Hindawi
Discrete Dynamics in Nature and Society
Volume 2020, Article ID 9738934, 12 pages
https://doi.org/10.1155/2020/9738934

Research Article

Mathematical Modeling, Analysis, and Optimal Control of Abstinence Behavior of Registration on the Electoral Lists

Omar Balatif[1], **Bouchaib Khajji**[2], **and Mostafa Rachik**[2]

[1] *Laboratory of Dynamical Systems, Mathematical Engineering Team (INMA), Department of Mathematics, Faculty of Sciences El Jadida, Chouaib Doukkali University, El Jadida, Morocco*
[2] *Laboratory of Analysis, Modeling and Simulation, Department of Mathematics and Computer Science, Faculty of Sciences Ben M'Sik, Hassan II University Mohammedia, Sidi Othman, Casablanca, Morocco*

Correspondence should be addressed to Omar Balatif; balatif.maths@gmail.com

Received 13 August 2019; Revised 15 November 2019; Accepted 26 November 2019; Published 3 January 2020

Academic Editor: Zhengqiu Zhang

We propose a mathematical model that describes the dynamics of citizens who have the right to register on the electoral lists and participate in the political process and the negative influence of abstainers, who abstain from registration on the electoral lists, on the potential electors. By using Routh–Hurwitz criteria and constructing Lyapunov functions, the local stability and the global stability of abstaining-free equilibrium and abstaining equilibrium are obtained. We also study the sensitivity analysis of the model parameters to know the parameters that have a high impact on the reproduction number $\mathfrak{R}_0$. In addition, we propose an optimal strategy for an awareness program that helps politicians and officials to increase the rate of citizens registered on the electoral lists with an optimal effort. Pontryagin's maximum principle is used to characterize the optimal controls, and the optimality system is solved by an iterative method. Finally, some numerical simulations are performed to verify the theoretical analysis using Matlab.

1. Introduction

Participation in political life is one of the most important indicators of democracy in a country's political system. Political participation takes different forms: joining political parties, participating in elections as well as the rate of registration on the electoral lists, which all aim at contributing to the different elections for electing people's representatives in the different political positions.

The higher rate of registered citizens on the electoral lists means that the majority of people are willing to participate in the electoral process. It also indicates that a large group of people is convinced of the possibility of change and reform through political participation. But, when these rates are low, it often means there are some administrative or cultural impediments that make registration on the electoral lists difficult. These low rates can also denote the existence of a behavior of political boycotting and nonregistration on the electoral lists due to electoral and political corruption or the citizens' lack of confidence in the political class.

Thus, countries seeking to improve their image and political position are in urgent need of improving many democratic indicators including increasing the rate of registration on the electoral lists. For example, many Arab and African countries suffer from a decline in the registration rate on the electoral lists (see Table 1 (data from The International Institute for Democracy and Electoral Assistance. https://www.idea.int/data-tools/2019) below).

According to these data and other data, these countries and others need to conduct more studies and research using a variety of scientific approaches for understanding the reasons and predicting the evolvement of these rates in the future as well as formulating strategies and programs for improving registration rates on the electoral lists.

Many studies and research in social sciences have focused on this topic and other related topics ([1–6] and the references cited therein). But, the mathematical studies and research on this topic are still limited and most of them have focused on the statistical aspect of the phenomenon ([7–10]).

Table 1

Country	Election type	Year	Registration rate with respect to voting age (%)	Registration rate with respect to the total population (%)
Bahrain	Parliamentary	2018	34.12	25.33
Burkina Faso	Parliamentary	2015	60.63	29.14
Cote d'Ivoire	Parliamentary	2016	47.37	26.40
Djibouti	Parliamentary	2018	34.76	21.94
Kuwait	Parliamentary	2016	25.41	17.06
Libya	Parliamentary	2014	37.45	24.17
Morocco	Parliamentary	2016	67.89	46.66
Oman	Parliamentary	2015	27.39	15.99
Sudan	Parliamentary	2015	66.74	34.15

