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Research Article

Bounds on the α-Distance Energy and α-Distance Estrada Index of Graphs

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Let G be a simple undirected connected graph, then $D_\alpha(G) = \alpha Tr(G) + (1-\alpha)D(G)$ is called the α-distance matrix of G, where $\alpha \in [0,1]$, $D(G)$ is the distance matrix of G, and $Tr(G)$ is the vertex transmission diagonal matrix of G. In this paper, we study some bounds on the α-distance energy and α-distance Estrada index of G. Furthermore, we establish the relation between α-distance Estrada index and α-distance energy.

1. Introduction

1.1. α-Distance Energy of Graphs. In this paper, we suppose that G is a connected graph. Let $G = (V(G), E(G))$ be a graph with the vertex set $V(G) = \{v_1, \ldots, v_n\}$ and edge set $E(G)$. The distance between two vertices $v_i, v_j \in V(G)$ is the length of the shortest path between v_i and v_j, denoted by $d(v_i, v_j)$. The *Wiener index* $W(G)$ of the graph G is $W(G) = (1/2)\sum_{v_i, v_j \in V(G)} d(v_i, v_j)$. The matrix $D(G) = (d_{i,j}) \in \mathscr{R}^{n\times n}$ is called the *distance matrix* of G, where $d_{i,j} = d(v_i, v_j)$, $i, j \in \{1, 2, \ldots, n\}$. For some properties of distance matrix, see [1–3].

The *adjacency matrix* of the graph G is $\widetilde{A}(G) = (\widetilde{a}_{ij}) \in \mathscr{R}^{n\times n}$, where $\widetilde{a}_{ij} = 1$ if $(i, j) \in E(G)$ and $\widetilde{a}_{ij} = 0$ otherwise. The Laplacian matrix and signless Laplacian matrix of G are $\widetilde{L}(G) = \widetilde{D}(G) - \widetilde{A}(G)$ and $\widetilde{Q}(G) = \widetilde{D}(G) + \widetilde{A}(G)$, respectively, where $\widetilde{D}(G) = \mathrm{diag}(\widetilde{d}_{v_1}, \ldots, \widetilde{d}_{v_n}) \in \mathscr{R}^{n\times n}$ and $\widetilde{d}_{v_i}$ is the degree of v_i, $i = 1, 2, \ldots, n$.

In 2013, the study of Laplacian matrix and signless Laplacian matrix was extended to distance Laplacian matrices and distance signless Laplacian matrices defined as in equation (1) (see [4]). In 2016, the study of the spectrum of signless Laplacian matrix was generalized to a convex combination of $\widetilde{D}(G)$ and $\widetilde{A}(G)$ defined as $A_{(\alpha)}(G) = \alpha\widetilde{D}(G) + (1-\alpha)\widetilde{A}(G), \alpha \in [0,1]$ (see [5]). In [6], the above study was further extended to the α-distance matrices (see equation (2)).

Let $Tr(v_i) = \sum_{v_j \in V(G)} d(v_i, v_j)$ is called the *transmission* of v_i. Let

$$\begin{aligned} \mathscr{L}(G) &= Tr(G) - D(G), \\ \mathscr{Q}(G) &= Tr(G) + D(G), \end{aligned} \tag{1}$$

where $Tr(G) = \mathrm{diag}(Tr(v_1), \ldots, Tr(v_n))$. $\mathscr{L}(G)$ and $\mathscr{Q}$ are called the distance Laplacian matrix and distance signless Laplacian matrix of the graph G, respectively. A graph G is said to be *transmission regular* if the transmissions of all the vertices in $V(G)$ are equal (see [4]). For a transmission regular graph G, the characteristic polynomials of $\mathscr{L}(G)$ and $\mathscr{Q}(G)$ were characterized in [4]. For more properties of $\mathscr{L}(G)$ and $\mathscr{Q}(G)$, see [7–9].

In [6], the *α-distance matrix* of a graph G

$$D_{(\alpha)}(G) = \alpha Tr(G) + (1-\alpha)D(G), \quad \alpha \in [0,1], \tag{2}$$

was defined. Clearly, $D_{(0)}(G) = D(G)$, $D_{(1/2)}(G) = (1/2)\mathscr{Q}(G)$, $D_{(1)}(G) = Tr(G)$, and $D_{(\alpha)}(G) - D_{(\beta)}(G) = (\alpha - \beta)\mathscr{L}(G)$. The spectra of $D_{(\alpha)}(G)$ is called the *α-distance spectra* of G. Since $D_{(\alpha)}(G)$ is a real symmetric matrix, the eigenvalues of $D_{(\alpha)}(G)$ are real. Let $\sigma^\alpha_{(1)}(G) \geq \sigma^\alpha_{(2)}(G) \geq \cdots \geq \sigma^\alpha_{(n)}(G)$ be the eigenvalues of $D_{(\alpha)}(G)$. And let $\rho_{(\alpha)}(G)$ denote the *spectral radius* of $D_{(\alpha)}(G)$. From the

Perron–Frobenius theorem, we have $\sigma_{(1)}(G) = \rho_\alpha(G)$. The spectral properties of $D_{(\alpha)}(G)$ were recently studied including spectral radius, second largest eigenvalue, k-th smallest eigenvalue, and smallest eigenvalue (see [6, 10–12]).

Graph energy is an important graph invariant in graph theory; some graph energies $E_\pi(G) = \sum_{i=1}^{n}|\lambda_i(G)|$, $DE(G) = \sum_{i=1}^{n}|\gamma_i(G)|$, and $\mathrm{DSLE}(G) = \sum_{i=1}^{n}|\nu_i(G) - (2W(G)/n)|$ are called the energy (original energy), the distance energy, and the distance signless Laplacian energy, respectively, where $\lambda_i(G)$, $\gamma_i(G)$, and $\nu_i(G)$ denote the eigenvalues of $\widetilde{A}(G)$, $D(G)$, and $\mathcal{Q}(G)$, respectively, $i = 1, 2, \ldots, n$ and $n = |V(G)|$ (see [13–16]).

