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Research Article

Optimal Contracts for Agents with Adverse Selection

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Due to information asymmetry, adverse selection exists largely in the multiagent market. Aiming at these problems, we develop two models: pure adverse selection model and mixed adverse selection and moral hazard model. We make the assumption that a type of agent is discrete and effort level is continuous in the models. With these models, we investigate the characters that make an optimal contract as well as the conditions under which the utility of a principal and agents can be optimized. As a result, we show that, in the pure adverse selection model, the conditions to reach the optimal utility of a principal and individual agents are that a principal needs to design different contracts for different types of agents, and an individual agent chooses the corresponding type of contracts. For the mixed model, we show that incentive constraint for agents plays a very important role. In fact, we find that whether a principal provides high-type contract or a separating equilibrium contract depends on the probability of existence of low-type agents in the market. In general, if a separating equilibrium contract is issued, then information asymmetry will cause the utility of the high-type agents to be less than that of the case in full information.

1. Introduction

In modern economy and financial market, information asymmetry can cause adverse selection problems between a principal and agents. This means that a principal is not able to know which type of an agent is before (or after) a contract is signed (in this paper, we use the level of production costs or the level of investment income) to represent the type of agents. The agents with high production costs are defined low-type (L-type) agents, and the agents with low production costs are defined high-type (H-type) agents. If a principal makes the contract based on an average level of agent types in the market, then L-type agents intend to sign contracts, while H-type agents are unwilling to sign contracts, and they probably choose to withdraw from the labor market. This eventually reduces the number of H-type agents in the labor market and increases the chances that a principal gets L-type agents. Because of this, the beneficial measure taken by a principal is to reduce the contract. As a consequence, more H-agents will leave the labor market.

Aiming at the economic phenomena, Akerlof [1] first studied the adverse selection problem. By investigating the old car market, he concluded that information asymmetry between buyers and sellers may lead to adverse selection problems. After Akerlof, more scholars, such as Green and Kahn [2], Grossman and Hart [3], and Hart [4] added adverse selection to the labor contract problem and studied the impact of information asymmetry between workers and enterprises on their relationship from a macroview. They mainly discussed the problem of overemployment and underemployment caused by information asymmetry. Demski and Sappington [5] and Mcafee and Mcmillan [6] analyzed the optimal incentive problem in teams under asymmetric information: adverse selection and moral hazard were studied from a microview. They showed that workers' contracts are affected by the type of their coworkers even in the case that workers work independently.

The research of the above scholars is carried out in a static environment. It is clear that to consider dynamic contracts in continuous time is more realistic. Cvitanic et al. [7] extended Sannikov's [8] model to the case of adverse selection. They obtain the optimal contracts for both good managers and bad managers and find that agents exiting from the labor market may be optimal at varying levels of the managers' continuation values. Since then, Cvitanic and Zhang [9] studied several adverse selection models under continuous time. Their research shows that if an agent only controls the drift rate and the effort cost is quadratic, an optimal contract is a function of the final output (usually nonlinear); if an agent also controls the volatility, an optimal contract is nonincentive stochastic return. Compared with the adverse selection in static environment, the problem in continuous time is more complex and so needs more mathematical theory involved.

On the contrary, as the asymmetry of postevent information is likely to lead to moral hazard, the moral hazard problem is also widespread in the labor market. In recent years, scholars started to pay attention to the mixed problem of moral hazard and adverse selection in the labor market since it is more in line with the practice of the labor market. For example, Sung [10] adds the adverse selection problem to the pure moral hazard model (the most famous continuous-time principal-agent model, see [11–13]) in a continuous time and studies a mixed model of risk aversion agent controlling drift and volatility. The results show that in the mixed model, the monotonous conditions in the pure adverse selection problem need to be modified to ensure the incentive compatibility of information revelation. When this mixed model is applied for management compensation in management project selection and capital budget decision-making, it draws a conclusion contrary to the traditional view. According to Sung's research, although the processes in solving principals' problems are similar to those in the mixed model and pure moral hazard model, it is possible to end up with different conclusions in mixed models.

Based on the works of Cvitanic, Sung, and other scholars, this article studies a pure adverse selection model and a mixed model of moral hazard adverse selection in the continuous time frame. We assume that a principal employs multiple agents of unknown types to jointly manage multiple production projects (or risk projects). In this model, an agent only knows his/her own type, but the probability distribution of agent types is public information. In addition, we also suppose the agents have Bayesian Nash equilibrium behavior; that is, each agent calculates his/her own optimal response function according to the probability distribution of other agents and their strategy choices and then reaches an equilibrium state.

In the pure adverse selection model, due to information asymmetry, a principal has the motivation to provide "menu contracts" to ensure that each agent of a certain type accepts a contract designed for his/her type, rather than choosing types of other agents. This means that a principal provides menu contracts to satisfy agents telling the truth constraints. So, we need to find viable contracts that meet the "tell the truth" constraints. In this article, with Lagrange multiplier method and Kuhn–Tucker condition, we are able to show that the sufficient and necessary condition for telling the truth is that the first derivative of the constraint is equal to zero.

Compared with the pure adverse selection model, the mixed model is more complex since a principal cannot observe the agents' behavior. Generally, Lagrange multiplier method is no longer applicable. So, we use the backward stochastic differential equation (BSDE) theory and martingale representation theorem to get the deterministic equivalence of the agents' expected utility function.

This paper is organized as follows. We present a continuous time multiagent model in Section 2 and discuss the optimal contracts in the case under adverse selection in Section 3. We then discuss the design of the optimal contract with a moral hazard problem adding to an adverse selection problem and analyze the optimal contract for some special cases through the numerical simulation in Section 4. We conclude this paper in Section 5.

2. The Model

In this section, we first define the dynamics of the outcome process. Second, we define the objective function of each agent, as well as one of the principals.

In our paper, in order to obtain the exact solution of the optimal contracts, without loss of generality, we consider a model where a principal hire 2 agents to manage 2 different risk projects. For each agent, if they accept the contract provided by the principal, they will pay for the effort to influence the output of the project.

2.1. The Outcome Process. Similar to [14], we consider an organization with a principal and two agents, indexed by $i = 1, 2$. At the initial moment $t = 0$, the principal hires the agents to manage two risk projects, whose value dynamics Y_t^i are given by the following equation:

$$\begin{aligned} \mathrm{d}Y_t^1 &= \left(a_{11}u_t^{1,1} + a_{12}u_t^{1,2}\right)\mathrm{d}t + \sigma_1 \mathrm{d}W_t^1, \\ \mathrm{d}Y_t^2 &= \left(a_{21}u_t^{2,1} + a_{22}u_t^{2,2}\right)\mathrm{d}t + \sigma_2 \mathrm{d}W_t^2, \end{aligned} \tag{1}$$

where W_t^1 and W_t^2 are two independent Brownian motions in probability space $(\Omega, \mathscr{G}, \mathscr{G}_t, \mathbb{P}^0)$.

Here, $\mathscr{G}_t$ is the augmented filtration generated by W_t^1, W_t^2. Also, $u_t^{j,i}$, $(j = 1, 2)$ is the effort exerted by the agent i. The structure of our model implies that each agent will be assigned both projects, and each agent's effort can affect two projects' returns. We consider two information structures: full information and hidden action. Full information means that all the information in the model is symmetric and exact, while under hidden action, the principal can observe the outcome process Y_t but cannot observe the efforts of all agents or the uncertainty which impact the outcomes. So, the principal cannot distinguish the level of influence of effort and uncertainty on the output process.

The principals' information flow is generated by processes (Y_t^1, Y_t^2), and we denote them by $\mathscr{F}_t$. In a certain probability space $(\Omega, \mathscr{F}_t, \mathbb{P})$, we can find $\mathscr{F}_t-$ measurable Brownian motions (B_t^1, B_t^2), satisfying the following weak formulation setting (as in [11]):

$$\mathrm{d}Y_t = \sum_t \mathrm{d}B_t, \tag{2}$$

where $Y_t = (Y_t^1, Y_t^2)^\top$ is a two-dimensional outcome vector at time t, $B_t = (B_t^1, B_t^2)^\top$ are two independent dimensional Brownian motions on space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, and $\sum = \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix}$, and we assume that $\sum_t$ is bounded and invertible. With this construction, the contracts can be written on B_t processes. According to the analysis in [15], the agents' efforts can change the measure from $\mathbb{P}$ to $\mathbb{P}^u$ with the following dynamics:

$$\frac{\mathrm{d}\mathbb{P}^u}{\mathrm{d}\mathbb{P}} = \exp\left(\int_0^T \left(\sum^{-1} b(t, A, u_t)\right)^\top \mathrm{d}B_t - \frac{1}{2}\int_0^T \left|\sum^{-1} b(t, A, u_t)\right|^2 \mathrm{d}t\right)$$
$$\triangleq \Lambda\left(\int_0^T \sum^{-1} b(t, A, u_t)^\top \mathrm{d}B_t\right), \tag{3}$$

where $b(t, A, u_t) = \begin{pmatrix} b^1(t, A, u_t) \\ b^2(t, A, u_t) \end{pmatrix} = \begin{pmatrix} a_{11}u_t^{1,1} + a_{12}u_t^{1,2} \\ a_{21}u_t^{2,1} + a_{22}u_t^{2,2} \end{pmatrix}$ is a bounded and continuous function.

By Girsanov's theorem, under $\mathbb{P}^u$,

$$B_t^u = B_t - \int_0^T \sum^{-1} b(s, A, u_s)\mathrm{d}s, \quad 0 \le t \le T, \tag{4}$$

is a vector of Brownian motion, and we can rewrite (2) as

$$\mathrm{d}Y_t = b(t, A, u_t)\mathrm{d}t + \sum \mathrm{d}B_t^u. \tag{5}$$

That is,

$$\begin{aligned} \mathrm{d}Y_t^1 &= \left(a_{11}u_t^{1,1} + a_{12}u_t^{1,2}\right)\mathrm{d}t + \sigma_1 \mathrm{d}B_t^{u,1}, \\ \mathrm{d}Y_t^2 &= \left(a_{21}u_t^{2,1} + a_{22}u_t^{2,2}\right)\mathrm{d}t + \sigma_2 \mathrm{d}B_t^{u,2}, \end{aligned} \tag{6}$$

where the parameters $a_{i,j}$ represent the productivity of agent i on project j. The total output generated by the two agents is $Y_t^1 + Y_t^2$ at time t.

This is the so-called weak formulation of the hidden action model in continuous time. All random processes and effort processes now are adapted to $\mathcal{F}_t$, which is generated by observable processes (Y_t^1, Y_t^2). Here, we denote, by $\mathbb{E}^{\mathbb{P}^{(u^1,u^2)}}$, the expectation defined on the space $(\Omega, \mathbb{P}^{(u^1,u^2)})$, and u_t^i is the effort of the agent i. We are modeling collaboration between agents in that the outcome of each project depends on the effort levels of both agents.

2.2. The Objective Function of Principal. We consider a principal who wants to hire two agents (agent 1 and agent 2); agent 1 is endowed with an unknown ability type $\theta_i \in \{\theta_L, \theta_H\}$, and agent 2 is endowed with an unknown ability type $\eta_j \in \{\eta_L, \eta_H\}$, where $\theta_L, \theta_H, \eta_L, \eta_H$ are known to the principal. Only the agent knows his own type (the value of θ_i or η_j), and θ_i and η_j are independent. The principal does not know θ_i, η_j but has the prior probability of type θ_L which is $p \in [0,1]$, and the prior probability of type η_L is $q \in [0,1]$, while the agents also have the common knowledge prior probability of his coworker.

Next, we explain some symbols. Denote that agent 1 of type $i, l \in \{L, H\}$ and agent 2 of type $j, m \in \{L, H\}$. $C^1(\cdot, \theta_i, \eta_j)$ is the contract designed for $i-$ type agent 1 when his coworker is $j-$ type; similarly, $C^2(\cdot, \theta_i, \eta_j)$ is the contract designed for $j-$ type agent 2 with a $i-$ type coworker. $u_{l,m}^1(\theta_i) = u_{l,m}^1(\theta_i)(t)_{t\in[0,T]}$ represents that when a coworker is $m-$ type, $i-$ type agent 1 chooses to provide $l-$ type efforts; if $i = l$, we say agent 1 tells the truth, and we write $u_{l,m}^1(\theta_l) = u_{l,m}^1$. $u_{l,m}^2(\eta_j) = u_{l,m}^2(\eta_j)(t)_{t\in[0,T]}$ represents that when a coworker is $l-$ type, $j-$ type agent 2 chooses to provide $m-$ type efforts; if $j \neq m$, we say agent 2 is lying, and if $j = m$, we write $u_{l,m}^2(\eta_m) = u_{l,m}^2$. $Y_{l,m} = \begin{pmatrix} Y_{l,m}^1(T) \\ Y_{l,m}^2(T) \end{pmatrix}$ is the output (or income) under the joint production of $l-$ type effort and $m-$ type effort. Denote that $\mathbb{E}^{\mathbb{P}^u}$ is the expectation operator under probability measure $\mathbb{P}^u$.

