots & \vdots & \ddots & \vdots\\
-a & 0 & \cdots & a+(M-1)d & 0 & 0 & \cdots & -d & \cdots\cdots\cdots\cdots & 0 & 0 & \cdots & -d\\
-d_0 & 0 & \cdots & 0 & (N-1)a+(M-1)d_0 & -a & \cdots & -a & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & -d & \cdots & 0 & -a & a+(M-1)d & \cdots & 0 & \ddots & \vdots & \vdots & \vdots & \vdots\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots\\
0 & 0 & \cdots & -d & -a & 0 & \cdots & a+(M-1)d & \ddots & \vdots & \vdots & \vdots & \vdots\\
 & \vdots & & \vdots & \ddots & & & & \ddots & -d_0 & 0 & \cdots & 0\\
 & \vdots & & \vdots & & \ddots & & & \ddots & 0 & -d & \cdots & 0\\
 & \vdots & & \vdots & & & \ddots & & \ddots & \vdots & \vdots & \ddots & \vdots\\
 & \vdots & & \vdots & & & & \ddots & \ddots & 0 & 0 & \cdots & -d\\
-d_0 & 0 & \cdots & 0 & & & & & -d_0\ \ 0\ \cdots\ 0 & (N-1)a+(M-1)d_0 & -a & \cdots & -a\\
0 & -d & \cdots & 0 & \cdots & \cdots & \cdots & \cdots & 0\ \ -d\ \cdots\ 0 & -a & a+(M-1)d & \cdots & 0\\
\vdots & \vdots & \ddots & \vdots & \cdots & \cdots & \cdots & \cdots & \vdots\ \ \vdots\ \ddots\ \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \cdots & -d & & & & & 0\ \ 0\ \cdots\ -d & -a & 0 & \cdots & a+(M-1)d
\end{bmatrix}. \tag{8}$$

According to Lemma 1, one obtains the characteristic polynomial of matrix $\mathscr{L}$ as

$$|\lambda I-\mathscr{L}|=\begin{vmatrix}
\lambda-(N-1)a-Md_0 & a & a & a & \cdots & a & a\\
a & \lambda-a-Md & 0 & 0 & \cdots & 0 & 0\\
a & 0 & \lambda-a-Md & 0 & \cdots & 0 & 0\\
a & 0 & 0 & \lambda-a-Md & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
a & 0 & 0 & 0 & \cdots & \lambda-a-Md & 0\\
a & 0 & 0 & 0 & \cdots & 0 & \lambda-a-Md
\end{vmatrix}_{N\times N}^{M-1}
\begin{vmatrix}
\lambda-(N-1)a & a & a & a & \cdots & a & a\\
a & \lambda-a & 0 & 0 & \cdots & 0 & 0\\
a & 0 & \lambda-a & 0 & \cdots & 0 & 0\\
a & 0 & 0 & \lambda-a & \cdots & 0 & 0\\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\
a & 0 & 0 & 0 & \cdots & \lambda-a & 0\\
a & 0 & 0 & 0 & \cdots & 0 & \lambda-a
\end{vmatrix}_{N\times N}$$
$$=\lambda(\lambda-Na)(\lambda-a)^{N-2}(\lambda-a-Md)^{(N-2)(M-1)}\left(\lambda-\frac{Na+Md+Md_0+\sqrt{(Na+Md+Md_0)^2-4[(N-1)aMd+aMd_0+M^2dd_0]}}{2}\right)^{M-1}$$
$$\cdot\left(\lambda-\frac{Na+Md+Md_0-\sqrt{(Na+Md+Md_0)^2-4[(N-1)aMd+aMd_0+M^2dd_0]}}{2}\right)^{M-1}. \tag{9}$$

Let $\lambda_{\mathrm{I}}=((Na+Md+Md_0)-\sqrt{(Na+Md+Md_0)^2-4[(N-1)aMd+aMd_0+M^2dd_0]})/2$ and $\lambda_{\mathrm{II}}=((Na+Md+Md_0)+\sqrt{(Na+Md+Md_0)^2-4[(N-1)aMd+aMd_0+M^2dd_0]})/2$. Therefore, the eigenvalues of $\mathscr{L}$ are 0, Na, $\underbrace{a\cdots a}_{N-2}$, $\underbrace{a+Md\cdots a+Md}_{(N-2)(M-1)}$, $\underbrace{\lambda_{\mathrm{I}}\cdots\lambda_{\mathrm{I}}}_{M-1}$, $\underbrace{\lambda_{\mathrm{II}}\cdots\lambda_{\mathrm{II}}}_{M-1}$, where a is the $N-2$ multiple roots, $a+Md$ is the $(N-2)(M-1)$ multiple roots, and λ_{I} and λ_{II} are the $M-1$ multiple roots. Through simplification, one gets

$$\begin{aligned}\lambda_{\mathrm{I}} &= \frac{Na+Md+Md_0-\sqrt{(Na+Md+Md_0)^2-4[(N-1)aMd+aMd_0+M^2dd_0]}}{2}\\ &= \frac{Na+Md+Md_0-\sqrt{(Na+Md_0-Md)^2+4Ma(d-d_0)}}{2}\\ &= \frac{1}{2}Na\left[1+\frac{Md+Md_0}{Na}-\sqrt{\left(1+\frac{Md_0-Md}{Na}\right)^2+\frac{4Ma(d-d_0)}{(Na)^2}}\right],\quad N\gg a,d,d_0,M\\ &\approx \frac{1}{2}Na\left[1+\frac{Md+Md_0}{Na}-\sqrt{\left(1+\frac{Md_0-Md}{Na}\right)^2}\right],\quad N\gg a,d,d_0,M\\ &= Md.\end{aligned} \tag{10}$$

Therefore, for sufficiently large N, there is $\lambda_{\mathrm{I}}\longrightarrow Md$. In a similar way, there is

$$\lambda_{\mathrm{II}}=\frac{Na+Md+Md_0+\sqrt{(Na+Md+Md_0)^2-4[(N-1)aMd+aMd_0+M^2dd_0]}}{2}\longrightarrow Na+Md_0,\quad N\gg a,d,d_0,M. \tag{11}$$

For sufficiently large N, there are $\lambda_2=\min\{\lambda_{\mathrm{I}},a\}\longrightarrow\min\{Md,a\}$ and $R=(\lambda_{\max}/\lambda_2)=(\lambda_{\mathrm{II}}/\min\{\lambda_I,a\})\longrightarrow((Na+Md_0)/\min\{Md,a\})$. To begin with, according to the MSF theory, we study synchronizability of multilayer star networks with different interlayer coupling strength ($d_0\neq d$), as shown in Table 1.

3.2. The Eigenvalue Spectrum and Synchronizability of the Multilayer Star Networks with $d_0 = d$. When $d_0=d$, one gets the characteristic polynomial of the supra-Laplacian matrix $\mathscr{L}$:

$$|\lambda I-\mathscr{L}|=\lambda(\lambda-Na)(\lambda-a)^{N-2}(\lambda-a-Md)^{(N-2)(M-1)}(\lambda-Na-Md)^{(M-1)}(\lambda-Md)^{(M-1)}. \tag{12}$$

The eigenvalues of the networks are 0, Na, $\underbrace{a\cdots a}_{N-2}$, $\underbrace{a+Md\cdots a+Md}_{(N-2)(M-1)}$, $\underbrace{Na+Md\cdots Na+Md}_{M-1}$, $\underbrace{Md\cdots Md}_{M-1}$. According to the MSF theory, $\lambda_2=\min\{Md,a\}$ and $R=\lambda_{\max}/\lambda_2=((Na+Md)/\min\{Md,a\})$. Then, we study synchronizability of multilayer star networks with invariant interlayer coupling strength ($d_0=d$), as shown in Table 2.

It is worth noting that, when $M=2$, the results about the synchronizability varying with structural parameters of multilayer star networks are basically consistent with the literature [28], but our results are more general. According to Tables 1 and 2, for sufficiently large N and $d_0\neq d$, when the synchronous region is unbounded, the synchronizability of the multilayer star networks depends on the smaller of a and Md, when $a>Md$, the synchronizability is determined by Md, and when $a<Md$, the synchronizability is determined by a. Therefore, the synchronizability of the networks is not affected by the number of nodes N. For bounded synchronous region, the synchronizability depends on $N+(Md_0/a)$ or $(Na+Md_0)/Md$. When $a<Md$, the synchronizability is determined by $N+(Md_0/a)$, and the synchronizability is weakened with the increase of N, M, and d_0 and strengthened with the increase of a. When $a>Md$, the synchronizability of the networks is determined by $Na+Md_0/Md$, and the synchronizability of the network is

TABLE 1: When $d_0 \neq d$, the synchronizability of the multilayer star networks varies with N, a, d, d_0, and M.

	The multilayer star networks			
	The case of synchronous region unbounded		The case of synchronous region bounded	
	$a < Md$	$a > Md$	$a < Md$	$a > Md$
	Synchronizability			
With the increase of N	–	– (N is sufficiently large)	↓	↓
With the increase of M	–	↑	↓	↑
With the increase of a	↑	–	↑	↓
With the increase of d	–	↑	–	↑
With the increase of d_0	–	–	↓	↓

–: unchange; ↑: strengthen; ↓: weaken.

TABLE 2: When $d_0 = d$, the synchronizability of the multilayer star networks varies with N, a, d, and M.

	The multilayer star networks			
	The case of synchronous region unbounded		The case of synchronous region bounded	
	$a < Md$	$a > Md$	$a < Md$	$a > Md$
	Synchronizability			
With the increase of N	–	–	↓	↓
With the increase of M	–	↑	↓	↑
With the increase of a	↑	–	↑	↓
With the increase of d	–	↑	↓	↑

[1]–: unchange; ↑: strengthen; ↓: weaken.

strengthened with the increase of M and d and weakened with the increase of a, N, and d_0; hence, d and a have the opposite effect; when N is fixed and d is increased, the synchronizability of the networks can be maintained invariant by increasing a. For $d_0 = d$, the synchronizability varying with structural parameters is basically consistent with that under the different interlayer coupling strength; the difference is that, when $a < Md$, the synchronizability with invariant interlayer coupling strength is weakened with increasing d, while that with different interlayer coupling strength is invariant with increasing d, with the bounded synchronous region.

4. The Eigenvalue Spectrum and Synchronizability of Multilayer Star-Ring Networks

In this section, we explore the eigenvalue spectrum and synchronizability of the multilayer star-ring networks in the case of $d_0 \neq d$ and $d_0 = d$, respectively.