Graph energy has important applications in the fields of mathematics and chemistry. There are many research studies on the above kinds of graph energy. Scholars gave the bounds on the energy of graphs, for example, the McClelland's bounds [17], Koolen–Moulton's bounds [18] and so on [19]. In [16], the distance energy of some graphs was calculated.

In [11], Guo and Zhou extended the concept of graph energy to a more general form called *α-distance energy*:

$$\varsigma_{(\alpha)}(G) = \sum_{i=1}^{n}\left|\sigma_{(i)}^{\alpha}(G) - \frac{2\alpha W(G)}{n}\right|, \quad \alpha \in [0,1], \tag{3}$$

where $\sigma_{(i)}^{\alpha}(G)$ is the eigenvalue of $D_{(\alpha)}(G)$, $i = 1, 2, \ldots, n$, $n = |V(G)|$. Clearly, $\varsigma_{(0)}(G) = DE(G)$ and $\varsigma_{(1/2)}(G) = (1/2)\mathrm{DSLE}(G)$.

1.2. α-Distance Estrada Index of Graphs. In [20], a spectral quantity is put forward by Estrada. $EE(G) = \sum_{i=1}^{n} e^{\lambda_i(G)} = \sum_{i=1}^{n}\sum_{k=0}^{\infty}\lambda_i^k(G)/k!$ is called the Estrada index of G, where $\lambda_1(G), \lambda_2(G), \ldots, \lambda_n(G)$ denote the eigenvalues of $\widetilde{A}(G)$ (see [20]). It is well-known that the Estrada index plays an important role in the problem of characterizing the molecular structure [21] and complex networks [22–25]. In [26], the study was extended to distance matrices, and the distance Estrada index of G is $\mathrm{DEE}(G) = \sum_{i=1}^{n} e^{\gamma_i(G)} = \sum_{i=1}^{n}\sum_{k=0}^{\infty}\gamma_i^k(G)/k!$, where $\gamma_1(G), \gamma_2(G), \ldots, \gamma_n(G)$ are the eigenvalues of $D(G)$.

In this paper, we consider a more general Estrada index. Let

$$\mathrm{DEE}_{(\alpha)}(G) = \sum_{i=1}^{n} e^{\sigma_{(i)}^{\alpha}(G)} = \sum_{i=1}^{n}\sum_{k=0}^{\infty}\frac{\left(\sigma_{(i)}^{\alpha}(G)\right)^k}{k!}, \tag{4}$$

be the *α-distance Estrada index* of G, where $\sigma_{(1)}^{\alpha}(G), \ldots, \sigma_{(n)}^{\alpha}(G)$ are the eigenvalues of $D_{(\alpha)}(G)$. Clearly, $\mathrm{DEE}_{(0)}(G) = \mathrm{DEE}(G)$.

1.3. Main Work. In this paper, we study some bounds on the α-distance energy and α-distance Estrada index of graphs in terms of the parameter α and the vertex number, the transmission of vertices and Wiener index. Furthermore, we establish the relation between α-distance Estrada index and α-distance energy.

2. Some Bounds for the α-Distance Energy of Graphs

To begin with this section, we introduce some notations and propositions.

Proposition 1 (see [6]). *Let G be a graph with n vertices. Then,*

$$\sum_{i=1}^{n}\sigma_{(i)}^{\alpha}(G) = \sum_{i=1}^{n}\alpha Tr(v_i) = 2\alpha W(G),$$
$$\sum_{i=1}^{n}\left(\sigma_{(i)}^{\alpha}(G)\right)^2 = \alpha^2\sum_{i=1}^{n}Tr^2(v_i) + 2(1-\alpha)^2 S, \tag{5}$$

where $S = \sum_{1\le i<j\le n} d^2(v_i, v_j)$ and $\sigma_{(1)}^{\alpha}(G) \ge \sigma_{(2)}^{\alpha}(G) \ge \cdots \ge \sigma_{(n)}^{\alpha}(G)$ denotes the eigenvalues of $D_{(\alpha)}(G)$.

In the following, a new matrix is established:

$$U_{(\alpha)}(G) = \alpha Tr(G) + (1-\alpha)D(G) - \frac{2\alpha W(G)}{n}I_n, \quad \alpha \in [0,1], \tag{6}$$

where I_n denotes identity matrix of order n. Let $\eta_{(1)}^{\alpha}(G), \eta_{(2)}^{\alpha}(G), \ldots, \eta_{(n)}^{\alpha}(G)$ denote the eigenvalues of $U_{(\alpha)}(G)$. Obviously,

$$\varsigma_{(\alpha)}(G) = \sum_{i=1}^{n}\left|\eta_{(i)}^{\alpha}(G)\right|, \quad \alpha \in [0,1]. \tag{7}$$

Proposition 2. *Let G be a graph with n vertices. Then,*

$$\sum_{i=1}^{n}\eta_{(i)}^{\alpha}(G) = 0, \quad \sum_{i=1}^{n}\left(\eta_{(i)}^{\alpha}(G)\right)^2 = 2Z, \tag{8}$$

$$Z = -\sum_{1\le i<j\le n}\eta_{(i)}^{\alpha}(G)\eta_{(j)}^{\alpha}(G) = \left|\sum_{1\le i<j\le n}\eta_{(i)}^{\alpha}(G)\eta_{(j)}^{\alpha}(G)\right|, \tag{9}$$

where $Z = (1-\alpha)^2 S + (\alpha^2/2)\sum_{i=1}^{n}(Tr(v_i) - (2W(G)/n))^2$ and $\eta_{(1)}^{\alpha}(G), \eta_{(2)}^{\alpha}(G), \ldots, \eta_{(n)}^{\alpha}(G)$ denote the eigenvalues of $U_{(\alpha)}(G)$.

