Research Article

Existence of Periodic Solutions of Seasonally Forced SEIR Models with Pulse Vaccination

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In this paper, we are interested in finding the periodic oscillation of seasonally forced SEIR models with pulse vaccination. Many infectious diseases show seasonal patterns of incidence. Pulse vaccination strategy is an effective tool to control the spread of these infectious diseases. Assuming that the seasonally dependent transmission rate is a T -periodic forcing, we obtain the existence of positive T -periodic solutions of seasonally forced SEIR models with pulse vaccination by Mawhin's coincidence degree method. Some relevant numerical simulations are presented to illustrate the effectiveness of such pulse vaccination strategy.

1. Introduction

It is a common phenomenon that the incidence of many infectious diseases often changes periodically with the seasonal cycle, such as measles, chickenpox, mumps, rubella, pertussis, and influenza [1–3]. In order to understand the mechanisms responsible for seasonal disease incidence and the epidemiological consequences of seasonality, a large number of mathematical models of infectious diseases with periodic transmission rates have been established [4–7]. Dietz [8] was the first to investigate the effects of one-year periodic contact rate in the classical SIR and SEIR models. Dietz considered a periodical contact rate given by

$$\beta(t) = \beta_m(1 + A \cos(\omega t)). \quad (1)$$

The periodically forced nonlinear effects in epidemic models have been studied extensively in the mathematical literature [9, 10].

Pulse vaccination strategy (PVS) is an effective tool to control the spread of epidemics, for example, control of poliomyelitis and measles in Central [11] and South American [12] and the UK vaccination campaigns against measles in 1994 [13]. The theoretical study on pulse vaccination strategy was firstly presented by Agur et al. [14]. Shulgin et al. [15, 16] incorporated pulse vaccination into the

SIR epidemic model. Nokes and Swinton studied the control of childhood viral infections by pulse vaccination [17]. d'Onofrio applied the pulse vaccination method for SIR and SEIR epidemic models [18, 19]. PVS consists of periodical repetitions of impulsive multiage cohort vaccinations in a population [18, 19]. PVS proposes to vaccinate a constant fraction p of the entire susceptible people in a single pulse, which can be formulated as

$$S(t_i^+) := \lim_{h \rightarrow 0^+} S(t_i + h) = S(t_i) - pS(t_i), \quad (2)$$

where $S(t)$ is left continuous satisfying $S(t_i) = S(t_i^-) := \lim_{h \rightarrow 0^-} S(t_i + h)$. Pulse vaccination gives life-long immunity to pS susceptibles who are transferred to the “recovered” class of the population, which can be formulated as

$$R(t_i^+) = R(t_i) + pS(t_i). \quad (3)$$

This kind of vaccination is called impulsive since all the vaccine doses are applied in a time which is short considering the dynamics of the disease. PVS has been further developed, for example, in [20–23]. A comprehensive introduction on vaccination strategies can be found in [24].

Many infectious diseases do not die out, but become endemic. For autonomous epidemic models, the existence of the positive equilibrium plays an important role. A positive

periodic solution in the periodic model will play the same role as a positive equilibrium in the autonomous model [25, 26]. Recently, Katriel [27] proved that the seasonally forced SIR model with a T -periodic forcing has a periodic solution with period T by Leray–Schauder degree theory provided $(1/T) \int_0^T \beta(t) dt > \gamma + \mu$. Jódar et al. [26] obtained that a T -periodic solution exists for a more general system by the famous Mawhin's coincidence degree method, whenever the condition $\min_{t \in \mathbb{R}} \beta(t) > \gamma + \mu$ holds. Using Leray–Schauder degree theory, Zu and the author [28] established new results on the existence of at least one positive periodic solution for a seasonally forced SIR model with impact of media coverage. The author [29] proved the existence of positive periodic solutions of seasonally forced SIR models with impulse vaccination at fixed time by Mawhin's coincidence degree method if the basic reproductive number $\mathcal{R}_0 = (\beta/(\gamma + \mu)) > 1$. Coincidence degree theory has been applied to prove the existence of multiple periodic solutions of the epidemic model with seasonal periodic rate [30, 31]. There are some research activities about the existence of periodic solutions of impulsive differential equation [32–36].

The aim of this paper is to study the existence of periodic solution of seasonally forced SEIR models with pulse vaccination. The paper is organized as follows. A seasonally forced SEIR model with pulse vaccination is formulated and a suitable region to our problem is chosen in Section 2. The existence of periodic solutions of our impulsive systems is established in Section 3. Some numerical simulations are demonstrated to verify the effectiveness of our pulse vaccination strategy in Section 4. The relevant conclusion will be stated in Section 5.