In this work, we propose an epidemiological approach (see [11–14]) to describe and study the boycotting behavior to registration on the electoral lists. In epidemiology, we generally use compartment model to describe the spread of an infectious disease. In these epidemiological models, the population is divided into different classes according to people's status versus the disease (susceptible to catch the disease, infected, or removed) and the infection process depends on the contact with infectious individuals. Similarly, during the election process, the population can be divided into several classes (potential electors, voters for the opposition parties, voters for the majority parties, etc). Furthermore, the interaction between people is a key factor in the electoral process; it is very similar to the contagion phenomenon since abstainers can affect the potential registrants in their network (family, friends, and coworkers) to abstain from the electoral process. For this reason, the epidemic approach is more appropriate to model the boycotting behavior in an election process.

We propose a mathematical model that describes the dynamics of citizens who have the right to register on the electoral lists and the negative influence of abstainers who abstain from registration on the electoral lists and the electoral process. By using Routh–Hurwitz criteria and constructing Lyapunov functions, the local stability and the global stability of abstaining-free equilibrium and abstaining equilibrium are obtained. Since there are usually errors in data collection and assumed parameter values, we also study the sensitivity analysis of the model parameters to know the parameters that have a high impact on the reproduction number $\mathfrak{R}_0$. In addition, we propose an optimal strategy for an awareness program that helps politicians and officials to increase the rate of the registered citizens on the electoral lists with an optimal effort. To achieve this objective, we use some theoretical results of optimal control theory.

The paper is organized as follows. In Section 1, we present the proposed mathematical model and we give some basic properties of the model. In Section 2, we analyze the local stability and global stability and the problem of parameters' sensitivity and some numerical simulations. In Section 3, we present the optimal control problem for the proposed model where we give some results concerning the existence of the optimal controls and we characterize these optimal controls using Pontryagin's maximum principle. Also, in this section, numerical simulations are given. Finally, we conclude the paper in Section 4.

1.1. Mathematical Model and Basic Properties

1.1.1. Mathematical Model. We consider a mathematical model PAR that describes the dynamics of citizens who have the right to register on the electoral lists and the negative influence of abstainers, who abstain from registration on the electoral lists and the electoral process, on the potential electors.

We divide the population denoted by N into three compartments:

The potential electors (P) are entitled to participate in the elections and they are not yet registered on the electoral lists. The class of potential electors is increased by the recruitment of individuals into the compartment P at a rate Λ and it is decreased when potential electors register on the electoral lists at rate α. It is assumed that potential electors can acquire abstainer behavior (and become the abstainers of the elections) via effective contact with abstainers of the elections at a rate β. In other words, it is assumed that the acquisition of an abstainer behavior is analogous to acquiring disease infection. Finally, potential electors suffer natural death (at a rate μ).

The abstainers (A) have a position of abstaining from the elections and the registration on the electoral lists. The population of abstainers is increased when the potential electors abstain to register on the electoral lists via effective contact with abstainers at a rate β. It is decreased by natural death (at the rate μ) and when some abstainers withdraw their position of boycotting the registration on the electoral lists and turn to participate in the electoral process then become registrants at a rate γ.

The registered (R) are listed on the electoral lists and wish to vote in the elections. The compartment of the registered is increased at a rate α when potential electors register on the electoral lists and when the abstainers become registered individuals at a rate γ. This compartment is decreased by natural death (at the rate μ).

The variables $P(t)$, $A(t)$, and $R(t)$ are the numbers of the individuals in the three classes at time t, respectively. The unit of time can correspond to periods, years, months, or days; it depends on the frequency of the survey studies as needed.

The graphical representation of the proposed model is shown in Figure 1.

The total population size at time t is denoted by $N(t)$ with $N(t) = P(t) + A(t) + R(t)$. The dynamics of this model are governed by the following nonlinear system of differential equations:

$$\begin{cases} \dot{P} = \Lambda - \beta \dfrac{AP}{N} - (\alpha + \mu)P, \\ \dot{A} = \beta \dfrac{AP}{N} - (\gamma + \mu)A, \\ \dot{R} = \alpha P + \gamma A - \mu R, \end{cases} \quad (1)$$

where $P(0) \geq 0$, $A(0) \geq 0$, and $R(0) \geq 0$ are the given initial states.