Because the principal faces two agents who do not know the type, there are four possible scenarios for the principal. Based on the prior probability p and q, we defined the principal's expected utility function (in our paper, we use the negative exponential utility function to represent the principal and agents' utility functions; this representation is not uncommon; we can see Schattler and Sung [12], Williams [16], and so on) as follows:

$$\begin{aligned} U^P(Y_T, C) = &-\Big(p \cdot q \cdot \mathbb{E}^{\mathbb{P}^u}\left[e^{-r_P\|Y_{L,L} - C(\cdot,\theta_L,\eta_L)\|}\right] + p \cdot (1-q) \\ &\cdot \mathbb{E}^{\mathbb{P}^u}\left[e^{-r_P\|Y_{L,H} - C(\cdot,\theta_L,\eta_H)\|}\right] + (1-p) \cdot q \cdot \mathbb{E}^{\mathbb{P}^u} \\ &\cdot \left[e^{-r_P\|Y_{H,L} - C(\cdot,\theta_H,\eta_L)\|}\right] + (1-p) \cdot (1-q) \cdot \mathbb{E}^{\mathbb{P}^u} \\ &\cdot \left[e^{-r_P\|Y_{H,H} - C(\cdot,\theta_H,\eta_H)\|}\right]\Big) \\ =:{}& pq \cdot U_{LL}^P(Y_{LL}, C_{LL}) + p(1-q) \cdot U_{LH}^P(Y_{LH}, C_{LH}) \\ &+ (1-p)q \cdot U_{HL}^P(Y_{HL}, C_{HL}) + (1-p)(1-q) \\ &\cdot U_{HH}^P(Y_{HH}, C_{HH}), \end{aligned} \tag{7}$$

where $\|Y - C\| = Y^1 + Y^2 - C^1 - C^2$. (in this paper, the symbol $\mathbf{M}^n$ represents the sum of the n-th power of the elements of the matrix $\mathbf{M}$, meaning that $\|\mathbf{M}\|^n = \sum_{i,j} \mathbf{m}_{i,j}^n$, $n = 1, 2, \ldots,$), and $r_P > 0$ is the risk aversion of the principal.

If the agents choose a contract $(C^1(\theta_l), C^2(\eta_m))$, then they would provide effort $(u^1(\theta_l), u^2(\eta_m))$ and the outcome process would evolve as follows:

$$\mathrm{d}Y_t = b\left(t, A, u_t^1(\theta_l), u_t^2(\eta_m)\right)\mathrm{d}t + \sum \mathrm{d}B_t^{(\theta_l,\eta_m)}, \tag{8}$$

where $B_t^{(\theta_l,\eta_m)}$ is short for $B_t^{(u^1(\theta_l),u^2(\eta_m))}$. If we assume for each (θ_l, η_m), b and $\sum$ satisfy Lipschitz condition so that SDE (8) has a unique strong solution (the conclusion can be seen in Theorem 5.2.1 in [10, 17]).

2.3. The Objective Function of Agents. Before time zero, the principal offers a menu of contracts $C_T = (C_T^1(\theta_i), C_T^2(\eta_j))$. The menu can contain different contracts for agents with different abilities. Once a suitable contract $(C_T^1(\theta_l), C_T^2(\eta_m))$ is chosen (where θ_l, η_m may or may not be equal to agent 1 and agent 2 real type θ_i, η_j), the agents exert efforts to affect the outcome process during a continuous time period $[0, T]$. At time T, the agents are compensated depending on the contract.

We suppose that agents are constant absolute risk aversion with risk aversion coefficient $r_i > 0$. Therefore, we assume that the agents are characterized by exponential utility functions, and the agents' utility is composed of the payoff and the cost of effort.

Denote that $\mathbb{E}^{\mathbb{P}^{(u^1(\theta), u^2(\eta))}}$ (abbreviated as $\mathbb{E}^{\theta,\eta}$) is the expectation operator under $\mathbb{P}^{(u^1(\theta), u^2(\eta))}$, where $(u^1(\theta), u^2(\eta))$ represent the efforts provided by agent 1 and agent 2 when their, respectively, selecting $\theta-$ type and $\eta-$ type contract.

Then, we defined the utility function of agent 1 by

$$
\begin{aligned}
&U^1\left(\theta_l, C^1(\cdot, \theta_l, \eta_m), u^1(\theta_l), u^2(\eta_m)\right) \\
&\quad = \mathbb{E}^{\theta,\eta}\Big[-q \cdot e^{-r_1\left(C^1(\cdot,\theta_l,\eta_L) - G_T^1(\theta_l,\eta_L)\right)} - (1-q) \\
&\qquad \cdot e^{-r_1\left(C^1(\cdot,\theta_l,\eta_H) - G_T^1(\theta_l,\eta_H)\right)}\Big] \\
&\quad =: q \cdot U_{lL}^1\left(C_{lL}^1, G_{lL}^1\right) + (1-q) \cdot U_{lH}^1\left(C_{lH}^1, G_{lH}^1\right),
\end{aligned}
\tag{9}
$$

where G_T^1 is the cumulative disutility of effort for agent 1, which is defined as

$$
G_T^1(\theta_l, \eta_m) = \int_0^T g_1\left(u_{l,m}^1(t)\right)\mathrm{d}t = \frac{\theta_l}{2}\int_0^T\left(\left(u_{l,m}^{1,1}\right)^2 + \left(u_{l,m}^{2,1}\right)^2\right)\mathrm{d}t,
\tag{10}
$$

and the utility function of agent 2 is

$$
\begin{aligned}
&U^2\left(\eta_m, C^2(\cdot, \theta_l, \eta_m), u^1(\theta_l), u^2(\eta_m)\right) \\
&\quad = \mathbb{E}^{\theta,\eta}\Big[-p \cdot e^{-r_2\left(C^2(\cdot,\theta_L,\eta_m) - G_T^2(\theta_L,\eta_m)\right)} - (1-p) \\
&\qquad \cdot e^{-r_2\left(C^2(\cdot,\theta_H,\eta_m) - G_T^2(\theta_H,\eta_m)\right)}\Big] \\
&\quad =: p \cdot U_{Lm}^2\left(C_{Lm}^2, G_{Lm}^2\right) + (1-p) \cdot U_{Hm}^2\left(C_{Hm}^2, G_{Hm}^2\right),
\end{aligned}
\tag{11}
$$

where

$$
G_T^2(\theta_l, \eta_m) = \int_0^T g_2\left(u_{l,m}^2(t)\right)\mathrm{d}t = \frac{\eta_m}{2}\int_0^T\left(\left(u_{l,m}^{1,2}\right)^2 + \left(u_{l,m}^{2,2}\right)^2\right)\mathrm{d}t.
\tag{12}
$$

According to the conclusion (Theorem 2) in Mcafee and Mcmillan [6], the contract of each agent is affected both in his own ability and in the other agent's ability. Therefore, under the premise of accepting the contract, the utility brought by the contract is greater than or equal to the reserved utility. For each agent, their problem is to provide an optimal reaction effort to maximize their expected utility function. Given a contract and a prior probability, this situation is typically considered as an incomplete information noncooperative game between agents. Therefore, we should focus on Bayesian Nash equilibrium solutions.

Mathematically, we define Bayesian Nash equilibrium for the agents as follows.

Definition 1 (Bayesian Nash equilibrium (see, for instance, [18, 19])). An admissible strategy combination $\overline{u} = (\overline{u}^1(\theta_l), \overline{u}^2(\eta_m))$ is a Bayesian Nash equilibrium; if for all $u^1 \in \mathbf{A}_1(\theta_l)$ and $u^2 \in \mathbf{A}_2(\eta_m)$, we have

$$
\begin{aligned}
\overline{u}^1(\theta_l) &\in \arg\max_{u^1} \mathbb{E}^{\theta,\eta}\left[U^1\left(\theta_l, C^1(\cdot, \theta_l, \eta_m), u^1(\theta_l), u^2(\eta_m)\right)\right], \\
\overline{u}^2(\eta_m) &\in \arg\max_{u^2} \mathbb{E}^{\theta,\eta}\left[U^2\left(\eta_m, C^2(\cdot, \theta_l, \eta_m), u^1(\theta_l), u^2(\eta_m)\right)\right].
\end{aligned}
\tag{13}
$$

Shown above, if the contract can motivate agents to choose Bayesian Nash equilibrium, then the contract can satisfy the incentive constraint for two agents.

3. Second Best: Adverse Selection without Moral Hazard

In this section, we consider the optimal incentive contract under the case of adverse selection without moral hazard, i.e., in this case, we assume that the efforts u are observed by the principal, while θ and η are not. The principal's problem is described as follows:

Problem 1. Choose control pairs $u_{l,m} = (u_{l,m}^1, u_{l,m}^2)$ and a menu of contracts $C(\cdot, \theta_l, \eta_m) = (C^1(\cdot, \theta_l, \eta_m), C^2(\cdot, \theta_l, \eta_m))$ to maximising $U^P(Y_T, C)$ subject to the following five constraints:

(1) $\mathrm{d}Y_{l,m}(t) = b(t, A, u_{l,m})\mathrm{d}t + \sum \mathrm{d}B_t^u, l, m \in \{L, H\}$
(2) $\mathbb{E}^{\theta,\eta}U^1(\theta_l, C^1(u_{l,m}^1(\theta_l), \theta_l, \eta_m)) \geq \mathbb{E}^{\theta,\eta}U^1(\theta_i, C^1(u_{l,m}^1(\theta_i), \theta_l, \eta_m)); i \neq l$
(3) $\mathbb{E}^{\theta,\eta}U^2(\eta_m, C^2(u_{l,m}^2(\eta_m), \theta_l, \eta_m)) \geq \mathbb{E}^{\theta,\eta}U^2(\eta_j, C^2(u_{l,m}^2(\eta_j), \theta_l, \eta_m)); j \neq m$
(4) $\mathbb{E}^{\theta,\eta}U^1(\theta_l, C^1(u_{l,m}^1, \theta_l, \eta_m)) \geq U^1(L_1(\theta_l, \eta_m)) = -e^{-r_1 L_1(\theta_l,\eta_m)}; l, m \in \{L, H\}$
(5) $\mathbb{E}^{\theta,\eta}U^2(\eta_m, C^2(u_{l,m}^2, \theta_l, \eta_m)) \geq U^2(L_2(\theta_l, \eta_m)) = -e^{-r_2 L_2(\theta_l,\eta_m)}; l, m \in \{L, H\}$

The first constraint defines the dynamics of the output processes of each type when choosing the contracts designed for agents. The second and the third constraint, the truth-telling constraint (or incentive compatibility constraint), makes each agent optimally choose the contract designed from him. Finally, the fourth and the fifth constraint, the participation constraint, ensures that agents contract with the principal, where $U^1(L_1(\theta_l, \eta_m))$ and $U^2(L_2(\theta_l, \eta_m))$ are the reservation utility of agent 1 and agent 2, respectively.

According to the classic principal-agent theory, we can transform the pure adverse selection problem into the risk sharing problem (see [20, 21]). Therefore, the principal's relaxed problem is to maximize the Lagrangian

$$U^{P}\left(Y_{T},C\right)+\lambda_{1}\cdot\left[U^{1}\left(\theta_{L},C^{1}\left(u^{1}_{L,m}\left(\theta_{L}\right),\theta_{L},\eta_{m}\right)\right)-U^{1}\left(L_{1}\left(\theta_{L},\eta_{m}\right)\right)\right]+\lambda_{2}\cdot\left[U^{2}\left(\eta_{L},C^{2}\left(u^{2}_{l,L}\left(\eta_{L}\right),\theta_{l},\eta_{L}\right)\right)\right.$$
$$\left.-U^{2}\left(L_{2}\left(\theta_{l},\eta_{L}\right)\right)\right]+\lambda_{3}\cdot\left[U^{1}\left(\theta_{L},C^{1}\left(u^{1}_{L,m}\left(\theta_{L}\right),\theta_{L},\eta_{m}\right)\right)-U^{1}\left(\theta_{L},C^{1}\left(u^{1}_{H,m}\left(\theta_{L}\right),\theta_{H},\eta_{m}\right)\right)\right]+\lambda_{4}$$
$$\cdot\left[U^{1}\left(\theta_{H},C^{1}\left(u^{1}_{H,m}\left(\theta_{H}\right),\theta_{H},\eta_{m}\right)\right)-U^{1}\left(\theta_{H},C^{1}\left(u^{1}_{L,m}\left(\theta_{H}\right),\theta_{L},\eta_{m}\right)\right)\right]+\lambda_{5}\cdot\left[U^{2}\left(\eta_{L},C^{2}\left(u^{2}_{l,L}\left(\eta_{L}\right),\theta_{l},\eta_{L}\right)\right)\right.$$
$$\left.-U^{2}\left(\eta_{L},C^{2}\left(u^{1}_{l,H}\left(\eta_{L}\right),\theta_{l},\eta_{H}\right)\right)\right]+\lambda_{6}\cdot\left[U^{2}\left(\eta_{H},C^{2}\left(u^{2}_{l,H}\left(\eta_{H}\right),\theta_{l},\eta_{H}\right)\right)-U^{2}\left(\eta_{H},C^{2}\left(u^{1}_{l,L}\left(\eta_{H}\right),\theta_{l},\eta_{L}\right)\right)\right], \tag{14}$$

