Research Article

Periodic Property and Asymptotic Behavior for a Discrete Ratio-Dependent Food-Chain System with Delays

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In this paper, a discrete ratio-dependent food-chain system with delay is investigated. By using Gaines and Mawhin's continuation theorem of coincidence degree theory and the method of Lyapunov function, a set of sufficient conditions for the existence of positive periodic solutions and global asymptotic stability of the model are established.

1. Introduction

The past decades have witnessed a great deal of interest in the periodic phenomena of predator-prey systems. For example, Zhang and Tian [1] investigated the multiple periodic solutions of a generalized predator-prey system with exploited terms. Zhang and Wang [2] analyzed the existence and global attractivity of a positive periodic solution for a generalized delayed prey-predator system. Li et al. [3] studied multiple positive periodic solutions of n species delay competition systems with harvesting terms. Ding et al. [4] made a detailed discussion on the periodic solution of a Gause-type predator-prey systems with impulse. Shen and Li [5] obtained a set of sufficient conditions for the existence of at least one strictly positive periodic solution and the uniqueness and global attractivity of positive periodic solution for an impulsive predator-prey model with dispersion and time delays. For more knowledge about the periodic solutions of predator-prey models, one can see [6–12]. For the papers mentioned above, it shall be pointed out that many investigations have been performed to analyze the dynamical behavior on biological species by using continuous or impulsive mathematical models [6–9]. It

has been widely argued and accepted that difference equations often occur in numerous setting and forms, both in mathematics itself and in its applications to statistics, computing, electrical circuit analysis, dynamics, economics, biology, and other fields [13]. In recent years, Xu et al. [14] have studied the persistence and stability of the following ratio-dependent food-chain system with delay:

$$\begin{cases} \dot{x}_1(t) = x_1(t) \left[a_1 - a_{11}x_1(t - \tau_1) - \frac{a_{12}x_2(t)}{m_{12}x_2(t) + x_1(t)} \right], \\ \dot{x}_2(t) = x_2(t) \left[-a_2 + \frac{a_{21}x_1(t - \tau_2)}{m_{12}x_2(t - \tau_2) + x_1(t - \tau_2)} - \frac{a_{23}x_3(t)}{m_{23}x_3(t) + x_2(t)} \right], \\ \dot{x}_3(t) = x_3(t) \left[-a_3 + \frac{a_{32}x_2(t - \tau_3)}{m_{23}x_3(t - \tau_3) + x_2(t - \tau_3)} \right], \end{cases} \quad (1)$$

where $x_1(t)$, $x_2(t)$, and $x_3(t)$ denote densities of the prey, predator, and the top predator populations at time t , respectively, $\tau_1 \geq 0$ is constant time delay due to negative feedback of the prey, and $\tau_2 \geq 0$ and $\tau_3 \geq 0$ are constant time delays due to gestation. a_i ($i = 1, 2, 3$), a_{11} , a_{12} , a_{21} , a_{23} , a_{32} , m_{12} , and m_{23} are all positive constants. In detail, one can see [14].

In real life, many biological and environmental parameters do vary in time (for example, naturally subject to

seasonal fluctuations). However, Xu et al. [14] did not involve the varying parameters of the food-chain model. To describe the object relationship between predator population and prey population, we modify system (1) as the following nonautonomous ratio-dependent food-chain system with varying delay:

$$\begin{cases} \dot{x}_1(t) = x_1(t) \left[a_1(t) - a_{11}(t)x_1(t - \tau_1(t)) - \frac{a_{12}(t)x_2(t)}{m_{12}(t)x_2(t) + x_1(t)} \right], \\ \dot{x}_2(t) = x_2(t) \left[-a_2(t) + \frac{a_{21}(t)x_1(t - \tau_2(t))}{m_{12}(t)x_2(t - \tau_2(t)) + x_1(t - \tau_2(t))} - \frac{a_{23}(t)x_3(t)}{m_{23}(t)x_3(t) + x_2(t)} \right], \\ \dot{x}_3(t) = x_3(t) \left[-a_3(t) + \frac{a_{32}(t)x_2(t - \tau_3(t))}{m_{23}(t)x_3(t - \tau_3(t)) + x_2(t - \tau_3(t))} \right]. \end{cases} \quad (2)$$

Many authors [15–18] argue that discrete time models governed by difference equations are more appropriate to describe the dynamics relationship among populations than continuous ones when the populations have nonoverlapping generations. Moreover, discrete time models can also provide efficient models of continuous ones for numerical simulation.

In order to reveal the dynamic relationship of predator and prey and explain the stability law of both species by applying the computer simulations, we think that it is reasonable to study time ratio-dependent food-chain systems governed by difference equations. Following the lines of Wiener [19] and Fan and Wang [20], we obtain the discrete time analogue of system (2):

$$\begin{cases} x_1(k+1) = x_1(k) \exp \left\{ a_1(k) - a_{11}(k)x_1(k - \tau_1(k)) - \frac{a_{12}(k)x_2(k)}{m_{12}(k)x_2(k) + x_1(k)} \right\}, \\ x_2(k+1) = x_2(k) \exp \left\{ -a_2(k) + \frac{a_{21}(k)x_1(k - \tau_2(k))}{m_{12}(k)x_2(k - \tau_2(k)) + x_1(k - \tau_2(k))} - \frac{a_{23}(k)x_3(k)}{m_{23}(k)x_3(k) + x_2(k)} \right\}, \\ x_3(k+1) = x_3(k) \exp \left\{ -a_3(k) + \frac{a_{32}(k)x_2(k - \tau_3(k))}{m_{23}(k)x_3(k - \tau_3(k)) + x_2(k - \tau_3(k))} \right\}, \end{cases} \quad (3)$$

where $k = 0, 1, 2, \dots$ and all the variables and parameters have the same biological meanings as those in system (1).

The main task of this article is to discuss the dynamics of system (3). That is, applying Mawhin's continuous theorem [21] to study the existence of positive periodic solutions of (3) and investigating the global asymptotical stability of system (3) by means of the method of Lyapunov function. The main innovation point lies in better applying computer simulation to explain the changing law of biological population.

The remainder of the paper is organized as follows. In Section 2, a easily verifiable sufficient condition for the existence of positive solutions of difference equations is obtained by the continuation theorem and priori estimations. The sufficient condition for the global asymptotical stability of system (3) when all the delays are zero is presented in Section 3. In Section 4, we give some computer simulations.