Proof. In order to prove equation (8), let $\eta_{(1)}^{\alpha}(G), \eta_{(2)}^{\alpha}(G), \ldots, \eta_{(n)}^{\alpha}(G)$ denote the eigenvalues of $U_{(\alpha)}(G)$, by equations (5) and (6), we have

$$\sum_{i=1}^{n}\left(\eta_{(i)}^{\alpha}(G)\right)^2 = \mathrm{trace}\left(U_{(\alpha)}^2(G)\right) = 2(1-\alpha)^2 S + \alpha^2\sum_{i=1}^{n}\left(Tr(v_i) - \frac{2W(G)}{n}\right)^2. \tag{10}$$

By equation (5), we have

$$0 = \left(\sum_{i=1}^{n}\eta_{(i)}^{\alpha}(G)\right)^2 = 2Z + 2\sum_{1\le i<j\le n}\eta_{(i)}^{\alpha}(G)\eta_{(j)}^{\alpha}(G). \tag{11}$$

Then,

$$Z = -\sum_{1\le i<j\le n} \eta^{\alpha}_{(i)}(G)\eta^{\alpha}_{(j)}(G) = \left|\sum_{1\le i<j\le n} \eta^{\alpha}_{(i)}(G)\eta^{\alpha}_{(j)}(G)\right|. \tag{12}$$

In the following, we introduce some Lemmas which are helpful for the following proofs of theorems.

Lemma 1 (see [6]). *Let G be a graph with n vertices. Then,*

$$\rho_{\alpha}(G) \ge \frac{2W(G)}{n}, \tag{13}$$

the equality holds if and only if G is a transmission regular graph.

Lemma 2 (see [27]). *Let G be a graph with n vertices. Then,*

$$W(G) \ge \frac{n(n-1)}{2}, \tag{14}$$

the equality holds if and only if $G \cong K_n$. K_n denotes a complete graph with n vertices.

Next, we give some bounds for the α-distance energy of a graph by using the parameter α and the vertex number.

Theorem 1. *Let G be a connected graph with n vertices. Then,*

$$\varsigma_{\alpha}(G) \ge 2(1-\alpha)(n-1), \tag{15}$$

the equality holds if and only if $G \cong K_n$.

Proof. Let $\sigma^{\alpha}_{(1)}(G) \ge \sigma^{\alpha}_{(2)}(G) \ge \ldots \ge \sigma^{\alpha}_{(n)}(G)$ be the eigenvalues of $D_{\alpha}(G)$.

By Lemma 1 and $\alpha \in [0,1]$, we know that $\sigma^{\alpha}_{(1)}(G) \ge (2W(G)/n) \ge (2W(G)\alpha/n)$. Suppose that ι is the largest number such that $\sigma^{\alpha}_{(\iota)}(G) \ge (2W(G)\alpha/n)$. It follows from equation (5) that

$$\begin{aligned}
\varsigma_{(\alpha)}(G) &= \sum_{i=1}^{\iota}\left(\sigma^{\alpha}_{(i)}(G) - \frac{2W(G)\alpha}{n}\right) + \sum_{i=\iota+1}^{n}\left(\frac{2W(G)\alpha}{n} - \sigma^{\alpha}_{(i)}(G)\right)\\
&= \sum_{i=1}^{\iota}\sigma^{\alpha}_{(i)}(G) - \iota\frac{2W(G)\alpha}{n} + (n-\iota)\frac{2W(G)\alpha}{n} - \sum_{i=\iota+1}^{n}\sigma^{\alpha}_{(i)}(G)\\
&= \sum_{i=1}^{\iota}\sigma^{\alpha}_{(i)}(G) - \iota\frac{2W(G)\alpha}{n} - \iota\frac{2W(G)\alpha}{n} + \sum_{i=1}^{n}\sigma^{\alpha}_{(i)}(G)\\
&\quad - \sum_{i=\iota+1}^{n}\sigma^{\alpha}_{(i)}(G)\\
&= 2\sum_{i=1}^{\iota}\left(\sigma^{\alpha}_{(i)}(G) - \frac{2W(G)\alpha}{n}\right)\\
&\ge 2\left(\sigma^{\alpha}_{(1)}(G) - \frac{2W(G)\alpha}{n}\right).
\end{aligned} \tag{16}$$

From Lemmas 1 and 2, we have

$$\begin{aligned}
2\left(\sigma^{\alpha}_{(1)}(G) - \frac{2W(G)\alpha}{n}\right) &\ge 2\left(\frac{2W(G)}{n} - \alpha\frac{2W(G)}{n}\right)\\
&= 4(1-\alpha)\frac{W(G)}{n} \ge 2(1-\alpha)(n-1).
\end{aligned} \tag{17}$$

The above three inequalities are the equality holds if and only if $G \cong K_n$. □

We give some bounds for α-distance energy through the order n, the transmission of vertex and the parameter α based on Cauchy–Schwarz inequalities in the following.

Theorem 2. *Let G be a graph with n vertices. Then,*

$$\sqrt{2Z + n(n-1)p^{(2/n)}} \le \varsigma_{(\alpha)}(G) \le \sqrt{2nZ}, \tag{18}$$

where $p = det|U_{(\alpha)}(G)|$ and $Z = (1-\alpha)^2 S + (\alpha^2/2)\sum_{i=1}^{n}(Tr(v_i) - (2W(G)/n))^2$.

