Research Article

Asymptotic Behavior of Multigroup SEIR Model with Nonlinear Incidence Rates under Stochastic Perturbations

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In this paper, the asymptotic behavior of a multigroup SEIR model with stochastic perturbations and nonlinear incidence rate functions is studied. First, the existence and uniqueness of the solution to the model we discuss are given. Then, the global asymptotical stability in probability of the model with $R_0 < 1$ is established by constructing Lyapunov functions. Next, we prove that the disease can die out exponentially under certain stochastic perturbation while it is persistent in the deterministic case when $R_0 > 1$. Finally, several examples and numerical simulations are provided to illustrate the dynamic behavior of the model and verify our analytical results.

1. Introduction

The history of human beings is full of struggle against diseases which cause great disaster to humans. At present, many countries and people around the world are suffering from the COVID-19, which has seriously affected people's lives and brought huge losses to the economy. Epidemiology is the subject to study the spread of diseases and formulate the strategies and measures for controlling and eliminating diseases. Mathematical modeling has been widely used in epidemiology to depict the mechanism of disease transmission and study the behavior of disease. One of the classic epidemic models is the SIR model which divides the host population into three parts, the susceptible, the infective, and the removed, and records their sizes by $S(t)$, $I(t)$, and $R(t)$ at time t, respectively. However, many diseases do not break out immediately, and there will be a latent period of time, so SEIR models with latent period have been widely studied. In SEIR models, the size of the exposed individuals is labeled by $E(t)$ at time t.

Many models have considered the case of only one group; however, groups in different communities, regions, or with different cultural backgrounds have various lifestyles, dietary habits, and so on, which will make the disease have different ways of transmission. Therefore, considering different contact patterns, transmission, or geographic distributions, it is more reasonable to divide the host population into several subgroups and study the disease interactions among different subgroups. This is known as the multigroup model. One of the earliest works on the multigroup disease model was done by Lajmanovich and Yorke [1], who discussed a class of SIS multigroup models for the transmission of gonorrhea and used Lyapunov functions to prove the stability of the unique endemic equilibrium. Since then, there has been a great quantity of literature on the multigroup model, such as [2–8].

In the classic SEIR models, the incidence function takes the bilinear form. A premise for this form is that the host population is homogeneously mixed, and everyone has the same possibility to be infected when the infectives are introduced to the group. In real life, however, the population may not be homogeneously mixed, and the immunity of each person may be different such that the chances of being infected are disparate, so extending bilinear incidence to nonlinear functions can conform to the actual situation better. Many scholars have studied the epidemic models with nonlinear incidence rate, such as [4, 8–11] and the reference therein. Also, many scholars investigated the epidemic

models with time delays, such as [12, 13]. In [4], the authors discussed the global stability of the multigroup epidemic model with nonlinear incidence rates of the form $f_{kj}(S_k, I_j)$, which satisfies the following assumptions:

(i) (H1) for $0 < S_k \leq S_k^0$, it has $0 < \lim_{I_j \to 0^+} (f_{kj}(S_k, I_j)/I_j) = C_{kj}(S_k)$, where S_k^0 is the positive solution of certain function.

(ii) (H2) $f_{kj}(S_k, I_j) \leq C_{kj}(S_k)I_j$ for any $I_j > 0$.

(iii) (H3) $C_{kj}(S_k) \leq C_{kj}(S_k^0)$, for $0 < S_k \leq S_k^0$.

This research intends to study this general form of incidence function and assumes further that $(C_{kj}(S_k)/S_k) \leq K$, where for K is a positive constant. The above incidence rate functions $f_{kj}(S_k, I_j)$ include some special cases which can be seen in some literature, for example,

$$
\begin{aligned}
f_{kj}(S_k, I_j) &= S_k I_j, \\
f_{kj}(S_k, I_j) &= S_k^q I_j, \quad q \geq 1, \\
f_{kj}(S_k, I_j) &= \frac{S_k I_j}{1 + \alpha I_j^2}, \\
f_{kj}(S_k, I_j) &= \frac{S_k I_j}{\varphi(I_j)}.
\end{aligned}
\tag{1}
$$

The multigroup SEIR model with above incidence functions can be obtained:

$$
\begin{cases}
\dfrac{dS_k}{dt} = \Lambda_k - \displaystyle\sum_{j=1}^n \beta_{kj} f_{kj}(S_k, I_j) - d_k^S S_k, \\[2mm]
\dfrac{dE_k}{dt} = \displaystyle\sum_{j=1}^n \beta_{kj} f_{kj}(S_k, I_j) - (\epsilon_k + d_k^E) E_k, \\[2mm]
\dfrac{dI_k}{dt} = \epsilon_k E_k - (\alpha_k + d_k^I + \gamma_k) I_k, \\[2mm]
\dfrac{dR_k}{dt} = \gamma_k I_k - d_k^R R_k.
\end{cases}
\tag{2}
$$

What the parameters mean can be summarized in the following list:

Λ_k: the influx of individuals in the kth group.

β_{kj}: the transmission rate between S_k and I_j.

$d_k^S, d_k^E, d_k^I,$ and d_k^R: the natural death rate of $S, E, I,$ and R in the kth group, respectively.

ϵ_k: the rate of becoming infectious in the kth group.

α_k: the death rate caused by disease in the kth group.

γ_k: the cure rate in the kth group.

The parameters above are all nonnegative. In particular, when $\beta_{kj} = 0$, it means that there is no disease transmission between S_k and I_j. The matrix $B = (\beta_{kj})_{n \times n}$ reflects transmission mechanism of disease among different subgroups built in the model. In this paper, we assume that the matrix B is irreducible.

Since that $R_k, k = 1, 2, \ldots, n$, do not appear in the first three equations of model (2) but only in the fourth equation, their properties and behaviors can be solved easily if $I_k, k = 1, 2, \ldots, n$, are known; they can be omitted when analyzed. Therefore, the model can be simplified into the following form:

$$
\begin{cases}
\dfrac{dS_k}{dt} = \Lambda_k - \displaystyle\sum_{j=1}^n \beta_{kj} f_{kj}(S_k, I_j) - d_k^S S_k, \\[2mm]
\dfrac{dE_k}{dt} = \displaystyle\sum_{j=1}^n \beta_{kj} f_{kj}(S_k, I_j) - (\epsilon_k + d_k^E) E_k, \\[2mm]
\dfrac{dI_k}{dt} = \epsilon_k E_k - (\alpha_k + d_k^I + \gamma_k) I_k.
\end{cases}
\tag{3}
$$

In the epidemic models, the basic reproduction number R_0, which represents the number of second generations produced by a single infected individual, plays an important role in the spread of disease for the long time. According to [4, 14], $R_0 = \rho(M_0)$, where $M_0 = (\beta_{kj} \epsilon_k C_{kj}(S_k^0)/(\alpha_k + d_k^I + \gamma_k)(\epsilon_k + d_k^E))_{n \times n}$, $S_k^0 = \Lambda_k/d_k^S$, and ρ is the spectral radius of the matrix M_0. If $R_0 < 1$, there is only disease-free equilibrium P_0, where $P_0 = ((\Lambda_1/d_1^S), 0, 0, \ldots, (\Lambda_n/d_n^S), 0, 0)$. When $R_0 > 1$, then P_0 is unstable, and the model has an endemic equilibrium P^* which means the disease will be persistent. In this situation, our concern is whether there is a way to exterminate the disease.