2. Existence of Positive Periodic Solutions

For convenience and simplicity in the following discussion, we always use the following notations throughout the paper:

$$\begin{aligned} I_\omega &:= \{0, 1, 2, \dots, \omega - 1\}, \\ \bar{f} &:= \frac{1}{\omega} \sum_{k=0}^{\omega-1} f(k), \\ f^L &:= \min_{k \in \mathbb{Z}} \{f(k)\}, \\ f_M &:= \max_{k \in \mathbb{Z}} \{f(k)\}, \end{aligned} \quad (4)$$

where $f(k)$ is an ω -periodic sequence of real numbers defined for $k \in \mathbb{Z}$. Let $\mathbb{Z}$ denote the integer number, $\mathbb{R}$ denote the real number, $\mathbb{R}^+$ denote the nonnegative real number, and $\mathbb{R}^3$ denote the three-dimensional real vector.

We always assume that

(H1) a_i ($i = 1, 2, 3$), $a_{11}, a_{12}, a_{21}, a_{23}, a_{32}, m_{12}, m_{23}$:
 $\mathbb{Z} \rightarrow \mathbb{R}^+$ are ω periodic, i.e.,

$$\begin{aligned} a_i(k + \omega) &= a_i(k) \quad (i = 1, 2, 3), \\ a_{11}(k + \omega) &= a_{11}(k), \\ a_{12}(k + \omega) &= a_{12}(k), \\ a_{21}(k + \omega) &= a_{21}(k), \\ a_{23}(k + \omega) &= a_{23}(k), \\ m_{12}(k + \omega) &= m_{12}(k), \\ m_{23}(k + \omega) &= m_{23}(k), \quad \text{for any } k \in \mathbb{Z}. \end{aligned} \quad (5)$$

In order to explore the existence of positive periodic solutions of (3) and for the reader's convenience, we shall first introduce a few concepts and results without proof, borrowing from Gaines and Mawhin [21].

Let X and Y be normed vector spaces, $L: \text{Dom } L \subset X \rightarrow Y$ is a linear mapping, and $N: X \rightarrow Y$ is a continuous mapping. The mapping L will be called a Fredholm mapping of index zero if $\dim \text{Ker } L = \text{codim Im } L < +\infty$ and $\text{Im } L$ is closed in Y . If L is a Fredholm mapping of index zero and there exist continuous projectors $P: X \rightarrow X$ and $Q: Y \rightarrow Y$ such that $\text{Im } P = \text{Ker } L$ and $\text{Im } L = \text{Ker } Q = \text{Im } (I - Q)$, It follows that $L|_{\text{Dom } L \cap \text{Ker } P}: (I - P)X \rightarrow \text{Im } L$ is invertible. We denote the inverse of that map by K_P . If Ω is an open-bounded subset of X , the mapping N will be called L -compact on $\overline{\Omega}$ if $QN(\overline{\Omega})$ is bounded and $K_P(I - Q)N: \overline{\Omega} \rightarrow X$ is compact. Since $\text{Im } Q$ is isomorphic to $\text{Ker } L$, there exist isomorphisms $J: \text{Im } Q \rightarrow \text{Ker } L$.

Lemma 1 (see [21], continuation theorem). Let L be a Fredholm mapping of index zero, and let N be L -compact on $\overline{\Omega}$. Suppose

- (a) For each $\lambda \in (0, 1)$, every solution x of $Lx = \lambda Nx$ is such that $x \notin \partial\Omega$
- (b) $QNx \neq 0$ for each $x \in \text{Ker } L \cap \partial\Omega$, and $\deg\{JQN, \Omega \cap \text{Ker } L, 0\} \neq 0$

Then, the equation $Lx = Nx$ has at least one solution lying in $\text{Dom } L \cap \Omega$.

Lemma 2 (see [20]). Let $g: \mathbb{Z} \rightarrow \mathbb{R}$ be ω periodic, i.e., $g(k + \omega) = g(k)$; then, for any fixed $k_1, k_2 \in I_\omega$ and any $k \in \mathbb{Z}$, one has

$$\begin{aligned} g(k) &\leq g(k_1) + \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|, \\ g(k) &\geq g(k_2) - \sum_{s=0}^{\omega-1} |g(s+1) - g(s)|. \end{aligned} \quad (6)$$

Lemma 3. Assume that $\bar{a}_2 < \bar{a}_{21}$ and $\bar{a}_3 < \bar{a}_{32}$; then, the system algebraic equations

$$\begin{cases} \bar{a}_1 - \bar{a}_{11}v_1 = 0, \\ \bar{a}_2 - \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{21}(k)v_1}{m_{12}(k)v_2 + v_1} = 0, \\ \bar{a}_3 - \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{32}(k)v_2}{m_{23}(k)v_3 + v_2} = 0, \end{cases} \quad (7)$$

have a unique solution (v_1^*, v_2^*, v_3^*) with $v_i^* > 0$, $i = 1, 2, 3$.

Proof. Obviously, $v_1 = \bar{a}_1/\bar{a}_{11} > 0$. Substituting $v_1 = \bar{a}_1/\bar{a}_{11}$ into the second equation of system (7) and simplifying, we obtain

$$\bar{a}_2 - \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{21}(k)}{(m_{12}(k)\bar{a}_{11}/\bar{a}_1)v_2 + 1} = 0. \quad (8)$$

In the following, we define the function:

$$f(\theta) = \bar{a}_2 - \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{21}(k)}{(m_{12}(k)\bar{a}_{11}/\bar{a}_1)\theta + 1}, \quad \theta \geq 0. \quad (9)$$

It is easy to see that

$$f(0) = \bar{a}_2 - \frac{1}{\omega} \sum_{k=0}^{\omega-1} a_{21}(k) = \bar{a}_2 - \bar{a}_{21} < 0, \quad (10)$$

$$\lim_{\theta \rightarrow +\infty} f(\theta) = \bar{a}_2 > 0. \quad (11)$$

Then, it follows from the zero-point theorem and monotonicity of $f(\theta)$ that there exists a unique $v_2^* > 0$ such that $f(v_2^*) = 0$. Similarly, substituting v_2^* into the third equation of system (7) and simplifying, we have

$$\bar{a}_3 - \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{32}(k)}{(m_{23}(k)/v_2^*)v_3 + 1} = 0. \quad (12)$$

We define the function

$$g(\gamma) = \bar{a}_3 - \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{32}(k)}{(m_{23}(k)/v_2^*)\gamma + 1}, \quad \gamma \geq 0. \quad (13)$$

Clearly,

$$g(0) = \bar{a}_3 - \frac{1}{\omega} \sum_{k=0}^{\omega-1} a_{32}(k) = \bar{a}_3 - \bar{a}_{32} < 0, \quad (14)$$

$$\lim_{\theta \rightarrow +\infty} g(\gamma) = \bar{a}_3 > 0. \quad (15)$$

Then, it follows from the zero-point theorem and monotonicity of $g(\gamma)$ that there exists a unique $v_3^* > 0$ such that $g(v_3^*) = 0$. The proof is complete.