2. The Model and Preliminaries

2.1. The Seasonally Forced SEIR Model with Pulse Vaccination. In this paper, we focus on the existence of periodic solution of seasonally forced SEIR models with pulse vaccination; we consider models of the form

$$\begin{cases} \frac{dS(t)}{dt} = \mu - \beta(t)S(t)I(t) - \mu S(t), \\ \frac{dE(t)}{dt} = \beta(t)S(t)I(t) - (\epsilon + \mu)E(t), \\ \frac{dI(t)}{dt} = \epsilon E(t) - (\mu + \gamma)I(t), \\ \frac{dR(t)}{dt} = \gamma I(t) - \mu R(t), \\ \Delta S|_{t=nT+t_i} = -J_i(S(t), I(t))|_{t=nT+t_i}, \\ \Delta R|_{t=nT+t_i} = J_i(S(t), I(t))|_{t=nT+t_i}, \end{cases} \quad (4)$$

modeling the spread of infectious diseases with PVS under the following hypotheses:

- (i) The population is divided into four classes; $S(t)$, $E(t)$, $I(t)$, and $R(t)$ denote, respectively, the fractions of the susceptible, exposed/latent, infective, and recovered population, and $S(t) + E(t) + I(t) + R(t) = 1$ is invariant with $S(t), E(t), I(t), R(t) \geq 0$, for all $t \geq 0$.
- (ii) μ , ϵ , and γ denote the birth (death) rate, the rate of latent individuals becoming infectious, and recovery rate, respectively, which are positive constants.
- (iii) $\beta(t)$ is the seasonally dependent transmission rate, which is a positive continuous T -periodic function.
- (iv) The susceptible population can be divided into many groups and all the groups cannot be vaccinated at the same time, and the susceptible population will be vaccinated for several times:

$$\begin{aligned} \Delta S|_{t=nT+t_i} &= S(nT + t_i^+) - S(nT + t_i), \\ 0 \leq t_1 < \dots < t_k < T, \quad n \in \mathbb{N}. \end{aligned} \quad (5)$$

Our vaccination strategies concern the impact of infected population, which can be formulated as

$$\begin{aligned} J_i(S(t), I(t))|_{t=nT+t_i} &= p_i \left(1 - e^{-\alpha I(nT+t_i)} \right) S(nT + t_i), \\ i &= 1, \dots, k, \end{aligned} \quad (6)$$

where $0 \leq p_i < 1$ and the sensitivity coefficient $\alpha > 0$ is sufficiently large.

Denote the basic reproduction number:

$$\mathcal{R}_0 = \frac{\min_{t \in \mathbb{R}} \beta(t) \epsilon}{(\epsilon + \mu)(\gamma + \mu)}. \quad (7)$$

Obviously, we have $(dS(t)/dt) + (dE(t)/dt) + (dI(t)/dt) + (dR(t)/dt) \equiv 0$. Since $S(t)$, $E(t)$, $I(t)$, and $R(t)$ are fractions of the population, we have $S(t) + E(t) + I(t) + R(t) = 1$ for all t . Because $R(t)$ does not appear in the first three equations in (4), system (4) reduces to the following 3-dimensional system:

$$\begin{cases} \frac{dS(t)}{dt} = \mu - \beta(t)S(t)I(t) - \mu S(t), \\ \frac{dE(t)}{dt} = \beta(t)S(t)I(t) - (\epsilon + \mu)E(t), \\ \frac{dI(t)}{dt} = \epsilon E(t) - (\mu + \gamma)I(t), \\ \Delta S|_{t=nT+t_i} = -J_i(S(t), I(t))|_{t=nT+t_i}, \end{cases} \quad (8)$$

with

$$S(t) \geq 0, E(t) \geq 0, I(t) \geq 0, S(t) + E(t) + I(t) \leq 1. \quad (9)$$

Obviously, finding the periodic solution of (8) is equivalent to finding the solutions of the following periodic boundary value problem:

$$\left\{ \begin{array}{l} \frac{dS(t)}{dt} = \mu - \beta(t)S(t)I(t) - \mu S(t), \\ \frac{dE(t)}{dt} = \beta(t)S(t)I(t) - (\epsilon + \mu)E(t), \\ \frac{dI(t)}{dt} = \epsilon E(t) - (\mu + \gamma)I(t), \\ \Delta S|_{t=t_i} = -J_i(S(t), I(t))|_{t=t_i}, \end{array} \right. \quad (10)$$

with periodic boundary condition

$$\left\{ \begin{array}{l} S(0) = S(T), \\ E(0) = E(T), \\ I(0) = I(T). \end{array} \right. \quad (11)$$

2.2. The Suitable Region to Our Problem. In order to prove the existence of periodic solutions of (8), we consider the following auxiliary problem:

$$\left\{ \begin{array}{l} \frac{dS(t)}{dt} = \lambda(\mu - \beta(t)S(t)I(t) - \mu S(t)), \\ \frac{dE(t)}{dt} = \lambda(\beta(t)S(t)I(t) - (\epsilon + \mu)E(t)), \\ \frac{dI(t)}{dt} = \lambda(\epsilon E(t) - (\mu + \gamma)I(t)), \\ \Delta S|_{t=t_i} = -\lambda J_i(S(t_i), I(t_i)), \end{array} \right. \quad (12)$$

where $\lambda \in [0, 1]$ and $t \in [0, T]$.

Let D be an open bounded subset of X (will be denoted in Section 2.4) satisfying

$$D = \{(S(t), E(t), I(t)) \in X | S(t) > 0, E(t) > 0, I(t) > 0, S(t) + E(t) + I(t) < 1\}. \quad (13)$$

Proposition 1. $\bar{D}$ is an invariant region with respect to (12). The disease-free equilibrium $(S_0, E_0, I_0) = (1, 0, 0)$ is the unique periodic solution of (12) satisfying $(S(t), E(t), I(t)) \in \partial D$, $0 < \lambda \leq 1$.

Proof. First, we will prove that $\bar{D}$ is an invariant region.