The reality is filled with randomness, and the epidemic models are often influenced by random environments. For example, there are a lot of natural disasters in reality, such as storm and earthquake. If these randomnesses happen, the parameters and the transmission mechanism in the model are likely to be affected. Thus, the deterministic model has some limitations to fully describe transmission of disease. Many scholars have studied the epidemic model with stochastic perturbations depicted by Brownian motion, and a lot of literature studies have been published; we refer the readers to [5, 7, 10, 12, 13, 15–17]. In [18–20], the authors studied the SIR or SIRS model with Markovian switching, and they gave some conditions on extinction or ergodicity of the model.

Influenced by the work of predecessors, we use the similar method of Dalal et al. and Witbooi [21, 22] to construct stochastic perturbations, that is, we replace the parameters d_k^E and d_k^I by $d_k^E - \sigma_{1k} dB_k$ and $d_k^I - \sigma_{2k} dW_k$, where the stochastic perturbations B_k and W_k are independent standard Brownian motions. The reason that not all parameters but only some of them are disturbed by stochastic perturbations may be the uncertainty of stochastic factors and the change of behavior of the infected.

For all we know, the papers that discuss asymptotic behaviors of stochastic multigroup SEIR models with nonlinear incidence rate functions are relatively few. In this paper, we will study the following stochastic multigroup SEIR model:

$$\begin{cases} \dot{S}_k = \left(\Lambda_k - \sum_{j=1}^{n} \beta_{kj} f_{kj}\left(S_k, I_j\right) - d_k^S S_k \right) dt, \\[2mm] \dot{E}_k = \left(\sum_{j=1}^{n} \beta_{kj} f_{kj}\left(S_k, I_j\right) - (\epsilon_k + d_k^E) E_k \right) dt + \sigma_{1k} E_k dB_k(t), \\[2mm] \dot{I}_k = \left(\epsilon_k E_k - (\alpha_k + d_k^I + \gamma_k) I_k \right) dt + \sigma_{2k} I_k dW_k(t), \end{cases}$$

$$(4)$$

where σ_{ik}, $i = 1, 2$, are the intensities of stochastic perturbation.

Because the incidence rate functions $f_{kj}(S_k, I_j)$ are general and can be of different types in one model, which increase the difficulty of research, we will overcome it by some inequality techniques. This paper is organized as follows. Section 2 presents some background knowledge and lemmas which will be used afterwards. In Section 3, we prove that there is a unique positive solution to the model for any initial value. Section 4 proves that the disease-free equilibrium is globally asymptotically stable in probability when $R_0 < 1$ by constructing Lyapunov functions. In Section 5, the disease will die out exponentially under certain stochastic perturbations when $R_0 > 1$, and in Section 6, we provide some numerical simulations of the model to verify our analytical results.

2. Preliminaries

Throughout the paper, unless otherwise specified, $(\Omega, \{\mathcal{F}_t\}_{t \geq 0}, P)$ denotes a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is right continuous, and $\mathcal{F}_0$ contains all P-null sets). Denote

$$\mathbb{R}_+^n = \{x \in R^n : x_i > 0 \quad \text{for all} \quad 1 \leq i \leq n\}. \tag{5}$$

In general, let X be a regular homogeneous Markov process in R^n; consider the stochastic differential equation

$$dX(t) = b(X(t))dt + \sum_{k=1}^{d} \sigma_k(X(t))dB_k(t), \tag{6}$$

with initial value $X(t_0) = x_0 \in R^n$ and $B_k(t), 1 \leq k \leq d$, are standard Brownian motions. Define the differential operator $\mathcal{L}$ associated with the above equation by

$$\mathcal{L} = \sum_{k=1}^{n} b_k(x) \frac{\partial}{\partial x_k} + \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\sigma^T(x)\sigma(x)\right]_{ij} \frac{\partial^2}{\partial x_i \partial x_j}. \tag{7}$$

If $\mathcal{L}$ acts on a function $V \in C^{2,1}(E^l \times \mathbb{R}_+; \mathbb{R}_+)$, then by Itô formula,

$$dV(X, t) = \mathcal{L}V(X, t)dt + \sum_{r=1}^{d} V_x(X, t)\sigma_r(X(t))dB_r(t). \tag{8}$$

where

$$\mathcal{L}V(X, t) = V_t(X, t) + \sum_{k=1}^{n} b_k(x) \frac{\partial V}{\partial x_k}$$

$$+ \frac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{n} \left[\sigma^T(x)\sigma(x)\right]_{ij} \frac{\partial^2 V}{\partial x_i \partial x_j}. \tag{9}$$

Next, we introduce some definitions about stability and lemmas which will be used latter. Assume that $b(0) = 0$ and $\sigma_k(0) = 0, k = 1, 2, \ldots, d$; then, $X(t) \equiv 0$ is the trivial solution to (6).

Definition 1. The trivial solution is called to be

(i) Stable in probability if for any $\epsilon > 0$ and the solution $X(t, x_0)$ with initial value $X(0) = x_0$, then

$$\lim_{x_0 \longrightarrow 0} \mathbb{P}\left(\sup_{t \geq 0} |X(t, x_0)| \geq \epsilon\right) = 0. \tag{10}$$

(ii) Globally asymptotically stable in probability if it is stable in probability, and for any $x_0 \in \mathbb{R}^n$,

$$\mathbb{P}\left(\lim_{t \longrightarrow \infty} X(t, x_0) = 0\right) = 1. \tag{11}$$

Lemma 1 (cf. [23]). *If there is a positive definite function $V(t, x) \in \mathscr{C}^2(\mathbb{R}^n)$ with an infinitesimal upper limit such that the function $\mathcal{L}V$ is negative definite, then the trivial solution is globally asymptotically stable in probability.*

Lemma 2 (Perron–Frobenius). *If $A = (a_{ij})_{n \times n}$ is irreducible and nonnegative, then the spectral radius $\rho(A)$ of A is a single eigenvalue, and there is a positive eigenvector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ corresponding to $\rho(A)$ of A. Moreover, $\rho(A)$ satisfies the inequality*

$$\min_i \sum_j a_{ij} \leq \rho(A) \leq \max_i \sum_j a_{ij}. \tag{12}$$

Remark 1. From our previous description in Introduction, we know that $R_0 = \rho(M_0) < 1$ will lead to the extinction of disease in deterministic model (3). Combining the expression of R_0 with the estimation of $\rho(M_0)$ in (12), we can infer that if transmission rate β_{kj} decreases, $\rho(M_0)$ will become smaller, which provides the possibility of eliminating disease. A very important way to reduce β_{kj} is to isolate people at home and restrict them from going out. This measure is being taken in many countries to combat COVID-19.

Lemma 3 (cf. [24]). *Let $M = \{M_t\}_{t \geq 0}$ be a real-valued continuous local martingale vanishing at $t = 0$. Then,*

$$\lim_{t \to \infty} \langle M, M \rangle_t = \infty \quad \text{a.s.} \Longrightarrow \lim_{t \to \infty} \frac{M_t}{\langle M, M \rangle_t} = 0 \text{ a.s.}$$

$$\limsup_{t \to \infty} \frac{\langle M, M \rangle_t}{t} < \infty \quad \text{a.s.} \Longrightarrow \lim_{t \to \infty} \frac{M_t}{t} = 0 \text{ a.s.}.$$

$$(13)$$

3. The Existence and Uniqueness of the Solution to Model (4)

The first question we concern is whether the system has a solution or not. In this section, we prove that the system has a global and positive solution for any initial value.