Define

$$I_3 = \{z = \{z(k)\}: z(k) = (z_1(k), z_2(k), z_3(k))^T \in R^3, k \in Z\}. \quad (16)$$

Let $I^\omega \subset I_3$ denote the subspace of all ω periodic sequences equipped with the usual supremum norm $\|\cdot\|$, i.e., $\|z\| = |z_1(k)| + |z_2(k)| + |z_3(k)|$ for any $z = \{z(k): k \in Z\} \in I^\omega$. It is easy to show that I^ω is a finite-dimensional Banach space.

Let

$$I_0^\omega = \left\{ z = \{z(k)\} \in I^\omega: \sum_{k=0}^{\omega-1} z(k) = 0 \right\}, \quad (17)$$

$$I_c^\omega = \{z = \{z(k)\} \in I^\omega: z(k) = h \in R^3, k \in Z\}.$$

Then, it follows that I_0^ω and I_c^ω are both closed linear subspaces of I^ω and

$$I^\omega = I_0^\omega + I_c^\omega, \dim I_c^\omega = 3. \quad (18)$$

In the following, we will ready to establish our result. $\square$

Theorem 1. Let S_1 be defined by (37). Under condition (H1), suppose that the following conditions

$$(H2) \bar{a}_1 > \left(\frac{\bar{a}_{12}}{m_{12}} \right),$$

$$\bar{a}_1 \exp(-S_1) > \bar{a}_{11}, \quad (19)$$

$$(H3) \bar{a}_2 < \bar{a}_{21}, \bar{a}_3 < \bar{a}_{32},$$

hold; then, system (3) has at least an ω positive periodic solution.

Proof. First, we make the change of variables $x_i(t) = \exp(u_i(t))$ ($i = 1, 2, 3$); then, (3) can be reformulated as

$$\begin{cases} u_1(k+1) - u_1(k) = f_1(u_1, u_2, u_3), \\ u_2(k+1) - u_2(k) = f_2(u_1, u_2, u_3), \\ u_3(k+1) - u_3(k) = f_3(u_1, u_2, u_3), \end{cases} \quad (20)$$

where

$$\begin{aligned} f_1(u_1, u_2, u_3) &= a_1(k) - a_{11}(k) \exp(u_1(k - \tau_1(k))) - \frac{a_{12}(k) \exp(u_2(k))}{m_{12}(k) \exp(u_2(k)) + \exp(u_1(k))}, \\ f_2(u_1, u_2, u_3) &= -a_2(k) + \frac{a_{21}(k) \exp(u_1(k - \tau_2(k)))}{m_{12}(k) \exp(u_2(k - \tau_2(k))) + \exp(u_1(k - \tau_2(k)))} \\ &\quad - \frac{a_{23}(k) \exp(u_3(k))}{m_{23}(k) \exp(u_3(k)) + \exp(u_2(k))}, \\ f_3(u_1, u_2, u_3) &= -a_3(k) + \frac{a_{32}(k) \exp(u_2(k - \tau_3(k)))}{m_{23}(k) \exp(u_3(k - \tau_3(k))) + \exp(u_2(k - \tau_3(k)))}. \end{aligned} \quad (21)$$

Let $X = Y = I^\omega$,

$$\begin{aligned} (Lu)(k) &= u(k+1) - u(k) \\ (Nu)(k) &= \begin{pmatrix} f_1(u_1, u_2, u_3) \\ f_2(u_1, u_2, u_3) \\ f_3(u_1, u_2, u_3) \end{pmatrix}, \end{aligned} \quad (22)$$

where $u \in X, k \in Z$. Then, it is trivial to see that L is a bounded linear operator and

$$\text{Ker } L = I_c^\omega, \quad \text{Im } L = I_0^\omega, \quad (23)$$

$$\dim \text{Ker } L = 3 = \text{codim Im } L. \quad (24)$$

Then, it follows that L is a Fredholm mapping of index zero. Define

$$Pu = \frac{1}{\omega} \sum_{s=0}^{\omega-1} u(s), \quad u \in X, \quad (25)$$

$$Qz = \frac{1}{\omega} \sum_{s=0}^{\omega-1} z(s), \quad z \in Y.$$

It is not difficult to show that P and Q are continuous projectors such that

$$\begin{aligned} \text{Im } P &= \text{Ker } L, \\ \text{Im } L &= \text{Ker } Q = \text{Im } (I - Q). \end{aligned} \quad (26)$$

Furthermore, the generalized inverse (to L) k_p : $\text{Im } L \rightarrow \text{Ker } P \cap \text{Dom } L$ exists and is given by

$$K_p(z) = \sum_{s=0}^{\omega-1} z(s) - \frac{1}{\omega} \sum_{s=0}^{\omega-1} (\omega - s)z(s). \quad (27)$$

Obviously, QN and $K_p(I - Q)N$ are continuous. Since X is a finite-dimensional Banach space, using the Ascoli–Arzela theorem, it is not difficult to show that $\overline{K_p(I - Q)N(\bar{\Omega})}$ is compact for any open-bounded set $\Omega \subset X$. Moreover, $QN(\bar{\Omega})$ is bounded. Thus, N is L -compact on $\bar{\Omega}$ with any open-bounded set $\Omega \subset X$.