4.1. The Eigenvalue Spectrum and Synchronizability of Multilayer Star-Ring Networks with $d_0 \neq d$. For multilayer star-ring networks, when $d_0 \neq d$, one obtains the supra-Laplacian matrix $\mathscr{L}$ of the networks:

$$
\mathscr{L} = \begin{bmatrix}
* & -a & -a & \cdots & -a & -a & -d_0 & 0 & 0 & \cdots & 0 & 0 & \cdots & & \cdots & & \cdots & \cdots & -d_0 & 0 & 0 & \cdots & 0 & 0 \\
-a & \otimes & -a & \cdots & 0 & -a & 0 & -d & 0 & \cdots & 0 & 0 & \cdots & & \cdots & & \cdots & \cdots & 0 & -d & 0 & \cdots & 0 & 0 \\
-a & -a & \otimes & \cdots & 0 & 0 & 0 & 0 & -d & \cdots & 0 & 0 & & & & & & & 0 & 0 & -d & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & & \cdots & & \cdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
-a & 0 & 0 & \cdots & \otimes & -a & 0 & 0 & 0 & \cdots & -d & 0 & & & & & & & 0 & 0 & 0 & \cdots & -d & 0 \\
-a & -a & 0 & \cdots & -a & \otimes & 0 & 0 & 0 & \cdots & 0 & -d & \cdots & & \cdots & & \cdots & \cdots & 0 & 0 & 0 & \cdots & 0 & -d \\
-d_0 & 0 & 0 & \cdots & 0 & 0 & * & -a & -a & \cdots & -a & -a & & & & & & & & \vdots & & & \vdots & \vdots \\
0 & -d & 0 & \cdots & 0 & 0 & -a & \otimes & -a & \cdots & 0 & -a & \ddots & & & & & & & & & & & \\
0 & 0 & -d & \cdots & 0 & 0 & -a & -a & \otimes & \cdots & 0 & 0 & & \ddots & & & & & & \vdots & & & \vdots & \vdots \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & & & \ddots & & & & & & & & & \\
0 & 0 & 0 & \cdots & -d & 0 & -a & 0 & 0 & \cdots & \otimes & -a & & & & & & & & \vdots & & & \vdots & \vdots \\
0 & 0 & 0 & \cdots & 0 & -d & -a & -a & 0 & \cdots & -a & \otimes & & & & & & & & & & & & \\
& \vdots & \vdots & & \vdots & \vdots & \ddots & & \ddots & & & & & & & & & & -d_0 & 0 & 0 & \cdots & 0 & 0 \\
& & & & & & & & & & & & & & & & & & 0 & -d & 0 & \cdots & 0 & 0 \\
& \vdots & \vdots & & \vdots & \vdots & & \ddots & & \ddots & & & & & & & & & 0 & 0 & -d & \cdots & 0 & 0 \\
& & & & & & & & \ddots & & & & & & & & & & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
& & & & & & & & & & & & & & & & & & 0 & 0 & 0 & \cdots & -d & 0 \\
& \vdots & \vdots & & \vdots & \vdots & & & & & \ddots & & & & & & & & 0 & 0 & 0 & \cdots & 0 & -d \\
-d_0 & 0 & 0 & \cdots & 0 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & -d_0 & 0 & 0 & \cdots & 0 & 0 & * & -a & -a & \cdots & -a & -a \\
0 & -d & 0 & \cdots & 0 & 0 & & & & & & & 0 & -d & 0 & \cdots & 0 & 0 & -a & \otimes & -a & \cdots & 0 & -a \\
0 & 0 & -d & \cdots & 0 & 0 & & & & & & & 0 & 0 & -d & \cdots & 0 & 0 & -a & -a & \otimes & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -d & 0 & & & & & & & 0 & 0 & 0 & \cdots & -d & 0 & -a & 0 & 0 & \cdots & \otimes & -a \\
0 & 0 & 0 & \cdots & 0 & -d & & & & & & & 0 & 0 & 0 & \cdots & 0 & -d & -a & -a & 0 & \cdots & -a & \otimes
\end{bmatrix}, \quad (13)
$$

where $* \triangleq (N-1)a + (M-1)d_0$ and $\otimes \triangleq 3a + (M-1)d$. According to Lemma 1, the characteristic polynomial of $\mathscr{L}$ is

$$|\lambda I - \mathscr{L}| = \begin{vmatrix} \lambda-(N-1)a-Md_0 & a & a & a & \cdots & a & a \\ a & \lambda-3a-Md & a & 0 & \cdots & 0 & a \\ a & a & \lambda-3a-Md & a & \cdots & 0 & 0 \\ a & 0 & a & \lambda-3a-Md & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & 0 & 0 & 0 & \cdots & \lambda-3a-Md & a \\ a & a & 0 & 0 & \cdots & a & \lambda-3a-Md \end{vmatrix}_{N\times N}^{M-1} \begin{vmatrix} \lambda-(N-1)a & a & a & a & \cdots & a & a \\ a & \lambda-3a & a & 0 & \cdots & 0 & a \\ a & a & \lambda-3a & a & \cdots & 0 & 0 \\ a & 0 & a & \lambda-3a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & 0 & 0 & 0 & \cdots & \lambda-3a & a \\ a & a & 0 & 0 & \cdots & a & \lambda-3a \end{vmatrix}_{N\times N}. \tag{14}$$

Let

$$|C|_{N\times N}^{M-1} = \begin{vmatrix} \lambda-(N-1)a-Md_0 & a & a & a & \cdots & a & a \\ a & \lambda-3a-Md & a & 0 & \cdots & 0 & a \\ a & a & \lambda-3a-Md & a & \cdots & 0 & 0 \\ a & 0 & a & \lambda-3a-Md & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & 0 & 0 & 0 & \cdots & \lambda-3a-Md & a \\ a & a & 0 & 0 & \cdots & a & \lambda-3a-Md \end{vmatrix}_{N\times N}^{M-1},$$

$$|D|_{N\times N} = \begin{vmatrix} \lambda-(N-1)a & a & a & a & \cdots & a & a \\ a & \lambda-3a & a & 0 & \cdots & 0 & a \\ a & a & \lambda-3a & a & \cdots & 0 & 0 \\ a & 0 & a & \lambda-3a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ a & 0 & 0 & 0 & \cdots & \lambda-3a & a \\ a & a & 0 & 0 & \cdots & a & \lambda-3a \end{vmatrix}_{N\times N}. \tag{15}$$

By simplifying,

$$\begin{aligned} |C|_n^{M-1} = {} & \left(\lambda - \frac{Na + Md + Md_0 + \sqrt{(Na + Md + Md_0)^2 - 4\left[(N-1)aMd + aMd_0 + M^2dd_0\right]}}{2}\right)^{M-1} \\ & \cdot \left(\lambda - \frac{Na + Md + Md_0 - \sqrt{(Na + Md + Md_0)^2 - 4\left[(N-1)aMd + aMd_0 + M^2dd_0\right]}}{2}\right)^{M-1} |H|_{(N-1)\times(N-1)}^{M-1}, \end{aligned} \tag{16}$$

where

$$|H| = \begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ a & \lambda - 3a - Md & a & 0 & \cdots & 0 & 0 \\ 0 & a & \lambda - 3a - Md & a & \cdots & 0 & 0 \\ 0 & 0 & a & \lambda - 3a - Md & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda - 3a - Md & a \\ a & 0 & 0 & 0 & \cdots & a & \lambda - 3a - Md \end{vmatrix}_{(N-1)\times(N-1)}.$$

$$|D|_{N\times N} = \lambda(\lambda - Na)|Q|_{(N-1)\times(N-1)} = \lambda(\lambda - Na)\begin{vmatrix} 1 & 1 & 1 & 1 & \cdots & 1 & 1 \\ a & \lambda - 3a & a & 0 & \cdots & 0 & 0 \\ 0 & a & \lambda - 3a & a & \cdots & 0 & 0 \\ 0 & 0 & a & \lambda - 3a & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \lambda - 3a & a \\ a & 0 & 0 & 0 & \cdots & a & \lambda - 3a \end{vmatrix}_{(N-1)\times(N-1)}, \quad (17)$$

When N is an odd number, the values of λ in $|Q|$ are

$$5a, \underbrace{a + 4a\sin^2((k\pi)/(2(N-1)))}_{2}, \quad k = 2, 4, 6, \ldots, N-3. \quad (18)$$

When N is an even number, the values of λ in $|Q|$ are

$$\underbrace{a + 4a\sin^2((k\pi)/(2(N-1)))}_{2}, \quad k = 2, 4, 6, \ldots, N-2. \quad (19)$$

When N is an odd number, the values of λ in $|H|$ are

$$Md + 5a, \underbrace{Md + a + 4a\sin^2((k\pi)/(2(N-1)))}_{2}, \quad k = 2, 4, 6, \ldots, N-3. \quad (20)$$

When N is an even number, the values of λ in $|H|$ are

$$\underbrace{a + 4a\sin^2((k\pi)/(2(N-1)))}_{2}, \quad k = 2, 4, 6, \ldots, N-2. \quad (21)$$

Thus, the eigenvalues of $\mathscr{L}$ are as follows:
When N is an odd number:

$$0, 5a, Na, \underbrace{a + 4a\sin^2\left(\frac{k\pi}{2(N-1)}\right)}_{2}, \quad (k = 2, 4, 6, \ldots, N-3), \underbrace{Md + 5a}_{M-1}, \underbrace{Md + a + 4a\sin^2\left(\frac{k\pi}{2(N-1)}\right)}_{2(M-1)}, \quad (k = 2, 4, 6, \ldots, N-3),$$
$$\underbrace{\frac{Na + Md + Md_0 - \sqrt{(Na + Md + Md_0)^2 - 4[(N-1)aMd + aMd_0 + M^2dd_0]}}{2}}_{M-1},$$
$$\underbrace{\frac{Na + Md + Md_0 + \sqrt{(Na + Md + Md_0)^2 - 4[(N-1)aMd + aMd_0 + M^2dd_0]}}{2}}_{M-1}. \quad (22)$$

Table 3: When $d_0 \neq d$, the synchronizability of the multilayer star-ring networks varies with N, a, d, d_0, and M.

	The multilayer star-ring networks			
	The case of synchronous region unbounded		The case of synchronous region bounded	
	$a + 4a\sin^2(\pi/N-1) < Md$	$a + 4a\sin^2(\pi/N-1) > Md$	$a + 4a\sin^2(\pi/N-1) < Md$	$a + 4a\sin^2(\pi/N-1) > Md$
	Synchronizability			
With the increase of N	First ↓, then −	− (N is sufficiently large)	↓	↓
With the increase of M	−	↑	↓	↑
With the increase of a	↑	−	↑	↓
With the increase of d	−	↑	−	↑
With the increase of d_0	−	−	↓	↓

[1]−: unchange; ↑: strengthen; ↓: weaken.