where $\lambda_n \geq 0, n = 1, 2, \ldots, 6$, denote that

$$\begin{aligned}
U^{1}\left(\theta_{L},C^{1}\left(u^{1}_{H,m}\left(\theta_{L}\right),\theta_{H},\eta_{m}\right)\right) &\coloneqq q\cdot\widetilde{U}^{1}_{HL}+(1-q)\cdot\widetilde{U}^{1}_{HH}\\
&=-q\cdot e^{-r_{1}\left(C^{1}_{HL}-\int_{0}^{T}\left(\theta_{L}/2\right)\left(u^{1}_{HL}\right)^{2}\mathrm{d}s\right)}-(1-q)\cdot e^{-r_{1}\left(C^{1}_{HH}-\int_{0}^{T}\left(\theta_{L}/2\right)\left(u^{1}_{HH}\right)^{2}\mathrm{d}s\right)},\\
U^{1}\left(\theta_{H},C^{1}\left(u^{1}_{L,m}\left(\theta_{H}\right),\theta_{L},\eta_{m}\right)\right) &\coloneqq q\cdot\widetilde{U}^{1}_{LL}+(1-q)\cdot\widetilde{U}^{1}_{LH}\\
&=-q\cdot e^{-r_{1}\left(C^{1}_{LL}-\int_{0}^{T}\left(\theta_{H}/2\right)\left(u^{1}_{LL}\right)^{2}\mathrm{d}s\right)}-(1-q)\cdot e^{-r_{1}\left(C^{1}_{LH}-\int_{0}^{T}\left(\theta_{H}/2\right)\left(u^{1}_{LH}\right)^{2}\mathrm{d}s\right)}.
\end{aligned} \tag{15}$$

For $m \in \{L, H\}$, take the first-order conditions for the optimization problem of the principal which are

$$(1-p)\frac{\partial U^{P}_{Hm}}{\partial C^{1}_{Hm}}-\lambda_{3}\frac{\partial\widetilde{U}^{1}_{Hm}}{\partial C^{1}_{Hm}}+\lambda_{4}\frac{\partial U^{1}_{Hm}}{\partial C^{1}_{Hm}}=0, \tag{16}$$

$$p\frac{\partial U^{P}_{Lm}}{\partial C^{1}_{Lm}}+\lambda_{1}\frac{\partial U^{1}_{Lm}}{\partial C^{1}_{Lm}}+\lambda_{3}\frac{\partial U^{1}_{Lm}}{\partial C^{1}_{LH}}-\lambda_{4}\frac{\partial\widetilde{U}^{1}_{Lm}}{\partial C^{1}_{Lm}}=0, \tag{17}$$

$$(1-p)\frac{\partial U^{P}_{Hm}}{\partial u^{1}_{Hm}}-\lambda_{3}\frac{\partial\widetilde{U}^{1}_{Hm}}{\partial u^{1}_{Hm}}+\lambda_{4}\frac{\partial U^{1}_{Hm}}{\partial u^{1}_{Hm}}=0, \tag{18}$$

$$p\frac{\partial U^{P}_{Lm}}{\partial u^{1}_{Lm}}+\lambda_{1}\frac{\partial U^{1}_{Lm}}{\partial u^{1}_{Lm}}+\lambda_{3}\frac{\partial U^{1}_{Lm}}{\partial C^{1}_{LH}}-\lambda_{4}\frac{\partial\widetilde{U}^{1}_{Lm}}{\partial u^{1}_{Lm}}=0. \tag{19}$$

We write linear equations as matrix multiplication:

$$\begin{pmatrix}
0 & -\dfrac{\partial\widetilde{U}^{1}_{Hm}}{\partial C^{1}_{Hm}} & \dfrac{\partial U^{1}_{Hm}}{\partial C^{1}_{Hm}}\\
\dfrac{\partial U^{1}_{Lm}}{\partial C^{1}_{Lm}} & \dfrac{\partial U^{1}_{Lm}}{\partial C^{1}_{LH}} & -\dfrac{\partial\widetilde{U}^{1}_{Lm}}{\partial C^{1}_{Lm}}\\
0 & -\dfrac{\partial\widetilde{U}^{1}_{Hm}}{\partial u^{1}_{Hm}} & \dfrac{\partial U^{1}_{Hm}}{\partial u^{1}_{Hm}}\\
\dfrac{\partial U^{1}_{Lm}}{\partial u^{1}_{Lm}} & \dfrac{\partial U^{1}_{Lm}}{\partial C^{1}_{LH}} & -\dfrac{\partial\widetilde{U}^{1}_{Lm}}{\partial u^{1}_{Lm}}
\end{pmatrix}\cdot\begin{pmatrix}\lambda_{1}\\ \lambda_{3}\\ \lambda_{4}\end{pmatrix}=\begin{pmatrix}
-(1-p)\dfrac{\partial U^{P}_{Hm}}{\partial C^{1}_{Hm}}\\
-p\dfrac{\partial U^{P}_{Lm}}{\partial C^{1}_{Lm}}\\
-(1-p)\dfrac{\partial U^{P}_{Hm}}{\partial u^{1}_{Hm}}\\
-p\dfrac{\partial U^{P}_{Lm}}{\partial u^{1}_{Lm}}
\end{pmatrix}$$
$$\coloneqq\begin{pmatrix}0 & d_{12} & d_{13}\\ d_{21} & d_{22} & d_{23}\\ 0 & d_{32} & d_{33}\\ d_{41} & d_{42} & d_{43}\end{pmatrix}\cdot\begin{pmatrix}\lambda_{1}\\ \lambda_{3}\\ \lambda_{4}\end{pmatrix}=\begin{pmatrix}b_{1}\\ b_{2}\\ b_{3}\\ b_{4}\end{pmatrix}. \tag{20}$$

Through the elementary transformation of the matrix, we have

$$\begin{pmatrix}
0 & 1 & \dfrac{d_{13}}{d_{12}}\\
1 & 0 & D_{23}\\
0 & 0 & \dfrac{d_{33}}{d_{32}}-\dfrac{d_{13}}{d_{12}}\\
0 & 0 & D_{43}
\end{pmatrix}\cdot\begin{pmatrix}\lambda_{1}\\ \lambda_{3}\\ \lambda_{4}\end{pmatrix}=\begin{pmatrix}
\dfrac{b_{1}}{d_{12}}\\
D_{24}\\
\dfrac{b_{3}}{d_{32}}-\dfrac{b_{1}}{d_{12}}\\
D_{44}
\end{pmatrix}, \tag{21}$$

where

$$\begin{aligned}
D_{23}&=-\frac{d_{22}}{d_{21}}\cdot\left(\frac{d_{13}}{d_{12}}-\frac{d_{23}}{d_{22}}\right),\\
D_{24}&=-\frac{d_{22}}{d_{21}}\cdot\left(\frac{b_{1}}{d_{12}}-\frac{b_{2}}{d_{22}}\right),\\
D_{43}&=\left(\frac{d_{33}}{d_{32}}-\frac{d_{43}}{d_{42}}\right)-\frac{d_{41}}{d_{42}}\cdot D_{23},\\
D_{44}&=\left(\frac{b_{3}}{d_{32}}-\frac{b_{4}}{d_{42}}\right)-\frac{d_{41}}{d_{42}}\cdot D_{24}.
\end{aligned} \tag{22}$$

According to the uniqueness of the solution, there is one and only one of the following three conditions established:

(1) $(d_{33}/d_{32})-(d_{13}/d_{12})=(b_3/d_{32})-(b_1/d_{12})=0$
(2) $D_{43}=D_{44}=0$
(3) $D_{44}\cdot((d_{33}/d_{32})-(d_{13}/d_{12}))=D_{43}\cdot((b_3/d_{32})-(b_1/d_{12}))$

Then, we have

$$\begin{cases} \lambda_1 = D_{24} - D_{23} \cdot \lambda_4, \\ \lambda_3 = \dfrac{b_1}{d_{12}} - \dfrac{d_{13}}{d_{12}} \cdot \lambda_4, \\ \lambda_4 = \left(\dfrac{d_{33}}{d_{32}} - \dfrac{d_{13}}{d_{12}}\right)^{-1} \cdot \left(\dfrac{b_3}{d_{32}} - \dfrac{b_1}{d_{12}}\right) \\ \text{or } \lambda_4 = \dfrac{D_{44}}{D_{43}}. \end{cases} \tag{23}$$

If we assume that $\lambda_4 = 0$, then $\lambda_3 = (b_1/d_{12})$ but $b_1 > 0$ and $d_{12} < 0$ which contradict $\lambda_3 \geq 0$. Therefore, $\lambda_4 > 0$. According to the Kuhn–Tacker conditions, the truth-telling constraint of $H-$type agent 1 is tight. Moreover, we assume that $\lambda_3 > 0$, then the truth-telling constraint of $L-$type agent 1 is tight also, i.e.,

$$\begin{aligned} &-q \cdot e^{-r_1\left(C_{HL}^1 - \int_0^T (\theta_H/2)(u_{HL}^1)^2 ds\right)} - (1-q) \cdot e^{-r_1\left(C_{HH}^1 - \int_0^T (\theta_H/2)(u_{HH}^1)^2 ds\right)} \\ &\quad = -q \cdot e^{-r_1\left(C_{LL}^1 - \int_0^T (\theta_H/2)(u_{LL}^1)^2 ds\right)} - (1-q) \cdot e^{-r_1\left(C_{LH}^1 - \int_0^T (\theta_H/2)(u_{LH}^1)^2 ds\right)}, \\ &-q \cdot e^{-r_1\left(C_{LL}^1 - \int_0^T (\theta_L/2)(u_{LL}^1)^2 ds\right)} - (1-q) \cdot e^{-r_1\left(C_{LH}^1 - \int_0^T (\theta_L/2)(u_{LH}^1)^2 ds\right)} \\ &\quad = -q \cdot e^{-r_1\left(C_{HL}^1 - \int_0^T (\theta_L/2)(u_{HL}^1)^2 ds\right)} - (1-q) \cdot e^{-r_1\left(C_{HH}^1 - \int_0^T (\theta_L/2)(u_{HH}^1)^2 ds\right)}. \end{aligned} \tag{24}$$

It is not hard to get $u_{LL}^1 = u_{HL}^1$ and $u_{LH}^1 = u_{HH}^1$. This shows that $H-$type and $L-$type agent 1 received the same contract and provided the same effort. This contradicts the original intention that the principal wants to distinguish between the $H-$type and the $L-$type. Therefore, we can assert that $\lambda_3 = 0$, i.e.,

$$\begin{cases} \lambda_1 = \dfrac{1}{d_{21}}\left(b_2 - \dfrac{b_1 d_{23}}{d_{13}}\right) > 0, \\ \lambda_3 = 0, \\ \lambda_4 = \dfrac{b_1}{d_{13}} > 0. \end{cases} \tag{25}$$

According to $\lambda_1 > 0$ and the Kuhn–Tacker condition, we know that the participation constraint of the $L-$type agent 1 is tight, that is, the expected utility of the $L-$type agent 1 is equal to his reserved utility.

Similarly, for agent 2, we can get the Lagrange multiplier $\lambda_2 > 0$, $\lambda_5 = 0$, and $\lambda_6 > 0$.