Now, we are at the point to search for an appropriate open-bounded subset Ω for the application of the continuation theorem. Corresponding to the operator equation $Lu = \lambda Nu$, $\lambda \in (0, 1)$, we have

$$\begin{cases} u_1(k+1) - u_1(k) = \lambda f_1(u_1, u_2, u_3), \\ u_2(k+1) - u_2(k) = \lambda f_2(u_1, u_2, u_3), \\ u_3(k+1) - u_3(k) = \lambda f_3(u_1, u_2, u_3). \end{cases} \quad (28)$$

Suppose that $u(k) = (u_1(k), u_2(k), u_3(k))^T \in X$ is an arbitrary solution of system (28) for a certain $\lambda \in (0, 1)$; summing both sides of (28) from 0 to $\omega - 1$ with respect to k , respectively, we obtain

$$\begin{cases} \sum_{k=0}^{\omega-1} \left[a_{11}(k) \exp(u_1(k - \tau_1(k))) + \frac{a_{12}(k) \exp(u_2(k))}{m_{12}(k) \exp(u_2(k)) + \exp(u_1(k))} \right] = \bar{a}_1 \omega, \\ \sum_{k=0}^{\omega-1} \left[\frac{a_{21}(k) \exp(u_1(k - \tau_2(k)))}{m_{12}(k) \exp(u_2(k - \tau_2(k))) + \exp(u_1(k - \tau_2(k)))} - \frac{a_{23}(k) \exp(u_3(k))}{m_{23}(k) \exp(u_3(k)) + \exp(u_2(k))} \right] = \bar{a}_2 \omega, \\ \sum_{k=0}^{\omega-1} \left[\frac{a_{32}(k) \exp(u_2(k - \tau_3(k)))}{m_{23}(k) \exp(u_3(k - \tau_3(k))) + \exp(u_2(k - \tau_3(k)))} \right] = \bar{a}_3 \omega. \end{cases} \quad (29)$$

In view of the hypothesis that $u = \{u(k)\} \in X$, there exist $\xi_i, \eta_i \in I_\omega$ ($i = 1, 2, 3$) such that

$$\begin{aligned} u_i(\xi_i) &= \min_{k \in I_\omega} \{u_i(k)\}, \\ u_i(\eta_i) &= \max_{k \in I_\omega} \{u_i(k)\}. \end{aligned} \quad (30)$$

It follows from (28) and (29) that

$$\begin{aligned} \sum_{k=0}^{\omega-1} |u_1(k+1) - u_1(k)| &\leq \lambda \left| \sum_{k=0}^{\omega-1} a_1(k) - a_{11}(k) \exp(u_1(k - \tau_1(k))) - \frac{a_{12}(k) \exp(u_2(k))}{m_{12}(k) \exp(u_2(k)) + \exp(u_1(k))} \right| \\ &< 2\bar{a}_1 \omega, \end{aligned} \quad (31)$$

$$\begin{aligned} \sum_{k=0}^{\omega-1} |u_2(k+1) - u_2(k)| &\leq \lambda \left| \sum_{k=0}^{\omega-1} \left[-a_2(k) + \frac{a_{21}(k) \exp(u_1(k - \tau_2(k)))}{m_{12}(k) \exp(u_2(k - \tau_2(k))) + \exp(u_1(k - \tau_2(k)))} - \frac{a_{23}(k) u_3(k)}{m_{23}(k) \exp(u_3(k)) + \exp(u_2(k))} \right] \right| \\ &< 2\bar{a}_2 \omega, \end{aligned} \quad (32)$$

$$\begin{aligned} \sum_{k=0}^{\omega-1} |u_3(k+1) - u_3(k)| &\leq \lambda \left| \sum_{k=0}^{\omega-1} \left[-a_3(k) + \frac{a_{32}(k) \exp(u_2(k - \tau_3(k)))}{m_{23}(k) \exp(u_3(k - \tau_3(k))) + \exp(u_2(k - \tau_3(k)))} \right] \right| < 2\bar{a}_3 \omega. \end{aligned} \quad (33)$$

By the first equation of (29), we have

$$\begin{aligned} & \sum_{k=0}^{\omega-1} [a_{11}(k) \exp(u_1(k - \tau_1(k)))] \\ & < \bar{a}_1 \omega, \sum_{k=0}^{\omega-1} \left[a_{11}(k) \exp(u_1(k - \tau_1(k))) + \frac{a_{12}(k)}{m_{12}(k)} \right] \geq \bar{a}_1 \omega, \end{aligned} \quad (34)$$

which leads to

$$u_1(\xi_1) < \ln \left[\frac{\bar{a}_1}{\bar{a}_{11}} \right], u_1(\eta_1) > \ln \left[\frac{\bar{a}_1 - (\bar{a}_{12}/m_{12})}{\bar{a}_{11}} \right]. \quad (35)$$

By (31) and (35) and Lemma 2, we obtain

$$\begin{aligned} u_1(k) & \leq u(\xi_1) + \sum_{s=0}^{\omega-1} |u_1(s+1) - u_1(s)| \\ & \leq \ln \left[\frac{\bar{a}_1}{\bar{a}_{11}} \right] + 2\bar{a}_1 \omega := M_1, \\ u_1(k) & \geq u_1(\eta_1) - \sum_{s=0}^{\omega-1} |u_1(s+1) - u_1(s)| \\ & \geq \ln \left[\frac{\bar{a}_1 - (\bar{a}_{12}/m_{12})}{\bar{a}_{11}} \right] - 2\bar{a}_1 \omega := m_1. \end{aligned} \quad (36)$$

Thus,

$$\max_{k \in I_\omega} \{u_1(k)\} < \max\{|m_1|, |M_1|\} := S_1. \quad (37)$$

In view of the second equation of (29) and (37), it is easy to obtain

$$\sum_{k=0}^{\omega-1} \left[\frac{a_{21}(k) \exp(u_1(k - \tau_2(k)))}{m_{12}(k) \exp(u_2(k - \tau_2(k)))} \right] > \bar{a}_2 \omega, \quad (38)$$

then

$$\sum_{k=0}^{\omega-1} \left[\frac{a_{21}(k) \exp(S_1)}{m_{12}(k) \exp(u_2(\xi_2))} \right] > \bar{a}_2 \omega. \quad (39)$$

Thus,

$$u_2(\xi_2) < \ln \left[\frac{(\bar{a}_{21}/m_{12}) \exp(S_1)}{\bar{a}_2} \right]. \quad (40)$$

From the first equation of (29) and (37), we obtain

$$\sum_{k=0}^{\omega-1} \left[a_{11}(k) \exp(S_1) + \frac{a_{12}(k) \exp(u_2(\eta_2))}{\exp(-S_1)} \right] > \bar{a}_1 \omega. \quad (41)$$

Then,

$$u_2(\eta_2) > \ln \left[\frac{\bar{a}_1 \exp(-S_1) - \bar{a}_{11}}{\bar{a}_{12}} \right]. \quad (42)$$