When N is an even number:

$$
\begin{gathered}
0, Na, \underbrace{a + 4a\sin^2\left(\frac{k\pi}{2(N-1)}\right)}_{2}, \quad k = 2,4,6,\ldots,N-2, \\
\underbrace{Md + a + 4a\sin^2\left(\frac{k\pi}{2(N-1)}\right)}_{2(M-1)}, \quad k = 2,4,6,\ldots,N-2, \\
\underbrace{\frac{Na + Md + Md_0 - \sqrt{(Na + Md + Md_0)^2 - 4[(N-1)aMd + aMd_0 + M^2dd_0]}}{2}}_{M-1}, \\
\underbrace{\frac{Na + Md + Md_0 + \sqrt{(Na + Md + Md_0)^2 - 4[(N-1)aMd + aMd_0 + M^2dd_0]}}{2}}_{M-1}.
\end{gathered} \tag{23}
$$

Let $\lambda_{III} = Na + Md + Md_0 - \sqrt{(Na + Md + Md_0)^2 - 4[(N-1)aMd + aMd_0 + M^2dd_0]}/2$ and $\lambda_{IV} = Na + Md + Md_0 + \sqrt{(Na + Md + Md_0)^2 - 4[(N-1)aMd + aMd_0 + M^2dd_0]}/2$. Whether N is an odd or even number, λ max $= \lambda_{IV}$ and $\lambda_2 = \min\{\lambda_{III}, a + 4a\sin^2(\pi/(N-1))\}$. Similar to Section 3.1, for sufficiently large N, $\lambda_{III} \longrightarrow Md$ and $\lambda_{IV} \longrightarrow Na + Md_0$; therefore, $\lambda_2 = \min\{\lambda_{III}, a + 4a\sin^2(\pi/N-1)\} \longrightarrow \min\{Md, a + 4a\sin^2(\pi/N-1)\}$ and $R = \lambda_{\max}/\lambda_2\ (\lambda_{IV}/\min\{\lambda_{III}, a + 4a\sin^2(\pi/N-1)\}) \longrightarrow (Na + Md_0)/\min\{Md, a + 4a\sin^2(\pi/N-1)\}$. According to the MSF theory, we can summarize the relationship between the synchronizability and structural parameters under the different interlayer coupling strength, as shown in Table 3.

4.2. The Eigenvalue Spectrum and Synchronizability of Multilayer Star-Ring Networks with $d_0 = d$. Similarly, when $d_0 = d$, the eigenvalues of $\mathscr{L}$ are as follows:

When N is an odd number: 0, $5a$, Na, $\underbrace{a + 4a\sin^2(k\pi/2(N-1))}_{2}$, $(k = 2,4,6,\ldots,N-3)$, $\underbrace{Md}_{M-1}$, $\underbrace{Md + 5a}_{M-1}$, $\underbrace{Md + a + 4a\sin^2(k\pi/2(N-1))}_{2(M-1)}$, $(k = 2,4,6,\ldots,N-3)$, $\underbrace{Na + Md}_{M-1}$. When N is an even number: 0, Na, $\underbrace{a + 4a\sin^2(k\pi/2(N-1))}_{2}$, $(k = 2,4,6,\ldots,N-2)$, $\underbrace{Md}_{M-1}$, $\underbrace{Md + a + 4a\sin^2(k\pi/2(N-1))}_{2(M-1)}$, $(k = 2,4,6,\ldots,N-2)$, $\underbrace{Na + Md}_{M-1}$. Hence, λ max $= Na + Md$ and $\lambda_2 = \min\{Md, a + 4a\sin^2(\pi/N-1)\}$. According to the MSF theory, we study synchronizability of multilayer star-ring networks with invariant interlayer coupling strength ($d_0 = d$), as shown in Table 4.

The synchronizability of two kinds of multilayer networks varying with the structural parameters under different ($d_0 \neq d$) interlayer coupling strength is basically the same as that under the same ($d_0 = d$) interlayer coupling strength.

5. Numerical Simulation

In this section, we will investigate the synchronizability of multilayer star and star-ring networks under different interlayer coupling strength ($d_0 > d$, $d_0 < d$, and $d_0 = d$) through a large number of numerical simulation experiments.

5.1. The Synchronizability of Multilayer Star Networks. For different interlayer coupling strength (i.e., $d_0 > d$, $d_0 < d$, and $d_0 = d$.), the synchronizability of multilayer star networks varying with the structural parameters is shown in Figures 2–8.

Table 4: When $d_0 = d$, the synchronizability of the multilayer star-ring networks varies with N, a, d, and M.

	The multilayer star-ring networks			
	The case of synchronous region unbounded		The case of synchronous region bounded	
	$a + 4a\sin^2(\pi/N-1) < Md$	$a + 4a\sin^2(\pi/N-1) > Md$	$a + 4a\sin^2(\pi/N-1) < Md$	$a + 4a\sin^2(\pi/N-1) > Md$
	Synchronizability			
With the increase of N	First ↓, then –	–	↓	↓
With the increase of M	–	↑	↓	↑
With the increase of a	↑	–	↑	↓
With the increase of d	–	↑	↓	↑

[1]–: unchange; ↑: strengthen; ↓: weaken.

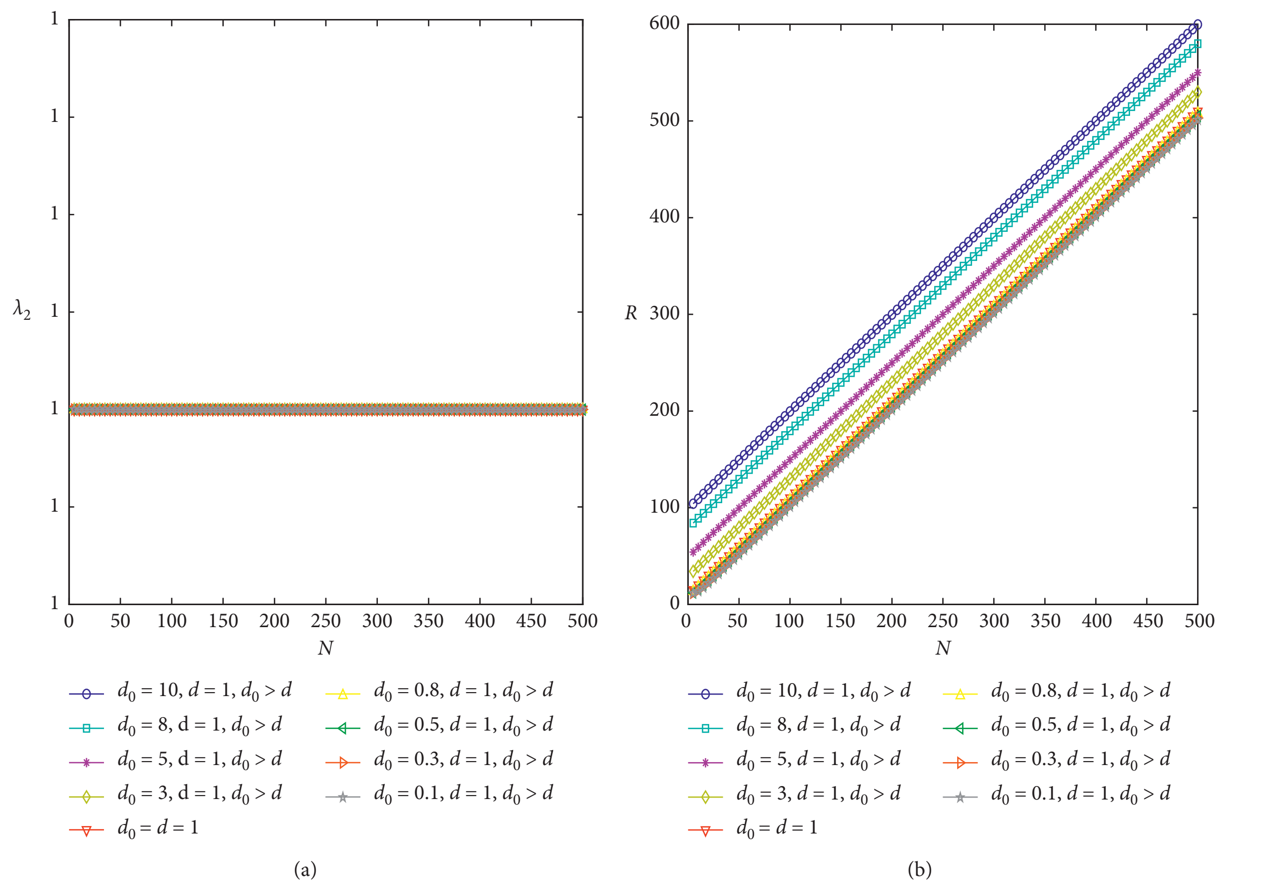


Figure 2: The synchronizability of multilayer star networks *vs.* varying the number of nodes N ($a = 1$, $M = 10$, $d = 1$, and $a < Md$). (a) λ_2 with respect to varying N for different d_0. (b) R with respect to varying N for different d_0.

When $a < Md$, it can be seen from the left panel of Figure 2 that λ_2 remains invariant at a specific value $\lambda_2 = a = 1$ with increasing N for different d_0 and different values of d_0 have no effect on the synchronizability of the networks, with the unbounded synchronous region. The right panel reveals that R increases with increasing N, and the observation reveals that the synchronizability is weakened with increasing the number of nodes in each layer. The smaller d_0, the stronger the synchronizability, with the bounded synchronous region. When $a > Md$, from Figure 3(a), λ_2 remains invariant at a specific value $\lambda_2 = Md = 10$ with increasing N for $d_0 = d$; λ_2 increases at first and then almost levels off at an upper bound value $\lambda_2 = Md = 10$ for $d_0 < d$; λ_2 decreases at first and then almost levels off at a lower bound value $\lambda_2 = Md = 10$ for $d_0 > d$, with the unbounded synchronous region. From Figure 3(b), R increases with increasing N, the synchronizability is weakened with increasing N, and different values of d_0 have no effect on the synchronizability of the networks.