Therefore, we have the following conclusion:

Proposition 1. *The optimal contracts and optimal efforts of Problem 1 satisfy the following conditions, for $l, m \in \{L, H\}$:*

(1) $U^1(\theta_L, C^1(u_{L,m}^1(\theta_L), \theta_L, \eta_m)) = U^1(L_1(\theta_L, \eta_m))$

(2) $U^2(\eta_L, C^2(u_{l,L}^2(\eta_L), \theta_l, \eta_L)) = U^2(L_2(\theta_l, \eta_L))$

(3) $U^1(\theta_H, C^1(u_{H,m}^1(\theta_H), \theta_H, \eta_m)) = U^1(\theta_H, C^1(u_{L,m}^1(\theta_H), \theta_L, \eta_m))$

(4) $U^2(\eta_H, C^2(u_{l,H}^2(\eta_H), \theta_l, \eta_H)) = U^2(\eta_H, C^2(u_{l,L}^1(\eta_H), \theta_l, \eta_L))$

(5) $(\partial U_{Hm}^P/\partial u_{Hm}^1)/(\partial U_{Hm}^P/\partial C_{Hm}^1) = (\partial U_{Hm}^1/\partial u_{Hm}^1)/(\partial U_{Hm}^1/\partial C_{Hm}^1)$

(6) $(\partial U_{lH}^P/\partial U_{lH}^2)/(\partial U_{lH}^P/\partial C_{lH}^2) = (\partial U_{lH}^2/\partial u_{lH}^2)/(\partial U_{lH}^2/\partial C_{lH}^2)$

(7) $(\partial U_{Lm}^P/\partial u_{Lm}^1)/(\partial U_{Lm}^P/\partial C_{Lm}^1) = (\lambda_1(\partial U_{Lm}^1/\partial C_{Lm}^1) - \lambda_4(\partial \tilde{U}_{Lm}^1/\partial C_{Lm}^1))/(\lambda_1(\partial U_{Lm}^1/\partial u_{Lm}^1) - \lambda_4(\partial \tilde{U}_{Lm}^1/\partial u_{Lm}^1))$

(8) $(\partial U_{lL}^P/\partial u_{lL}^2)/(\partial U_{lL}^P/\partial C_{lL}^2) = (\lambda_2(\partial U_{lL}^2/\partial C_{lL}^2) - \lambda_6(\partial \tilde{U}_{lL}^2/\partial C_{lL}^2))/(\lambda_2(\partial U_{lL}^2/\partial C_{lL}^2) - \lambda_6(\partial \tilde{U}_{lL}^2/\partial u_{lL}^2))$

Proposition 1 shows that the optimal contracts and the optimal efforts have the following properties. First, according to (1)–(4), for $H-$type agents, due to asymmetric information (private information), the expected utility obtained is higher than its reserved utility. And the $L-$type agents only obtained the reserved utility. Second, $L-$type agents' optimal contracts satisfied the truth-telling constraints. Third, the optimal contracts of $H-$type agent are efficient because under full information, (6) and (7) also hold. Finally, from (7) and (8), we can see

$$\begin{aligned} \frac{\partial U_{Lm}^P/\partial u_{Lm}^1}{\partial u_{Lm}^P/\partial C_{Lm}^1} &< \frac{\partial U_{Lm}^1/\partial C_{Lm}^1}{\partial U_{Lm}^1/\partial u_{Lm}^1}, \\ \frac{\partial U_{lL}^P/\partial u_{lL}^2}{\partial U_{lL}^P/\partial C_{lL}^2} &< \frac{\partial U_{lL}^2/\partial C_{lL}^2}{\partial U_{lL}^2/\partial C_{lL}^2}, \end{aligned} \tag{26}$$

i.e., the $L-$type agents' efficiency conditions are distorted. This means that the principal lost efficiency relative to the $L-$type agents, but he paid less information rent to the $H-$type agents *(here, our conclusion is the same as an adverse selection model with only one risk-neutral (but unknown type) agent (see [21])).*

4. Third Best: Moral Hazard and Adverse Selection

In real life, due to asymmetric information, moral hazard and adverse selection are often inseparable. In this section, we consider the model in the presence of adverse selection and moral hazard. Obviously, the problem faced by the principal is more complicated when the principal cannot observe the agents' efforts. Therefore, we will restate the problem of the principal.

4.1. The Principal's Problem. We follow the marks in the Section 2.2. Then, the principal's problem is described as follows:

Problem 2. Choose control pairs $u_{l,m} = (u^1_{m,l}, u^2_{m,l})$ and a menu of contracts $C(\cdot, \theta_l, \eta_m) = (C^1(\cdot, \theta_l, \eta_m), C^2(\cdot, \theta_l, \eta_m))$ to maximising

$$
\begin{aligned}
U^P(Y_T, C) = -\Big(& p \cdot q \cdot \mathbb{E}^{\mathbb{P}^u}\left[e^{-r_P \| Y_{L,L} - C(\cdot, \theta_L, \eta_L) \|}\right] \\
& + p \cdot (1-q) \cdot \mathbb{E}^{\mathbb{P}^u}\left[e^{-r_P \| Y_{L,H} - C(\cdot, \theta_L, \eta_H) \|}\right] \\
& + (1-p) \cdot q \cdot \mathbb{E}^{\mathbb{P}^u}\left[e^{-r_P \| Y_{H,L} - C(\cdot, \theta_H, \eta_L) \|}\right] \\
& + (1-p) \cdot (1-q) \cdot \mathbb{E}^{\mathbb{P}^u}\left[e^{-r_P \| Y_{H,H} - C(\cdot, \theta_H, \eta_H) \|}\right]\Big),
\end{aligned} \tag{27}
$$

subject to

(1) $dY_{l,m}(t) = b(t, A, u_{l,m})dt + \sum dB^u_t, l, m \in \{L, H\}$
(2) $u^1_{l,m} \in \arg\max_{u^1} \mathbb{E}^{\theta,\eta} U^1(\theta_l, C^1(u^1, \theta_l, \eta_m))$
(3) $u^2_{l,m} \in \arg\max_{u^2} \mathbb{E}^{\theta,\eta} U^2(\eta_m, C^2(u^2, \theta_l, \eta_m))$
(4) $\mathbb{E}^{\theta,\eta} U^1(\theta_l, C^1(u^1_{l,m}(\theta_l), \theta_l, \eta_m)) \geq \mathbb{E}^{\theta,\eta} U^1(\theta_i, C^1(u^1_{l,m}(\theta_i), \theta_l, \eta_m)); i \neq l$
(5) $\mathbb{E}^{\theta,\eta} U^2(\eta_m, C^2(u^2_{l,m}(\eta_m), \theta_l, \eta_m)) \geq \mathbb{E}^{\theta,\eta} U^2(\eta_j, C^2(u^2_{l,m}(\eta_j), \theta_l, \eta_m)); j \neq m$
(6) $\mathbb{E}^{\theta,\eta} U^1(\theta_l, C^1(u^1_{l,m}, \theta_l, \eta_m)) \geq U^1(L_1(\theta_l, \eta_m)) = -e^{-r_1 L_1(\theta_l, \eta_m)}; l, m \in \{L, H\}$
(7) $\mathbb{E}^{\theta,\eta} U^2(\eta_m, C^2(u^2_{l,m}, \theta_l, \eta_m)) \geq U^2(L_2(\theta_l, \eta_m)) = -e^{-r_2 L_2(\theta_l, \eta_m)}; l, m \in \{L, H\}$

where $\|Y - C\| = Y^1 + Y^2 - C^1 - C^2$.

The first constraint defines the dynamics of the output processes of each type when choosing the contracts designed for agents. The second and the third constraint express that agents maximise their expected utility (incentive constraint). The fourth and the fifth constraint, the truth-telling constraint (or incentive compatibility constraint), makes each agent optimally select the contract designed from him. Finally, the sixth and the seventh constraint, the participation constraint, ensures that the agents contract with the principal.

4.2. The Agents' Problem. Given a menu of contracts $C^1(\cdot, \theta_l, \eta_m)$, for agent 1, his problem is to choose the suitable effort $\overline{u}^1(\theta_i) = (\overline{u}^1_{i,L}(\theta_i), \overline{u}^1_{i,H}(\theta_i)), i \in \{L, H\}$ to maximize his expected utility based on the type of agent 2., i.e.,

$$
\begin{aligned}
\overline{u}^1(\theta_i) &\in \arg\max_{u^1} \mathbb{E}^{\theta,\eta} U^1\left(\theta_i, C^1\left(u^1, \theta_i, \eta\right)\right) \\
&= \max_{u^1} \mathbb{E}^{\theta,\eta}\left[-q \cdot e^{-r_1\left(C^1(\theta_i,\eta_L) - \int_0^T (\theta_i/2)\cdot (u^1_{i,L})^2 ds\right)} - (1-q) \cdot e^{-r_1\left(C^1(\theta_i,\eta_H) - \int_0^T (\theta_i/2)\cdot (u^1_{i,H})^2 ds\right)}\right].
\end{aligned} \tag{28}
$$

The following result is slightly adapted from Elie and Possamai [15] and Mastrolia [22].

Lemma 1. *Denote that $\overline{C}^1(\cdot, \theta_i, \eta_m)$ is the optimal contract provided by the principal to the i− type agent 1. There exists a pair $(Z^1_{i,L}, \overline{X}^1_0(\theta_i, \eta_L))$ and $(Z^1_{i,H}, \overline{X}^1_0(\theta_i, \eta_L))$ such that contracts satisfy the following conditions:*

$$
\begin{aligned}
\overline{X}^1_0(\theta_i, \eta_L) &= \overline{C}^1(\theta_i, \eta_L) - \int_0^T Z^1_{i,L} \sum dB_s + \int_0^T \left[-\frac{r_1}{2}\left\|Z^1_{i,L}\sum\right\|^2 + Z^1_{i,L} b\left(s, A, \overline{u}^1_{i,L}, \overline{u}^2_{i,L}\right) - \frac{\theta_i}{2}\left(\overline{u}^1_{i,L}\right)^2\right] ds, \\
\overline{X}^1_0(\theta_i, \eta_H) &= \overline{C}^1(\theta_i, \eta_H) - \int_0^T Z^1_{i,H} \sum dB_s + \int_0^T \left[-\frac{r_1}{2}\left\|Z^1_{i,H}\sum\right\|^2 + Z^1_{i,H} b\left(s, A, \overline{u}^1_{i,H}, \overline{u}^2_{i,H}\right) - \frac{\theta_i}{2}\left(\overline{u}^1_{i,H}\right)^2\right] ds.
\end{aligned} \tag{29}
$$

For agent 1's optimization problem (28), According to Borch theorem [23], we can can obtain the optimal solution of agent 1 as follows

Lemma 2 (Borch's theorem). *A risk exchange $(\overline{X}^1_0(\theta_i, \eta_L), \overline{X}^1_0(\theta_i, \eta_H))$ maximizes problem (28) if and only if $q \cdot r_1 \cdot e^{-r_1 \overline{X}^1_0(\theta_i, \eta_L)} = (1-q) \cdot r_1 \cdot e^{-r_1 \overline{X}^1_0(\theta_i, \eta_H)}$.*

Therefore, we have

$$p \cdot e^{-r_1\left(\overline{C}^1(\theta_i,\eta_L)-\int_0^T(\theta_i/2)\left(\overline{u}_{i,L}^1\right)^2\mathrm{d}s\right)} = (1-p)\cdot e^{-r_1\left(\overline{C}^1(\theta_i,\eta_H)-\int_0^T(\theta_i/2)\left(\overline{u}_{i,H}^1\right)^2\mathrm{d}s\right)}, \tag{30}$$

i.e.,

$$\max_{u^1}\mathbb{E}^{\theta,\eta}U^1\left(\theta_i, C^1\left(u^1,\theta_i,\eta\right)\right) = 2p\cdot e^{-r_1\left(\overline{C}^1(\theta_i,\eta_L)-\int_0^T(\theta_i/2)\left(\overline{u}_{i,L}^1\right)^2\mathrm{d}s\right)}. \tag{31}$$

Next, we will solve the following optimization problem:

$$\overline{u}_{i,L}^1(\theta_i) \in \arg\max_{u_{i,L}^1}\mathbb{E}^{\theta,\eta}\left[-e^{-r_1\left(C^1(\theta_i,\eta_L)-\int_0^T(\theta_i/2)\cdot\left(u_{i,L}^1\right)^2\mathrm{d}s\right)}\right]. \tag{32}$$

According to Lemma 1, we get the contract form of i–type agent 1:

$$C^1\left(u_{i,L}^1,\theta_i,\eta_L\right) = X_0^1(\theta_i,\eta_L) - \int_0^T h^1\left(s,\theta_i,Z_{i,L}^1\right)\mathrm{d}s + \int_0^T Z_{i,L}^1\sum \mathrm{d}B_s, \tag{33}$$

where $h^1(s,\theta_i,Z_{i,L}^1) = -(r_1/2)\|Z_{i,L}^1\sum\|^2 + Z_{i,L}^1 b(s,A,u_{i,L}^1, u_{i,L}^2) - (\theta_i/2)(u_{i,L}^1)^2$.

Suppose that l– type agent 1 has chosen $C^1(u_{i,L}^1,\theta_i,\eta_L)$ satisfied equation (33). The optimal contract should satisfy the truth-telling constraint, i.e.,

$$C^1\left(u_{i,L}^1,\theta_i,\eta_L\right) - \int_0^T\frac{\theta_l}{2}\cdot\left(u_{i,L}^1\right)^2\mathrm{d}s \le C^1\left(u_{l,L}^1,\theta_l,\eta_L\right) - \int_0^T\frac{\theta_l}{2}\cdot\left(u_{l,L}^1\right)^2\mathrm{d}s. \tag{34}$$

When agent 1 tells the truth $(l=i)$, the equal sign is established.