1.2. Basic Properties. System (1) describes human population and therefore it is necessary to prove that all solutions of system (1) with positive initial data will remain positive for all times $t > 0$ and are bounded. This will be established by the following theorem and lemma.

1.2.1. Positivity of the Model's Solutions

Theorem 1. *If $P(0) \geq 0$, $A(0) \geq 0$, and $R(0) \geq 0$, the solutions $P(t)$, $A(t)$, and $R(t)$ of system (1) are positive for all $t \geq 0$.*

Proof. It follows from the first equation of system (1) that

$$\frac{dP}{dt} + \left(\beta \frac{A(t)}{N} + (\alpha + \mu)\right) P(t) \geq 0. \quad (2)$$

Multiplying the inequality (2) by $\exp[\int_0^t (\beta((A(s))/N) + (\alpha + \mu))ds]$, we have

$$\frac{dP}{dt} \exp\left[\int_0^t \left(\beta \frac{A(s)}{N} + (\alpha + \mu)\right) ds\right] + \left[\beta \frac{A(t)}{N} + (\alpha + \mu)\right] \cdot P(t) \exp\left[\int_0^t \left(\beta \frac{A(s)}{N} + (\alpha + \mu)\right) ds\right] \geq 0. \quad (3)$$

So,

$$\frac{d}{dt}\left[P(t) \exp\left(\int_0^t \left(\beta \frac{A(s)}{N} + (\alpha + \mu)\right) ds\right)\right] \geq 0. \quad (4)$$

Integrating (4) gives

$$P(t) \geq P(0) \exp\left(\int_0^t \left(-\beta \frac{A(s)}{N} - (\alpha + \mu)\right) ds\right). \quad (5)$$

So, the solution $P(t)$ is positive.

Similarly, from the second and third equations of (1), we have

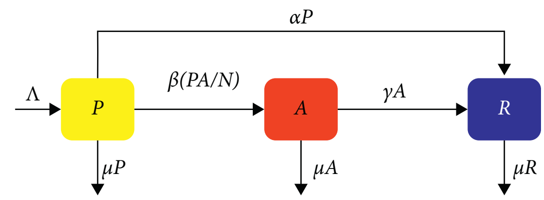


FIGURE 1

$$A(t) \geq A(0) \exp(-(\gamma + \mu)t) \geq 0, \quad (6)$$

and

$$R(t) \geq R(0) \exp(-\mu t) \geq 0. \quad (7)$$

Therefore, we can see that the solutions $P(t)$, $A(t)$, and $R(t)$ of system (1) are positive for all $t \geq 0$. □

1.2.2. Invariant Region

Lemma 1. *The feasible region Ω defined by*

$$\Omega = \left\{(P(t), R(t), R(t)) \in \mathbb{R}_+^3,\ P(t) + A(t) + R(t) \leq \frac{\Lambda}{\mu}\right\}, \quad (8)$$

with initial conditions, $P(0) \geq 0$, $A(0) \geq 0$, and $R(0) \geq 0$ are positive invariants for system (1).

Proof. By adding the equations of system (1), we obtain

$$\frac{dN}{dt} \leq \Lambda - (\alpha + \mu)N. \quad (9)$$

So,

$$N(t) \leq N(0) + \Lambda t + \int_0^t -\mu N(s) ds. \quad (10)$$

By using a Gronwall lemma, we have

$$N(t) \leq N(0) \exp(-\mu t) - \frac{\Lambda}{\mu}(1 - \exp(-\mu t)), \quad (11)$$

where $N(0)$ represents the initial values of the total population. Thus, $\lim_{t \to \infty} \sup N(t) = \Lambda/\mu$. It implies that the region Ω is a positively invariant set for system (1). So, we only need to consider the dynamics of the system on the set Ω. □

2. Stability Analysis and Sensitivity of the Model Parameters

In this section, we will study the stability behavior of system (1) at an abstaining-free equilibrium point and an abstaining equilibrium point. System (1) has the following two equilibrium points:

(i) Abstaining-free equilibrium given by $E^0 = ((\Lambda/(\alpha + \mu)), 0, (\alpha\Lambda/(\mu(\alpha + \mu))))$.