Theorem 1. *Given any initial value* $(S_1(0), E_1(0), I_1(0) \cdots S_n(0), E_n(0), I_n(0)) \in \mathbb{R}_+^{3n}$, *then model (4) has a unique solution on* $t \geq 0$, *and the solution will remain in* $\mathbb{R}_+^{3n}$ *with probability one, that is,* $(S_1(t), E_1(t), I_1(t) \cdots S_n(t), E_n(t), I_n(t)) \in \mathbb{R}_+^{3n}$ *for* $t \geq 0$ *almost surely.*

Proof. Since the coefficients of the model are locally Lipschitz continuous, there is a unique local solution $(S_1(t), E_1(t), I_1(t) \cdots S_n(t), E_n(t), I_n(t))$ on $t \in [0, \tau_e]$, where τ_e is the explosion time (cf. [24]). In order to illustrate the solution is global, we only need to prove $\tau_e = \infty$. Assume c_0 is sufficiently large so that $S_1(0), E_1(0), I_1(0) \cdots S_n(0), E_n(0), I_n(0)$ lie within the interval $[(1/c_0), c_0]$. For $c \geq c_0$, define the stopping time

$$\tau_c = \inf \left\{ t \in [0, \tau_e], \min_{1 \leq k \leq n} \{ S_k(t), E_k(t), I_k(t) \} \leq \frac{1}{c}, \right.$$

$$\left. \text{or} \quad \max_{1 \leq k \leq n} \{ S_k(t), E_k(t), I_k(t) \} \geq c \right\}.$$

$$(14)$$

We set $\inf \varnothing = \infty$ (where $\varnothing$ denotes the empty set). Clearly, τ_c is increasing as $c \to \infty$. Let $\tau_\infty = \lim_{c \to \infty} \tau_c$, and $\tau_\infty \leq \tau_e$ a.s. If we can prove $\tau_\infty = \infty$ a.s., then equality $\tau_e = \infty$ holds true, and the conclusion can be obtained. If the assertion is false, then there exist two constants $T > 0$ and $\epsilon \in (0, 1)$ such that

$$\mathbb{P}(\tau_\infty \leq T) > \epsilon. \tag{15}$$

Hence, there exists a positive integer $c_1 \geq c_0$ such that

$$\mathbb{P}(\tau_c \leq T) > \epsilon, \quad \text{for all} \quad c \geq c_1. \tag{16}$$

Then, we define a function $V: R_+^{3n} \longrightarrow R$ by

$$V(S_k, E_k, I_k) := \sum_{k=1}^{n} \left[\left(S_k - a_k - a_k \ln \frac{S_k}{a_k} \right) \right.$$

$$+ (E_k - 1 - \ln E_k) \tag{17}$$

$$\left. + (I_k - 1 - \ln I_k) \right],$$

where $a_k, k = 1, 2, \cdots n$, are constants which will be determined later. Using Itô's formula, we can get

$$dV = \sum_{k=1}^{n} \left(1 - \frac{a_k}{S_k} \right) \left[\left(\Lambda_k - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - d_k^S S_k \right) dt \right]$$

$$+ \frac{1}{2} \sum_{k=1}^{n} \left[\sigma_{1k}^2 + \sigma_{2k}^2 \right]$$

$$+ \sum_{k=1}^{n} \left(1 - \frac{1}{E_k} \right) \left[\left(\sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - (\epsilon_k + d_k^E) E_k \right) dt \right.$$

$$\left. + \sigma_{1k} E_k dB_k(t) \right]$$

$$+ \sum_{k=1}^{n} \left(1 - \frac{1}{I_k} \right) \left[\left(\epsilon_k E_k - (\alpha_k + d_k^I + \gamma_k) I_k \right) dt \right.$$

$$\left. + \sigma_{2k} I_k dW_k(t) \right] dt$$

$$= \mathcal{L} V dt + \sum_{k=1}^{n} \left[\sigma_{1k}(E_k - 1) dB_k(t) + \sigma_{2k}(I_k - 1) dW_k(t) \right],$$

$$(18)$$

where

$$\mathcal{L} V = \sum_{k=1}^{n} \left[\Lambda_k - d_k^S S_k - \frac{a_k}{S_k} \Lambda_k + \frac{a_k}{S_k} \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) \right.$$

$$+ a_k d_k^S - d_k^E E_k$$

$$- \frac{1}{E_k} \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - (\alpha_k + d_k^I + \gamma_k) I_k - \frac{\epsilon_k E_k}{I_k} \right]$$

$$+ \epsilon_k + d_k^E + \alpha_k + d_k^I + \gamma_k + \frac{1}{2} \sum_{k=1}^{n} \left[\sigma_{1k}^2 + \sigma_{2k}^2 \right]$$

$$\leq \sum_{k=1}^{n} \left[\Lambda_k + a_k \sum_{j=1}^{n} \beta_{kj} K I_j + a_k d_k^S - (\alpha_k + d_k^I + \gamma_k) I_k \right]$$

$$+ \epsilon_k + d_k^E + \alpha_k + d_k^I + \gamma_k + \frac{1}{2} \sum_{k=1}^{n} \left[\sigma_{1k}^2 + \sigma_{2k}^2 \right].$$

$$(19)$$

Notice that

$$\sum_{k=1}^{n} \sum_{j=1}^{n} a_k \beta_{kj} K I_j - \sum_{k=1}^{n} (\alpha_k + d_k^I + \gamma_k) I_k$$

$$= \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_k \beta_{kj} K \right) I_j - \sum_{j=1}^{n} (\alpha_j + d_j^I + \gamma_j) I_j \tag{20}$$

$$= \sum_{j=1}^{n} \left[\sum_{k=1}^{n} K a_k \beta_{kj} - (\alpha_j + d_j^I + \gamma_k) \right] I_j.$$

We choose appropriate numbers $a_k, 1 \leq k \leq n$, such that $\sum_{k=1}^{n} K a_k \beta_{kj} - (\alpha_j + d_j^I + \gamma_j) = 0$; then, $\mathscr{L}V \leq M$, where M is a positive constant. Therefore,

$$dV \leq Mdt + \sum_{k=1}^{n} \left[\sigma_{1k}(E_k - 1)dB_k(t) + \sigma_{2k}(I_k - 1)dW_k(t) \right].$$

(21)

Integrate both sides of (21) from 0 to $\tau_c \wedge T$ and take expectation; then,

$$\mathbb{E}V\left(S_k(\tau_c \wedge T), E_k(\tau_c \wedge T), I_k(\tau_c \wedge T)\right)$$

$$\leq V\left(S_k(0), E_k(0), I_k(0)\right) + \mathbb{E} \int_0^{\tau_c \wedge T} M dt$$

(22)

$$\leq V\left(S_k(0), E_k(0), I_k(0)\right) + MT.$$

Set $\Omega_c = \{\tau_c \leq T\}$; we have $\mathbb{P}(\Omega_c) \geq \epsilon$. Notice that, for every $\omega \in \Omega_c$, there exists at least one of $S(\tau_c, \omega)$, $E(\tau_c, \omega), I(\tau_c, \omega)$ which equals either c or $1/c$. Therefore,

$$V\left(S_k(\tau_c \wedge T), E_k(\tau_c \wedge T), I_k(\tau_c \wedge T)\right)$$

$$\geq \min_{1 \leq k \leq n} \left\{ c - a_k - a_k \ln \frac{c}{a_k}, \frac{1}{c} - a_k - a_k \ln \frac{1}{a_k c} \right\}$$

(23)

$$\wedge \left(c - 1 - \ln c\right) \wedge \left(\frac{1}{c} - 1 - \ln \frac{1}{c}\right).$$