Similarly, for agent 2, given a menu of contracts $C^2(\cdot,\theta_l,\eta_m)$, his problem is to choose the suitable effort $\overline{u}^2(\eta_j) = (\overline{u}_{L,j}^2(\eta_j), \overline{u}_{H,j}^1(\eta_j))$, $j\in\{L,H\}$ to maximize his expected utility based on the type of agent 1, i.e.,

$$\begin{aligned}\overline{u}^2\left(\eta_j\right) &\in \arg\max_{u^2}\mathbb{E}^{\theta,\eta}U^2\left(\eta_j, C^2\left(u^2,\theta,\eta_j\right)\right)\\ &= \max_{u^2}\mathbb{E}^{\theta,\eta}\left[-p\cdot e^{-r_2\left(C^2(\theta_L,\eta_j)-\int_0^T(\eta_j/2)\cdot\left(u_{L,j}^2\right)^2\mathrm{d}s\right)} - (1-p)\cdot e^{-r_2\left(C^2(\theta_H,\eta_j)-\int_0^T(\eta_j/2)\cdot\left(u_{H,j}^2\right)^2\mathrm{d}s\right)}\right]\\ &= 2q\cdot e^{-r_2\left(\overline{C}^2(\theta_L,\eta_j)-\int_0^T(\eta_j/2)\cdot\left(\overline{u}_{L,j}^2\right)^2\mathrm{d}s\right)},\end{aligned} \tag{35}$$

where $\overline{C}^2(\theta_L,\eta_j)$ is a optimal contract for j–type agent 2 with L–type agent 1. Therefore, the optimization problem for agent 2 is as follows:

$$\overline{u}_{L,j}^2\left(\eta_j\right) \in \arg\max_{u_{L,j}^2}\mathbb{E}^{\theta,\eta}\left[-e^{-r_2\left(C^2(\theta_L,\eta_j)-\int_0^T(\eta_j/2)\cdot\left(u_{L,j}^2\right)^2\mathrm{d}s\right)}\right]. \tag{36}$$

According to Lemma 1, there exists a pair $(X_0^2(\theta_L,\eta_j), Z_{L,j}^2)$ such that

$$C^2\left(u_{L,j}^2,\theta_L,\eta_j\right) = X_0^2\left(\theta_L,\eta_j\right) - \int_0^T h^2\left(s,\theta_i,Z_{L,j}^2\right)\mathrm{d}s + \int_0^T Z_{L,j}^2\sum\mathrm{d}B_s, \tag{37}$$

where $h^2(s,\eta_j,Z_{L,j}^2) = -(r_2/2)\|Z_{L,j}^2\sum\|^2 + Z_{L,j}^2 b(s,A,u_{L,j}^1, u_{L,j}^2) - (\eta_j/2)(u_{L,j}^2)^2$. According to the truth-telling constraint, the optimal contract $C^2(u_{L,m}^2,\theta_L,\eta_m)$ for m–type agent 2 should satisfy

$$C^2\left(u_{L,m}^2,\theta_L,\eta_m\right) - \int_0^T\frac{\eta_m}{2}\left(u_{L,m}^2\right)^2\mathrm{d}s \ge C^2\left(u_{L,j}^2,\theta_L,\eta_j\right) - \int_0^T\frac{\eta_m}{2}\left(u_{L,j}^2\right)^2\mathrm{d}s. \tag{38}$$

When $m=j$, the equal sign is established.

4.3. Optimal Contracts. In this subsection, we solve explicit solutions to optimal contracts and optimal efforts. At first, we introduce the definition of two different equilibrium contracts.

Definition 2 (see [21]). If there is only one contract accepted by two different types of agents, that is, the equilibrium contract of different types of agents is the same, the equilibrium contract is called the pooling equilibrium contract. If there are different equilibrium contracts for different types of agents, it is called separating equilibrium contracts.

We assume that the pooling equilibrium contract exists, i.e., $C^1(\cdot,\theta_L,\eta_j) = C^1(\cdot,\theta_H,\eta_j)$ and $C^2(\cdot,\theta_i,\eta_L) = C^2(\cdot,\theta_i,\eta_H)$. From the truth-telling constraints for equations (34) and (38), we have $u_{Hj}^1 = u_{Lj}^1$ and $u_{iH}^2 = u_{iL}^2$. But, from the incentive constraints, obviously, the optimal efforts of different types of agents are different, i.e., $u_{Hj}^1 \neq u_{Lj}^1$ and $u_{iH}^2 \neq u_{iL}^2$. Therefore, we can assert that pooling equilibrium contracts do not exist.

Next, we will derive the specific form of separation equilibrium contracts.

For the principal, we use Borch's theorem again, and then we have the following conclusion:

$$\max_C U^P\left(Y_T,C\right) = \max_C\left(-4p\cdot q\cdot\mathbb{E}^{\mathbb{P}^u}\left[e^{-r_P\left\|Y_{L,L}-C\left(\cdot,\theta_L,\eta_L\right)\right\|}\right]\right) \Leftrightarrow \max_C\mathbb{E}^{\mathbb{P}^u}\left[-e^{-r_P\left\|Y_{i,j}-C\left(\cdot,\theta_i,\eta_j\right)\right\|}\right],\quad i,j\in\{L,H\}. \tag{39}$$

According to the analysis in Section 4.2, we only need to analyze the optimal contract in the case of i−type agent 1 and j−type agent 2 $(i, j \in \{L, H\})$, and then we can derive the optimal contract in other cases.

For i−type agent 1 and j−type agent 2, from equations (33) and (37), we know that contracts $C^1(u^1_{i,j}, \theta_i, \eta_j)$ and $C^2(u^2_{i,j}, \theta_i, \eta_j)$ are linear quadratic about $u^1_{i,j}$ and $u^2_{i,j}$, respectively. From the first-order condition with respect to $u^1_{i,j}$ and $u^2_{i,j}$, we get the form of optimal efforts as follows:

$$\overline{u}_{i,j} = \left(\overline{u}^1_{i,j}\ \ \overline{u}^2_{i,j}\right) = \begin{pmatrix} \overline{u}^{1,1}_{i,j} & \overline{u}^{1,2}_{i,j} \\ \overline{u}^{2,1}_{i,j} & \overline{u}^{2,2}_{i,j} \end{pmatrix} = \begin{pmatrix} \dfrac{a_{11}}{\theta_i} Z^{1,1}_{i,j} & \dfrac{a_{12}}{\eta_j} Z^{1,2}_{i,j} \\ \dfrac{a_{21}}{\theta_i} Z^{2,1}_{i,j} & \dfrac{a_{22}}{\eta_j} Z^{2,2}_{i,j} \end{pmatrix}. \tag{40}$$

Combining with equations (33) and (37) and substituting into the principal's expected utility function, then

$$\begin{aligned}
U^P(Y_T, C) &= \mathbb{E}^{\mathbb{P}^{\overline{u}(Z)}}\left[-e^{-r_P\left(\int_0^T \left\|b\left(s, A, \overline{u}_{i,j}\right)\right\| \mathrm{d}s + \int_0^T \left\|\sum_s\right\| \mathrm{d}B^{\overline{u}}_s\right)} \cdot e^{-r_P\left(-\left\|X_0\left(\theta_i,\eta_j\right)\right\| - (1/2)\int_0^T \left(r\left\|Z_{i,j}\Sigma_s\right\|^2 + k\left\|\overline{u}_{i,j}\right\|^2\right)\mathrm{d}s - \int_0^T \left\|Z_{i,j}\sum_s\right\| \mathrm{d}B^{\overline{u}}_s\right)}\right] \\
&= \mathbb{E}^{\mathbb{P}^{\overline{u}(Z)}}\left[-e^{-r_P\left(\int_0^T \left[\left\|\sum_s\right\| - \left\|Z_{i,j}\sum_s\right\|\right] \mathrm{d}B^{\overline{u}}_s + (r_P/2)\int_0^T \left(\left\|\sum_s\right\| - \left\|Z_{i,j}\sum_s\right\|\right)^2 \mathrm{d}s\right)}\right. \\
&\quad \left. \cdot e^{-r_P\int_0^T \left(\left\|b\left(s, A, \overline{u}_{i,j}\right)\right\| - \left\|X_0\left(\theta_i, \eta_j\right)\right\| - (1/2)\left[r\left\|Z_{i,j}\sum_s\right\|^2 + k\left\|\overline{u}_{i,j}\right\|^2\right] - (r_P/2)\left[\left\|\sum_s\right\| - \left\|\sum_{i,j}\sum_s\right\|\right]^2\right)\mathrm{d}s}\right] \\
&= \mathbb{E}^{\mathbb{P}^{\overline{u}(Z)}}\left[-e^{-r_P\left(\int_0^T \left[\left\|\sum_s\right\| - \left\|Z_{i,j}\sum_s\right\|\right] \mathrm{d}B_s - (r_P/2)\int_0^T \left(\left\|\Sigma_s\right\| - \left\|Z_{i,j}\Sigma_s\right\|\right)^2 \mathrm{d}s\right)}\right. \\
&\quad \left. \cdot e^{-r_P\int_0^T \left(\left\|b\left(s, A, \overline{u}_{i,j}\right)\right\| - \left\|X_0\right\| - (1/2)\left[r\left\|Z_{i,j}\sum_s\right\|^2 + k\left\|\overline{u}_{i,j}\right\|^2\right] - (r_P/2)\left[\left\|\sum_s\right\| - \left\|Z_{i,j}\sum_s\right\|\right]^2\right)\mathrm{d}s}\right] \\
&= \mathbb{E}^{\mathbb{P}^{\overline{u}(Z)}}\left[-\Lambda\left(-r_P \cdot \int_0^T \left(\left\|\sum_s\right\| - \left\|Z_{i,j}\Sigma_s\right\|\right)\mathrm{d}B_s\right) \cdot e^{-r_P \cdot \int_0^T g\left(s, Z_{i,j}\right)\mathrm{d}s}\right],
\end{aligned} \tag{41}$$

where

$$g\left(s, Z_{i,j}\right) = -\left(X_0^1\left(\theta_i, \eta_j\right) + X_0^2\left(\theta_i, \eta_j\right)\right) + \frac{a_{11}^2}{\theta_i} Z_{i,j}^{1,1} + \frac{a_{12}^2}{\eta_j} Z_{i,j}^{1,2} + \frac{a_{21}^2}{\theta_i} Z_{i,j}^{2,1} + \frac{a_{22}^2}{\eta_j} Z_{i,j}^{2,2} - \frac{a_{11}^2}{2\theta_i}\left(Z_{i,j}^{1,1}\right)^2 - \frac{a_{12}^2}{2\eta_j}\left(Z_{i,j}^{1,2}\right)^2 - \frac{a_{21}^2}{2\theta_i}\left(Z_{i,j}^{2,1}\right)^2 - \frac{a_{22}^2}{2\eta_j}\left(Z_{i,j}^{2,2}\right)^2 - \frac{\sigma_1^2}{2}\left[r_1\left(Z_{i,j}^{1,1}\right)^2 + r_2\left(Z_{i,j}^{1,2}\right)^2\right] - \frac{\sigma_2^2}{2}\left[r_1\left(Z_{i,j}^{2,1}\right)^2 + r_2\left(Z_{i,j}^{2,2}\right)^2\right] - \frac{r_P}{2}\left[\sigma_1^2\left(Z_{i,j}^{1,1} + Z_{i,j}^{1,2} - 1\right)^2 + \sigma_2^2\left(Z_{i,j}^{2,1} + Z_{i,j}^{2,2} - 1\right)^2\right],$$

$$Z_{i,j} = \begin{pmatrix} Z_{i,j}^1 & Z_{i,j}^2 \end{pmatrix} = \begin{pmatrix} Z_{i,j}^{1,1} & Z_{i,j}^{1,2} \\ Z_{i,j}^{2,1} & Z_{i,j}^{2,2} \end{pmatrix}. \tag{42}$$

We show that the principal maximizes his utility function equivalent to maximizing the function g and easy to compute that g is concave on $Z_{i,j}^1$ and $Z_{i,j}^2$. Taking the first-order conditions, then we can get the exact expression of $Z_{i,j}^1$ and $Z_{i,j}^2$, for $i, j \in \{L, H\}$.