This equilibrium corresponds to the case when there are no abstainers in the population.

(ii) Abstaining equilibrium point, if $\mathfrak{R}_0 > 1$, given by $E^* = (P^*, A^*, R^*)$, where $P^* = ((\Lambda(\gamma+\mu))/(\mu\beta))$, $A^* = (\Lambda(\alpha+\mu)(\mathfrak{R}_0 - 1))/(\mu\beta)$, and $R^* = (\alpha\Lambda^2 (\gamma+\mu)^2 + \gamma\Lambda(\alpha+\mu)(\gamma+\mu)(\mathfrak{R}_0 - 1))/(\mu\beta(\gamma+\mu))$.

This equilibrium corresponds to the case when the behavior of abstaining from the registration on the electoral lists is able to invade the population.

Here, $\mathfrak{R}_0$ is the basic reproduction number given by:

$$\mathfrak{R}_0 = \frac{\mu\beta}{(\alpha+\mu)(\gamma+\mu)}. \tag{12}$$

In epidemiology, the basic reproduction number $\mathfrak{R}_0$ is defined as the average number of secondary infections produced by an infected individual in a completely susceptible population.

In the context of our work, this threshold indicates the average number of persons that an abstainer will “infect” during his “infection” period within the potential elector population, so that the infected individuals will enter the compartment of abstainers. We obtained this number by using the next-generation matrix method formulated in [15, 16].

Indeed, letting $x = (A, R, P)$, then system (1) can be written as

$$\frac{dx}{dt} = \mathcal{F}(x) - \vartheta(x), \tag{13}$$

where

$$\mathcal{F}(x) = \begin{pmatrix} \beta\frac{AP}{N} \\ 0 \\ 0 \end{pmatrix},$$
$$\vartheta(x) = \begin{pmatrix} (\gamma+\mu)A \\ -\alpha P - \gamma A + \mu R \\ -\Lambda + \beta\frac{AP}{N} + (\alpha+\mu)P \end{pmatrix}. \tag{14}$$

The Jacobian matrices of $\mathcal{F}(x)$ and $\vartheta(x)$ at the free equilibrium E^0 are, respectively,

$$D\mathcal{F}(E^0) = \begin{pmatrix} F_{2x2} & & 0 \\ & & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$D\vartheta(E^0) = \begin{pmatrix} V_{2x2} & & 0 \\ & & -\alpha \\ \frac{\mu\beta}{(\alpha+\mu)} & 0 & 0 \end{pmatrix}, \tag{15}$$

where

$$F = \begin{pmatrix} \frac{\mu\beta}{(\alpha+\mu)} & 0 \\ 0 & 0 \end{pmatrix},$$
$$V = \begin{pmatrix} \gamma+\mu & 0 \\ -\gamma & \mu \end{pmatrix}. \tag{16}$$

Finally, we have

$$\mathfrak{R}_0 = \rho\left(FV^{-1}\right) = \frac{\mu\beta}{(\alpha+\mu)(\gamma+\mu)}. \tag{17}$$

2.1. Local Stability Analysis. In this section, we analyze the local stability of the abstaining-free equilibrium and the abstaining equilibrium.

Theorem 2. *The abstaining-free equilibrium E^0 is locally asymptotically stable if $\mathfrak{R}_0 < 1$, whereas E^0 is unstable if $\mathfrak{R}_0 > 1$.*

Proof. The Jacobian matrix at E^0 is given by

$$J_{E^0} = \begin{pmatrix} -(\alpha+\mu) & -\frac{\mu\beta}{(\alpha+\mu)} & 0 \\ 0 & \frac{\mu\beta}{(\alpha+\mu)} - (\gamma+\mu) & 0 \\ \alpha & \gamma & -\mu \end{pmatrix}. \tag{18}$$

Therefore, Eigen values of the characteristic equation of J_{E_0} are

$$\begin{aligned} \lambda_1 &= -\mu, \\ \lambda_2 &= -(\alpha+\mu), \\ \lambda_3 &= (\gamma+\mu)(\mathfrak{R}_0 - 1). \end{aligned} \tag{19}$$