Combining (22) with (23), we can obtain that

$$V\left(S_k(0), E_k(0), I_k(0)\right) + MT \geq \mathbb{E}\left[1_{\Omega_c(\omega)} V\left(S_k(\tau_c \wedge T), E_k(\tau_c \wedge T), I_k(\tau_c \wedge T)\right)\right]$$

$$\geq \epsilon \left\{ \min_{1 \leq k \leq n} \left(c - a_k - a_k \ln \frac{c}{a_k}, \frac{1}{c} - a_k - a_k \ln \frac{1}{a_k c} \right) \right.$$

(24)

$$\left. \wedge \left(c - 1 - \ln c\right) \wedge \left(\frac{1}{c} - 1 - \ln \frac{1}{c}\right) \right\},$$

where $1_{\Omega_m(\omega)}$ is the indicator function of Ω_m. Letting $m \longrightarrow \infty$ leads to the contradiction that $\infty > V\left(S_k(0), E_k(0), I_k(0)\right) + MT = \infty$. So, $\tau_e = \infty$ a.s. The proof is completed. □

Corollary 1. *For S_k, $k = 1, 2 \cdots n$, in model (4), there exists a set of M_k such that $S_k \leq M_k$. Furthermore, the set $\Gamma = \{S_k: S_k > 0, S_k \leq (\Lambda_k/d_k^S)\}$ is the invariant set, that is to say, if the initial value $S_k(0) \in \Gamma$, then $S_k(t) \in \Gamma$, for $t \geq 0$ almost surely.*

Proof. For the first equation of model (4), we have $dS_k \leq (\Lambda_k - d_k^S S_k)dt$. By the method of variation of constants, we get that

$$S_k(t) \leq \frac{\Lambda_k}{d_k^S} + \left(S_k(0) - \frac{\Lambda_k}{d_k^S}\right)e^{-d_k^S t}.$$

(25)

If $S_k(0) \leq (\Lambda_k/d_k^S)$, then $S_k(t) \leq (\Lambda_k/d_k^S)$. If $S_k(0) > (\Lambda_k/d_k^S)$, then $S_k(t) \leq S_k(0)$. Let $M_k = \max\{(\Lambda_k/d_k^S), S_k(0)\}$. The proof is complete.

The assumption $S_k(0) \leq (\Lambda_k/d_k^S)$ will be used in the rest of the paper. □

4. The Behavior of the Model with $R_0 < 1$

In the deterministic SEIR model, P_0 is the disease-free equilibrium, and it is globally stable which means that the disease will die out with any initial value when $R_0 < 1$. In this section, we will discuss the asymptotic behavior of the stochastic model with $R_0 < 1$.

Theorem 2. *Let $(S_1(t), E_1(t), I_1(t), \ldots, S_n(t), E_n(t), I_n(t))$ be the solution to model (4) with the initial valueinitial value $(S_1(0), E_1(0), I_1(0), \ldots, S_n(0), E_n(0), I_n(0) \in \mathbb{R}_+^{3n}$. If $B = (\beta_{kj})_{n \times n}$ is irreducible and $R_0 = \rho(M_0) < 1$, then P_0 is the unique equilibrium of model (4), and it is globally asymptotically stable in probability.*

Proof. According to the assumption, B is irreducible and nonnegative; then, by Lemma 2, M_0 has a single eigenvalue $\rho(M_0)$ and a positive eigenvector $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$ corresponding to $\rho(M_0)$ such that

$$(\omega_1, \omega_1, \ldots, \omega_n) M_0 = (\omega_1, \omega_1, \ldots, \omega_n) \rho(M_0). \qquad (26)$$

Let $V_1 = \sum_{k=1}^{n} (1/2) a_k ((\Lambda_k/d_k^S) - S_k)^2$ and $V_2 = \sum_{k=1}^{n} c_k$ $(E_k + ((\epsilon_k + d_k^E)/\epsilon_k) I_k)$, where $c_k = \omega_k \epsilon_k / (d_k^E + \epsilon_k) (\alpha_k$

$+ d_k^I + \gamma_k)$ and a_k will be determined later. Using Itô's formula, we can obtain that

$$
\begin{aligned}
\mathscr{L}V_1 &= -\sum_{k=1}^{n} a_k \left(\frac{\Lambda_k}{d_k^S} - S_k \right) \left[\Lambda_k - \sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - d_k^S S_k \right] \\
&\leq -\sum_{k=1}^{n} a_k d_k^S \left(\frac{\Lambda_k}{d_k^S} - S_k \right)^2 + \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \beta_{kj} C_{kj}(S_k) I_j \left(S_k - \frac{\Lambda_k}{d_k^S} \right) \\
&\leq -\sum_{k=1}^{n} a_k d_k^S \left(\frac{\Lambda_k}{d_k^S} - S_k \right)^2 + \frac{\epsilon}{2} \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \beta_{kj} C_{kj}(S_k^0) \left(\frac{\Lambda_k}{d_k^S} - S_k \right)^2 \\
&\quad + \frac{1}{2\epsilon} \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \beta_{kj} C_{kj}(S_k^0) I_j^2 \\
&= -\sum_{k=1}^{n} a_k \left[d_k^S - \frac{\epsilon}{2} \sum_{j=1}^{n} \beta_{kj} C_{kj}(S_k^0) \right] \left(\frac{\Lambda_k}{d_k^S} - S_k \right)^2 \\
&\quad + \frac{1}{2\epsilon} \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \beta_{kj} C_{kj}(S_k^0) I_j^2.
\end{aligned}
\tag{27}
$$

Here, the second inequality holds true because of the inequality $ab \leq (\epsilon/2) a^2 + (1/2\epsilon) b^2$. Similarly, we use Itô's formula to V_2 to get

$$
\begin{aligned}
\mathscr{L}V_2 &= \sum_{k=1}^{n} c_k \left[\sum_{j=1}^{n} \beta_{kj} f_{kj}(S_k, I_j) - \frac{(\epsilon_k + d_k^E)(\alpha_k + d_k^I + \gamma_k)}{\epsilon_k} I_k \right] \\
&\leq \sum_{j=1}^{n} \sum_{k=1}^{n} \omega_k \frac{\epsilon_k \beta_{kj} C_{kj}(S_k^0)}{(\epsilon_k + d_k^E)(\alpha_k + d_k^I + \gamma_k)} I_j - \sum_{k=1}^{n} \omega_k I_k \\
&= (\rho_0 - 1) \sum_{k=1}^{n} \omega_k I_k.
\end{aligned}
\tag{28}
$$

Define Lyapunov function $V(t)$ by $V(t) = V_1(t) + V_2(t)$, and according to Theorem 1, $V(t)$ is positive definite; then,

$$
\begin{aligned}
\mathscr{L}V &\leq -\sum_{k=1}^{n} a_k \left[d_k^S - \frac{\epsilon}{2} \sum_{j=1}^{n} \beta_{kj} C_{kj}(S_k^0) \right] \left(\frac{\Lambda_k}{d_k^S} - S_k \right)^2 \\
&\quad + \frac{1}{2\epsilon} \sum_{k=1}^{n} \sum_{j=1}^{n} a_k \beta_{kj} C_{kj}(S_k^0) I_j^2 + (\rho_0 - 1) \sum_{k=1}^{n} \omega_k I_k.
\end{aligned}
\tag{29}
$$

We can choose small ϵ such that $d_k^S - (\epsilon/2) \sum_{j=1}^{n} \beta_{kj}$ $C_{kj}(S_k^0) > 0$, a_k are chosen to be sufficiently small, and because of $R_0 < 1$, we have $\mathscr{L}V < 0$. Hence, applying Lemma 1, we arrive at the desired assertion. The proof is completed. □

5. The Influence of Large Noise on Disease

In this section, we will discuss the influence of large noises on disease when $R_0 > 1$. Before we give the theorem, an inequality is presented first.