Proof. Let $\eta^{\alpha}_{(1)}(G), \eta^{\alpha}_{(2)}(G), \ldots, \eta^{\alpha}_{(n)}(G)$ denote the eigenvalues of $U_{(\alpha)}(G)$. From Cauchy–Schwarz inequality, we have

$$\left(\varsigma_{(\alpha)}(G)\right)^2 = \left(\sum_{i=1}^{n}\left|\eta^{\alpha}_{(i)}(G)\right| \cdot 1\right)^2 \le \sum_{i=1}^{n}\left|\eta^{\alpha}_{(i)}(G)\right|^2 \sum_{i=1}^{n} 1. \tag{19}$$

Using equations (7) and (8), we have

$$\left(\varsigma_{(\alpha)}(G)\right)^2 \le 2nZ. \tag{20}$$

So,

$$\varsigma_{(\alpha)}(G) \le \sqrt{2nZ}. \tag{21}$$

Similarly, from equation (11), we know

$$\begin{aligned}
\left(\varsigma_{(\alpha)}(G)\right)^2 &= \left(\sum_{i=1}^{n}\left|\eta^{\alpha}_{(i)}(G)\right|\right)^2 = \sum_{i=1}^{n}\left|\eta^{\alpha}_{(i)}(G)\right|^2\\
&\quad + \sum_{i\ne j}\left|\eta^{\alpha}_{(i)}(G)\right|\left|\eta^{\alpha}_{(j)}(G)\right|.
\end{aligned} \tag{22}$$

According to arithmetic-geometric inequality, we have

$$\begin{aligned}
\frac{1}{n(n-1)}\sum_{i\ne j}\left|\eta^{\alpha}_{(i)}(G)\right|\left|\eta^{\alpha}_{(j)}(G)\right| &\ge \prod_{i\ne j}\left(\left|\eta^{\alpha}_{(i)}(G)\right|\left|\eta^{\alpha}_{(j)}(G)\right|\right)^{(1/n(n-1))}\\
&= \left(\prod_{i=1}^{n}\left|\eta^{\alpha}_{(i)}(G)\right|^{(n-1)}\prod_{j=1}^{n}\left|\eta^{\alpha}_{(j)}(G)\right|^{(n-1)}\right)^{(1/n(n-1))}\\
&= \left(\prod_{i=1}^{n}\left|\eta^{\alpha}_{(i)}(G)\right|^{2(n-1)}\right)^{(1/n(n-1))} = \left|\prod_{i=1}^{n}\eta^{\alpha}_{(i)}(G)\right|^{(2/n)}.
\end{aligned} \tag{23}$$

By equations (7) and (9), we have

$$\begin{aligned}
\left(\varsigma_{(\alpha)}(G)\right)^2 &= \sum_{i=1}^{n}\left|\eta^{\alpha}_{(i)}(G)\right|^2 + \sum_{i\ne j}\left|\eta^{\alpha}_{(i)}(G)\right|\left|\eta^{\alpha}_{(j)}(G)\right|\\
&\ge 2Z + n(n-1)p^{(2/n)},
\end{aligned} \tag{24}$$

where $p = \det|U_{(\alpha)}(G)|$.

So,

$$\sqrt{2Z + n(n-1)p^{(2/n)}} \le \varsigma_{(\alpha)}(G) \le \sqrt{2nZ}. \tag{25}$$

In the following, we can obtain another lower bound in terms of the vertex number and the maximum value of $|\eta^{\alpha}_{(i)}(G)|$ of $U_{(\alpha)}(G)$. □

Corollary 1. *Let G be a graph with n vertices, then*

$$\varsigma_{(\alpha)}(G) \le \delta^{\alpha}_{(1)}(G) + \sqrt{(n-1)\left(2Z - \delta^{\alpha}_{(1)}(G)^2\right)}, \tag{26}$$

where $\delta^{\alpha}_{(1)}(G) = max\left\{|\eta^{\alpha}_{(i)}(G)|\right\}$ *and* $Z = (1-\alpha)^2 S + (\alpha^2/2)\sum_{i=1}^{n}(Tr(v_i) - (2W(G)/n))^2$.

Proof. Let $\delta^{\alpha}_{(1)}(G) \ge \delta^{\alpha}_{(2)}(G) \ge, \ldots, \ge \delta^{\alpha}_{(n)}(G)$ be a nonincreasing sequence of $|\eta^{\alpha}_{(i)}(G)|$. From Cauchy–Schwarz inequality, we have

$$\left(\sum_{i=2}^{n} \delta^{\alpha}_{(i)}(G) \cdot 1\right)^2 \le \sum_{i=2}^{n}\left(\delta^{\alpha}_{(i)}(G)\right)^2 \sum_{i=2}^{n} 1. \tag{27}$$

By equation (7), we have

$$\left(\varsigma_{(\alpha)}(G) - \delta^{\alpha}_{(1)}(G)\right)^2 \le (n-1)\left(2Z - \left(\delta^{\alpha}_{(1)}(G)\right)^2\right). \tag{28}$$

So,

$$\varsigma_{(\alpha)}(G) \le \delta^{\alpha}_{(1)}(G) + \sqrt{(n-1)\left(2Z - \left(\delta^{\alpha}_{(1)}(G)\right)^2\right)}. \tag{29}$$

□

In the following, we obtained some new bounds for α-distance energy through the Ozeki [28] and Polya's [29] inequality, respectively.

Lemma 3 (see [29]). *Suppose a_i and b_i are real numbers for $1 \le i \le n$, then*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) \le \frac{1}{4}\left(\sqrt{\frac{M_1 M_2}{m_1 m_2}} + \sqrt{\frac{m_1 m_2}{M_1 M_2}}\right)^2 \left(\sum_{i=1}^{n} a_i b_i\right)^2, \tag{30}$$

where $M_1 = \max_{1\le i\le n} a_i$, $M_2 = \max_{1\le i\le n} b_i$, $m_1 = \min_{1\le i\le n} a_i$, *and* $m_2 = \min_{1\le i\le n} b_i$.

Lemma 4 (see [28]). *If a_i and b_i are real numbers for $1 \le i \le n$, then*

$$\left(\sum_{i=1}^{n} a_i^2\right)\left(\sum_{i=1}^{n} b_i^2\right) - \left(\sum_{i=1}^{n} a_i b_i\right)^2 \le \frac{n^2}{4}(M_1 M_2 - m_1 m_2)^2, \tag{31}$$

where $M_1 = \max_{1\le i\le n} a_i$, $M_2 = \max_{1\le i\le n} b_i$, $m_1 = \min_{1\le i\le n} a_i$, *and* $m_2 = \min_{1\le i\le n} b_i$.