Therefore, all the Eigen values of the characteristic equation are negative if $\mathfrak{R}_0 < 1$. Hence, the equilibrium point E^0 is locally asymptotically stable if $\mathfrak{R}_0 < 1$ and unstable if $\mathfrak{R}_0 > 1$. □

Theorem 3. *If $\mathfrak{R}_0 \geq 1$, the abstaining equilibrium E^* is locally asymptotically stable.*

Proof. The Jacobian matrix at E^* is given by

$$J_{E^*} = \begin{pmatrix} -\beta\frac{A^*}{N^*} - (\alpha+\mu) & -\beta\frac{P^*}{N^*} & 0 \\ \beta\frac{A^*}{N^*} & \beta\frac{P^*}{N^*} - (\gamma+\mu) & 0 \\ \alpha & \gamma & -\mu \end{pmatrix}, \tag{20}$$

and its characteristic equation is

$$\lambda^3 + a_1\lambda^2 + a_2\lambda + a_3 = 0, \tag{21}$$

where

$$\begin{aligned} a_1 &= \frac{\mu\beta}{(\alpha+\mu)} + \mu, \\ a_2 &= \frac{\mu^2\beta}{(\alpha+\mu)} + (\alpha+\mu)(\gamma+\mu)(\mathfrak{R}_0 - 1), \\ a_3 &= \frac{\mu^3\beta}{(\alpha+\mu)N} + \mu(\alpha+\mu)(\gamma+\mu)(\mathfrak{R}_0 - 1). \end{aligned} \tag{22}$$

By Routh-Hurwitz Criterion, system (1) is locally asymptotically stable if $a_1 > 0$, $a_3 > 0$, and $a_1a_2 > a_3$.

Thus, E^* is locally asymptotically stable if $\mathfrak{R}_0 \geq 1$. □

2.2. Global Stability Analysis. Now, we are concerned with the global asymptotic stability of abstaining-free equilibrium E^0 and abstaining equilibrium E^* of model (1), respectively.

Theorem 4. *If $\mathfrak{R}_0 \leq 1$, then the free equilibrium E^0 of system (1) is globally asymptotically stable on Ω.*

Proof. To prove the global stability of the free equilibrium E^0, we construct the following Lyapunov function V: $\Omega \longrightarrow R$

$$V(P,A) = \frac{1}{2}\left(\left(P - P^0\right) + A\right)^2 + \frac{(\alpha+\gamma+2\mu)N}{\beta}A. \tag{23}$$

Then, the time derivative of V is

$$\begin{aligned} \dot{V}(P,A) &= \left(P - P^0 + A\right)(\Lambda - (\alpha+\mu)P - (\gamma+\mu)A) \\ &\quad + \frac{(\alpha+\gamma+2\mu)N}{\beta}\dot{A}. \end{aligned} \tag{24}$$

Since $\Lambda = P^0(\alpha+\mu)$, (24) becomes

$$\begin{aligned} \dot{V}(P,A) &= \left(P - P^0 + A\right)\left(-(\alpha+\mu)\left(P - P^0\right) - (\gamma+\mu)A\right) \\ &\quad + \frac{(\alpha+\gamma+2\mu)N}{\beta}\dot{A} \\ &= -(\alpha+\mu)\left(P - P^0\right)^2 - (\gamma+\mu)A^2 - (\alpha+\gamma+2\mu) \\ &\quad \cdot \frac{(\gamma+\mu)N}{\beta}(1 - \mathfrak{R}_0)A. \end{aligned} \tag{25}$$

Thus, $\dot{V}(P,A) \leq 0$ for $\mathfrak{R}_0 \leq 1$.

In addition,

If $\mathfrak{R}_0 \leq 1$, then $\dot{V}(P,A) = 0 \Longleftrightarrow P = P^0$ and $A = 0$.