Lemma 4. For a_k, b_k, c_k, d_k, $k = 1, 2 \cdots, n$, the following inequality holds true:

$$
\left[\sum_{k=1}^{n} (a_k b_k + c_k d_k) \right]^2 \leq \left(\sum_{k=1}^{n} a_k^2 + \sum_{k=1}^{n} c_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 + \sum_{k=1}^{n} d_k^2 \right).
\tag{30}
$$

Proof. We prove it by transforming it into an inner product in space $\mathbb{R}^n$. Let $\mathbf{a} := (a_1, a_2, \ldots, a_n)^T \in \mathbb{R}^n$, and the vectors $\mathbf{b}$, $\mathbf{c}$, and $\mathbf{d}$ are defined in a similar way. Then,

$$
\begin{aligned}
\left(\mathbf{a}^T \mathbf{b} + \mathbf{c}^T \mathbf{d} \right)^2 &= ((\mathbf{a}, \mathbf{b}) + (\mathbf{c}, \mathbf{d}))^2 \leq \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \|\mathbf{d}\|^2 \\
&\quad + 2\|\mathbf{a}\| \|\mathbf{b}\| \|\mathbf{c}\| \|\mathbf{d}\| \\
&\leq \|\mathbf{a}\|^2 \|\mathbf{b}\|^2 + \|\mathbf{c}\|^2 \|\mathbf{d}\|^2 + \|\mathbf{a}\|^2 \|\mathbf{c}\|^2 + \|\mathbf{b}\|^2 \|\mathbf{d}\|^2 \\
&\leq \left(\|\mathbf{a}\|^2 + \|\mathbf{c}\|^2 \right) \left(\|\mathbf{b}\|^2 + \|\mathbf{d}\|^2 \right).
\end{aligned}
\tag{31}
$$

The proof is completed. □

Theorem 3. If $B = (\beta_{kj})_{1 \leq k, j \leq n}$ is irreducible, then we have

$$\max_{1 \le k \le n} \left\{ \lim_{t \to \infty} \sup \frac{1}{t} \ln E_k(t), \lim_{t \to \infty} \sup \frac{1}{t} \ln I_k(t) \right\}$$

$$\le (R_0 - 1) \max_{1 \le k \le n} \left\{ \alpha_k + d_k^I + \gamma_k \right\} \tag{32}$$

$$- \frac{1}{2 \sum_{i=1}^n \left(\left(1/\sigma_{1k}^2 \right) + \left(1/\sigma_{2k}^2 \right) \right)}, \quad \text{a.s.}$$

Proof. We define a C^2 function $V(E_k, I_k)$ by

$$V = \sum_{k=1}^n c_k \left(E_k + \frac{d_k^E + \epsilon_k}{\epsilon_k} I_k \right), \tag{33}$$

where $c_k = \omega_k \epsilon_k / (d_k^E + \epsilon_k)(\alpha_k + d_k^I + \gamma_k)$. By calculation, we can get that

$$dV = \left[\sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} f_{kj}(S_k, I_j) - \sum_{k=1}^n \omega_k I_k \right] dt$$

$$+ \sum_{k=1}^n c_k \sigma_{1k} E_k dB_k(t) \tag{34}$$

$$+ \sum_{k=1}^n c_k \sigma_{2k} \frac{d_k^E + \epsilon_k}{\epsilon_k} I_k dW_k(t).$$

Using Ito's formula, we arrive at

$$d \ln V = \frac{1}{V} \left[\sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} f_{kj}(S_k, I_j) - \sum_{k=1}^n \omega_k I_k \right] dt$$

$$- \frac{1}{2V^2} \sum_{k=1}^n c_k^2 \left[\sigma_{1k}^2 E_k^2 + \sigma_{2k}^2 \frac{(d_k^E + \epsilon_k)^2}{\epsilon_k^2} I_k^2 \right] dt$$

$$+ \frac{1}{V} \sum_{k=1}^n c_k \sigma_{1k} E_k dB_k(t) + \frac{1}{V} \sum_{k=1}^n c_k \sigma_{2k} \frac{d_k^E + \epsilon_k}{\epsilon_k} I_k dW_k(t).$$

$$\le \frac{1}{V} \left[\sum_{k=1}^n \sum_{j=1}^n c_k \beta_{kj} C_{kj}(S_k^0) I_j - \sum_{k=1}^n \omega_k I_k \right] dt$$

$$- \frac{1}{2V^2} \sum_{k=1}^n c_k^2 \left[\sigma_{1k}^2 E_k^2 + \sigma_{2k}^2 \frac{(d_k^E + \epsilon_k)^2}{\epsilon_k^2} I_k^2 \right] dt$$

$$+ \frac{1}{V} \sum_{k=1}^n c_k \sigma_{1k} E_k dB_k(t) + \frac{1}{V} \sum_{k=1}^n c_k \sigma_{2k} \frac{d_k^E + \epsilon_k}{\epsilon_k} I_k dW_k(t).$$

$$=: V_1(t) dt + V_2(t) dt + V_3(t) + V_4(t). \tag{35}$$

For $V_1(t)$, from the expression of eigenvector of R_0, i.e., $(\omega_1, \omega_1 \cdots \omega_n) M_0 = R_0 (\omega_1, \omega_1 \cdots \omega_n)$, we obtain that

$$V_1(t) = \frac{1}{V} (R_0 - 1) \sum_{k=1}^n \omega_k I_k$$

$$\le \frac{R_0 - 1}{\sum_{k=1}^n c_k (d_k^E + \epsilon_k/\epsilon_k) I_k} \sum_{k=1}^n \omega_k I_k \tag{36}$$

$$\le \max_{1 \le k \le n} \left\{ \alpha_k + d_k^I + \gamma_k \right\} (R_0 - 1).$$

According to the expression of V, utilizing Lemma 4 yields

$$V^2 = \left[\sum_{k=1}^n \left(c_k \sigma_{1k} E_k \frac{1}{\sigma_{1k}} + c_k \frac{\sigma_{2k}(d_k^E + \epsilon_k) I_k}{\epsilon_k} \frac{1}{\sigma_{2k}} \right) \right]^2$$

$$\le \left[\sum_{k=1}^n c_k^2 \left(\sigma_{1k}^2 E_k^2 + \frac{\sigma_{2k}^2 (d_k^E + \epsilon_k)^2 I_k^2}{\epsilon_k^2} \right) \right] \left[\sum_{k=1}^n \left(\frac{1}{\sigma_{1k}^2} + \frac{1}{\sigma_{2k}^2} \right) \right]. \tag{37}$$

Hence, $V_2(t)$ satisfies the inequality

$$V_2(t) \le \frac{1}{2 \sum_{k=1}^n \left(\left(1/\sigma_{1k}^2 \right) + \left(1/\sigma_{2k}^2 \right) \right)}. \tag{38}$$

Because

$$\lim_{t \to \infty} \sup \frac{1}{t} \int_0^t \frac{\sum_{k=1}^n c_k^2 \sigma_{1k}^2 E_k^2}{V^2} dt < \infty, \tag{39}$$

applying Lemma 3 to $V_3(t)$ yields

$$\lim_{t \to \infty} \sup \frac{1}{t} \int_0^t \sum_{k=1}^n V^{-1} c_k \sigma_{1k} E_k dB_k(t) = 0. \tag{40}$$

$V_4(t)$ can be done in the same way. Therefore,

$$\lim_{t \to \infty} \sup \frac{\ln V(t)}{t} \le \max_{1 \le k \le n} \left\{ d_k^I + \gamma_k \right\} (R_0 - 1)$$

$$- \frac{1}{2 \sum_{k=1}^n \left(\left(1/\sigma_{1k}^2 \right) + \left(1/\sigma_{2k}^2 \right) \right)}. \tag{41}$$