In fact, it follows from model (12) that

$$\frac{dS}{dt}|_{S=0} = \lambda\mu > 0, \quad \frac{dE}{dt}|_{E=0} \geq \lambda\beta(t)SI \geq 0, \quad (14)$$

$$\frac{dI}{dt}|_{I=0} = \lambda\epsilon E \geq 0, \quad \frac{d(S+E+I)}{dt}|_{S+E+I=1} = -\lambda\gamma I \leq 0.$$

Since there is no impulsive motion for E, I , and

$$S(t_i^+) = \left(1 - \lambda p_i \left(1 - e^{-\alpha I(t_i)}\right)\right) S(t_i), \quad (15)$$

it is easy to conclude that every possible solution will remain in the region $\bar{D}$ ultimately.

Second, we will prove that the disease-free equilibrium $(S_0, E_0, I_0) \equiv (1, 0, 0)$ is the unique periodic solution of (12) satisfying $(S, E, I) \in \partial D$.

We assume that $(S(t), E(t), I(t)) \in \partial D$ is a solution of (12); this means at least one of the following conditions holds:

- (i) There exists $t_0 \in [0, T]$ so that $I(t_0) = 0$
- (ii) There exists $t_0 \in [0, T]$ so that $S(t_0) = 0$
- (iii) There exists $t_0 \in [0, T]$ so that $E(t_0) = 0$
- (iv) There exists $t_0 \in [0, T]$ so that $S(t_0) + E(t_0) + I(t_0) = 1$

We now consider each of these four cases:

In case of (i), we have $I(t_0) = 0$ and $I'(t_0) = \lambda\epsilon E(t_0) \geq 0$. If $E(t_0) = 0$, $I'(t) \equiv 0$ and $\Delta S(t_i) = 0, i = 1, \dots, k$. Thus, the only possible periodic solution of $S' = \lambda\mu(1 - S)$ is $S \equiv 1$. On the other hand, if $E(t_0) \neq 0$, $I'(t_0) > 0$. Thus, it is easy to obtain that $I(t) < 0$ for $t < t_0$ sufficiently close to t_0 , which contradicts that $\bar{D}$ is an invariant region.

In case of (ii), we have $S(t_0) = 0$ and $S'(t_0) = \lambda\mu > 0$. Thus, it is easy to obtain that $S(t) < 0$ for $t < t_0$ sufficiently close to t_0 , which contradicts that $\bar{D}$ is an invariant region.

In case of (iii), we have $E(t_0) = 0$ and $E'(t_0) = \lambda\beta(t_0)S(t_0)I(t_0) \geq 0$. Because the cases $I(t_0) = 0$ and $S(t_0) = 0$ have been discussed above, we have $E'(t_0) > 0$ which again contradicts that $\bar{D}$ is an invariant region.

In case of (iv), we get

$$\begin{aligned} (S + E + I)'(t_0) &= \lambda\mu(1 - S(t_0) - E(t_0) - I(t_0)) \\ &\quad - \lambda\gamma I(t_0) = -\lambda\gamma I(t_0) \leq 0. \end{aligned} \quad (16)$$

Because $I(t_0) = 0$ has been discussed, we only discuss $S(t_0) + E(t_0) + I(t_0) = 1$, $(S + E + I)'(t_0) < 0$, which contradicts that $\bar{D}$ is an invariant region. $\square$

Remark 1. If the impulsive motion is not influenced by I , which means $S(t_i^+) = (1 - \lambda p_i)S(t_i)$, the system $E \equiv 0, I \equiv 0, S' = \mu(1 - S)$ can have a nonconstant periodic solution on ∂D , which is hard to handle. In fact, if there are no infectious patients, it is often meaningless to vaccinate the susceptible people.

In order to use continuity method, it is necessary to choose an open bounded set $\Omega \subset D$, such that there is no solution (S, E, I) of (12) satisfying $(S(t), E(t), I(t)) \in \partial\Omega$ for any $\lambda \in (0, 1)$. Motivated by the idea of Katriel [27], we choose Ω to be the open subset of D given by

$$\Omega = \left\{ (S(t), E(t), I(t)) \in D \mid \min_{t \in [0, T]} S(t) < \delta \right\}, \quad (17)$$

where $\delta \in (0, 1)$ is to be fixed.

Proposition 2. Let $\mathcal{R}_0 > 1$, $\delta \in ((1/\mathcal{R}_0), 1)$. Then, there exists no solution $(S(t), E(t), I(t))$ of (12) satisfying $(S(t), E(t), I(t)) \in \partial\Omega$, for any $\lambda \in (0, 1]$.

Proof. Suppose $(S(t), E(t), I(t)) \in \partial\Omega$. Then, either $\{(S(t), E(t), I(t)) \in \partial D | \min_{t \in [0, T]} S(t) < \delta\}$ or $\{(S(t), E(t), I(t)) \in D | \min_{t \in [0, T]} S(t) = \delta\}$.

In the first case, $S(t) \equiv 1$ and Proposition 1 imply that there is no solution of (12) on $\{(S, E, I) \in \partial D | \min_{t \in [0, T]} S(t) < \delta\}$.