Hence, by LaSalle's invariance principle [17], the free equilibrium point E^0 is globally asymptotically stable on Ω. □

Theorem 5. *If $\mathfrak{R}_0 > 1$, then the abstaining equilibrium E^* of the system is globally asymptotically stable on Ω.*

Proof. For the global stability of the abstaining equilibrium E^0, we construct the Lyapunov function V: $\Omega \longrightarrow R$ given by

$$V(P,A) = X_1\left(P - P^*\ln\left(\frac{P}{P^*}\right)\right) + X_2\left(A - A^*\ln\left(\frac{A}{A^*}\right)\right), \tag{26}$$

where X_1 and X_2 are positive constants to be chosen latter.

Then, the time derivative of the Lyapunov function is given by

$$\dot{V}(P,A) = \frac{\beta}{N}(X_2 - X_1)(A - A^*)(P - P^*) - \Lambda X_1\frac{(P-P^*)^2}{PP^*}. \tag{27}$$

For $X_1 = X_2 = 1$, we have

$$\dot{V}(P,A) = -\Lambda\frac{(P-P^*)^2}{PP^*} \leq 0. \tag{28}$$

Also, we obtain

$$\dot{V}(P,A) = 0 \Longleftrightarrow P = P^*. \tag{29}$$

Hence, by LaSalle's invariance principle [17], the free equilibrium point E^* is globally asymptotically stable on Ω. □

2.3. Numerical Simulation. In this section, we present some numerical simulations of system (1) to illustrate our results. By choosing $\Lambda = 9\times10^5$, $\alpha = 0.01$, $\beta = 0.1$, $\gamma = 0.06$, $\mu = 0.04$, $t_f = 365$, and different initial values for each variable of state, we have the abstaining-free equilibrium $E^0 = (18\times10^6, 0, 4.5\times10^5)$ and $\mathfrak{R}_0 = 0.8 < 1$. In this case, according to theorem (5), the abstaining-free equilibrium E^0 of system (1) is globally asymptotically stable on Ω (see Figure 2).

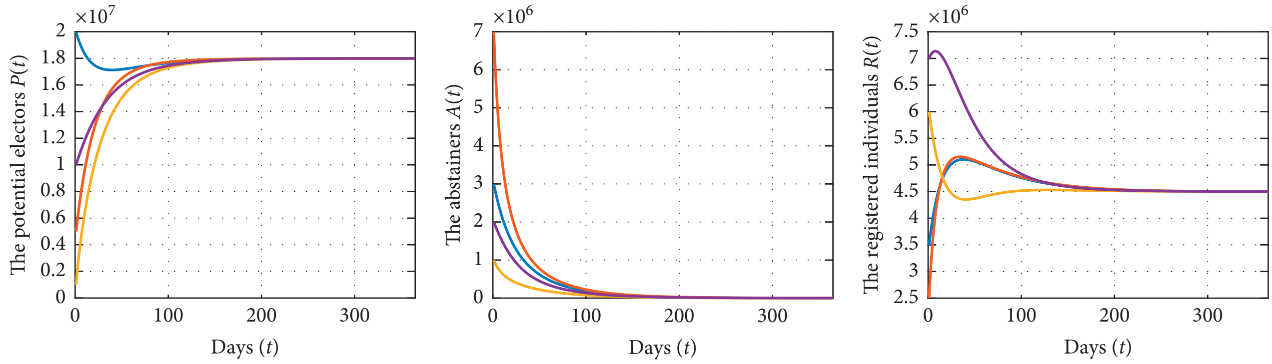
Also, for $\Lambda = 9\times10^5$, $\alpha = 0.04$, $\beta = 0.3$, $\gamma = 0.08$, $\mu = 0.04$, $t_f = 365$, we have the abstaining equilibrium $E^* = (9\times10^6, 1.5\times10^6, 1.2\times10^7)$ and $\mathfrak{R}_0 = 1.25 > 1$. In this case, according to theorem (6), the abstaining equilibrium E^* of system (1) is globally asymptotically stable on Ω (see Figure 3).

2.4. Sensitivity Analysis of Model the Parameters. Sensitivity analysis is commonly used to determine the model robustness to parameter values, that is, to help us know the parameters that have a high impact on the reproduction number $\mathfrak{R}_0$ (because there are usually errors in data collection and assumed parameter values). A sensitivity analysis of the model (1) is carried out in the sense of [18, 19].