Since

$$\frac{1}{t} \ln \left(\sum_{k=1}^n c_k E_k \right) \le \frac{1}{t} \left[\ln \left(\max_{1 \le k \le n} \{c_k\} n \max_{1 \le k \le n} \{E_k\} \right) \right]$$

$$\le \frac{1}{t} \left[\ln \left(\max_{1 \le k \le n} \{c_k\} n \right) + \ln \left(\max_{1 \le k \le n} \{E_k\} \right) \right],$$

$$\frac{1}{t} \ln \left(\sum_{k=1}^n c_k E_k \right) \ge \frac{1}{t} \left[\ln \left(\min_{1 \le k \le n} \{c_k\} \max_{1 \le k \le n} \{E_k\} \right) \right]$$

$$\ge \frac{1}{t} \left[\ln \left(\min_{1 \le k \le n} \{c_k\} \right) + \ln \left(\max_{1 \le k \le n} \{E_k\} \right) \right], \tag{42}$$

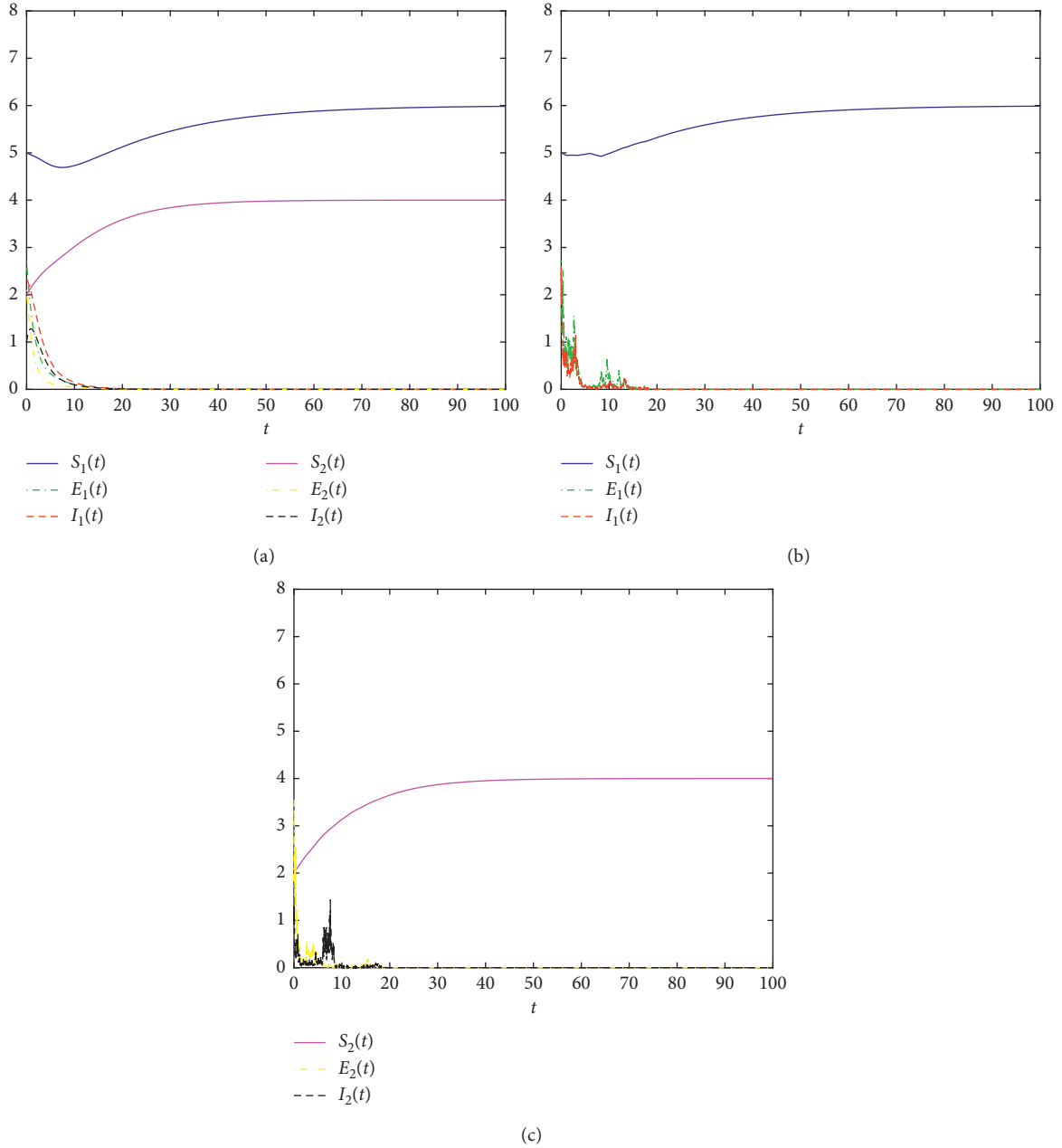taking the upper limit yields

(a)

(b)

(c)

Figure 1: The trajectories with $R_0 < 1$ and initial value $S_1(0) = 5, E_1(0) = 2.7, I_1(0) = 2.3; S_2(0) = 2, E_2(0) = 2.1, I_2(0) = 1$: (a) the trajectory without stochastic perturbation; (b, c) the trajectories with stochastic perturbations where parameters are shown in Example 1.

$$\limsup_{t \longrightarrow \infty} \frac{1}{t} \ln \left(\sum_{k=1}^{n} c_k E_k \right) = \max_{1 \le k \le n} \left\{ \limsup_{t \longrightarrow \infty} \frac{1}{t} \ln E_k \right\}. \quad (43)$$

Making use of the same method, we can obtain that

$$\limsup_{t \longrightarrow \infty} \frac{1}{t} \ln \left(\sum_{k=1}^{n} \frac{c_k \left(d_k^E + \epsilon_k \right)}{\epsilon_k} I_k \right) = \max_{1 \le k \le n} \left\{ \limsup_{t \longrightarrow \infty} \frac{1}{t} \ln I_k \right\}. \quad (44)$$

Combining (43) and (44) yields

$$\limsup_{t \longrightarrow \infty} \frac{1}{t} \ln V(t) \ge \max_{1 \le k \le n} \left\{ \limsup_{t \longrightarrow \infty} \frac{1}{t} \ln E_k, \limsup_{t \longrightarrow \infty} \frac{1}{t} \ln I_k \right\}. \quad (45)$$