Theorem 3. *Let G be a graph with n vertices. Then,*

$$\varsigma_{(\alpha)}(G) \ge \frac{2\sqrt{2nZ}\sqrt{\delta^{\alpha}_{(1)}(G)\delta^{\alpha}_{(n)}(G)}}{\delta^{\alpha}_{(1)}(G) + \delta^{\alpha}_{(n)}(G)}, \tag{32}$$

where $\delta^{\alpha}_{(1)}(G)$ *and* $\delta^{\alpha}_{(n)}(G)$ *are the largest and the smallest of* $|\eta^{\alpha}_{(i)}(G)|$, *respectively, and* $Z = (1-\alpha)^2 S + (\alpha^2/2)\sum_{i=1}^{n}(Tr(v_i) - (2W(G)/n))^2$.

Proof. Let $\delta^{\alpha}_{(1)}(G) \ge \delta^{\alpha}_{(2)}(G) \ge, \ldots, \ge \delta^{\alpha}_{(n)}(G)$ be a nonincreasing sequence of $|\eta^{\alpha}_{(i)}(G)|$ and let $a_i = \delta^{\alpha}_{(i)}(G)$ and $b_i = 1$, where $i = 1, \ldots, n$. By Lemma 3, we have

$$\left(\sum_{i=1}^{n}\left(\delta^{\alpha}_{(i)}(G)\right)^2\right)\left(\sum_{i=1}^{n} 1\right) \le \frac{1}{4}\left(\sqrt{\frac{\delta^{\alpha}_{(1)}(G)}{\delta^{\alpha}_{(n)}(G)}} + \sqrt{\frac{\delta^{\alpha}_{(n)}(G)}{\delta^{\alpha}_{(1)}(G)}}\right)^2 \cdot \left(\sum_{i=1}^{n} \delta^{\alpha}_{(i)}(G)\right)^2. \tag{33}$$

By equation (8), we have

$$2nZ \le \frac{1}{4}\left(\sqrt{\frac{\delta^{\alpha}_{(1)}(G)}{\delta^{\alpha}_{(n)}(G)}} + \sqrt{\frac{\delta^{\alpha}_{(n)}(G)}{\delta^{\alpha}_{(1)}(G)}}\right)^2 \left(\varsigma_{(\alpha)}(G)\right)^2. \tag{34}$$

Thus,

$$\varsigma_{(\alpha)}(G) \ge \frac{2\sqrt{2nZ}\sqrt{\delta^{\alpha}_{(1)}(G)\delta^{\alpha}_{(n)}(G)}}{\delta^{\alpha}_{(1)}(G) + \delta^{\alpha}_{(n)}(G)}. \tag{35}$$

□

Theorem 4. *Let G be a graph with n vertices. Then,*

$$\varsigma_{(\alpha)}(G) \ge \sqrt{2nZ - \frac{n^2}{4}\left(\delta^{\alpha}_{(1)}(G) - \delta^{\alpha}_{(n)}(G)\right)^2}, \tag{36}$$

where $\delta^{\alpha}_{(1)}(G)$ *and* $\delta^{\alpha}_{(n)}(G)$ *are the largest and the smallest of* $|\eta^{\alpha}_{(i)}(G)|$, *respectively, and* $Z = (1-\alpha)^2 S + (\alpha^2/2)\sum_{i=1}^{n}(Tr(v_i) - (2W(G)/n))^2$.

Proof. Let $\delta^{\alpha}_{(1)}(G) \ge \delta^{\alpha}_{(2)}(G) \ge, \ldots, \ge \delta^{\alpha}_{(n)}(G)$ be a nonincreasing sequence of $|\eta^{\alpha}_{(i)}(G)|$. According to Lemma 4, let $a_i = \delta^{\alpha}_{(i)}(G)$ and $b_i = 1$, where $i = 1, \ldots, n$, we have

$$\left(\sum_{i=1}^{n}\left(\delta^{\alpha}_{(i)}(G)\right)^2\right)\left(\sum_{i=1}^{n} 1\right) - \left(\sum_{i=1}^{n} \delta^{\alpha}_{(i)}(G) \cdot 1\right)^2 \le \frac{n^2}{4}\left(\delta^{\alpha}_{(1)}(G) - \delta^{\alpha}_{(n)}(G)\right)^2. \tag{37}$$

By equation (7), we have

$$2nZ - \left(\varsigma_{(\alpha)}(G)\right)^2 \le \frac{n^2}{4}\left(\delta^{\alpha}_{(1)}(G) - \delta^{\alpha}_{(n)}(G)\right)^2. \tag{38}$$

Then,

$$\varsigma_{(\alpha)}(G) \ge \sqrt{2nZ - \frac{n^2}{4}\left(\delta^{\alpha}_{(1)}(G) - \delta^{\alpha}_{(n)}(G)\right)^2}. \tag{39}$$

□

Lemma 5 (see [30]). *Let $x_1 > x_2 \ge \ldots \ge x_n > 0$ be n real numbers. Then,*

$$\sum_{i=1}^{n} \left| x_i - M \right| < \frac{n}{2} x_1, \tag{40}$$

where $M = (\sum_{1=1}^{n} x_i/n)$.

It follows from the above Proposition the following result holds directly.

Proposition 3. *For a graph G with n vertices, let* $\eta_{(1)}^{\alpha}(G)$ *be the largest eigenvalue of* $D_\alpha(G)$. *For* $\alpha \in [(1/2), 1)$,

$$\varsigma_{(\alpha)}(G) \leq \frac{n}{2} \eta_{(1)}^{\alpha}(G). \tag{41}$$

3. Bounds for the α-Distance Estrada Index of Graphs

In this section, some bounds for α-distance Estrada index are obtained in terms of Wiener index, the transmission of the vertex, spectral radius of $D_\alpha(G)$, and the vertex number. Furthermore, we give the relation between α-distance Estrada index and α-distance energy.