Figure 4 shows the synchronizability of multilayer star networks varying with the intralayer coupling strength a. From Figure 4(a), it is obvious that λ_2 increases linearly from zero and then remains invariant; this means that the synchronizability is strengthened at first and then remains

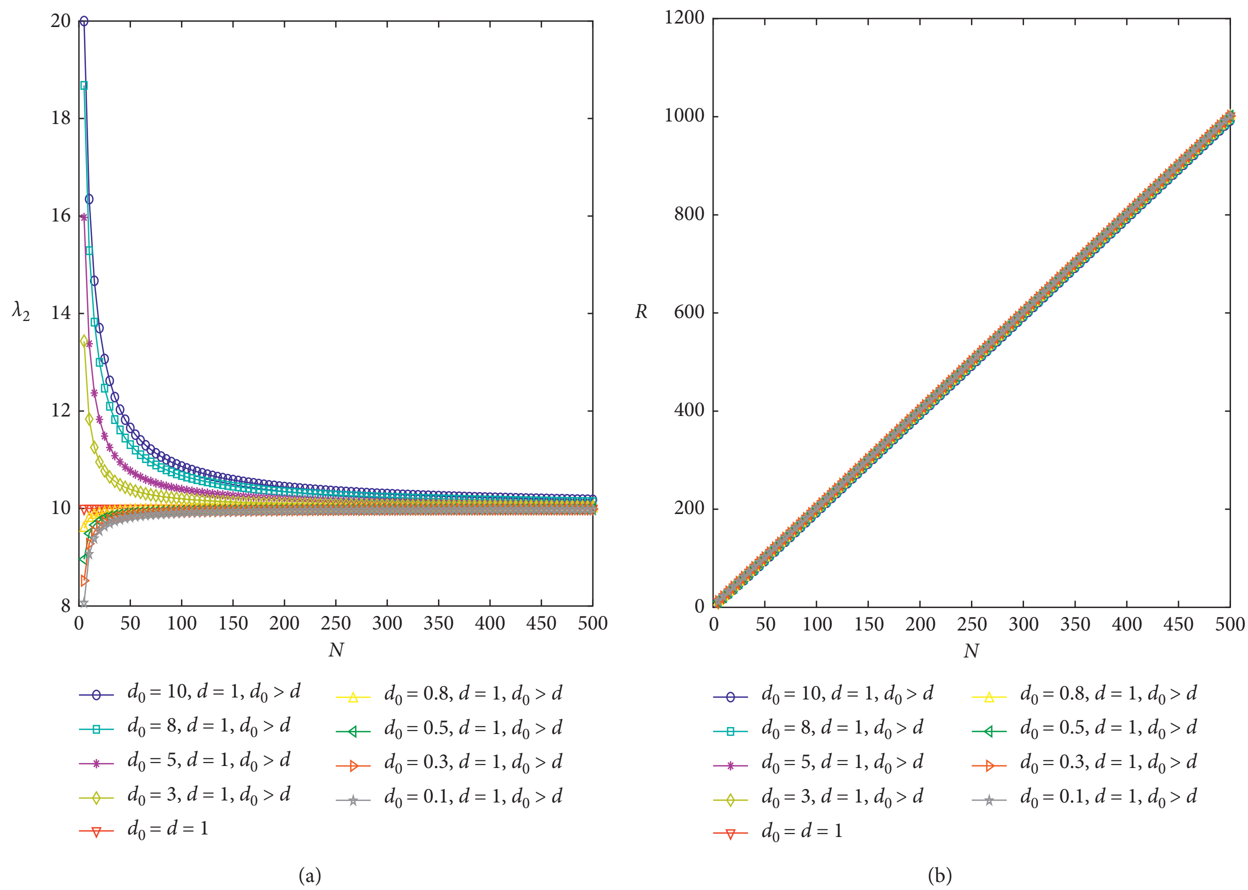


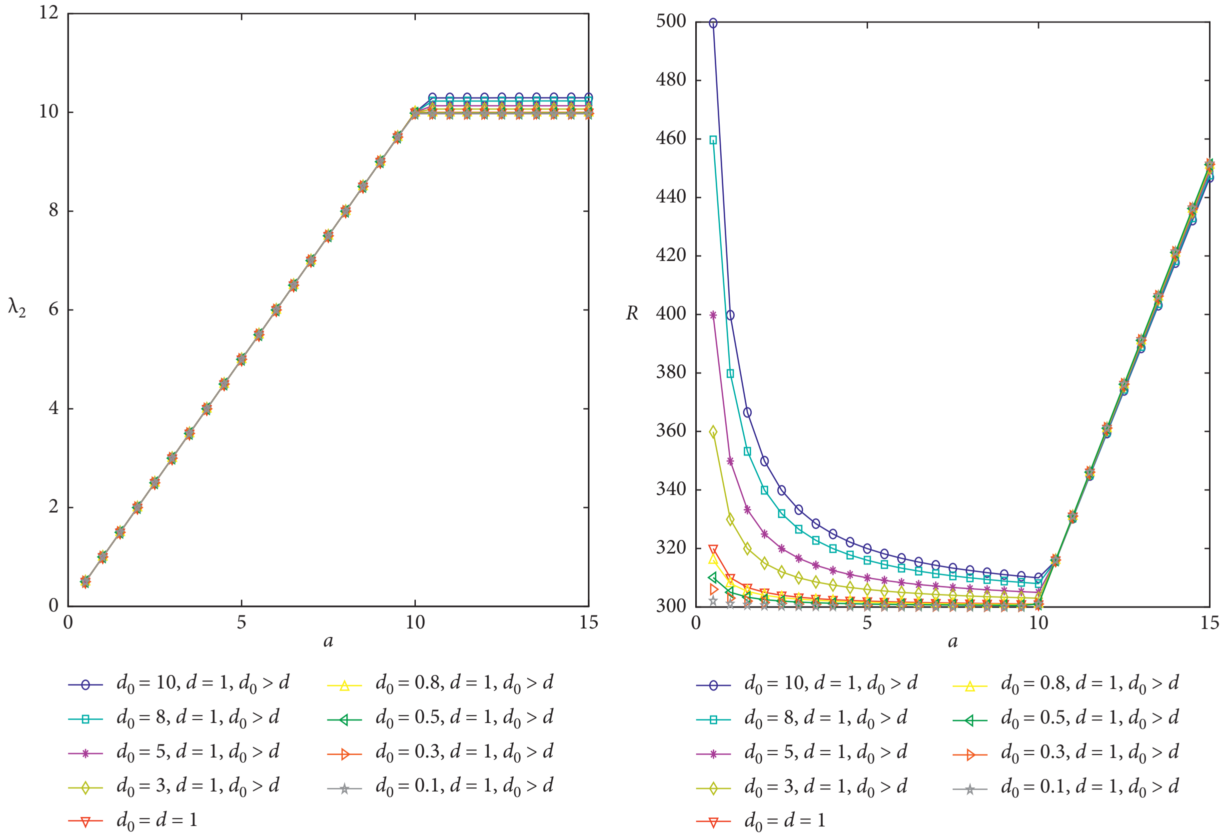
Figure 3: The synchronizability of multilayer star networks *vs*. varying the number of nodes N ($a = 20$, $M = 10$, $d = 1$, and $a > Md$). (a) λ_2 with respect to varying N for different d_0. (b) R with respect to varying N for different d_0.

invariant with increasing a. When $a < Md$, different values of d_0 have no effect on the synchronizability of the networks; when $a > Md$, the smaller d_0, the worse the synchronizability, with the unbounded synchronous region. From Figure 4(b), one can observe that, with increasing a, R first decreases (when $a < Md$) and then increases sharply (when $a > Md$), and the synchronizability is strengthened at first ($a < Md$) and then weakened with increasing intralayer coupling strength ($a > Md$). When $a < Md$, the smaller d_0, the stronger the synchronizability, and when $a > Md$, different values of d_0 have no effect on the synchronizability, with the bounded synchronous region. This is because when the intralayer coupling strength far exceeds the interlayer coupling strength, the structure of the multiplex network is similar to the community structure of multiple clusters; therefore, the synchronizability is weakened.

Figures 5(a) and 5(b) show the synchronizability is strengthened at first (when $a > Md$) and then remains invariant (when $a < Md$) with increasing d. That is to say, whether the synchronous region is unbounded or bounded, when $d < a/M$, the synchronizability of the networks is strengthened, and when $d > a/M$, the synchronizability of the networks remains basically invariant; therefore, there exists an optimal value of $d = a/M$ for interlayer coupling strength to maximizing synchronizability of multilayer star networks. When d is too large, it not conducive to synchronization.

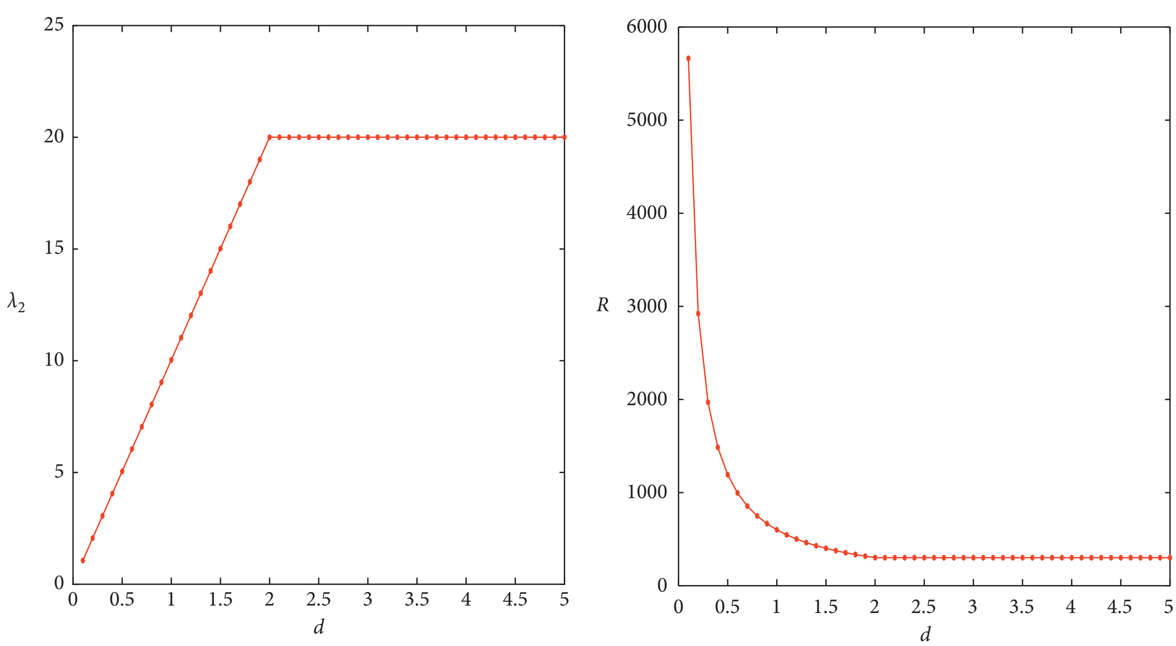
As is shown in Figure 6, when $a < Md$, panel (a) plots that the synchronizability is not affected by the interlayer coupling strength d_0, with the unbounded synchronous region; panel (b) plots that the synchronizability is weakened with d_0, with the bounded synchronous region. From Figure 7(a), when $a > Md$, the synchronizability is strengthened slightly and basically maintained at $\lambda_2 = Md = 10$, with the unbounded synchronous region; from Figure 7(b), the synchronizability is also strengthened slightly, with the bounded synchronous region. Figure 8(a) shows that increasing the number of layers M increases λ_2 (increases gradually at first and then approaches Md), and therefore, the synchronizability is strengthened at first and then basically maintained invariant, with the unbounded synchronous region. Figure 8(b) shows that the synchronizability is strengthened at first and then weakened slightly with increasing the number of layers, with the bounded synchronous region.