Along with (41), we arrive at the desired assertion. The proof is complete. □
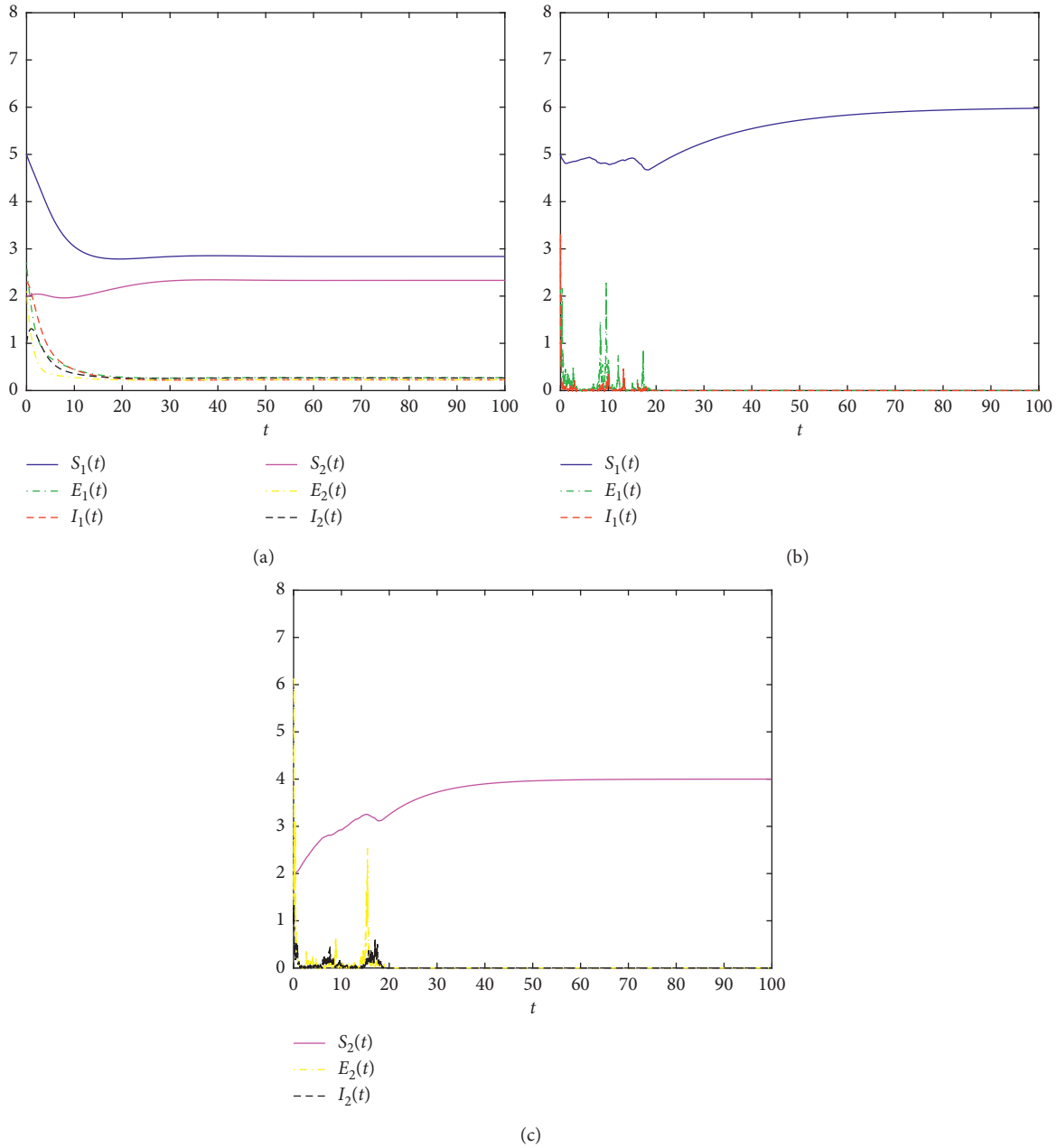
(a)

(b)

(c)

FIGURE 2: The trajectories with $R_0 > 1$: (a) the trajectory without stochastic perturbation; (b, c) the trajectories with stochastic perturbations where parameters are shown in Example 2.

Corollary 2. *For the solution to model (4), $E_k(t)$ and $I_k(t)$, $k = 1, 2 \cdots n$, decay exponentially to zero almost surely if*

$$(R_0 - 1) \max_{1 \le k \le n} \left\{ \alpha_k + d_k^I + \gamma_k \right\} < \frac{1}{2 \sum_{i=1}^{n} \left(\left(1/\sigma_{1k}^2 \right) + \left(1/\sigma_{2k}^2 \right) \right)}. \tag{46}$$

Remark 2. From (46), we know that the right side of the inequality increases with the increase of σ_{1k} and σ_{2k}; therefore, the inequality above holds true for certain α_k, d_k^I, γ_k, and sufficiently large σ_{1k} and σ_{2k} even if $R_0 > 1$, which

makes the disease extinct. It reflects that stochastic perturbations play an important role in disease control. Compared with the deterministic model in [4], the SEIR model with stochastic perturbations can show more properties and different behaviors.

Remark 3. We can see from many literature studies that the incidence function of the multigroup SEIR model is single one, such as $S_k(t)I_j(t)$ in [5, 7] and $S_k(t)I_j(t)/(1 + \alpha_k I_j(t))$ in [9]. These may have some limitations and cannot reflect the actual situation well. Incidence functions $f_{kj}(S_k(t), I_j(t))$ in this paper can be expressed in different forms, which can better
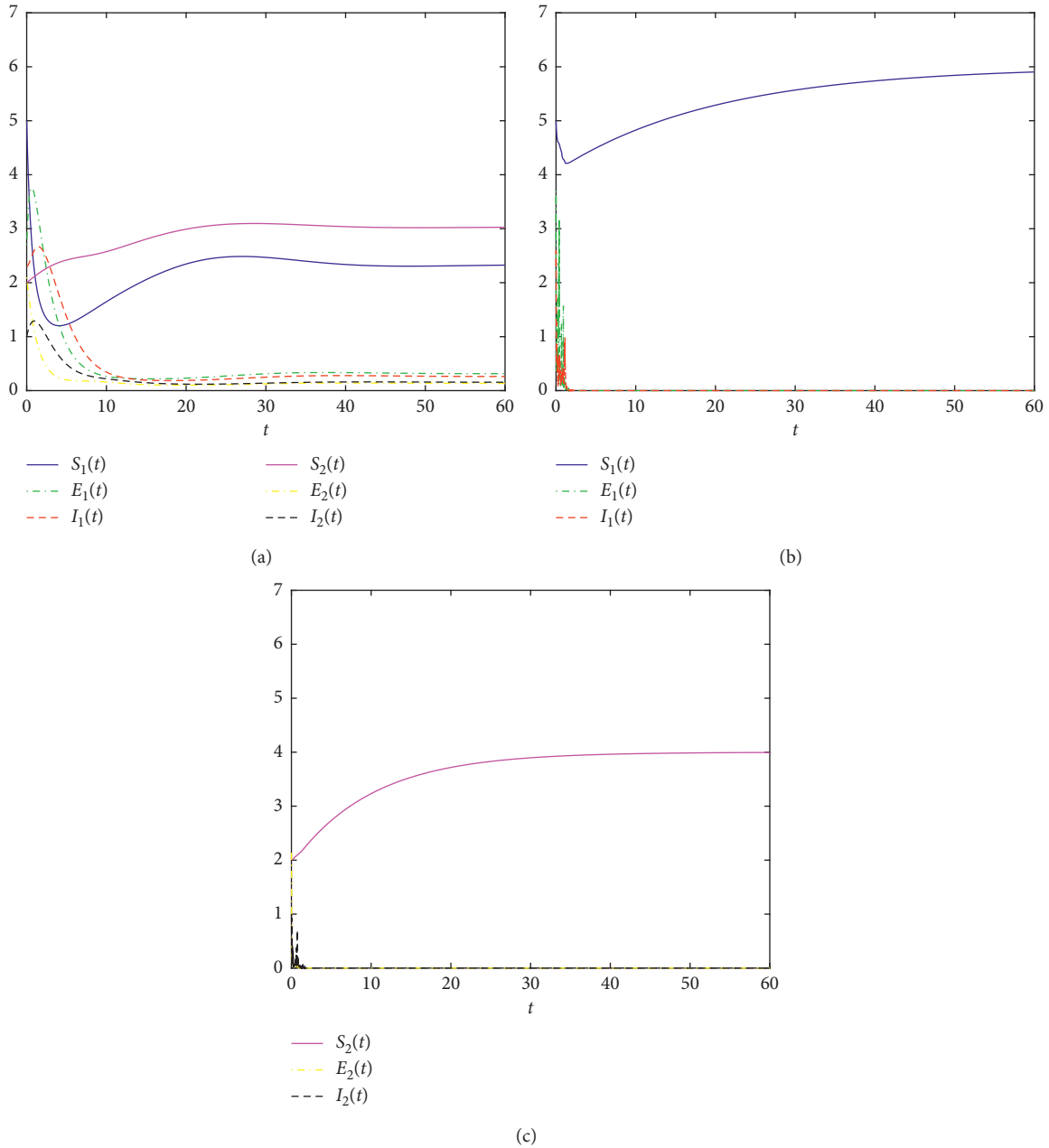
(a)

(b)

(c)

FIGURE 3: The trajectories with $R_0 > 1$ and two different incidence functions: (a) the trajectory without stochastic perturbation; (b, c) the trajectories with stochastic perturbations where parameters are shown in Example 3.

describe the reality of life. We will provide different examples to illustrate the results in Section 6.