Next, we establish some bounds on the α-distance Estrada index.

Lemma 6 (see [31]). *Let* $x_1, x_2, \ldots, x_n$ *be nonnegative real numbers. Then, for* $k \geq 2$,

$$\sum_{i=1}^{n} x_i^k \leq \left(\sum_{i=1}^{n} x_i^2 \right)^{(k/2)}. \tag{42}$$

Theorem 5. *Let G be a graph with n vertices. Then,*

$$\mathrm{DEE}_{(\alpha)}(G) \leq n + 2\alpha W(G) - 1 - \omega + e^{\omega}, \tag{43}$$

where $\omega = \sqrt{\alpha^2 \sum_{i=1}^{n} Tr^2(v_i) + 2(1-\alpha)^2 S}$.

Proof. Let $\sigma_{(1)}^{\alpha}(G), \ldots, \sigma_{(n)}^{\alpha}(G)$ be the eigenvalues of $D_\alpha(G)$. From equation (5) and Lemma 6, we have

$$\begin{aligned}
\mathrm{DEE}_{(\alpha)}(G) &= n + 2\alpha W(G) + \sum_{i=1}^{n} \sum_{k=2}^{\infty} \frac{\left(\sigma_{(i)}^{\alpha}(G)\right)^k}{k!} \\
&\leq n + 2\alpha W(G) + \sum_{i=1}^{n} \sum_{k=2}^{\infty} \frac{\left|\sigma_{(i)}^{\alpha}(G)\right|^k}{k!} \\
&= n + 2\alpha W(G) + \sum_{k=2}^{\infty} \frac{1}{k!} \sum_{i=1}^{n} \left|\sigma_{(i)}^{\alpha}(G)\right|^k \\
&\leq n + 2\alpha W(G) + \sum_{k=2}^{\infty} \frac{1}{k!} \left(\sum_{i=1}^{n} \left(\sigma_{(i)}^{\alpha}(G)\right)^2 \right)^{(k/2)} \\
&= n + 2\alpha W(G) + \sum_{k=2}^{\infty} \frac{1}{k!} \left(\alpha^2 \sum_{i=1}^{n} Tr^2(v_i) + 2(1-\alpha)^2 S \right)^{(k/2)} \\
&= n + 2\alpha W(G) - 1 - \omega + \sum_{k=0}^{\infty} \frac{1}{k!} \omega^k \\
&= n + 2\alpha W(G) - 1 - \omega + e^{\omega},
\end{aligned} \tag{44}$$

where $\omega = \sqrt{\alpha^2 \sum_{i=1}^{n} Tr^2(v_i) + 2(1-\alpha)^2 S}$. □

In the following, we obtained a lower bound on the α-distance Estrada index by arithmetic-geometric inequality.

Theorem 6. *Let G be a graph with n vertices. Then,*

$$\mathrm{DEE}_{(\alpha)}(G) \geq \sqrt{n + 4\alpha W(G) + n(n-1) e^{(4\alpha W(G)/n)}}. \tag{45}$$

Proof. Let $\sigma_{(1)}^{\alpha}(G), \ldots, \sigma_{(n)}^{\alpha}(G)$ be the eigenvalues of $D_\alpha(G)$. Then,

$$\left(\mathrm{DEE}_{(\alpha)}(G)\right)^2 = \sum_{i=1}^{n} e^{2\sigma_{(i)}^{\alpha}(G)} + 2 \sum_{1 \leq i < j \leq n} e^{\sigma_{(i)}^{\alpha}(G)} e^{\sigma_{(j)}^{\alpha}(G)}. \tag{46}$$

From arithmetic-geometric inequality and equation (5), we obtain

$$\begin{aligned}
2 \sum_{1 \leq i < j \leq n} e^{\sigma_{(i)}^{\alpha}(G)} e^{\sigma_{(j)}^{\alpha}(G)} &\geq n(n-1) \left(\prod_{1 \leq i < j \leq n} e^{\sigma_{(i)}^{\alpha}(G)} e^{\sigma_{(j)}^{\alpha}(G)} \right)^{(2/n(n-1))} \\
&= n(n-1) \left(\left(\prod_{i=1}^{n} e^{\sigma_{(i)}^{\alpha}(G)} \right)^{n-1} \right)^{(2/n(n-1))} \\
&= n(n-1) \left(e^{\sum_{i=1}^{n} \sigma_{(i)}^{\alpha}(G)} \right)^{(2/n)} \\
&= n(n-1) e^{(4\alpha W(G)/n)}.
\end{aligned} \tag{47}$$

By means of a power-series expansion, we have

$$\begin{aligned}
\sum_{i=1}^{n} e^{2\sigma_{(i)}^{\alpha}(G)} &= \sum_{i=1}^{n} \sum_{k=0}^{\infty} \frac{\left(2\sigma_{(i)}^{\alpha}(G)\right)^k}{k!} \\
&= n + 4\alpha W(G) + \sum_{i=1}^{n} \sum_{k=2}^{\infty} \frac{\left(2\sigma_{(i)}^{\alpha}(G)\right)^k}{k!} \\
&\geq n + 4\alpha W(G).
\end{aligned} \tag{48}$$

□

By substituting equations (47) and (48) in equation (46), we see that

$$\mathrm{DEE}_{(\alpha)}(G) \geq \sqrt{n + 4\alpha W(G) + n(n-1) e^{(4\alpha W(G)/n)}}. \tag{49}$$

Theorem 7. *Let G be a graph with n vertices. Then,*

$$\mathrm{DEE}_{(\alpha)}(G) \geq e^{(2W(G)/n)} + (n-1) + 2\alpha W(G) - \frac{2W(G)}{n}. \tag{50}$$