In the second case, we have $I(t) > 0, E(t) > 0$ and $S(t) \geq \delta, \forall t \in [0, T]$. Since there is no impulsive motion for I and $I(0) = I(T)$, after integrating the third equation of (12) over $[0, T]$, we obtain

$$(\mu + \gamma) \frac{1}{T} \int_0^T I(t) dt = \epsilon \frac{1}{T} \int_0^T E(t) dt. \quad (18)$$

Since there is no impulsive motion for E and $E(0) = E(T)$, after integrating the second equation of (12) over $[0, T]$, we obtain

$$(\epsilon + \mu) \frac{1}{T} \int_0^T E(t) dt = \frac{1}{T} \int_0^T \beta(t) S(t) I(t) dt. \quad (19)$$

With the help of (17)–(19), we conclude that

$$\begin{aligned} \frac{(\gamma + \mu)(\epsilon + \mu)}{\epsilon} \frac{1}{T} \int_0^T I(t) dt &= \frac{1}{T} \int_0^T \beta(t) S(t) I(t) dt \\ &\geq \delta \min_{t \in \mathbb{R}} \beta(t) \frac{1}{T} \int_0^T I(t) dt. \end{aligned} \quad (20)$$

For $I(t) > 0$, we get immediately

$$\frac{(\gamma + \mu)(\epsilon + \mu)}{\epsilon} \geq \delta \min_{t \in \mathbb{R}} \beta(t), \quad (21)$$

which is a contradiction to the assumption $\delta > (1/\mathcal{R}_0)$. $\square$

2.3. Outline of Mawhin's Coincidence Degree Method. We introduce a few definitions and recall the continuation theorem which will help us to prove the existence of positive solutions of system (10).

Consider the operator equation:

$$Lx = Nx, \quad (22)$$

where $L: \text{dom} L \subset X \rightarrow Z$ is a linear bounded operator, $N: X \rightarrow Z$ is a continuous operator, and X and Z are Banach spaces.

Definition 1. (see [37]). The linear mapping L is called a Fredholm mapping of index zero if

- (1) $\text{Index} L = \dim \text{Ker} L - \text{codim} \text{Im} L = 0$
- (2) $\text{Im} L$ is closed in Z

If $L: \Omega \subset X \rightarrow Z$ is a Fredholm mapping of index zero, there exist continuous projectors $P: X \rightarrow X$ and $Q: Z \rightarrow Z$ such that

- (1) $\text{Im} P = \text{Ker} L$
- (2) $\text{Ker} Q = \text{Im} L = \text{Im}(I - Q)$
- (3) $X = \text{Ker} L \oplus \text{Ker} P$
- (4) $Z = \text{Im} L \oplus \text{Im} Q$

It follows that $L|_{\text{dom} L \cap \text{Ker} P}: (I - P)X \rightarrow \text{Im} L$ is invertible. We denote the inverse of that map by K_p .

Definition 2 (see [37]). Let Ω be an open bounded subset of X . The mapping N is called L -compact on $\overline{\Omega}$, if

- (1) $QN(\overline{\Omega})$ is bounded
- (2) $K_p(1 - Q)N: \overline{\Omega} \rightarrow X$ is compact

Since $\text{Im} Q$ is isomorphic to $\text{Ker} L$, there exists an isomorphism $\Lambda: \text{Im} Q \rightarrow \text{Ker} L$.

Theorem 1 (see [37]) (Mawhin's continuation theorem). Let $\Omega \subset X$ be an open bounded set. Let L be a Fredholm mapping of index zero and be L -compact on Ω . Assume that

- (1) For each $\lambda \in (0, 1)$, $x \in \partial\Omega \cap \text{dom} L$, $Lx \neq \lambda Nx$
- (2) For each $x \in \partial\Omega \cap \text{Ker} L$, $QNx \neq 0$
- (3) $\deg(\Lambda QN, \Omega \cap \text{Ker} L, 0) \neq 0$

Then, the equation $Lx = Nx$ has at least one solution in $\text{dom} L \cap \overline{\Omega}$.

Lemma 4 (Arzela–Ascoli) (see [38]). Let D be a compact subset of $\mathbb{R}^n$ and $L(D, \mathbb{R}^n)$ be the linear space of continuous functions which take D into $\mathbb{R}^n$; any uniformly bounded equicontinuous sequence of functions $\phi_n, n = 1, 2, \dots$ in $L(D, \mathbb{R}^n)$ has a subsequence which converges uniformly on D .

2.4. Notations. We now prepare the setting to apply Mawhin's continuation theorem. Define $C^j[0, T; t_1, \dots, t_k] = x: [0, T] \rightarrow |\mathbb{R}|x^{(m)}(t)$ exists for $t \neq t_1, \dots, t_k$ and $x^{(m)}(t_i^+)$, $x^{(m)}(t_i^-)$ exist with $x^{(m)}(t_i) := x^{(m)}(t_i^+)$, $i = 1, \dots, k$ where j is a non-negative integer and $m \in 0, \dots, j$, $i = 1, \dots, k$, $m = 0, \dots, j$

Define a Banach space

$$\begin{aligned} X = \{ & (x_1, x_2, x_3) | x_1 \in C[0, T; t_1, \dots, t_k], x_2, x_3 \in C[0, T], \\ & x_1(0) = x_1(T), x_2(0) = x_2(T), x_3(0) = x_3(T) \}, \end{aligned} \quad (23)$$

with the norm

$$\|(x_1, x_2, x_3)\|_X = \max_{t \in [0, T]} (|x_1(t)| + |x_2(t)| + |x_3(t)|). \quad (24)$$