Therefore, we have the following result:

Proposition 2. *For $i, j \in \{L, H\}$, the optimal equilibrium effort for the i–agent 1 and the j– type agent 2 is as follows:*

$$\overline{u}_{i,j} = \begin{pmatrix} \overline{u}_{i,j}^1 & \overline{u}_{i,j}^2 \end{pmatrix} = \begin{pmatrix} \overline{u}_{i,j}^{1,1} & \overline{u}_{i,j}^{1,2} \\ \overline{u}_{i,j}^{2,1} & \overline{u}_{i,j}^{2,2} \end{pmatrix} = \begin{pmatrix} \dfrac{a_{11}}{\theta_i} Z_{i,j}^{1,1} & \dfrac{a_{12}}{\eta_j} Z_{i,j}^{1,2} \\ \dfrac{a_{21}}{\theta_i} Z_{i,j}^{2,1} & \dfrac{a_{22}}{\eta_j} Z_{i,j}^{2,2} \end{pmatrix}. \tag{43}$$

The optimal utility of the principal is

$$U^P\left(Y_T, C\right) = -\left(pq \cdot e^{-r_P T g\left(Z_{L,L}\right)} + p(1-q) \cdot e^{-r_P T g\left(Z_{L,H}\right)} + (1-p)q \cdot e^{-r_P T g\left(Z_{H,L}\right)} + (1-p)(1-q) \cdot e^{-r_P T g\left(Z_{H,H}\right)}\right). \tag{44}$$

The optimal contract given to i–type agent 1 is

$$C^1\left(u_{i,j}^1, \theta_i, \eta_j\right) = X_0^1\left(\theta_i, \eta_j\right) + \frac{r_1 T}{2} \cdot \left[\left(\sigma_1 Z_{i,j}^{1,1}\right)^2 + \left(\sigma_2 Z_{i,j}^{2,1}\right)^2\right] + \frac{T}{2\theta_i} \cdot \left[\left(a_{11} Z_{i,j}^{1,1}\right)^2 + \left(a_{21} Z_{i,j}^{2,1}\right)^2\right] + Z_{i,j}^{1,1} Y_{i,j}^1 + Z_{i,j}^{2,1} Y_{i,j}^2 - T \cdot \left[Z_{i,j}^{1,1}\left(\frac{a_{11}^2}{\theta_i} Z_{i,j}^{1,1} + \frac{a_{12}^2}{\eta_j} Z_{i,j}^{1,2}\right) + Z_{i,j}^{2,1}\left(\frac{a_{21}^2}{\theta_i} Z_{i,j}^{2,1} + \frac{a_{22}^2}{\eta_j} Z_{i,j}^{2,2}\right)\right], \tag{45}$$

and optimal contract given to j–type agent 2 is

$$C^2\left(u_{i,j}^2, \theta_i, \eta_j\right) = X_0^2\left(\theta_i, \eta_j\right) + \frac{r_2 T}{2} \cdot \left[\left(\sigma_1 Z_{i,j}^{1,2}\right)^2 + \left(\sigma_2 Z_{i,j}^{2,2}\right)^2\right] + \frac{T}{2\eta_j} \cdot \left[\left(a_{12} Z_{i,j}^{1,2}\right)^2 + \left(a_{22} Z_{i,j}^{2,2}\right)^2\right] + Z_{i,j}^{1,2} Y_{i,j}^1 + Z_{i,j}^{2,2} Y_{i,j}^2 - T \cdot \left[Z_{i,j}^{1,2}\left(\frac{a_{11}^2}{\theta_i} Z_{i,j}^{1,1} + \frac{a_{12}^2}{\eta_j} Z_{i,j}^{1,2}\right) + Z_{i,j}^{2,2}\left(\frac{a_{21}^2}{\theta_i} Z_{i,j}^{2,1} + \frac{a_{22}^2}{\eta_j} Z_{i,j}^{2,2}\right)\right], \tag{46}$$

where $Z_{i,j}$ satisfy the first-order conditions of equation (42).

Also, the certainty equivalent $X_0^1(\theta_i, \eta_j)$ and $X_0^2(\theta_i, \eta_j)$ for agents satisfy the following conditions:

$$\begin{cases} X_0^1\left(\theta_L, \eta_L\right) = L_1\left(\theta_L, \eta_L\right), \\ X_0^1\left(\theta_H, \eta_L\right) = L_1\left(\theta_L, \eta_L\right) + \dfrac{T \cdot \left(\theta_L - \theta_H\right)}{2} \cdot \left(u_{H,L}^1\right)^2, \\ X_0^1\left(\theta_L, \eta_H\right) = L_1\left(\theta_L, \eta_H\right), \\ X_0^1\left(\theta_H, \eta_H\right) = L_1\left(\theta_L, \eta_H\right) + \dfrac{T \cdot \left(\theta_L - \theta_H\right)}{2} \cdot \left(u_{H,H}^1\right)^2, \end{cases} \tag{47}$$

and

$$\begin{cases} X_0^2(\theta_L,\eta_L) = L_2(\theta_L,\eta_L), \\ X_0^2(\theta_L,\eta_H) = L_2(\theta_L,\eta_H) + \dfrac{T\cdot(\eta_L-\eta_H)}{2}\cdot\left(u_{L,H}^2\right)^2, \\ X_0^2(\theta_H,\eta_L) = L_2(\theta_H,\eta_L), \\ X_0^2(\theta_H,\eta_H) = L_2(\theta_H,\eta_L) + \dfrac{T\cdot(\eta_L-\eta_H)}{2}\cdot\left(u_{H,H}^2\right)^2. \end{cases} \tag{48}$$

Next, we will explain that equations (47) and (48) are established. Referring to Conclusion 4.2 of Macho-Stadler and Perez-Castrillo [21], for different types of agent 1, the separating equilibrium contracts $(C^1(\cdot,\theta_L,\eta_j), C^1(\cdot,\theta_H,\eta_j))$ should satisfy the following two constraints:

(1) And for L−type agent 1, the optimal contract optimal satisfied

$$\begin{aligned} &\mathbb{E}^{\theta,\eta}U^1\left(\theta_L, C^1\left(u_{L,j}^1(\theta_L),\theta_L,\eta_j\right)\right) \\ &\quad = \mathbb{E}^{\theta,\eta}U^1\left(\theta_L, C^1\left(u_{H,j}^1(\theta_L),\theta_H,\eta_j\right)\right) = U^1\left(L_1\left(\theta_L,\eta_j\right)\right), \end{aligned} \tag{49}$$

i.e.,

$$\begin{aligned} &-q\cdot e^{-r_1X_0^1(\theta_L,\eta_L)} - (1-q)\cdot e^{-r_1X_0^1(\theta_L,\eta_H)} \\ &= -q\cdot e^{-r_1\cdot\left[C^1\left(u_{L,L}^1(\theta_L),\theta_L,\eta_L\right) - \int_0^T(\theta_L/2)\left(u_{L,L}^1(\theta_L)\right)^2 ds\right]}, \\ &- (1-q)\cdot e^{-r_1\cdot\left[C^1\left(u_{L,H}^1(\theta_L),\theta_L,\eta_H\right) - \int_0^T(\theta_L/2)\left(u_{L,H}^1(\theta_L)\right)^2 ds\right]} \\ &= -q\cdot e^{-r_1\cdot\left[X_0^1(\theta_H,\eta_L) + \int_0^T(\theta_H/2)\left(u_{H,L}^1(\theta_L)\right)^2 ds - \int_0^T(\theta_L/2)\left(u_{H,L}^1(\theta_L)\right)^2 ds\right]}, \\ &- (1-q)\cdot e^{-r_1\cdot\left[X_0^1(\theta_H,\eta_H) + \int_0^T(\theta_H/2)\left(u_{H,H}^1(\theta_L)\right)^2 ds - \int_0^T(\theta_L/2)\left(u_{H,H}^1(\theta_L)\right)^2 ds\right]} \\ &= -q\cdot e^{-r_1L_1(\theta_L,\eta_L)} - (1-q)\cdot e^{-r_1L_1(\theta_L,\eta_H)}. \end{aligned} \tag{50}$$

That is, the truth-telling constraint of L-type agent 1 is tight (the inequality is taken equal sign). And his optimal expected utility equals his reservation utility. Therefore, equation (47) is established.

(2) For H−type agent 1, the optimal contract always satisfies the truth-telling constraint. Then, in this case, the optimal contract for H−type agent 1 does not satisfy Borch's theorem. Therefore, H−type agent 1 lost optimal expected utility due to asymmetric information. That is, the expected utility of H−type agent 1 when there is an adverse selection is lower than the expected utility when there is no adverse selection.

Similarly, for agent 2, we also have the same conclusion, i.e., equations (48) and (51) are established.

According to Conclusion 4.3 of Macho-Stadler and Perez-Castrillo [21], we have to follow the conclusion.

Proposition 3. *There exists a probability* $q^* < \left[1+\exp\left\{(r_1T(\theta_L-\theta_H)/2)\cdot\left[(u_{HH}^1)^2-(u_{HL}^1)^2\right]\right\}\right]^{-1} \in (0,1),$ *and a probability* $p^* < \left[1+\exp\left\{(r_2T(\eta_L-\eta_H)/2)\cdot\left[(u_{HH}^2)^2-(u_{LH}^2)^2\right]\right\}\right]^{-1}.$

(1) If $q \in (q^*,1)$ and $p \in (p^*,1)$, then the optimal contracts in Proposition 2 are called the separating equilibrium contracts.

(2) If $q \in (0,q^*)$ and $p \in (0,p^*)$, for the principal, there is no equilibrium contract. This means that the principal only provides H−type contracts although there is a risk that the agents are L−type (this is a small probability event).

But, the $H-$type agents satisfy Broch's theorem, i.e.,

$$\begin{cases} X_0^1(\theta_H,\eta_L) - \dfrac{\ln q}{r_1} = X_0^1(\theta_H,\eta_H) - \dfrac{\ln(1-q)}{r_1}, \\ X_0^2(\theta_L,\eta_H) - \dfrac{\ln p}{r_2} = X_0^2(\theta_H,\eta_H) - \dfrac{\ln(1-p)}{r_2}. \end{cases} \quad (51)$$

For $H-$type agent 1, according to equation (47), if the optimal contract satisfies Borch's theorem, we have

$$L_1(\theta_L,\eta_H) = L_1(\theta_L,\eta_L) + \frac{T\cdot(\theta_L-\theta_H)}{2}\cdot\left(\left(u_{H,L}^1\right)^2 - \left(u_{H,H}^1\right)^2\right) + \frac{1}{r_1}\ln\left(\frac{1-q}{q}\right). \quad (52)$$

According to our assumption $L_1(\theta_L,\eta_H) \geq L_1(\theta_L,\eta_L)$; therefore, we should have

$$\frac{T\cdot(\theta_L-\theta_H)}{2}\cdot\left(\left(u_{H,L}^1\right)^2 - \left(u_{H,H}^1\right)^2\right) + \frac{1}{r_1}\ln\left(\frac{1-q}{q}\right) \geq 0, \quad (53)$$

i.e.,

$$0 \leq q \leq q^* < \left[1 + \exp\left\{\frac{r_1 T(\theta_L-\theta_H)}{2}\cdot\left[\left(u_{HH}^1\right)^2 - \left(u_{HL}^1\right)^2\right]\right\}\right]^{-1}. \quad (54)$$

Similarly, we can easily get

$$p^* < \left[1 + \exp\left\{\frac{r_2 T(\eta_L-\eta_H)}{2}\cdot\left[\left(u_{HH}^2\right)^2 - \left(u_{LH}^2\right)^2\right]\right\}\right]^{-1}. \quad (55)$$

In the next section, we will find a suitable p^* and q^* through a specific example, such that, when $q \in (0,q^*)$ and $p \in (0,p^*)$, the principal prefers to provide $H-$type contracts compared to providing separate equilibrium contracts.

Thus, we have obtained an explicit solution to the optimal contracts where the agents' type is discrete. The solution process for this case is similar to solving the optimal contract with only moral hazard. Next, we use numerical simulation to present our conclusions more intuitively.

4.4. Numerical Simulation. In this section, we graphically simulate the conclusions in Proposition 2. For simplicity, we assume that the parameters of the two agents are the same (here, we still think that agents need to guess each other's type). Therefore, in the following, we have an obligation to simulate one agent's optimal contract and optimal efforts.

Firstly, we analyze the impact of uncertainty $\sum$ on the optimal contract. The parameters are $a_{11} = a_{12} = a_{21} = a_{22} = 1$, $r_1 = r_2 = r_P = 1/2$, $T = 1$, $L_1(\theta_L,\eta_L) = L_2(\theta_L,\eta_L) = 0$ $\theta_L,\eta_L = 2$, $\theta_H,\eta_H = 1$, and $p = q = 1/2$. Therefore,

$$\begin{cases} Z_{H,H}^{1,1} = Z_{H,H}^{1,2} = \dfrac{\sigma_1^4 + 6\sigma_1^2 + 8}{3\sigma_1^4 + 16\sigma_1^2 + 16}, \\ Z_{H,H}^{2,1} = Z_{H,H}^{2,2} = \dfrac{\sigma_2^4 + 6\sigma_2^2 + 8}{3\sigma_2^4 + 16\sigma_2^2 + 16}, \\ Z_{L,H}^{1,1} = \dfrac{\sigma_1^4 + 4\sigma_1^2 + 4}{3\sigma_1^4 + 10\sigma_1^2 + 4}, \\ Z_{L,H}^{1,2} = \dfrac{\sigma_1^4 + 4\sigma_1^2 + 2}{3\sigma_1^4 + 10\sigma_1^2 + 4}, \\ Z_{L,H}^{2,1} = \dfrac{\sigma_2^4 + 4\sigma_2^2 + 4}{3\sigma_2^4 + 10\sigma_2^2 + 4}, \\ Z_{L,H}^{2,2} = \dfrac{\sigma_2^4 + 4\sigma_2^2 + 2}{3\sigma_2^4 + 10\sigma_2^2 + 4}, \end{cases}$$

$$\begin{cases} Z_{L,L}^{1,1} = Z_{L,L}^{1,2} = \dfrac{\sigma_1^4 + 2\sigma_1^2 + 1}{3\sigma_1^4 + 4\sigma_1^2 + 1}, \\ Z_{L,L}^{2,1} = Z_{L,L}^{2,2} = \dfrac{\sigma_2^4 + 2\sigma_2^2 + 1}{3\sigma_2^4 + 4\sigma_2^2 + 1}, \\ Z_{H,L}^{1,1} = \dfrac{\sigma_1^4 + 4\sigma_1^2 + 2}{3\sigma_1^4 + 10\sigma_1^2 + 4}, \\ Z_{H,L}^{1,2} = \dfrac{\sigma_1^4 + 4\sigma_1^2 + 4}{3\sigma_1^4 + 10\sigma_1^2 + 4}, \\ Z_{H,L}^{2,1} = \dfrac{\sigma_2^4 + 4\sigma_2^2 + 2}{3\sigma_2^4 + 10\sigma_2^2 + 4}, \\ Z_{H,L}^{2,2} = \dfrac{\sigma_2^4 + 4\sigma_2^2 + 4}{3\sigma_2^4 + 10\sigma_2^2 + 4}. \end{cases} \quad (56)$$

Then, we can draw the efforts of agents, as shown in Figure 1.