Proof. Let $f(x) = (x-1) - \ln x$, where $x > 0$. Obviously, $f(x)$ is a decreasing function when $x \in (0, 1]$, and $f(x)$ is increasing when $x \in [1, +\infty)$. Then, $f(x) \geq f(1) = 0$, that is,

$$x \geq 1 + \ln x, \quad x > 0, \tag{51}$$

and the equality holds if and only if $x = 1$. So, by this function and equation (5), we have

$$\begin{aligned}\mathrm{DEE}_{(\alpha)}(G) &\geq e^{\sigma^{\alpha}_{(1)}(G)} + (n-1) + \sum_{k=2}^{n} \ln e^{\sigma^{\alpha}_{(k)}(G)} \\ &= e^{\sigma^{\alpha}_{(1)}(G)} + (n-1) + \sum_{k=2}^{n} \sigma^{\alpha}_{(k)}(G) \\ &= e^{\sigma^{\alpha}_{(1)}(G)} + (n-1) + 2\alpha W(G) - \sigma^{\alpha}_{(1)}(G),\end{aligned} \tag{52}$$

where $\sigma^{\alpha}_{(1)}(G), \ldots, \sigma^{\alpha}_{(n)}(G)$ are the eigenvalues of $D_{\alpha}(G)$.

Let $\Gamma(x) = e^{x} + (n-1) + 2\alpha W(G) - x$, where $x > 0$. Clearly, $\Gamma(x)$ is an increasing function when $x \in (0, +\infty)$. From Lemma 1, we have

$$\sigma^{\alpha}_{(1)}(G) \geq \frac{2W(G)}{n} \geq 0. \tag{53}$$

Then,

$$\Gamma\left(\sigma^{\alpha}_{(1)}(G)\right) \geq \Gamma\left(\frac{2W(G)}{n}\right). \tag{54}$$

Hence,

$$\mathrm{DEE}_{(\alpha)}(G) \geq e^{(2W(G)/n)} + (n-1) + 2\alpha W(G) - \frac{2W(G)}{n}. \tag{55}$$

From Theorem 7, we have the following result. □

Corollary 2. *Let G be a transmission regular graph with n vertices. Let $Tr(u) = r$ for each $u \in V(G)$. Then,*

$$\mathrm{DEE}_{(\alpha)}(G) \geq e^{r} + (n-1) + \alpha n r - r. \tag{56}$$

We are inspired by literature [32], and we give Theorems 8 and 9 as follows.

Lemma 7 (see [33]). *For $a_1, a_2, \ldots, a_n \geq 0$ and $p_1, p_2, \ldots, p_n \geq 0$ such that $\sum_{i=1}^{n} p_i = 1$. Then,*

$$\sum_{i=1}^{n} p_i a_i - \prod_{i=1}^{n} a_i^{p_i} \geq nT\left(\frac{1}{n}\sum_{i=1}^{n} a_i - \prod_{i=1}^{n} a_i^{(1/n)}\right), \tag{57}$$

where $T = \min\{p_1, p_2, \ldots, p_n\}$. Equality holds if and only if $a_1 = a_2 = , \ldots, = a_n$.

Theorem 8. *Let G be a graph with n vertices. Then,*

$$\mathrm{DEE}_{(\alpha)}(G) \geq e^{\sigma^{\alpha}_{(1)}(G)} + 2(n-1)\Delta - (n-1)e^{(2\alpha W(G)/n)}, \tag{58}$$

where $\Delta = e^{(2(n-1)\alpha W(G) + n(\sigma^{\alpha}_{(2)}(G) + \ldots + \sigma^{\alpha}_{(n)}(G))/2n(n-1))}$. Equality holds if and only if $G \cong nK_1$.

Proof. Let $p_1 = (1/2n)$, $p_i = (2n - 1/2n(n-1))$ for $i = 2, \ldots, n$, $a_i = e^{\sigma^{\alpha}_{(i)}(G)}$ for $i = 2, \ldots, n$. Obviously, $T = \min\{(1/2n), (2n - 1/2n(n-1))\} = (1/2n)$, and according to Lemma 7, we have

$$\begin{aligned}&\frac{e^{\sigma^{\alpha}_{(1)}(G)}}{2n} + \frac{2n-1}{2n(n-1)}\sum_{i=2}^{n} e^{\sigma^{\alpha}_{(i)}(G)} - \Delta \\ &\geq \frac{1}{2}\left(\frac{1}{n}\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)} - \prod_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)/n}\right),\end{aligned} \tag{59}$$

where $\Delta = e^{(\sigma^{\alpha}_{(1)}(G)/2n)} \prod_{i=2}^{n} e^{((2n-1)\sigma^{\alpha}_{(i)}(G)/2n(n-1))}$.

By equation (5), we have

$$\begin{aligned}\Delta &= e^{\left(\sigma^{\alpha}_{(1)}(G)/2n\right)} \prod_{i=2}^{n} e^{\left((2n-1)\sigma^{\alpha}_{(i)}(G)/2n(n-1)\right)} \\ &= e^{\left((n-1)\left(2\alpha W(G) - \sum_{i=2}^{n}\sigma^{\alpha}_{(i)}(G)\right)/2n(n-1)\right)} \\ &\quad \cdot e^{\left((2n-1)\sum_{i=2}^{n}\sigma^{\alpha}_{(i)}(G)/2n(n-1)\right)} \\ &= e^{\left(2(n-1)\alpha W(G) + n\left(\sigma^{\alpha}_{(2)}(G) + \ldots + \sigma^{\alpha}_{(n)}(G)\right)/2n(n-1)\right)}.\end{aligned} \tag{60}$$

Since

$$\begin{aligned}&\frac{e^{\sigma^{\alpha}_{(1)}(G)}}{2n} + \frac{2n-1}{2n(n-1)}\left(\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G) - e^{\sigma^{\alpha}_{(1)}(G)}}\right) - \Delta \\ &\geq \frac{1}{2n}\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)} - \frac{1}{2}e^{(2\alpha W(G)/n)},\end{aligned} \tag{61}$$

then

$$\begin{aligned}\mathrm{DEE}_{(\alpha)}(G) &= \sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)} \geq e^{\sigma^{\alpha}_{(1)}(G)} + 2(n-1)\Delta \\ &\quad - (n-1)e^{(2\alpha W(G)/n)}.\end{aligned} \tag{62}$$