5.2. The Synchronizability of Multilayer Star-Ring Networks. For different interlayer coupling strength (i.e., $d_0 > d$, $d_0 < d$, and $d_0 = d$), the synchronizability of the multilayer star-ring networks varies with the structural parameters, as shown in Figures 9–15.



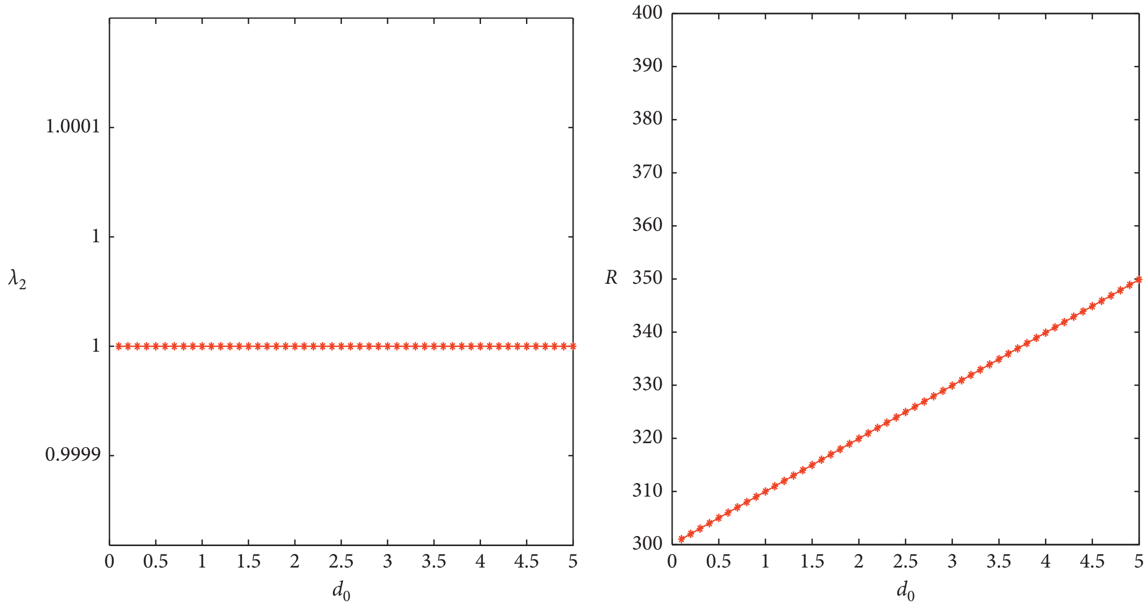
(a) (b)

FIGURE 4: The synchronizability of multilayer star networks *vs.* varying the intralayer coupling strength a ($N = 300$, $M = 10$, and $d = 1$). (a) λ_2 with respect to varying a for different d_0. (b) R with respect to varying a for different d_0.



(a) (b)

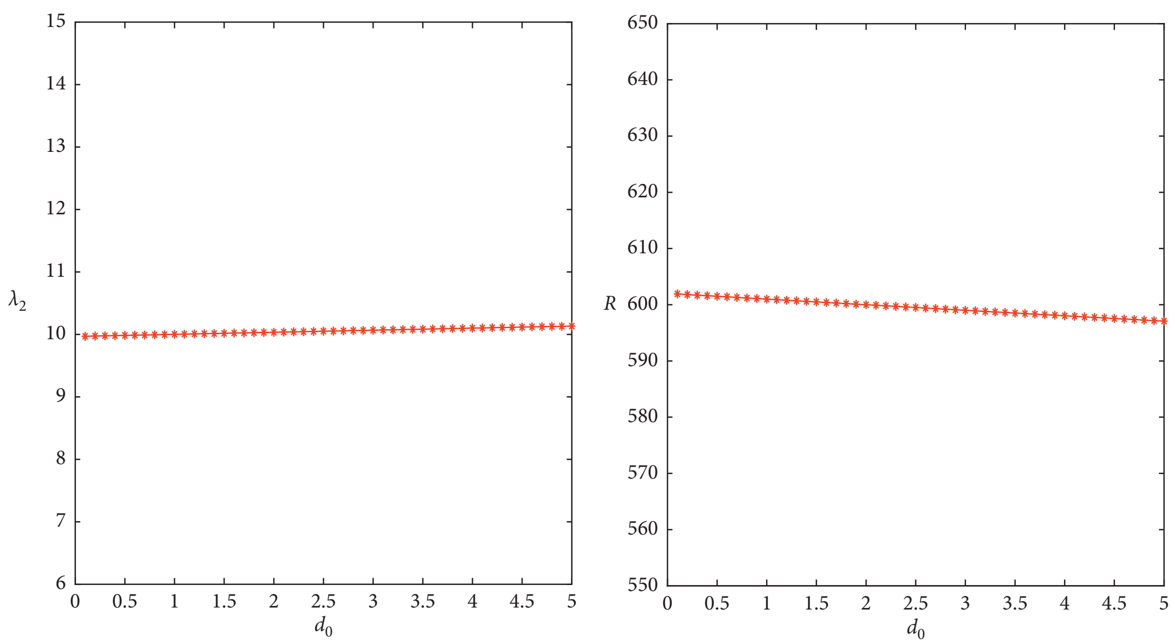
FIGURE 5: The synchronizability of multilayer star networks *vs.* varying the interlayer coupling strength d ($N = 300$, M = 10, $a = 20$, and $d_0 = 2$). (a) λ_2 with respect to varying d. (b) R with respect to varying d.



(a)

(b)

FIGURE 6: The synchronizability of multilayer star networks *vs*. varying the interlayer coupling strength d_0 ($N = 300$, $M = 10$, $a = 1$, $d = 1$, and $a < Md$). (a) λ_2 with respect to varying d_0. (b) R with respect to varying d_0.



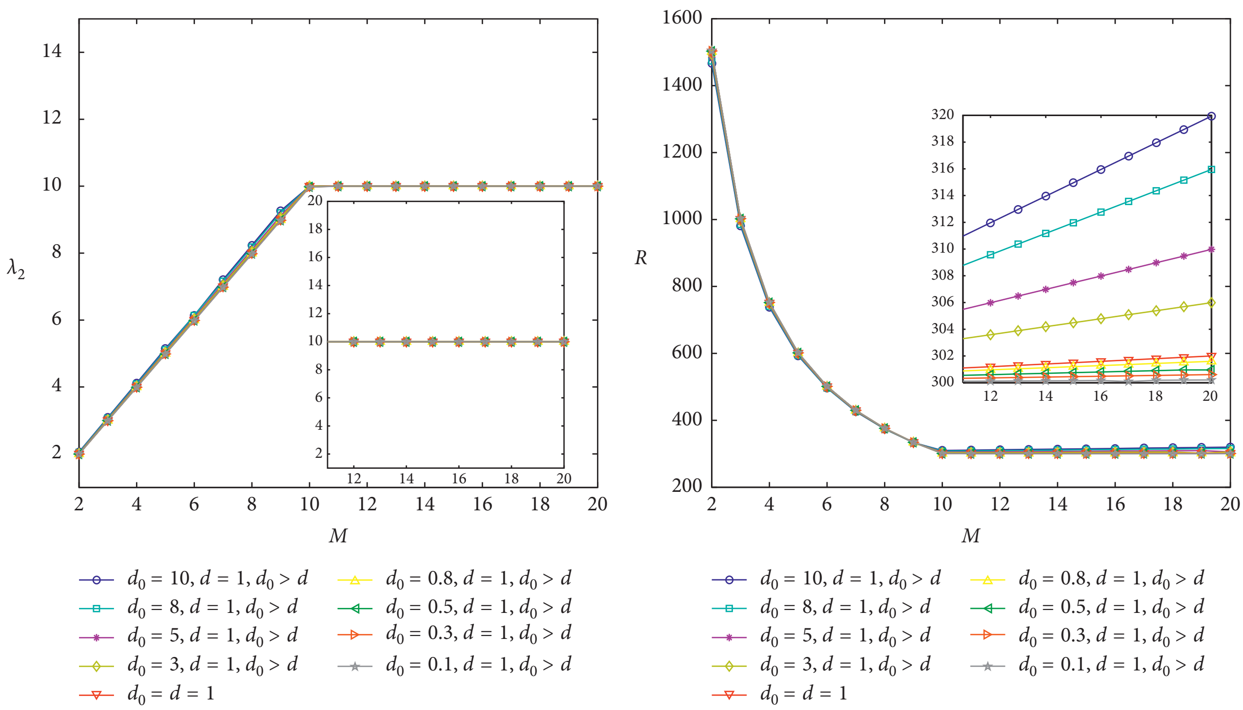
(a)

(b)

FIGURE 7: The synchronizability of multilayer star networks *vs*. varying the interlayer coupling strength d_0 ($N = 300$, $M = 10$, $a = 20$, $d = 1$, and $a > Md$). (a) λ_2 with respect to varying d_0. (b) R with respect to varying d_0.