Figure 1 shows a general conclusion that effort is a decreasing function of volatility (uncertainty). Furthermore, the type of coworker will affect the agent's efforts. The conclusion in Figure 1 shows that first, the H-type agent provides more effort than the L-type agent provides; second, the H-type agent will prompt his coworker to provide higher effort (whether the coworker is H-type or L-type). Among them, the second conclusion is the unique conclusion in the multiagent problem.

Figures 2 and 3 show the optimal contract for the $L-$type agent and the $H-$type agent, respectively. We can see from

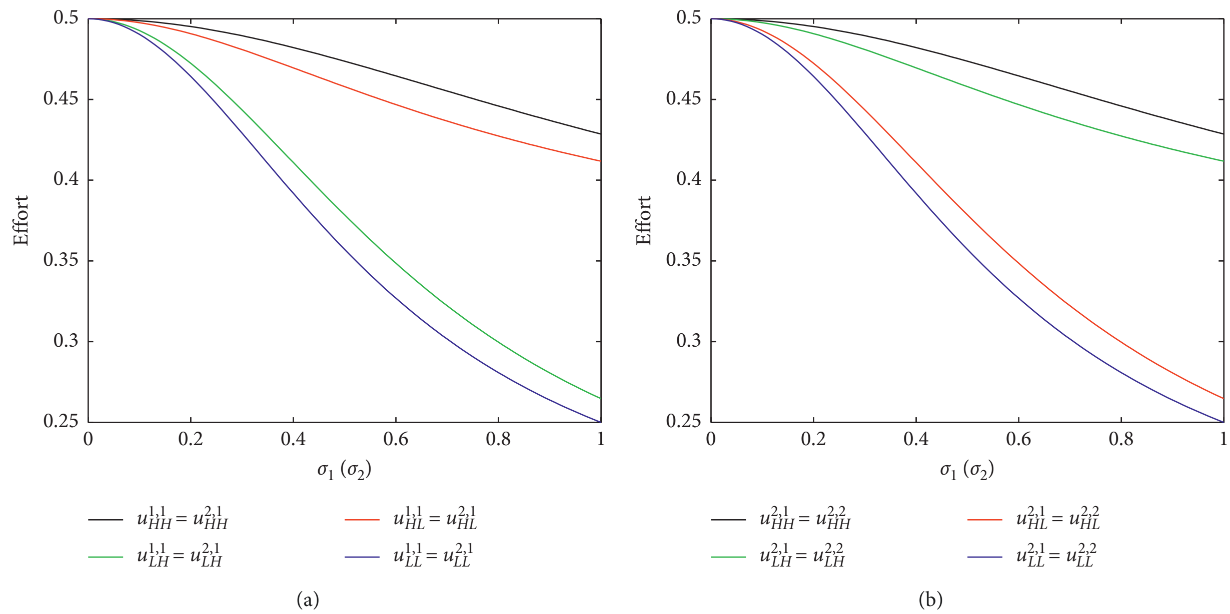

Figure 1: The optimal efforts for agent 1 and agent 2.

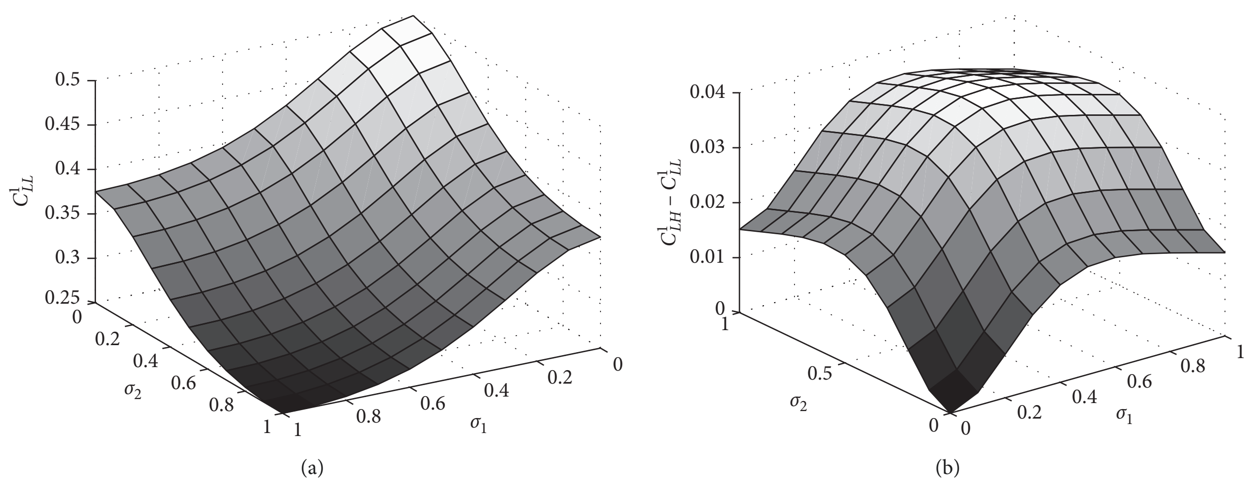

Figure 2: The optimal contract for L−type agent 1.

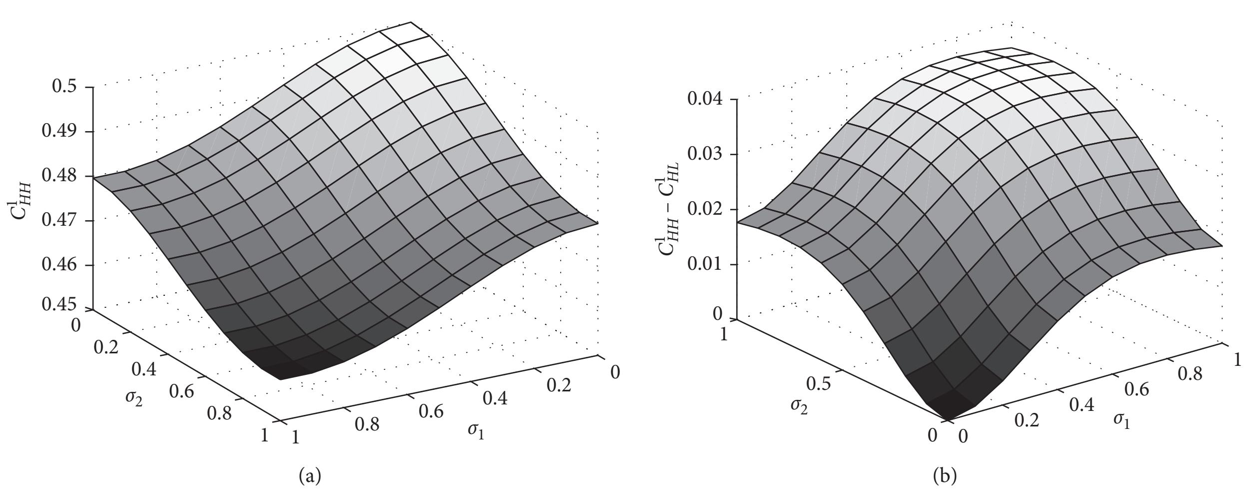

Figure 3: The optimal contract for H−type agent 1.

Figures 2 and 3 that the optimal incentive contract of each agent is affected by his own type and the type of coworker. In detail, each agent's contract is increasing both in his type and his coworker's type (we can see the same conclusion in [6]). Moreover, the H-type optimal incentive contract is larger than the L-type optimal incentive contract, regardless of whether the agent is H-type or the coworker is H-type. This conclusion is a generalization of the conclusion of the single-agent adverse selection problem.

The trend in Figure 4 is similar to Figure 3. That is, the greater the uncertainty, the lower the expected utility of the principal. This shows that when there are moral hazard and adverse selection, the expected utility of both of them has been reduced, i.e., corporate management is inefficient. This is a general conclusion in the principal-agent problem. The influence of uncertainty on contract is also a topic worth studying in the principal-agent problem (such as [24]).

Secondly, we analyze the influence of prior distribution on the optimal contract. Similarly, we also have to fix some parameters, $a_{11} = a_{12} = a_{21} = a_{22} = 1$, $r_1 = r_2 = r_P = 1/2$, $T = 1$, $\Sigma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $\theta_L, \eta_L = 2$, $\theta_H, \eta_H = 1$, $L_1(\theta_L, \eta_j) = L_2(\theta_i, \eta_L) = 0$, and $L_1(\theta_H, \eta_j) = L_2(\theta_i, \eta_H) = 0.18$. According to the conclusions in Proposition 1 and Proposition 2, $H-$type agents' utility exceeds his reserved utility, i.e., $L_1(\theta_H, \eta_j) < L_1(\theta_L, \eta_j) + (T \cdot (\theta_L - \theta_H)/2) \cdot (u^1_{H,j})^2$. Therefore, in this case, we assume that $L_1(\theta_H, \eta_j) = (1/2) \cdot [L_1(\theta_L, \eta_j) + (T \cdot (\theta_L - \theta_H)/2) \cdot (u^1_{H,j})^2]$. Then,

$$\begin{cases} Z^{1,1}_{H,H} = Z^{1,2}_{H,H} = Z^{2,1}_{H,H} = Z^{2,2}_{H,H} = \dfrac{3}{7}, \\ Z^{1,1}_{L,H} = Z^{2,1}_{L,H} = \dfrac{9}{17}, \\ Z^{1,2}_{L,H} = Z^{2,2}_{L,H} = \dfrac{7}{17}, \end{cases}$$
$$\begin{cases} Z^{1,1}_{L,L} = Z^{1,2}_{L,L} = Z^{2,1}_{L,L} = Z^{2,2}_{L,L} = \dfrac{1}{2}, \\ Z^{1,1}_{H,L} = Z^{2,1}_{H,L} = \dfrac{7}{17}, \\ Z^{1,2}_{H,L} = Z^{2,2}_{H,L} = \dfrac{9}{17}. \end{cases} \tag{57}$$

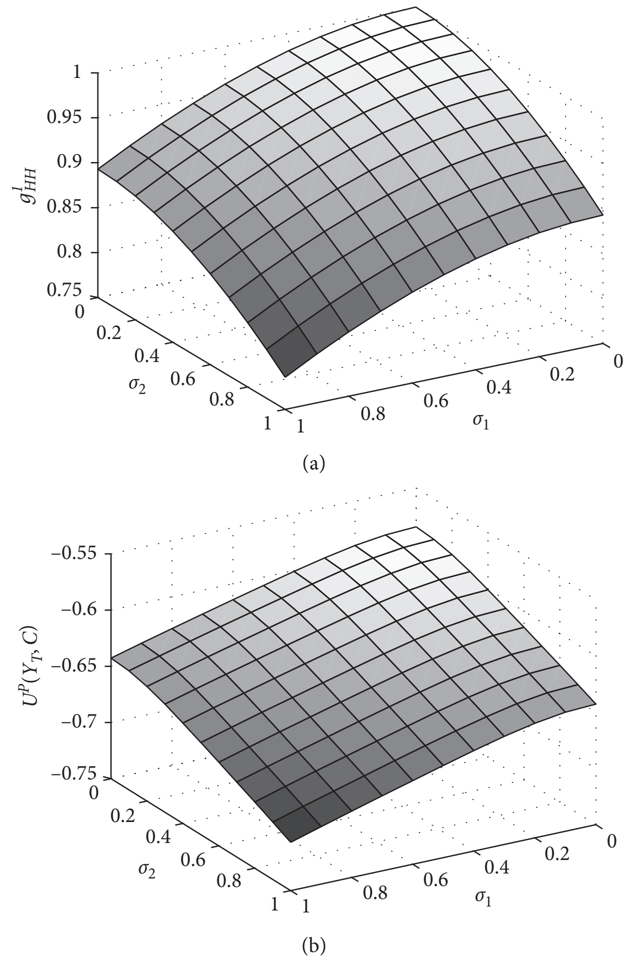

FIGURE 4: The function g and the expected utility for the principal.