Define another Banach space

$$Z = C[0, T; t_1, \dots, t_k] \times C[0, T] \times C[0, T] \times \mathbb{R}^k, \quad (25)$$

with the norm

$$\begin{aligned} \|(z_1, z_2, z_3, C_1, \dots, C_k)\|_Z = \max_{t \in [0, T]} (|z_1(t)| + |z_2(t)| \\ + |z_3(t)|) + |C_1| + \dots + |C_k|. \end{aligned} \quad (26)$$

Let

$$\begin{aligned} L: \text{dom}L &\longrightarrow Z, \\ (S, E, I) &\longrightarrow (S', E', I', \Delta S(t_1), \dots, \Delta S(t_k)), \end{aligned} \quad (27)$$

where

$$\begin{aligned} N: X &\longrightarrow Z, \\ (S, E, I) &\longrightarrow (f_1(\cdot, S, E, I), f_2(\cdot, S, E, I), f_3(\cdot, S, E, I), -J_1(S(t_1), I(t_1)), \dots, -J_k(S(t_k), I(t_k))), \end{aligned} \quad (29)$$

where

$$\begin{aligned} f_1(\cdot, S, E, I) &= \mu - \beta(t)SI - \mu S, \\ f_2(\cdot, S, E, I) &= \beta(t)SI - (\epsilon + \mu)E, \\ f_3(\cdot, S, E, I) &= \epsilon E - (\mu + \gamma)I. \end{aligned} \quad (30)$$

Obviously, system (10) can be written by $L(S, E, I) = N(S, E, I)$. By a simple calculation, we get

$$\text{Ker}L = \{(S, E, I) | (S, E, I) \equiv C \in \mathbb{R}^3\}, \quad (31)$$

It is easy to see that

$$\dim \text{Ker}L = \text{codim} \text{Im}L = 3. \quad (32)$$

Since $\text{Im}L$ is closed, L is Fredholm mapping of index 0. Let $P: X \longrightarrow X$ be the projector given by

$$\begin{aligned} P(S, E, I) &= \left(\frac{1}{T} \int_0^T S(\tau) d\tau, \frac{1}{T} \int_0^T E(\tau) d\tau, \frac{1}{T} \int_0^T I(\tau) d\tau \right), \\ &\cdot (S, E, I) \in X. \end{aligned} \quad (33)$$

Obviously,

$$\text{Im}P = \text{Ker}L = \mathbb{R}^3. \quad (34)$$

Let $Q: Z \longrightarrow Z$ be the projector given by

$$\begin{aligned} Q(z_1, z_2, z_3, C_1, \dots, C_k) &= \left(\frac{1}{T} \int_0^T z_1(\tau) d\tau + \frac{1}{T} \sum_{i=1}^k C_i, \frac{1}{T} \int_0^T z_2(\tau) d\tau, \frac{1}{T} \int_0^T z_3(\tau) d\tau, 0_{1 \times k} \right), \end{aligned} \quad (35)$$

$$\begin{aligned} \text{dom}L &= \{(S, E, I) | S \in C^1[0, T; t_1, \dots, t_k], E, I \in C^1[0, T], \\ &S(0) = S(T), E(0) = E(T), I(0) = I(T)\}. \end{aligned} \quad (28)$$

Let

Obviously,

$$\text{Ker}Q = \text{Im}L = \text{Im}(I - Q). \quad (36)$$

Furthermore, the generalized inverse (to L) $K_p: \text{Im}L \longrightarrow \text{Ker}P \cap \text{dom}L$ exists given by

$$\begin{aligned} K_p(z_1, z_2, z_3, C_1, \dots, C_k) &= \left(\int_0^t z_1(\tau) d\tau + \sum_{t > t_i} C_i - \frac{1}{T} \int_0^T \int_0^t z_1(\tau) d\tau dt - \sum_{i=1}^k C_i \int_0^t z_2(\tau) \right. \\ &\cdot d\tau - \frac{1}{T} \int_0^T \int_0^t z_2(\tau) d\tau dt, \int_0^t z_3 \\ &\cdot (\tau) d\tau - \frac{1}{T} \int_0^T \int_0^t z_3(\tau) d\tau dt \Big). \end{aligned} \quad (37)$$

Then, $QN: \longrightarrow XZ$ has the following form:

$$\begin{aligned} QN(S, E, I) &= \left(\frac{1}{T} \int_0^T f_1(\tau, S, E, I) d\tau - \frac{1}{T} \sum_{i=1}^k J_i(S(t_i), I(t_i)), \right. \\ &\frac{1}{T} \int_0^T f_2(\tau, S, E, I) d\tau, \frac{1}{T} \int_0^T f_3(\tau, S, E, I) d\tau, 0_{1 \times k} \Big). \end{aligned} \quad (38)$$