6. Examples and Numerical Simulations

In this section, we give some simulations of model (4) to confirm the analytical results above. By using Milstein's higher-order method [25], we obtain the corresponding discretization equation:

$$
\begin{cases}
S_{i,k+1} = S_{i,k} + \left(\Lambda_i - \sum_{j=1}^{n} \beta_{ij} f_{kj}(S_{i,k}, I_{j,k}) - d_i^S S_{i,k} \right)\Delta t, \\[3mm]
E_{i,k+1} = E_{i,k} + \left(\sum_{j=1}^{n} \beta_{ij} f_{kj}(S_{i,k}, I_{j,k}) - (\epsilon_i + d_i^E)E_{i,k} \right)\Delta t + \sigma_{1i} E_{i,k}\eta_{i,k}\sqrt{\Delta t} \\[2mm]
\quad + \dfrac{1}{2}\sigma_{1i}^2 E_{i,k}\left(\eta_{i,k}^2\Delta t - \Delta t\right), \\[3mm]
I_{i,k+1} = I_{i,k} + \left(\epsilon_i E_{i,k} - (\alpha_i + d_i^I + \gamma_i)I_{i,k}\right)\Delta t + \sigma_{2i}I_{i,k}\rho_{i,k}\sqrt{\Delta t} + \dfrac{1}{2}\sigma_{2i}^2 I_{i,k}\left(\rho_{i,k}^2\Delta t - \Delta t\right),
\end{cases}
\tag{47}
$$

where η_{ik}, ρ_{ik} are Gaussian random variables which follow the distribution $N(0,1)$. Let $n = 2$, i.e., we consider the interactions of diseases in two groups.

First, we give an example to verify Theorem 2.

Example 1. Assume that $f_{kj} = S_k I_j / (1 + 2I_j^2)$. We choose $\Lambda_1 = 0.3, \Lambda_2 = 0.4; \epsilon_1 = 0.5, \epsilon_2 = 0.6; \gamma_1 = 0.4, \gamma_2 = 0.3; \alpha_1 = 0.1, \alpha_2 = 0.07; \beta_{11} = 0.02, \beta_{12} = 0.05, \beta_{21} = 0.04, \beta_{22} = 0.02, d_1^S = 0.05, d_2^S = 0.1, d_1^E = 0.08, d_2^E = 0.12, d_1^I = 0.1, d_2^I = 0.15$ such that $R_0 = 0.4835 < 1$, which satisfies the condition of Theorem 2. Moreover, let $\sigma_{11} = 1$, $\sigma_{12} = 0.5$, $\sigma_{21} = 0.8$, and $\sigma_{22} = 1.5$. Its trajectory is shown in Figure 1.

From Figure 1(a), we can see that the diseases are extinct when stochastic perturbations are absent. From Figures 1(b) and 1(c), we can see the diseases in two groups are globally asymptotically stable.

Now, we move forward to verify Theorem 3. We will present two examples to illustrate the two cases of incidence functions. In Example 2, we give the same incidence function for two groups, and in Example 3, two different incidence functions are presented.

Example 2. Assume that $f_{kj} = S_k I_j / (1 + 2I_j^2)$. We choose $\Lambda_1 = 0.3, \Lambda_2 = 0.4; \epsilon_1 = 0.5, \epsilon_2 = 0.6; \gamma_1 = 0.4, \gamma_2 = 0.3; \alpha_1 = 0.1, \alpha_2 = 0.07; \beta_{11} = 0.1, \beta_{12} = 0.05, \beta_{21} = 0.12, \beta_{22} = 0.2, d_1^S = 0.05, d_2^S = 0.1, d_1^E = 0.08, d_2^E = 0.12, d_1^I = 0.1, d_2^I = 0.15$ such that $R_0 = 1.685 > 1$, which satisfies the condition of Theorem 3. Moreover, let $\sigma_{11} = 1, \sigma_{12} = 0.5, \sigma_{21} = 0.8, \sigma_{22} = 1.5$ so that $(R_0 - 1)\max_{1 \le k \le n}\{\alpha_k + d_k^I + \gamma_k\} < 1/2 \sum_{i=1}^{n}((1/\sigma_{1k}^2) + (1/\sigma_{2k}^2))$ is satisfied. Its trajectory is shown in Figure 2. From Figure 2(a), we can see that the diseases are persistent because of $R_0 > 1$ when stochastic perturbation is absent. We can see in Figures 2(b) and 2(c) that the diseases in two groups die out under certain stochastic perturbations and the exposed are the same results.

Example 3. Assume that $f_{1j} = S_1^2 I_j$, $f_{2j} = S_k I_j / (1 + 2I_j^2)$, $j = 1, 2$, such that

$$
M_0 = \begin{pmatrix}
\dfrac{\beta_{11}\epsilon_1(\Lambda_1/d_1^S)^2}{(\alpha_1 + d_1^I + \gamma_1)(\epsilon_1 + d_1^E)} & \dfrac{\beta_{12}\epsilon_1(\Lambda_1/d_1^S)^2}{(\alpha_1 + d_1^I + \gamma_1)(\epsilon_1 + d_1^E)} \\[5mm]
\dfrac{\beta_{21}\epsilon_2(\Lambda_2/d_2^S)}{(\alpha_2 + d_2^I + \gamma_2)(\epsilon_2 + d_2^E)} & \dfrac{\beta_{22}\epsilon_2(\Lambda_2/d_2^S)}{(\alpha_2 + d_2^I + \gamma_2)(\epsilon_2 + d_2^E)}
\end{pmatrix}.
\tag{48}
$$

We choose $\Lambda_1 = 0.3, \Lambda_2 = 0.4; \epsilon_1 = 0.5, \epsilon_2 = 0.6; \gamma_1 = 0.4, \gamma_2 = 0.3; \alpha_1 = 0.1, \alpha_2 = 0.07; \beta_{11} = 0.04, \beta_{12} = 0.15, \beta_{21} = 0.1, \beta_{22} = 0.05, d_1^S = 0.05, d_2^S = 0.1, d_1^E = 0.08, d_2^E = 0.12, d_1^I = 0.1, d_2^I = 0.15$ so that $R_0 = 3.61 > 1$ can be obtained. Moreover, let $\sigma_{11} = 4, \sigma_{12} = 4.5, \sigma_{21} = 2.8$, and $\sigma_{22} = 5$; then, the conditions in Theorem 3 are satisfied. Its trajectory is shown in Figure 3. From Figure 3(a), we can see that the diseases are persistent because $R_0 > 1$ without stochastic perturbation. We can see in Figures 3(b) and 3(c) that the exposed and infected in two groups die out under certain stochastic perturbations, which conform to the results of Theorem 3.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

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