Next, we will find the specific values of p^* and q^*. Given the principal only provides $H-$type contracts $C^1_{HH} = C^1(u^1_{HH}, \theta_H, \eta_H)$ and $C^2_{HH} = C^2(u^2_{HH}, \theta_H, \eta_H)$. According to the conclusion of Proposition 3, when $p \in (0, p^*)$ and $q \in (0, q^*)$, the $H-$type agents satisfied Borch's theorem; therefore, for $H-$type agents, regardless of the type of the coworker, choosing to provide effort u^n_{HH} $(n = 1, 2)$ is always optimal. We assume that $L-$type agents also accept $H-$type contracts and provide efforts, marked as $\tilde{u}^n_{HH}$. Because of the moral hazard, the principal cannot observe the agents' efforts. So, for $L-$type agents, $\tilde{u}^n_{HH} < u^n_{HH}$. Because of the certainty equivalent provided by the principal to $H-$type and $L-$type is the same, i.e.,

$$\begin{aligned}
&C^1\left(u^1_{H,H},\theta_H,\eta_H\right)\\
&\quad = X^1_0(\theta_H,\eta_H) + \frac{r_1T}{2}\cdot\left[\left(\sigma_1 Z^{1,1}_{H,H}\right)^2 + \left(\sigma_2 Z^{2,1}_{H,H}\right)^2\right] + \frac{T}{2\theta_H}\cdot\left[\left(a_{11}Z^{1,1}_{H,H}\right)^2 + \left(a_{21}Z^{2,1}_{H,H}\right)^2\right]\\
&\quad = X^1_0(\theta_H,\eta_H) + \frac{r_1T}{2}\cdot\left[\left(\sigma_1 \widetilde{Z}^{1,1}_{H,H}\right)^2 + \left(\sigma_2 \widetilde{Z}^{2,1}_{H,H}\right)^2\right] + \frac{T\theta_L}{2}\cdot\left[\left(\frac{a_{11}}{\theta_H}\widetilde{Z}^{1,1}_{H,H}\right)^2 + \left(\frac{a_{21}}{\theta_H}\widetilde{Z}^{2,1}_{H,H}\right)^2\right],
\end{aligned} \tag{58}$$

and for agent 2,

$$\begin{aligned}
&C^2\left(u^2_{H,H},\theta_H,\eta_H\right)\\
&\quad = X^2_0(\theta_H,\eta_H) + \frac{r_2T}{2}\cdot\left[\left(\sigma_1 Z^{1,2}_{H,H}\right)^2 + \left(\sigma_2 Z^{2,2}_{H,H}\right)^2\right] + \frac{T}{2\eta_H}\cdot\left[\left(a_{12}Z^{1,2}_{H,H}\right)^2 + \left(a_{22}Z^{2,2}_{H,H}\right)^2\right]\\
&\quad = X^2_0(\theta_H,\eta_H) + \frac{r_2T}{2}\cdot\left[\left(\sigma_1 \widetilde{Z}^{1,2}_{H,H}\right)^2 + \left(\sigma_2 \widetilde{Z}^{2,2}_{H,H}\right)^2\right] + \frac{T\eta_L}{2}\cdot\left[\left(\frac{a_{12}}{\eta_H}\widetilde{Z}^{1,2}_{H,H}\right)^2 + \left(\frac{a_{22}}{\eta_H}\widetilde{Z}^{2,2}_{H,H}\right)^2\right].
\end{aligned} \tag{59}$$

Therefore, in this case, we have

$$\begin{aligned}
\widetilde{u}^{1,1}_{HH} &= \frac{a_{11}}{\theta_H}\widetilde{Z}^{1,1}_{H,H} = \frac{3\sqrt{15}}{35},\\
\widetilde{u}^{2,1}_{HH} &= \frac{a_{21}}{\theta_H}\widetilde{Z}^{2,1}_{H,H} = \frac{3\sqrt{15}}{35},\\
\widetilde{u}^{1,2}_{HH} &= \frac{a_{12}}{\eta_H}\widetilde{Z}^{1,2}_{H,H} = \frac{3\sqrt{15}}{35},\\
\widetilde{u}^{2,2}_{HH} &= \frac{a_{22}}{\eta_H}\widetilde{Z}^{2,2}_{H,H} = \frac{3\sqrt{15}}{35}.
\end{aligned} \tag{60}$$

Denote that $\widetilde{U}^P(Y_T,C)$ represents the principal's optimal expected utility under the situation he only provided the H−type contract, and $U^P(Y_T,C)$ is his optimal expected utility under the situation with which he provided the separating equilibrium contracts. If the following inequality holds,

$$\begin{aligned}
U^P(Y_T,C) - \widetilde{U}^P(Y_T,C) = &-\Big(pq\cdot e^{-r_PTg(Z_{L,L})} + p(1-q)\cdot e^{-r_PTg(Z_{L,H})} + (1-p)q\cdot e^{-r_PTg(Z_{H,L})}\\
&+ (1-p)(1-q)\cdot e^{-r_PTg(Z_{H,H})}\Big) + \Big(pq\cdot e^{-r_PTg\left(\widetilde{Z}^1_{H,H},\widetilde{Z}^2_{H,H}\right)} + p(1-q)\cdot e^{-r_PTg\left(\widetilde{Z}^1_{H,H},Z^2_{H,H}\right)}\\
&+ (1-p)q\cdot e^{-r_PTg\left(Z^1_{H,H},\widetilde{Z}^2_{H,H}\right)} + (1-p)(1-q)\cdot e^{-r_PTg\left(Z^1_{H,H},Z^2_{H,H}\right)}\Big) < 0.
\end{aligned} \tag{61}$$

We should have $p < 0.1 = p^*$ and $q < 0.1 = q^*$. In other words, when the probability of the L−type agents is small, the principal only provides H−type contracts to obtain higher expected utility.

The left side of Figure 5 is $|U^P(Y_T,C) - \widetilde{U}^P(Y_T,C)|$, that is, the comparison of the principal's utility function in both cases. From the figure, we can see that when $p \in (0, 0.1)$ and $q \in (0, 0.1)$, we have $U^P(Y_T,C) < \widetilde{U}^P(Y_T,C)$, i.e., when the probability of the L−type agent is relatively small, the principal only provides the H−type contract, which can improve his expected utility. The right side of Figure 5 shows the entire expected utility of the principal. Obviously, the higher the probability of the H−type agent, the greater the expected utility of the principal.

Figure 5 depicts an interesting conclusion in this paper after the exact solution of the optimal contract, and the optimal effort is obtained through the fixed parameters. That is to say, in our model, we can find a probability value p^* and q^* that decides whether the principal provides H-type contract or separates equilibrium contract.

Figure 6 shows the separating equilibrium contracts for agent 1 and agent 2. The L−type agents' contract is obviously

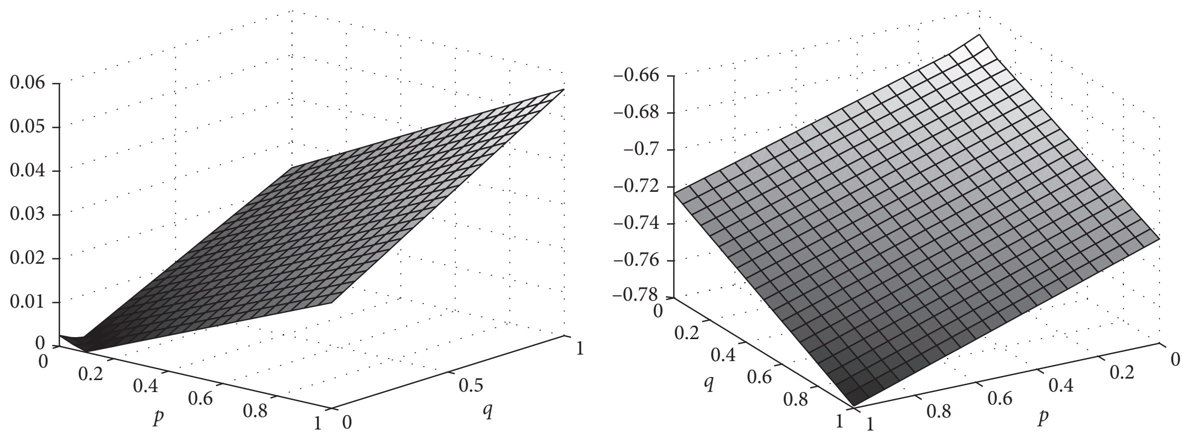

Figure 5: The expected utility for the principal.

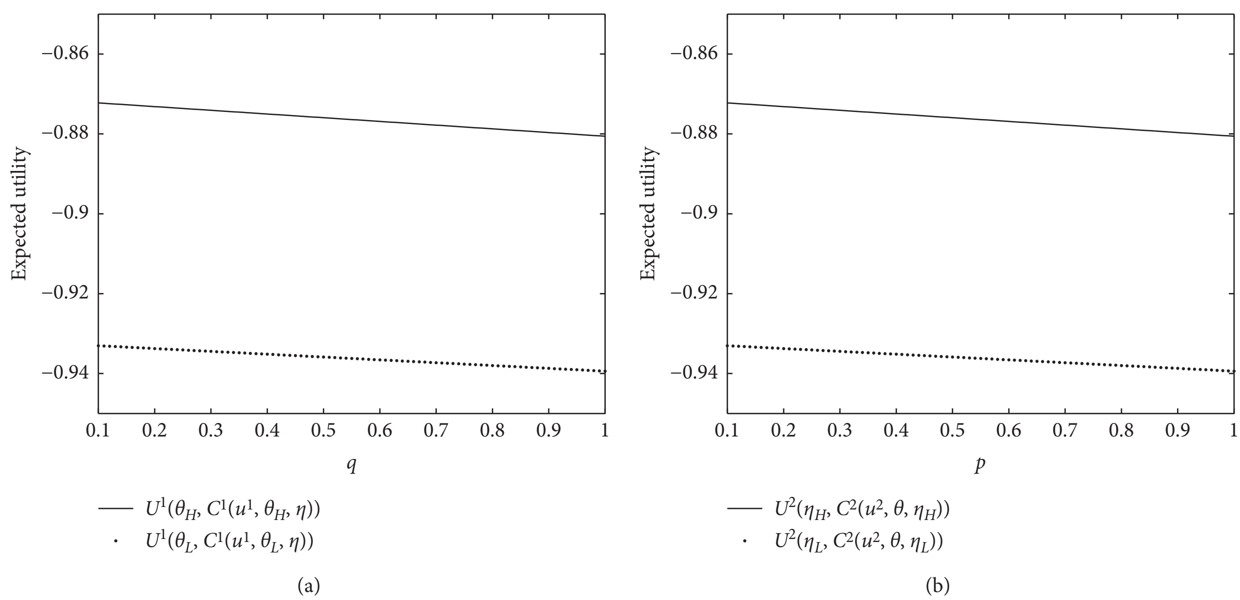

Figure 6: The optimal expected utility for the agents.

smaller than the H−type agents' contract. And when the probability of a coworker being an L−type agent is increasing, the agent's expected utility is diminishing.

Unlike the single-agent model, Figure 6 shows the interaction between agents. The types of agents and the efforts they provide will affect the expected utility of their coworkers. This conclusion is closer to the real situation in the labor market.

5. Conclusion

We develop a continuous-time model for modeling a multiagent relationship in the presence of adverse selection, with or without moral hazard. For principal, it is also satisfactory to have a model in which the firm's rate of return (or risk item yield) can be controlled by the agent, and the volatility is unknown (but fixed). And for principal, it is important to have models in which to employ multiple agents to provide the efforts to coproduce (or manage risk projects together). Moreover, it is important to allow for these effort actions to be dynamic. We have shown that the multiagent incentive problem can be transformed into risk sharing problem, in the case of pure reverse selection. Thus, solving the principal's problem becomes similar to solving a first-best problem as seen in Cvitanic and Zhang [25]. Given both the principal and the agents are risk averse, because of the nonlinearity of the utility functions, we need to solve the risk sharing problem with a method that is different from solving the linear problem.

In the third best case, we have the following conclusions: firstly, moral hazard and adverse selection in the market will lead to the absence of market equilibrium. Secondly, if market equilibrium exists, then it must be a separating equilibrium. That is, the market supports different contracts, each of which applies to one type of agent. Thirdly, although the team incentive problem in continuous time is much more complicated than the static single agency incentive problem, we get the conclusions are similar to the conclusions in the static model. Finally, there is an interesting conclusion that under the separating equilibrium, the $L-$type agents obtain the reservation utility. Because of the asymmetry of information, the expected utility of the $H-$type agents is less than that of the full information.

It should be noted that the continuous payment provided by the principal is not considered in our model. In addition, in order to get the exact solution of the optimal contracts, our model assumes only two types of two agents, which are obviously not very realistic. In many cases, it is assumed that the set of agents' type is a continuous space or that more agents in the team are more reasonable. These issues would be of significant interest for future research.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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