By a direct calculation, we have

$$\begin{aligned} K_p(1 - Q)N(S, E, I) &= K_p \left[\left(f_1(\cdot, S, E, I) - \frac{1}{T} \int_0^T f_1(\tau, S, E, I) d\tau + \frac{1}{T} \sum_{i=1}^k J_i(S(t_i), I(t_i)), f_1(\cdot, S, E, I) - \frac{1}{T} \right. \right. \\ &\cdot \int_0^T f_2(\tau, S, E, I) d\tau, f_3(\cdot, S, E, I) - \frac{1}{T} \int_0^T f_3(\tau, S, E, I) d\tau, J_1(S(t_1), I(t_1)), \dots, J_k(S(t_k), I(t_k)) \Big) \Big] \\ &= \left(\int_0^t f_1 d\tau - \frac{1}{T} \int_0^T \int_0^t f_1 d\tau dt + \left(\frac{1}{2} - \frac{t}{T} \right) \left(\int_0^T f_1 d\tau + \sum_{i=1}^k J_i(S(t_i), I(t_i)) \right) + \sum_{t > t_i} J_i(S(t_i), I(t_i)) - \sum_{i=1}^k J_i(S(t_i), I(t_i)), \right. \\ &\int_0^t f_2 d\tau - \frac{1}{T} \int_0^T \int_0^t f_2 d\tau dt + \left(\frac{1}{2} - \frac{t}{T} \right) \int_0^T f_2 d\tau, \int_0^t f_3 d\tau - \frac{1}{T} \int_0^T \int_0^t f_3 d\tau dt + \left(\frac{1}{2} - \frac{t}{T} \right) \int_0^T f_3 d\tau \Big). \end{aligned} \quad (39)$$

3. Results

The following theorem gives the main results of this paper.

Theorem 2. *Let $\mathcal{R}_0 > 1$; there exists at least one T -periodic solution $(S(t), E(t), I(t), R(t))$ of (1), all of whose components are positive.*

Remark 2. When $p_i \equiv 0, i = 1, \dots, k$, system (4) is the usual seasonally forced SEIR model without pulse vaccination.

$$\begin{aligned} \|QN(S, E, I)\|_Z &= \max_{t \in [0, T]} \left(\left| \frac{1}{T} \int_0^T f_1(\tau, S, E, I) d\tau - \frac{1}{T} \sum_{i=1}^k J_i((t_i), I(t_i)) \right| + \left| \frac{1}{T} \int_0^T f_2(\tau, S, E, I) d\tau \right| + \left| \frac{1}{T} \int_0^T f_3(\tau, S, E, I) d\tau \right| \right) \\ &\leq 3\mu + 2\bar{\beta} + 2\epsilon + \gamma + \frac{1}{T} \sum_{i=1}^k p_i < +\infty \end{aligned} \quad (40)$$

Then, it is easy to check that $K_p(I - Q)N: \bar{\Omega} \rightarrow X$ is uniform bounded and equicontinuous on each $[t_i, t_{i+1}]$. Assume that $\{(S_j, E_j, I_j)\}_{j=1}^\infty \subset \bar{\Omega}$. Using Lemma 4, there exists a uniformly convergent subsequence denoted by $K_p(I - Q)N(S_{j_1}, E_{j_1}, I_{j_1})$ on $[0, t_1]$. Using Arzela–Ascoli lemma again on $[t_1, t_2]$, we have a uniformly convergent subsequence $K_p(I - Q)N(S_{j_2}, E_{j_2}, I_{j_2})$ which is also uniformly convergent on $[0, t_1]$. Repeat it again and again, and we can prove that $K_p(I - Q)N(S_{j_{k+1}}, E_{j_{k+1}}, I_{j_{k+1}})$ is uniformly convergent on $[0, T]$. In this way, $K_p(I - Q)N: \bar{\Omega} \rightarrow X$ is compact.

Step 2: For each $\lambda \in (0, 1)$, $(S, E, I) \in \partial\Omega \cap \text{dom}L$, $L(S, E, I) \neq \lambda N(S, E, I)$, which has been proved by Proposition 2.

Step 3: For each $(S, E, I) \in \partial\Omega \cap \text{Ker}L$, $QN(S, E, I) \neq 0$. If $QN(S_1, E_1, I_1) = 0$, we have

$$\begin{cases} \frac{1}{T} \int_0^T f_1(\tau, S_1, E_1, I_1) d\tau - \frac{1}{T} \sum_{i=1}^k J_i(S_1(t_i), I_1(t_i)) = 0, \\ \frac{1}{T} \int_0^T f_2(\tau, S_1, E_1, I_1) d\tau = 0, \\ \frac{1}{T} \int_0^T f_3(\tau, S_1, E_1, I_1) d\tau = 0. \end{cases} \quad (41)$$

Assume $(S_1, E_1, I_1) \in \text{Ker}L$; we know that (S_1, E_1, I_1) is a constant vector in $\mathbb{R}^3$. Thus, (40) is equivalent to

The proof of Theorem 2 will be divided into 4 steps.

Proof

Step 1: N is L -compact on $\bar{\Omega}$.

First, it is obvious that $QN(\bar{\Omega})$ is bounded.