It follows from (40) and (42) and Lemma 2 that

$$u_2(k) \leq u(\xi_2) + \sum_{s=0}^{\omega-1} |u_2(s+1) - u_2(s)| \quad (43)$$

$$\leq \ln \left[\frac{(\bar{a}_{21}/m_{12}) \exp(S_1)}{\bar{a}_2} \right] + 2\bar{a}_2 \omega := M_2,$$

$$\begin{aligned} u_2(k) & \geq u_2(\eta_2) - \sum_{s=0}^{\omega-1} |u_2(s+1) - u_2(s)| \\ & \geq \ln \left[\frac{\bar{a}_1 \exp(-S_1) - \bar{a}_{11}}{\bar{a}_{12}} \right] - 2\bar{a}_2 \omega := m_2. \end{aligned} \quad (44)$$

Thus,

$$\max_{k \in I_\omega} \{u_2(k)\} < \max\{|m_2|, |M_2|\} := S_2. \quad (45)$$

By the third equation of (29), we obtain

$$\sum_{k=0}^{\omega-1} \left[\frac{a_{32}(k) \exp(u_2(k - \tau_3(k)))}{m_{23}(k) \exp(u_3(k - \tau_3(k)))} \right] > \bar{a}_3 \omega, \quad (46)$$

which leads to

$$\sum_{k=0}^{\omega-1} \left[\frac{a_{32}(k) \exp(S_2)}{m_{23}(k) \exp(u_3(\xi_3))} \right] > \bar{a}_3 \omega. \quad (47)$$

Then,

$$u_3(\xi_3) < \ln \left[\frac{(\bar{a}_{32}/m_{23}) \exp(S_2)}{\bar{a}_3} \right]. \quad (48)$$

By the third equation of (29), we also obtain

$$\sum_{k=0}^{\omega-1} \left[\frac{a_{32}(k) \exp(-S_2)}{m_{23}^M \exp(u_3(\eta_3)) + \exp(-S_2)} \right] < \bar{a}_3 \omega. \quad (49)$$

Thus, we obtain

$$u_3(\eta_3) > \ln \left[\frac{(\bar{a}_{32} - \bar{a}_3) \exp(-S_2)}{\bar{a}_3 m_{23}^M} \right]. \quad (50)$$

From (48) and (50) and Lemma 2, we derive

$$\begin{aligned} u_3(k) &\leq u(\xi_3) + \sum_{s=0}^{\omega-1} |u_3(s+1) - u_3(s)| \\ &\leq \ln \left[\frac{(\bar{a}_{32}/m_{23}) \exp(S_2)}{\bar{a}_3} \right] + 2\bar{a}_3\omega := M_3, \\ u_3(k) &\geq u_3(\eta_3) - \sum_{s=0}^{\omega-1} |u_2(s+1) - u_2(s)| \\ &\geq \ln \left[\frac{(\bar{a}_{32} - \bar{a}_3) \exp(-S_2)}{\bar{a}_3 m_{23}^M} \right] - 2\bar{a}_3\omega := m_3. \end{aligned} \quad (51)$$

Thus,

$$\max_{k \in I_\omega} \{u_3(k)\} < \max\{|m_3|, |M_3|\} := S_3. \quad (52)$$

Obviously, S_i ($i = 1, 2, 3$) are independent of the choice of $\lambda \in (0, 1)$. Take $S = \max\{S_1, S_2, S_3\} + S_0$, where S_0 is taken sufficiently large such that $|\ln v_1^*| + |\ln v_2^*| + |\ln v_3^*| < S_0$, where $(v_1^*, v_2^*, v_3^*)^T$ is the unique positive solution of (7).

Now, we have proved that any solution $u = \{u(k)\} = \{(u_1(k), u_2(k), u_3(k))^T\}$ of (28) in X satisfies $\|u\| < S, k \in \mathbb{Z}$.

Let $\Omega := \{u = \{u(k)\} \in X: \|u\| < S\}$; then, it is easy to see that Ω is an open-bounded set in X and verifies requirement (a) of Lemma 2. When $u \in \partial\Omega \cap \text{Ker } L$, $u = \{(u_1, u_2, u_3)^T\}$ is a constant vector in $\mathbb{R}^3$ with $\|u\| = |u_1| + |u_2| + |u_3| = S$. Then,

$$QN y = \begin{pmatrix} \bar{a}_1 - \bar{a}_{11} \exp(u_1) - \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{21}(k) \exp(u_1)}{m_{12}(k) \exp(u_2) + \exp(u_1)} \\ -\bar{a}_2 + \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{21}(k) \exp(u_1)}{m_{12}(k) \exp(u_2) + \exp(u_1)} - \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{23}(k) \exp(u_3)}{m_{23}(k) \exp(u_3) + \exp(u_3)} \\ -\bar{a}_3 + \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{32}(k) \exp(u_2)}{m_{23}(k) \exp(u_3) + \exp(u_2)} \end{pmatrix} \neq 0. \quad (53)$$

Now, let us consider homotopic $\phi(u_1, u_2, u_3, \mu) = \mu QNu + (1 - \mu)Gu, \mu \in [0, 1]$, where

$$Gz = \frac{1}{\omega} \sum_{k=0}^{\omega-1} \begin{pmatrix} \bar{a}_1 - \bar{a}_{11} \exp(u_1) \\ -\bar{a}_2 + \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{21}(k) \exp(u_1)}{m_{12}(k) \exp(u_2) + \exp(u_1)} \\ -\bar{a}_3 + \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{32}(k) \exp(u_2)}{m_{23}(k) \exp(u_3) + \exp(u_2)} \end{pmatrix}. \quad (54)$$

In fact, when $\phi(u_1, u_2, u_3, 0) = Gu$ and $\phi(u_1, u_2, u_3, 1) = QNu$, then ϕ is a homotopic mapping. Letting J be the identity mapping and by direct calculation, we obtain

$$\begin{aligned} &\deg\{JQN(u_1, u_2, u_3)^T; \Omega \cap \ker L; 0\} \\ &= \deg\{QN(u_1, u_2, u_3)^T; \Omega \cap \ker L; 0\} \\ &= \deg\{\phi(u_1, u_2, u_3, 1); \Omega \cap \ker L; 0\} \\ &= \deg\{\phi(u_1, u_2, u_3, 0); \Omega \cap \ker L; 0\} \\ &= \text{sign} \left\{ \det \begin{pmatrix} \chi_{11} & 0 & 0 \\ \chi_{21} & \chi_{22} & 0 \\ 0 & \chi_{32} & \chi_{33} \end{pmatrix} \right\}, \end{aligned} \quad (55)$$