Equality holds, that is, $e^{\sigma^{\alpha}_{(1)}(G)} = e^{\sigma^{\alpha}_{(2)}(G)} = , \ldots, = e^{\sigma^{\alpha}_{(n)}(G)}$ if and only if $G \cong nK_1$. □

Lemma 8 (see [34]). *For $a_1, a_2, \ldots, a_n \geq 0$. Then,*

$$\begin{aligned}&n\left(\frac{1}{n}\sum_{i=1}^{n} a_i - \left(\prod_{i=1}^{n} a_i\right)^{(1/n)}\right) \leq \Psi \\ &\leq n(n-1)\left(\frac{\sum_{i=1}^{n} a_i}{n} - \left(\prod_{i=1}^{n} a_i\right)^{(1/n)}\right),\end{aligned} \tag{63}$$

where $\Psi = n\sum_{i=1}^{n} a_i - (\sum_{i=1}^{n} \sqrt{a_i})^2$.

Theorem 9. *Let G be a graph with n vertices. Then,*

$$\begin{aligned}&\frac{\left(\sum_{i=1}^{n} e^{\left(\sigma^{\alpha}_{(i)}(G)/2\right)}\right)^2 - ne^{(2\alpha W(G)/n)}}{n-1} \leq \mathrm{DEE}_{(\alpha)}(G) \\ &\leq \left(\sum_{i=1}^{n} e^{\left(\sigma^{\alpha}_{(i)}(G)/2\right)}\right) - n(n-1)e^{(2\alpha W(G)/n)}.\end{aligned} \tag{64}$$

Proof. Let $a_i = e^{\sigma^{\alpha}_{(i)}(G)}$ for $i = 1, 2, \ldots, n$, by Lemma 8, we have

$$\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)} - n\left(\prod_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)}\right)^{(1/n)} \le \Psi \le n(n-1)\left(\frac{\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)}}{n} - \left(\prod_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)}\right)^{(1/n)}\right), \tag{65}$$

where $\Psi = n\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)} - \left(\sum_{i=1}^{n} e^{(\sigma^{\alpha}_{(i)}(G)/2)}\right)^2$.

Analyzing the left and right side of the previous inequality, respectively, by equation (5), we have

$$\begin{aligned}
\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)} - n\left(\prod_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)}\right)^{(1/n)} &\le n\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)} - \left(\sum_{i=1}^{n} e^{\left(\sigma^{\alpha}_{(i)}(G)/2\right)}\right)^2 \\
(n-1)\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)} &\ge \left(\sum_{i=1}^{n} e^{\left(\sigma^{\alpha}_{(i)}(G)/2\right)}\right)^2 - ne^{(2\alpha W(G)/n)} \\
\mathrm{DEE}_{(\alpha)}(G) &\ge \frac{\left(\sum_{i=1}^{n} e^{\left(\sigma^{\alpha}_{(i)}(G)/2\right)}\right)^2 - ne^{(2\alpha W(G)/n)}}{n-1}, \\
n\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)} - \left(\sum_{i=1}^{n} e^{\left(\sigma^{\alpha}_{(i)}(G)/2\right)}\right)^2 &\le n(n-1)\left(\frac{\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)}}{n} - \left(\prod_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G)}\right)^{(1/n)}\right) \\
\mathrm{DEE}_{(\alpha)}(G) &\le \left(\sum_{i=1}^{n} e^{\left(\sigma^{\alpha}_{(i)}(G)/2\right)}\right) - n(n-1)e^{(2\alpha W(G)/n)}.
\end{aligned} \tag{66}$$

□

Also, the relation between α-distance Estrada index and α-distance energy are established.

Theorem 10. *Let G be a graph with n vertices. Then,*

$$\mathrm{DEE}_{(\alpha)}(G) \le e^{(2\alpha W(G)/n)}\left(n - 1 - \varsigma_{(\alpha)}(G) + e^{\varsigma_{(\alpha)}(G)}\right). \tag{67}$$

Proof. By the definition of α-distance energy, we have

$$\begin{aligned}
\mathrm{DEE}_{(\alpha)}(G) &= e^{(2\alpha W(G)/n)}\left(\sum_{i=1}^{n} e^{\sigma^{\alpha}_{(i)}(G) - (2\alpha W(G)/n)}\right) \\
&= e^{(2\alpha W(G)/n)}\left(n + \sum_{i=1}^{n}\sum_{k=2}^{\infty}\frac{\left(\sigma^{\alpha}_{(i)}(G) - (2\alpha W(G)/n)\right)^k}{k!}\right) \\
&\le e^{(2\alpha W(G)/n)}\left(n + \sum_{i=1}^{n}\sum_{k=2}^{\infty}\frac{\left|\sigma^{\alpha}_{(i)}(G) - (2\alpha W(G)/n)\right|^k}{k!}\right) \\
&\le e^{(2\alpha W(G)/n)}\left(n + \sum_{k=2}^{\infty}\frac{1}{k!}\left(\varsigma_{(\alpha)}(G)\right)^k\right) \\
&= e^{(2\alpha W(G)/n)}\left(n - 1 - \varsigma_{(\alpha)}(G) + e^{\varsigma_{(\alpha)}(G)}\right).
\end{aligned} \tag{68}$$

□

4. Conclusion

It is well-known that the graph energy and the Estrada index of graphs are important topics in graph theory. The α-distance matrix is an extension of the distance matrix. And the α-distance energy and α-distance Estrada index are generalized distance energy and distance Estrada index of graphs, respectively. In this paper, we establish some bounds on α-distance energy and α-distance Estrada index of G. Furthermore, a new lower bound for the α-distance Estrada index in relation to the α-distance energy of the graph G is given.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare to have no conflicts of interest.

Authors' Contributions

All authors contributed equally to the writing of this paper. All authors read and approved the final manuscript.

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