For any $(S, E, I) \in \bar{\Omega}$,

$$\begin{cases} \mu - \bar{\beta}S_1I_1 - \mu S_1 - \frac{1}{T} \sum_{i=1}^k p_i \cdot (1 - e^{-\alpha I_1})S_1 = 0, \\ \bar{\beta}S_1I_1 - (\epsilon + \mu)E_1 = 0, \\ \epsilon E_1 - (\gamma + \mu)I_1 = 0. \end{cases} \quad (42)$$

We claim that there are exactly two solutions: $(1, 0, 0)$ and (S^*, E^*, I^*) in $\text{Ker}L$ satisfying

$$\begin{cases} S^* = \frac{(\epsilon + \mu)(\gamma + \mu)}{\bar{\beta}\epsilon}, \\ E^* = \left(\frac{\gamma + \mu}{\epsilon} \right) I^*, \\ \mu \left(\frac{\bar{\beta}\epsilon}{(\epsilon + \mu)(\gamma + \mu)} - 1 \right) - \bar{\beta}I^* - \frac{1}{T} \sum_{i=1}^k p_i (1 - e^{-\alpha I^*}) = 0. \end{cases} \quad (43)$$

According to the definition of Ω , we know that $(1, 0, 0) \notin \bar{\Omega}$. Denote

$$G(I) = \mu \left(\frac{\bar{\beta}\epsilon}{(\epsilon + \mu)(\gamma + \mu)} - 1 \right) - \bar{\beta}I - \frac{1}{T} \sum_{i=1}^k p_i (1 - e^{-\alpha I}). \quad (44)$$

We have

$$\begin{aligned}
G(0) &= \mu \left(\frac{\bar{\beta}\epsilon}{(\epsilon + \mu)(\gamma + \mu)} - 1 \right) > 0, \quad G(1) \\
&= \frac{-\bar{\beta}(\epsilon\gamma + \mu\gamma + \mu^2)}{(\epsilon + \mu)(\gamma + \mu)} - \mu - \frac{1}{T} \sum_{i=1}^k p_i (1 - e^{-\alpha}) < 0.
\end{aligned} \tag{45}$$

Since

$$\frac{\partial G(I)}{\partial I} = -\bar{\beta} - \frac{1}{T} \sum_{i=1}^k p_i \alpha e^{-\alpha I} < 0, \tag{46}$$

$G(I)$ is a monotonous function with respect to I . Thus, (S^*, E^*, I^*) is the unique solution of $QN(S, E, I) = 0$ in $KerL \cap \bar{\Omega}$. Since

$$S^* = \frac{(\epsilon + \mu)(\gamma + \mu)}{\bar{\beta}\epsilon} < \frac{(\epsilon + \mu)(\gamma + \mu)}{\min_{t \in \mathbb{R}} \beta(t)\epsilon} = \frac{1}{\mathcal{R}_0} < \delta, \tag{47}$$

we have $(S^*, E^*, I^*) \notin KerL \cap \partial\Omega$.

Consequently, for each $(S, E, I) \in \partial\Omega \cap KerL$, $QN(S, E, I) \neq 0$.

Step 4. There exists an isomorphism $\Lambda: ImQ \longrightarrow KerL$ such that

$$\begin{aligned}
\Lambda Q(z_1, z_2, C_1, \dots, C_k) &= \Lambda \left(\frac{1}{T} \int_0^T z_1(\tau) d\tau + \frac{1}{T} \sum_{i=1}^k C_i, \frac{1}{T} \int_0^T z_2(\tau) d\tau, \frac{1}{T} \int_0^T z_3(\tau) d\tau, 0_{1 \times k} \right) \\
&= \left(\frac{1}{T} \int_0^T z_1(\tau) d\tau + \frac{1}{T} \sum_{i=1}^k C_i, \frac{1}{T} \int_0^T z_2(\tau) d\tau, \frac{1}{T} \int_0^T z_3(\tau) d\tau \right).
\end{aligned} \tag{48}$$

We will prove that $deg(\Lambda QN, \Omega \cap KerL, 0) \neq 0$. On account of the discussion in step 3, we know that (S^*, E^*, I^*) is the unique solution of $\Lambda QN(S, E, I) = 0$ in $\Omega \cap KerL$.

A direct calculation shows that

$$\begin{aligned}
deg(\Lambda QN(S, E, I), \Omega \cap KerL, 0) &= deg \mu - \bar{\beta}SI - \mu S - \frac{1}{T} \sum_{i=1}^k p_i (1 - e^{-\alpha I}) S, \bar{\beta}SI - (\epsilon + \mu)E, \epsilon E - (\gamma + \mu)I, \Omega \cap KerL, 0 \\
&= Sign \begin{vmatrix} -\bar{\beta}I^* - \mu - \frac{1}{T} \sum_{i=1}^k p_i (1 - e^{-\alpha I^*}) & 0 & -\bar{\beta}S^* - \frac{1}{T} \sum_{i=1}^k p_i \alpha e^{-\alpha I^*} S^* \\ \bar{\beta}I^* & -(\epsilon + \mu) & \bar{\beta}S^* \\ 0 & \epsilon & -(\gamma + \mu) \end{vmatrix} \\
&= -1 \neq 0.
\end{aligned} \tag{49}$$

Thus, we conclude from Theorem 1 that the equation $L(S, E, I) = N(S, E, I)$ has at least one solution on $domL \cap \bar{\Omega}$.

From Proposition 2, we deduce that for each $\lambda = 1$, $(S, E, I) \in domL \cap \partial\Omega$, $L(S, E, I) \neq N(S, E, I)$. It follows that $L(S, E, I) = N(S, E, I)$ has at least one solution in $domL \cap \Omega$. $\square$

4. Simulation

In this section, some relevant numerical simulations about the T-periodic solution of the seasonally forced SEIR models

are presented to show the effectiveness of PVS. Furthermore, we will compare the effects of different parameters on the solutions of the seasonally forced SEIR models with PVS.