where

$$\begin{aligned} \chi_{11} &= -\bar{a}_{11} \exp(u_1^*) < 0, \\ \chi_{21} &= \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{21}(k) m_{12}(k) \exp(u_1^* + u_2^*)}{[m_{12}(k) \exp(u_2^*) + \exp(u_1^*)]^2} > 0, \\ \chi_{22} &= -\frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{21}(k) m_{12}(k) \exp(u_1^* + u_2^*)}{[m_{12}(k) \exp(u_2^*) + \exp(u_1^*)]^2} < 0, \\ \chi_{32} &= \frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{32}(k) m_{23}(k) \exp(u_2^* + u_3^*)}{[m_{23}(k) \exp(u_3^*) + \exp(u_2^*)]^2} > 0, \\ \chi_{33} &= -\frac{1}{\omega} \sum_{k=0}^{\omega-1} \frac{a_{32}(k) m_{23}(k) \exp(u_2^* + u_3^*)}{[m_{23}(k) \exp(u_3^*) + \exp(u_2^*)]^2} < 0. \end{aligned} \quad (56)$$

Thus,

$$\deg\{JQN(u_1, u_2, u_3)^T; \Omega \cap \ker L; 0\} = \text{sign}\{\chi_{11}\chi_{22}\chi_{33}\} = -1 \neq 0. \quad (57)$$

By now, we have proved that Ω verifies all requirements of Lemma 2; then, it follows that $Lu = Nu$ has at least one solution in $\text{Dom } L \cap \bar{\Omega}$, namely, (20) has at least one ω

FIGURE 1: The time histories of $t - x_1$, $t - x_2$, and $t - x_3$, where the red line represents x_1 , the blue line represents x_2 , and the green line represents x_3 .

periodic solution in $\text{Dom } L \cap \overline{\Omega}$, say $u^* = \{u^*(k)\} = \{(u_1^*(k), u_2^*(k), u_3^*(k))^T\}$; then, it follows that $(\exp\{u_1^*(k)\}, \exp\{u_2^*(k)\}, \exp\{u_3^*(k)\})^T$ is an ω periodic solution of system (3) with strictly positive components. The proof is complete. $\square$

3. Global Asymptotic Stability

In this section, we will present sufficient conditions for the globally asymptotical stability of system (3) when all the delays are zero.

Theorem 2. Let A_1^* , A_2^* , and A_3^* be defined by (67), (68), and (69), respectively. Assume that (H1)–(H3) are satisfied. Furthermore, suppose that there exist positive constants η , δ_1 , δ_2 , and δ_3 such that $A_i^* > 0$ ($i = 1, 2, 3$) and

$$\begin{aligned} \text{(i)} \quad & 1 - a_{11}(k)\exp(M_1) - \frac{a_{12}(k)\exp(M_2)}{m_{12}(k)\exp(m_2) + \exp(m_1)} + \frac{a_{12}(k)\exp(m_1 + m_2)}{(m_{12}(k)\exp(M_2) + \exp(M_1))^2} \geq \eta, \\ \text{(ii)} \quad & \frac{\exp(m_2) - a_{21}(k)\exp(M_1 + M_2) - a_{23}(k)\exp(M_2 + M_3)}{m_{12}(k)\exp(M_2) + \exp(M_1)} - \frac{a_{21}(k)a_{23}(k)\exp(M_1 + M_2 + M_3)}{(m_{12}(k)\exp(m_2) + \exp(m_1))^2} \geq \eta, \\ \text{(iii)} \quad & \frac{m_{23}(k)}{m_{23}(k)\exp(M_3) + \exp(M_2)} \leq 1. \end{aligned} \quad (58)$$

Then, the positive ω -periodic solution of system (3) is globally asymptotically stable.

Proof. In view of Theorem 2, there exists a positive periodic solution $\{x_1^*(k), x_2^*(k), x_3^*(k)\}$ of system (3). We prove below that it is uniformly asymptotically stable. First, we introduce the change of variables as follows:

$$\begin{aligned} N_1(k) &= x_1(k) - x_1^*(k), N_2(k) = x_2(k) - x_2^*(k), N_3(k) \\ &= x_3(k) - x_3^*(k). \end{aligned} \quad (59)$$

Then, it follows from (3) that

$$\begin{aligned}
N_1(k+1) &= x_1(k+1) - x_1^*(k+1) \\
&= x_1(k) \exp \left\{ a_1(k) - a_{11}(k)x_1(k) - \frac{a_{12}(k)x_2(k)}{m_{12}(k)x_2(k) + x_1(k)} \right\}, \\
&\quad - x_1^*(k) \exp \left\{ a_1(k) - a_{11}(k)x_1^*(k) - \frac{a_{12}(k)x_2^*(k)}{m_{12}(k)x_2^*(k) + x_1^*(k)} \right\} \\
&= \left\{ x_1(k) \exp \left[-a_{11}(k)N_1(k) - a_{12}(k) \left(\frac{x_2(k)}{m_{12}(k)x_2(k) + x_1(k)} \right. \right. \right. \\
&\quad \left. \left. \left. - \frac{x_2^*(k)}{m_{12}(k)x_2^*(k) + x_1^*(k)} \right) \right] - x_1^*(k) \right\} \frac{x_1^*(k+1)}{x_1^*(k)}, \\
&= \left\{ \left[1 - a_{11}(k)x_1^*(k) - \frac{a_{12}(k)x_2^*(k)}{m_{12}(k)x_2^*(k) + x_1^*(k)} + \frac{a_{12}(k)x_1^*(k)x_2^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} \right] \frac{N_1(k)}{x_1^*(k)} \right. \\
&\quad \left. - \frac{a_{12}(k)}{m_{12}(k)x_2^*(k) + x_1^*(k)} N_2(k) + \rho_1 \right\} x_1^*(k+1),
\end{aligned} \tag{60}$$