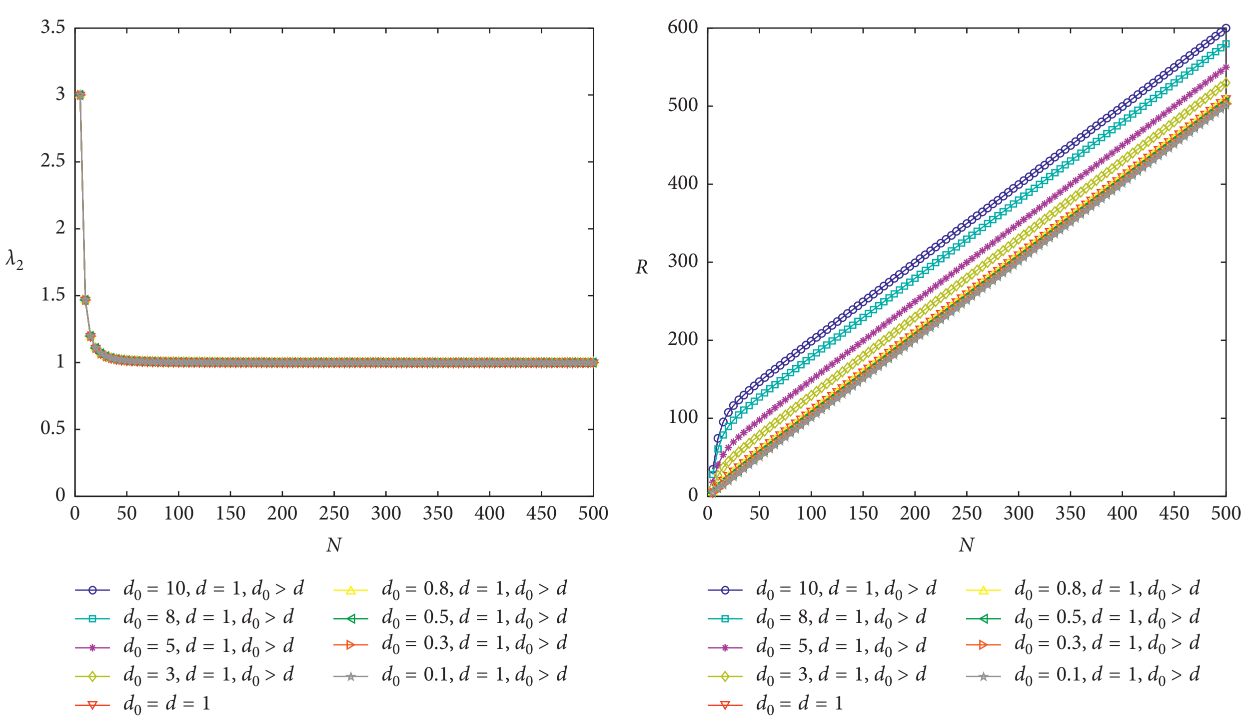
From Figure 9, when $a < Md$, the synchronizability of multilayer star-ring networks is weakened at first and then remains invariant with increasing N, with the unbounded synchronous region. When the synchronous region is bounded, the synchronizability is weakened with increasing the number of nodes in each layer N.

Figure 10 displays that the synchronizability remains invariant with increasing N for $d_0 = d$; the synchronizability



(a) (b)

FIGURE 8: The synchronizability of multilayer star networks *vs*. varying the number of the layers M ($N = 100$, $a = 10$, and $d = 1$). (a) λ_2 with respect to varying M for different d_0. (b) R with respect to varying M for different d_0.



(a) (b)

FIGURE 9: The synchronizability of multilayer star-ring networks *vs*. varying the number of nodes N ($a = 1$, $M = 10$, $d = 1$, and $a < Md$). (a) λ_2 with respect to varying N for different d_0. (b) R with respect to varying N for different d_0.

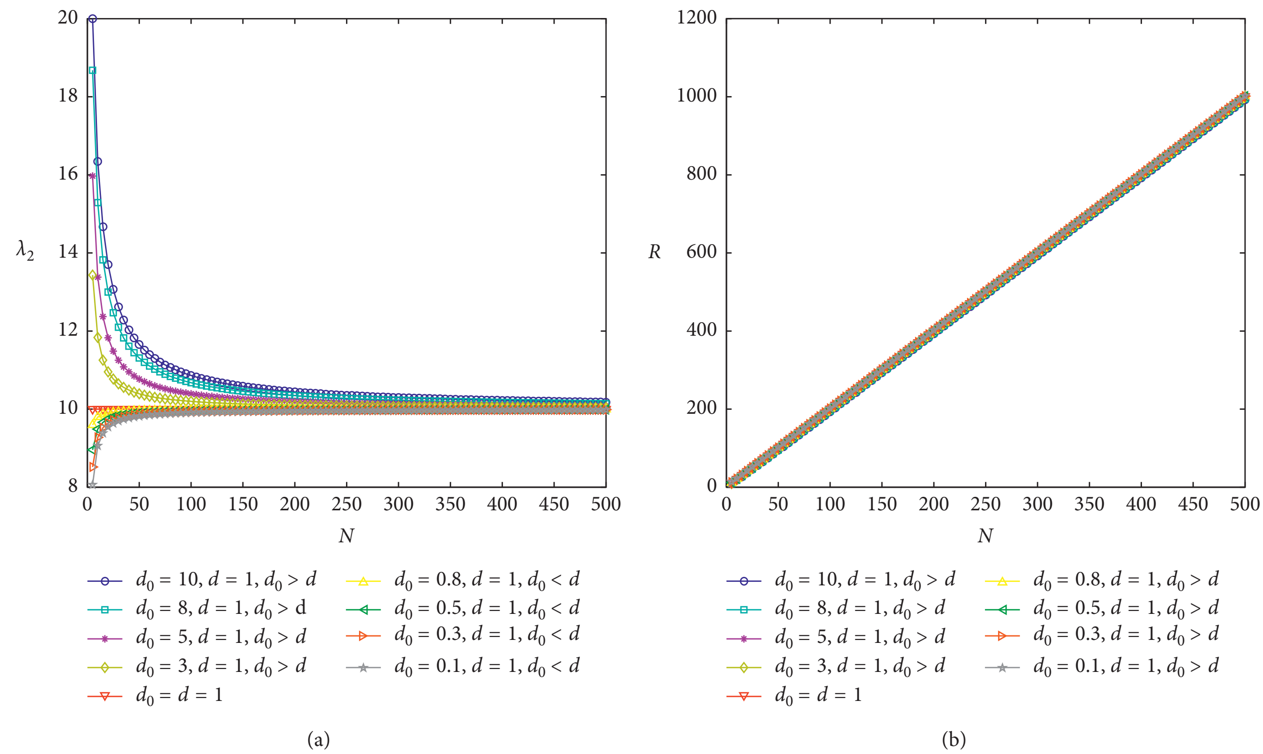


FIGURE 10: The synchronizability of multilayer star-ring networks *vs*. varying the number of nodes N ($a = 20$, $M = 10$, $d = 1$, and $a > Md$). (a) λ_2 with respect to varying N for different d_0. (b) R with respect to varying N for different d_0.

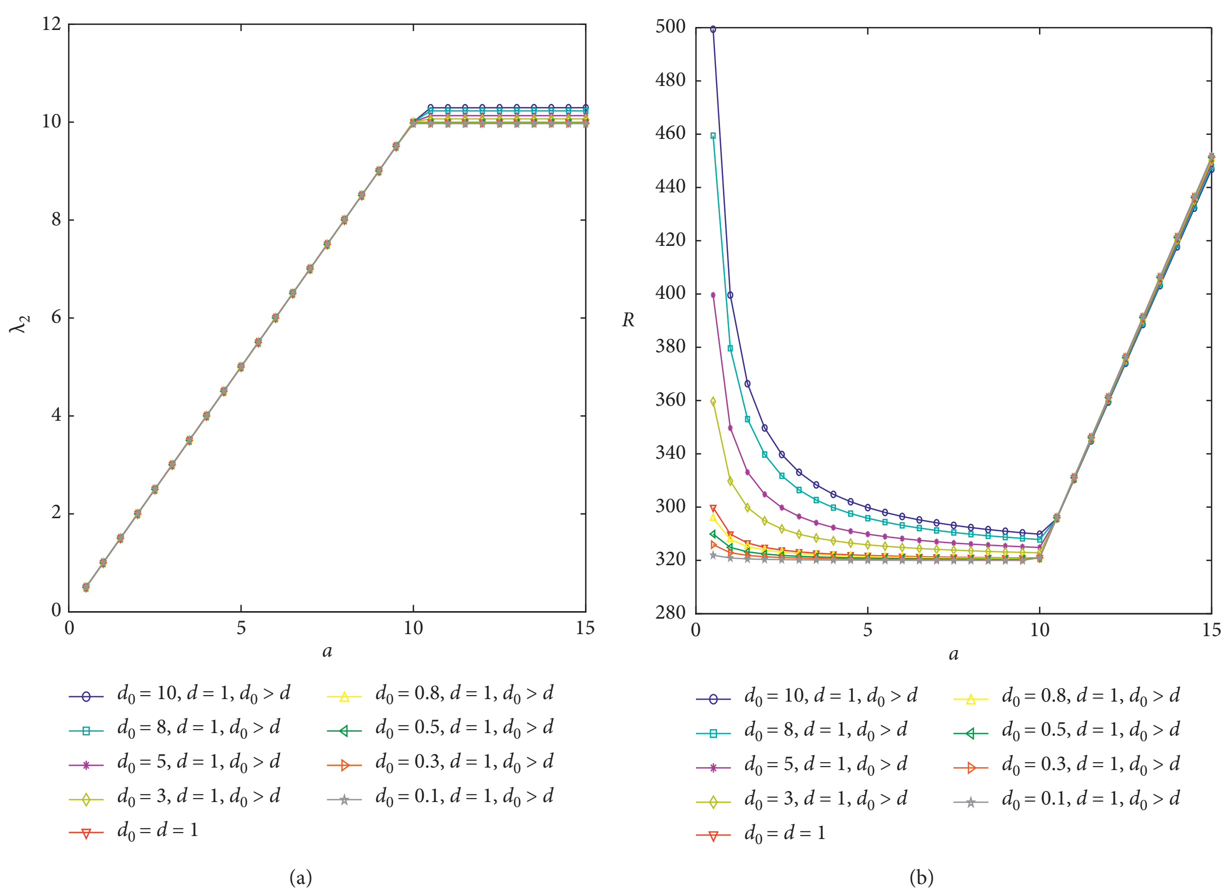


FIGURE 11: The synchronizability of multilayer star-ring networks *vs*. varying the intralayer coupling strength a ($N = 300$, $M = 10$, and $d = 1$). (a) λ_2 with respect to varying a for different d_0. (b) R with respect to varying a for different d_0.