With the period $T = 2\pi$ of the forcing representing one year, we take $\gamma = 14(2\pi/365)$ corresponding to a 2-week infectious period. We set $\beta(t) = \bar{\beta}(1 + 0.6 \cos(t))$. Assume that there are three impulsive points at fixed time $\pi/2$, $2\pi/2$, and $3\pi/2$ with p_i , $i = 1, 2, 3$. Let $[0, 2\pi]$ be divided into $k = 100$ intervals equally. Given the initial point $S^{**} = (\epsilon + \mu)(\gamma + \mu)/\bar{\beta}\epsilon$, $E^{**} = \mu(1/(\mu + \epsilon) - (\mu + \gamma)/\epsilon\bar{\beta})$, $I^{**} = \mu(\epsilon/(\mu + \epsilon)(\mu + \gamma) - 1/\bar{\beta})$, which is the endemic equilibrium of the SEIR model without periodic transmission rate and pulse

FIGURE 1: Positive periodic solutions of seasonally forced SEIR with pulse vaccination in 1 year and 10 years.

FIGURE 2: Infective and susceptible population with $p_i = 0.0$ and $p_i = 0.2$.

FIGURE 3: Exposed population, infective population, recovered population, and susceptible population with $p_i = 0.0$, $p_i = 0.2$, and $p_i = 0.4$.

FIGURE 4: Exposed population, infective population, recovered population, and susceptible population with $\bar{\beta} = 12\gamma$, $\bar{\beta} = 15\gamma$, and $\bar{\beta} = 18\gamma$.

FIGURE 5: Exposed population, infective population, recovered population, and susceptible population with $\mu = 0.8/2\pi$, $\mu = 0.6/2\pi$, and $\mu = 0.4/2\pi$.

vaccination. The periodic solutions of system (10) can be solved by the Newton iteration method in which we set $S(t_i + (2\pi/100)) - S(t_i) = p_i(1 - e^{-\alpha I(t_i)})S(t_i)$ at fixed time $\pi/2$, $2\pi/2$, and $3\pi/2$.

Simulation 1. Set $\bar{\beta} = 15\gamma$, $\epsilon = 0.9/2\pi$, $\mu = 0.8/2\pi$, and $p_i = 0.2$; we get the approximate susceptible population, exposed population, infective population, and recovered population of system (10) by Newton iteration. In Figure 1, we compare the T-periodic solution of the seasonally forced SEIR models with PVS in 1 year and 10 years. Figure 1 shows periodicity of the positive solution of the seasonally forced SEIR models with PVS.

Simulation 2. Set $\bar{\beta} = 15\gamma$, $\epsilon = 0.9/2\pi$, and $\mu = 0.8/2\pi$; we make 8 steps of Newton iteration to get the approximate

infective population of system (10) with both $p_i = 0$ (the surface at the bottom) and $p_i = 0.2$ (the surface at the top). Obviously, the infective population of system (10) with pulse is lower than the infective population of system (10) without pulse in Figure 2. The susceptible population of system (10) with $p_i = 0.2$ has periodic and impulsive properties. Thus, it is very effective to lower the infective population by PVS. Furthermore, Figure 2 shows the stability of the periodic solution by Newton iteration.

Simulation 3. Set $\bar{\beta} = 15\gamma$, $\epsilon = 0.9/2\pi$, and $\mu = 0.8/2\pi$; we simulate system (10) with $p_i = 0$, $p_i = 0.2$, and $p_i = 0.4$ by Newton iteration. Obviously, it is very effective to lower the exposed population and the infective population by PVS in Figure 3.

FIGURE 6: Exposed population, infective population, recovered population, and susceptible population with $\epsilon = 0.9/2\pi$, $\epsilon = 0.8/2\pi$, and $\epsilon = 0.7/2\pi$.

Simulation 4. Set $\epsilon = 0.9/2\pi$, $\mu = 0.8/2\pi$, and $p_i = 0.2$; we simulate system (10) with $\bar{\beta} = 12\gamma$, 15γ , and 18γ by Newton iteration. Figure 4 shows the impact on exposed population, infective population, recovered population, and susceptible population by different transmission rates. As $\bar{\beta}$ increases, the infective population increases while susceptible population decreases.

Simulation 5. Set $\bar{\beta} = 15\gamma$, $\epsilon = 0.9/2\pi$, and $p_i = 0.2$; we simulate system (10) with $\mu = 0.4/2\pi$, $0.6/2\pi$, and $0.8/2\pi$ by Newton iteration. Figure 5 shows the impact on exposed population, infective population, recovered population, and susceptible population by different birth rates. As μ increases, the infective population and susceptible population increase, respectively.

Simulation 6. Set $\bar{\beta} = 15\gamma$, $\mu = 0.8/2\pi$, and $p_i = 0.2$; we simulate system (10) with $\epsilon = 0.7/2\pi$, $0.8/2\pi$, and $0.9/2\pi$ by Newton iteration. Figure 6 shows the impact on systems by the different rates of latent individuals becoming infectious. As ϵ increases, the infective population increases while susceptible population decreases.

5. Conclusion

We obtain the existence of positive T-periodic solutions of seasonally forced SEIR models with pulse vaccination by Mawhin's coincidence degree method. Some relevant numerical simulations are presented to show the T-periodic solution of the seasonally forced epidemiological models and to illustrate the effectiveness of such pulse vaccination strategy.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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