$$\begin{aligned}
N_2(k+1) &= x_2(k+1) - x_2^*(k+1) \\
&= x_2(k) \exp \left\{ -a_2(k) + \frac{a_{21}(k)x_1(k)}{m_{12}(k)x_2(k) + x_1(k)} - \frac{a_{23}(k)x_3(k)}{m_{23}(k)x_3(k) + x_2(k)} \right\} \\
&\quad - x_2^*(k) \exp \left\{ -a_2(k) + \frac{a_{21}(k)x_1^*(k)}{m_{12}(k)x_2^*(k) + x_1^*(k)} - \frac{a_{23}(k)x_3^*(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right\} \\
&= \left[x_2(k) \exp \left\{ \left(\frac{a_{21}(k)x_1(k)}{m_{12}(k)x_2(k) + x_1(k)} - \frac{a_{21}(k)x_1^*(k)}{m_{12}(k)x_2^*(k) + x_1^*(k)} \right) \right. \right. \\
&\quad \left. \left. - \left(\frac{a_{23}(k)x_3(k)}{m_{23}(k)x_3(k) + x_2(k)} - \frac{a_{23}(k)x_3^*(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right) \right\} - x_2^*(k) \right] \frac{x_2^*(k+1)}{x_2^*(k)} \\
&= \left\{ \left[\frac{a_{21}(k)}{m_{12}(k)x_2^*(k) + x_1^*(k)} - \frac{a_{21}(k)x_1^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} \right] N_1(k) \right. \\
&\quad + \left[\frac{x_2^*(k)(1 - a_{21}(k)x_1^*(k) - a_{23}(k)x_3^*(k))}{m_{12}(k)x_2^*(k) + x_1^*(k)} - \frac{a_{21}(k)a_{23}(k)x_1^*(k)x_2^*(k)x_3^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} \right] \frac{N_2(k)}{x_2^*(k)} \\
&\quad \left. + \left[\frac{a_{21}(k)m_{23}(k)x_3^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} - \frac{m_{23}(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right] N_3(k) + \rho_2 \right\} x_2^*(k+1),
\end{aligned} \tag{61}$$

$$\begin{aligned}
N_3(k+1) &= x_3(k+1) - x_3^*(k+1) \\
&= x_3(k) \exp \left\{ -a_3(k) + \frac{a_{32}(k)x_2(k)}{m_{23}(k)x_3(k) + x_2(k)} \right\} \\
&\quad - x_3^*(k) \exp \left\{ -a_3(k) + \frac{a_{32}(k)x_2^*(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right\} \\
&= \left\{ x_3(k) \exp \left[a_{32}(k) \left(\frac{x_2(k)}{m_{23}(k)x_3(k) + x_2(k)} - \frac{x_2^*(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right) \right] - x_3^*(k) \right\} \frac{x_3^*(k+1)}{x_3^*(k)}, \\
&= \left\{ \left[a_{32}(k) - \frac{a_{32}(k)x_2^*(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right] N_2(k) + \left[\frac{a_{32}(k)x_2^*(k)x_3^*(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right. \right. \\
&\quad \left. \left. - \frac{a_{32}(k)x_2^*(k)x_3^*(k)x_3^*(k)m_{23}(k)}{(m_{23}(k)x_3^*(k) + x_2^*(k))^2} \right] \frac{N_3(k)}{x_3^*(k)} + \rho_3 \right\} x_3^*(k+1),
\end{aligned} \tag{62}$$

where $|\rho_i|/\|N\|$ ($i = 1, 2, 3$) converges, uniformly with respect to $k \in \mathbb{Z}^+$, to zero as $\|N\| \rightarrow 0$.

Define Lyapunov function as follows:

$$V(N(k)) = \delta_1 \left| \frac{N_1(k)}{x_1^*(k)} \right| + \delta_2 \left| \frac{N_2(k)}{x_2^*(k)} \right| + \delta_3 \left| \frac{N_3(k)}{x_3^*(k)} \right|, \quad (63)$$

where δ_1, δ_2 , and δ_3 are positive constants given in (67)–(69), respectively, and satisfy $A_i^* > 0$ ($i = 1, 2, 3$). Calculating the difference of V along the solution of systems (60)–(62) and using (i), (ii), and (iii), we have

$$\begin{aligned} \Delta V &= \delta_1 \left| \frac{N_1(k+1)}{x_1^*(k+1)} - \frac{N_1(k)}{x_1^*(k)} \right| + \delta_2 \left| \frac{N_2(k+1)}{x_2^*(k+1)} - \frac{N_2(k)}{x_2^*(k)} \right| + \delta_3 \left| \frac{N_3(k+1)}{x_3^*(k+1)} - \frac{N_3(k)}{x_3^*(k)} \right| \\ &\leq -\delta_1 \left\{ \left[a_{11}(k) + \frac{a_{12}(k)x_2^*(k)}{x_1^*(k)(m_{12}(k)x_2^*(k) + x_1^*(k))} - \frac{a_{12}(k)x_2^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} \right] |N_1(k)| \right. \\ &\quad + \delta_1 \frac{a_{12}(k)}{m_{12}(k)x_2^*(k) + x_1^*(k)} |N_2(k)| \\ &\quad - \delta_2 \left[\frac{a_{21}(k)a_{23}(k)x_1^*(k)x_3^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} - \frac{(1 - a_{21}(k)x_1^*(k) - a_{23}(k)x_3^*(k))}{m_{12}(k)x_2^*(k) + x_1^*(k)} \right] |N_2(k)| \\ &\quad + \delta_2 \left[\frac{a_{21}(k)}{m_{12}(k)x_2^*(k) + x_1^*(k)} - \frac{a_{21}(k)x_1^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} \right] |N_1(k)| \\ &\quad + \delta_2 \left[\frac{a_{21}(k)m_{23}(k)x_3^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} - \frac{m_{23}(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right] |N_3(k)| \\ &\quad - \delta_3 \left[\frac{a_{32}(k)x_2^*(k)(k)m_{23}(k)}{(m_{23}(k)x_3^*(k) + x_2^*(k))^2} - \frac{a_{32}(k)x_2^*(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right] |N_3(k)| \\ &\quad + \delta_3 \left[\left[a_{32}(k) - \frac{a_{32}(k)x_2^*(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right] |N_2(k)| + \sum_{i=1}^3 \delta_i |\rho_i| \right. \\ &\quad \left. - A_1 |N_1(k)| - A_2 |N_2(k)| - A_3 |N_3(k)| + \sum_{i=1}^3 \delta_i |\rho_i| \right], \end{aligned} \quad (64)$$