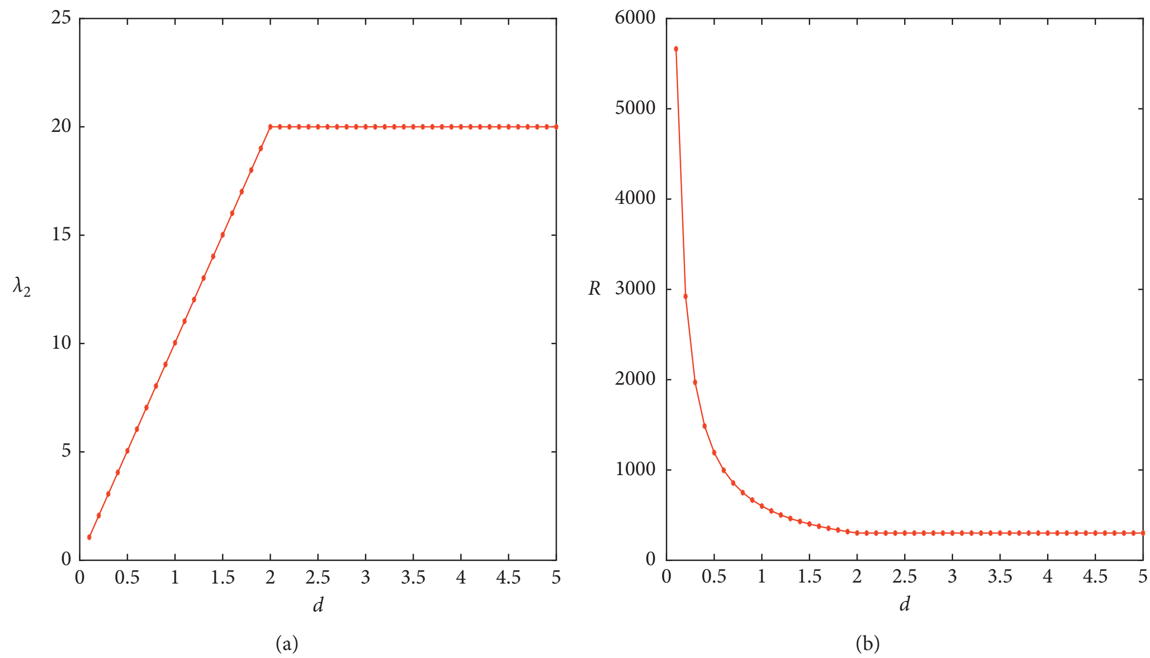


Figure 12: The synchronizability of multilayer star-ring networks *vs*. varying the interlayer coupling strength d ($N = 300$, $M = 10$, $a = 20$, and $d_0 = 2$). (a) λ_2 with respect to varying d. (b) R with respect to varying d.

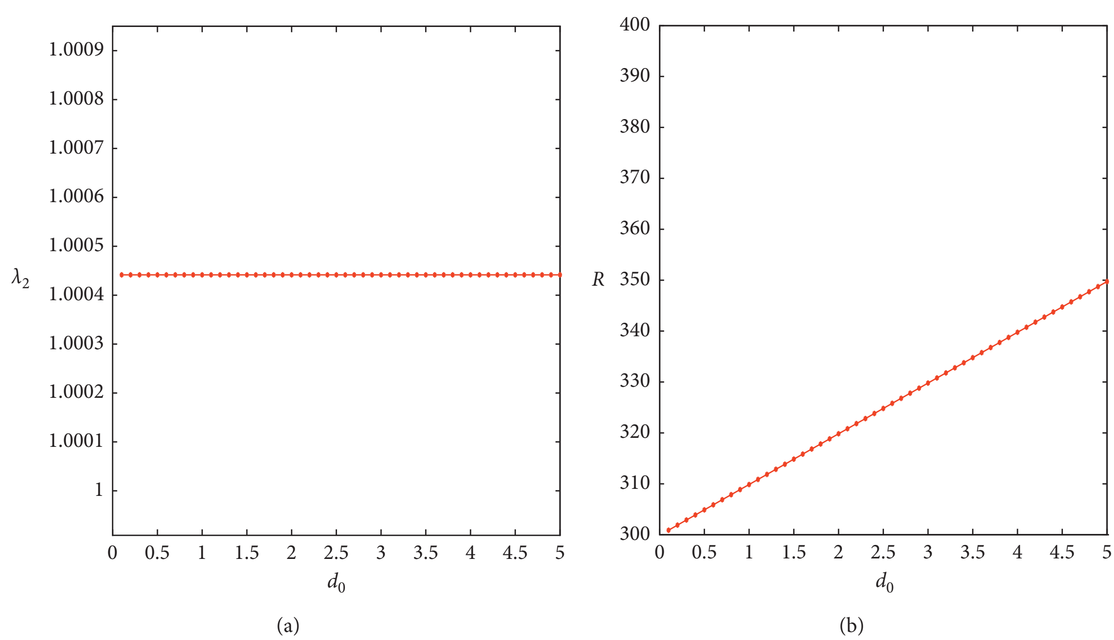
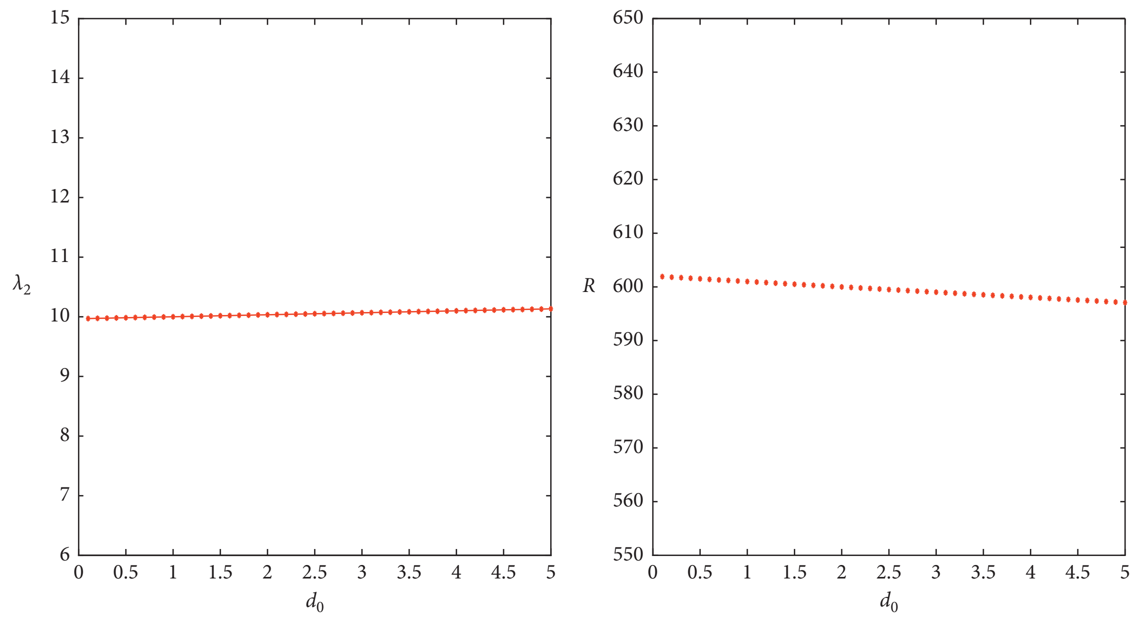


Figure 13: The synchronizability of multilayer star-ring networks *vs*. varying the interlayer coupling strength d_0 ($N = 300$, $M = 10$, $a = 1$, $d = 1$, and $a < Md$). (a) λ_2 with respect to varying d_0. (b) R with respect to varying d_0.

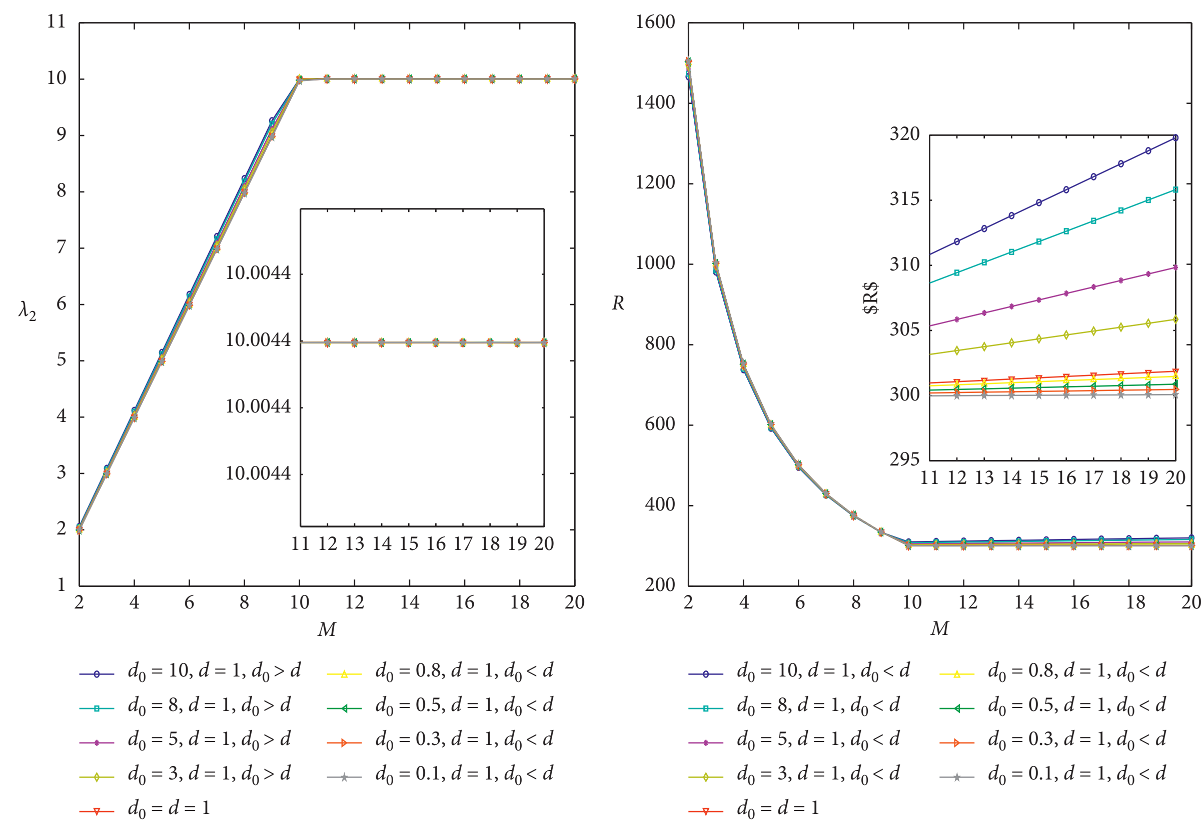
is strengthened at first and then almost remains invariant with increasing N for $d_0 < d$; and the synchronizability is weakened at first and then almost remains invariant with increasing N for $d_0 > d$, with the unbounded synchronous region. The influences of the number of nodes on synchronizability for the multilayer star-ring networks are similar to those for the multilayer star networks. When the synchronous region is bounded, the synchronizability is also weakened with increasing the number of nodes in each layer N.

As can be obtained from Figures 11–15, the influences of the structural parameters (i.e., the number of layers, the interlayer coupling strength, and the intralayer coupling strength) on synchronizability for the multilayer star-ring networks are basically similar to those for the multilayer star networks; a few different points between the two will be introduced in Section 5.3.



(a) (b)

FIGURE 14: The synchronizability of multilayer star-ring networks *vs.* varying the interlayer coupling strength d_0 ($N = 300$, $M = 10$, $a = 20$, $d = 1$, and $a > Md$) (a) λ_2 with respect to varying d_0. (b) R with respect to varying d_0.



(a) (b)

FIGURE 15: The synchronizability of multilayer star-ring networks *vs.* varying the number of the layers M ($N = 100$, $a = 10$, and $d = 1$). (a) λ_2 with respect to varying M for different d_0. (b) R with respect to varying M for different d_0.

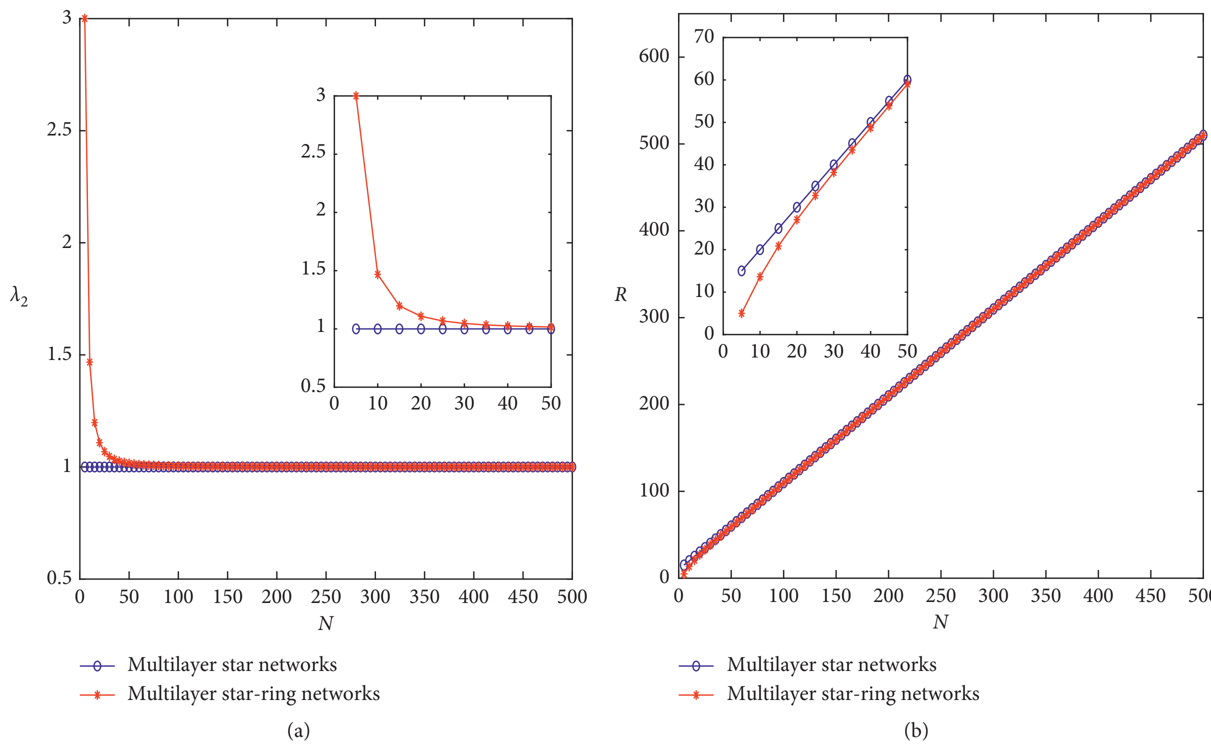


FIGURE 16: The synchronizability of multilayer star and star-ring networks, where $a = 1$, $d = 1$, $M = 10$, and $a < Md$. (a) λ_2 with respect to varying N for multilayer star and star-ring networks. (b) R with respect to varying N for multilayer star and star-ring networks.

5.3. Comparison of Two Types of Network Synchronizability. In this section, we will discuss the difference of synchronizability between multilayer star and star-ring networks under the same structural parameters.

Figure 16 shows λ_2 and eigenratio R numerically calculated from supra-Laplacian matrices of multilayer star and star-ring networks with varying scale size N in the case of $a < Md$. When N is small (approximately $N < 50$), as can be observed from Figure 16(a), λ_2 for multilayer star-ring networks is larger than that for multilayer star networks; Figure 16(b) shows R for multilayer star networks is larger than that for multilayer star-ring networks. This implies that, when N is small, the multilayer star-ring network has better synchronizability as that of the multilayer star network. When N is sufficiently large, the multilayer star-ring network has the same synchronizability as that of the multilayer star network, and the synchronizability of the two types of networks is nearly invariant with increasing N, implying that a larger size has little effect on the synchronizability of the two networks, for the unbounded synchronous region; while leads to linearly weakened synchronizability for two types of networks, for the bounded synchronous region. It is obvious that the simulation results are consistent with the theoretical results, for the case of $a < Md$; when N is small enough, $\lambda_2 = a + 4a\sin^2(\pi/N - 1)$ for multilayer star-ring networks is larger than $\lambda_2 = a$ for multilayer star networks; when N is sufficiently large, $\lambda_2 = a + 4a\sin^2(\pi/N - 1) \longrightarrow a$ of multilayer star-ring networks is almost equal to that of multilayer star networks.

Figure 17 shows λ_2 and eigenratio R numerically calculated from supra-Laplacian matrices of multilayer star and star-ring networks with varying scale size N in the case of $a > Md$. When $a > Md$, the multilayer star-ring network has the same synchronizability as that of the multilayer star network under taking the same structural parameter values. The synchronizability of the two networks is determined by the number of the layers M and the interlayer coupling strength d between leaf nodes and invariant with increasing N, for the unbounded synchronous region; the synchronizability of the two networks is weakened with increasing N, for the bounded synchronous region. The simulation results are consistent with the theoretical results.

6. Discussion and Conclusion

In this paper, the eigenvalue spectrum of the supra-Laplacian matrices of multilayer star and star-ring networks is strictly derived theoretically, and the synchronizability of two kinds of multilayer networks varying with the structural parameters is analyzed. In the case of unbounded synchronous region, the synchronizability of two types of networks is related to the intralayer coupling strength and interlayer coupling strength (the network scale is fixed) under different interlayer coupling strength. When the intralayer coupling strength a is relatively weak, the synchronizability of the networks only depends on the intralayer coupling strength a. When the interlayer coupling strength d between the leaf nodes is weak, the

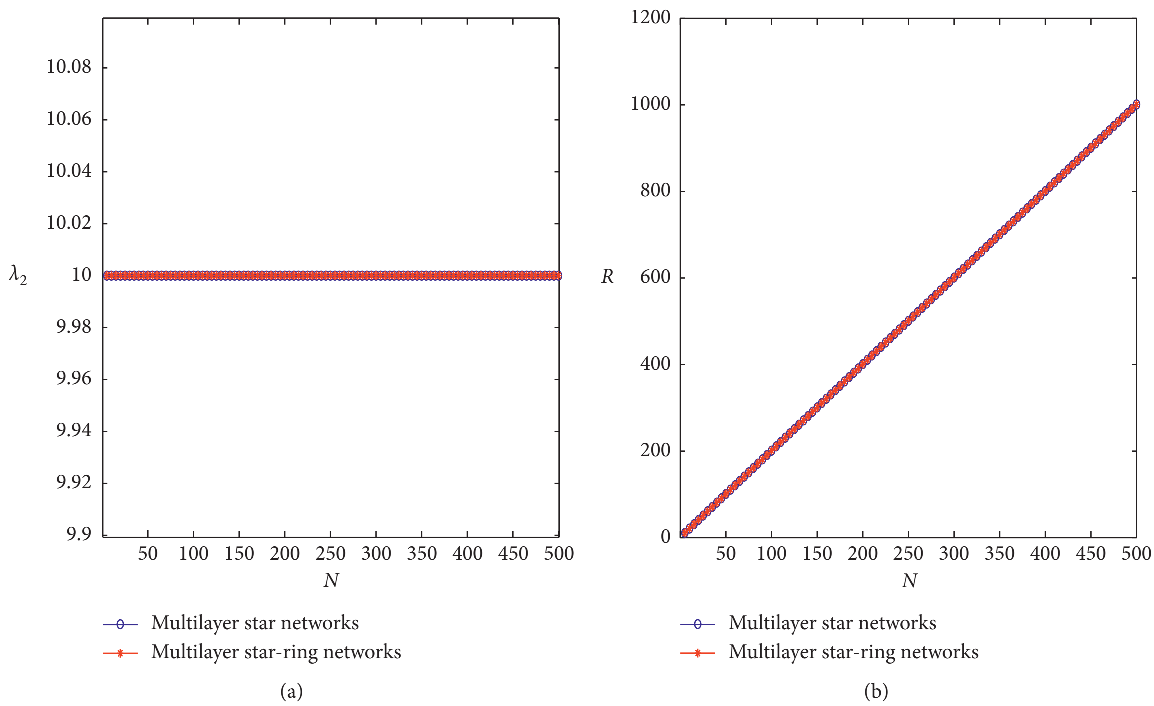


FIGURE 17: The synchronizability of multilayer star and star-ring networks, where a = 20, $d = 1$, $M = 10$, and $a > Md$. (a) λ_2 with respect to varying N for multilayer star and star-ring networks. (b) R with respect to varying N for multilayer star and star-ring networks.

synchronizability only depends on the interlayer coupling strength d, that is, the synchronizability is determined by the weaker of the two. In the case of the bounded synchronous region, when the intralayer coupling strength is weak, the increase of the intralayer coupling strength will enhance the synchronizability of the networks, while the increase of the interlayer coupling strength between the leaf nodes will weaken the synchronizability. When the interlayer coupling strength between the leaf nodes d is weak, the increase of the interlayer coupling strength between the leaf nodes will enhance the synchronizability of the networks, while the increases of the intralayer coupling strength will weaken the synchronizability. The same change of the intralayer coupling strength and the interlayer coupling strength between the leaf nodes has an opposite effect on the synchronizability of the two types of networks. Therefore, when the scale of the networks is fixed (M and N are unchanged), the interlayer coupling strength d increases, we can increase the intralayer coupling strength a to maintain the synchronizability. The multilayer star-ring networks have better synchronizability than that of multilayer star networks, for sufficiently small N.

So far, there are many unresolved issues in the research studies on the multilayer network. For example, for the multilayer star network and star-ring network proposed in this paper, how to adjust or control the relationship between parameters d, d_0, and a to maintain or improve the synchronizability of the networks? How to optimize the structural parameters of the networks to achieve the best synchronizability? All these need further study. In a word, many theoretical and practical problems have to be challenged.

Data Availability

The data used to support the findings of this study are included within the supplementary information file.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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Supplementary Materials

The procedures are MATLAB language code, and the data are obtained by running the code provided. (*Supplementary Materials*)

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