where

$$\begin{aligned} A_1 &= \delta_1 \left[a_{11}(k) + \frac{a_{12}(k)x_2^*(k)}{x_1^*(k)(m_{12}(k)x_2^*(k) + x_1^*(k))} - \frac{a_{12}(k)x_2^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} \right] \\ &\quad + \delta_2 \left[\frac{a_{21}(k)}{m_{12}(k)x_2^*(k) + x_1^*(k)} - \frac{a_{21}(k)x_1^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} \right], \\ A_2 &= \delta_2 \left[\frac{a_{21}(k)a_{23}(k)x_1^*(k)x_3^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} - \frac{(1 - a_{21}(k)x_1^*(k) - a_{23}(k)x_3^*(k))}{m_{12}(k)x_2^*(k) + x_1^*(k)} \right] \\ &\quad + \delta_1 \frac{a_{12}(k)}{m_{12}(k)x_2^*(k) + x_1^*(k)} + \delta_3 \left[a_{32}(k) - \frac{a_{32}(k)x_2^*(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right], \\ A_3 &= \delta_3 \left[\frac{a_{32}(k)x_2^*(k)(k)m_{23}(k)}{(m_{23}(k)x_3^*(k) + x_2^*(k))^2} - \frac{a_{32}(k)x_2^*(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right] \\ &\quad + \delta_2 \left[\frac{a_{21}(k)m_{23}(k)x_3^*(k)}{(m_{12}(k)x_2^*(k) + x_1^*(k))^2} - \frac{m_{23}(k)}{m_{23}(k)x_3^*(k) + x_2^*(k)} \right]. \end{aligned} \quad (65)$$

From (64), we have

$$\Delta V \leq -A_1^*|N_1(k)| - A_2^*|N_2(k)| - A_3^*|N_3(k)| + \sum_{i=1}^3 \delta_i |\rho_i|, \quad (66)$$

where

$$A_1^* = \delta_1 \left[a_{11}(k) + \frac{a_{12}(k)\exp(m_2)}{\exp(M_1)(m_{12}(k)\exp(M_2) + \exp(M_1))} - \frac{a_{12}(k)\exp(M_2)}{(m_{12}(k)\exp(m_2) + \exp(m_1))^2} \right] + \left| \delta_2 \left[\frac{a_{21}(k)}{m_{12}(k)\exp(m_2) + \exp(m_1)} - \frac{a_{21}(k)\exp(m_1)}{(m_{12}(k)\exp(M_2) + \exp(M_1))^2} \right] \right|, \quad (67)$$

$$A_2^* = \delta_2 \left[\frac{a_{21}(k)a_{23}(k)\exp(m_1 + m_3)}{(m_{12}(k)\exp(M_2) + \exp(M_1))^2} - \frac{(1 - a_{21}(k)\exp(m_1) - a_{23}(k)\exp(m_3))}{m_{12}(k)\exp(M_2) + \exp(M_1)} \right] + \delta_1 \frac{a_{12}(k)}{m_{12}(k)\exp(m_2) + \exp(m_1)} + \delta_3 \left[a_{32}(k) - \frac{a_{32}(k)\exp(m_2)}{m_{23}(k)\exp(M_3) + \exp(M_2)} \right], \quad (68)$$

$$A_3^* = \delta_3 \left[\frac{a_{32}(k)\exp(m_2)m_{23}(k)}{(m_{23}(k)\exp(M_3) + \exp(M_2))^2} - \frac{a_{32}(k)\exp(M_2)}{m_{23}(k)\exp(m_3) + \exp(m_2)} \right] + \delta_2 \left[\frac{a_{21}(k)m_{23}(k)\exp(M_3)}{(m_{12}(k)\exp(m_2) + \exp(m_1))^2} - \frac{m_{23}(k)}{m_{23}(k)\exp(m_3) + \exp(m_2)} \right]. \quad (69)$$

Since $|\rho_i|/\|N\|$ ($i = 1, 2, 3$) converges, uniformly with respect to $k \in \mathbb{Z}^+$, to zero as $\|N\| \rightarrow 0$, it follows from conditions (i), (ii), and (iii) that there is a positive constant φ such that, if k is sufficiently large and $\|N\| < \varphi$, then

$$\Delta V \leq -\frac{\eta}{3} \{|N_1(k)| + |N_2(k)| + |N_3(k)|\} = -\frac{\eta}{3} \|N\|. \quad (70)$$

It follows from Freedman [22] that the trivial solutions of (60)–(62) is uniformly asymptotically stable. Thus, the

solution $x^* = \{x^*(k)\} = \{(x_1^*(k), x_2^*(k), x_3^*(k))^T\}$ of (3) is uniformly asymptotically stable. According to Wang and Lu [23], we can conclude that the positive periodic solution of (3) is globally asymptotically stable. The proof ends. $\square$

4. Numerical Example

Given a discrete model as follows:

$$\begin{cases} x_1(k+1) = x_1(k)\exp\left\{a_1(k) - a_{11}(k)x_1(k - \tau_1(k)) - \frac{a_{12}(k)x_2(k)}{m_{12}(k)x_2(k) + x_1(k)}\right\}, \\ x_2(k+1) = x_2(k)\exp\left\{-a_2(k) + \frac{a_{21}(k)x_1(k - \tau_2(k))}{m_{12}(k)x_2(k - \tau_2(k)) + x_1(k - \tau_2(k))} - \frac{a_{23}(k)x_3(k)}{m_{23}(k)x_3(k) + x_2(k)}\right\}, \\ x_3(k+1) = x_3(k)\exp\left\{-a_3(k) + \frac{a_{32}(k)x_2(k - \tau_3(k))}{m_{23}(k)x_3(k - \tau_3(k)) + x_2(k - \tau_3(k))}\right\}, \end{cases} \quad (71)$$

where $a_1(k) = 0.8, a_2(k) = 0.3, a_3(k) = 0.4, a_{11}(k) = 0.3 + 0.2 \sin k\pi, a_{12}(k) = 0.2 + 0.2 \sin k\pi, a_{21}(k) = 0.8 + 0.4 \sin k\pi, a_{23}(k) = 0.6 + 0.4 \sin k\pi, a_{32}(k) = 0.9 + 0.2 \sin k\pi, m_{12}(k) = 0.4 + 0.4 \sin k\pi, m_{23}(k) = 0.5 + 0.3 \sin k\pi$, and $\tau_1(k) = \tau_2(k) = \tau_3(k) = 1$. Then, $\bar{a}_1 = 0.8, (\bar{a}_{12}/\bar{m}_{12}) = 0.5, S_1 = 0.5823, \bar{a}_{11} = 0.3, \bar{a}_2 = 0.3, \bar{a}_{21} = 0.8, \bar{a}_3 = 0.4, c$ and $\bar{a}_{32} = 0.9$. Thus, (H1)–(H3) hold. Therefore, system (71) has at least a positive two-periodic solution, which is shown in Figure 1.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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