Definition 1. The normalized forward sensitivity index of a variable v that depends differentiably on a parameter s is defined as

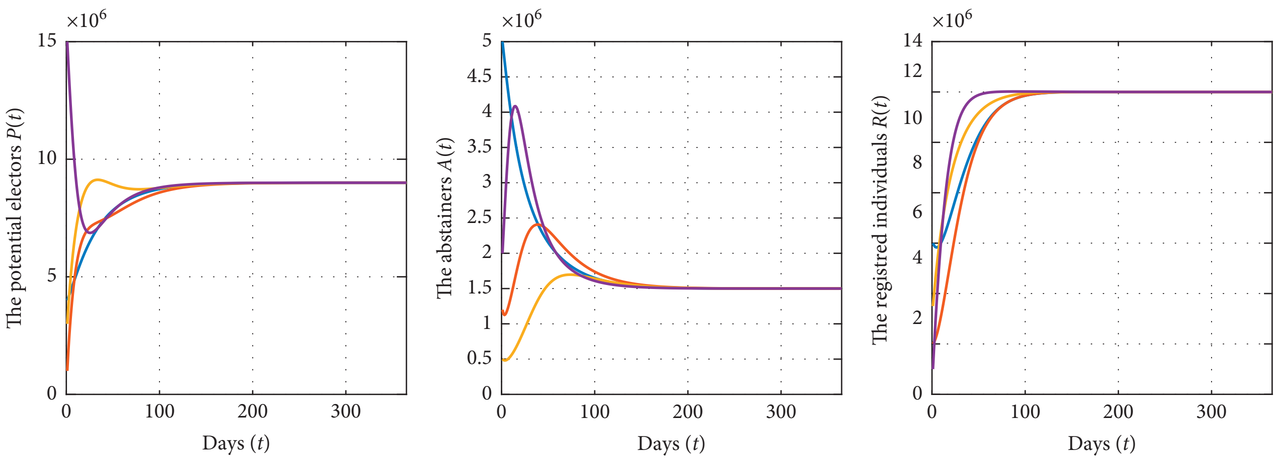
$$\Upsilon_s^v = \frac{\partial v}{\partial s}\cdot\frac{s}{v}. \tag{30}$$

In particular, sensitivity indices of the basic reproduction number $\mathfrak{R}_0$ with respect to the model parameters are computed as follows:



(a) (b) (c)

FIGURE 2



(a) (b) (c)

FIGURE 3

$$\begin{cases} \Upsilon_{\beta}^{\mathfrak{R}_0} = \dfrac{\partial \mathfrak{R}_0}{\partial \beta} \cdot \dfrac{\beta}{\mathfrak{R}_0} = 1, \\ \Upsilon_{\alpha}^{\mathfrak{R}_0} = \dfrac{\partial \mathfrak{R}_0}{\partial \alpha} \cdot \dfrac{\alpha}{\mathfrak{R}_0} = -\dfrac{\alpha}{\alpha + \mu}, \\ \Upsilon_{\gamma}^{\mathfrak{R}_0} = \dfrac{\partial \mathfrak{R}_0}{\partial \gamma} \cdot \dfrac{\gamma}{\mathfrak{R}_0} = -\dfrac{\gamma}{\gamma + \mu}, \\ \Upsilon_{\mu}^{\mathfrak{R}_0} = \dfrac{\partial \mathfrak{R}_0}{\partial \mu} \cdot \dfrac{\mu}{\mathfrak{R}_0} = \dfrac{\alpha\gamma - \mu^2}{(\alpha + \mu)(\gamma + \mu)}. \end{cases} \quad (31)$$

The positive sign of sensitivity index (S.I) of the basic reproduction number $\mathfrak{R}_0$ to the model parameters shows that an increase (or decrease) in the value of each of the parameters in this case will lead to an increase (or decrease) in the basic reproduction number of the disease. For example, $\Upsilon_{\beta}^{\mathfrak{R}_0} = 1$ suggests that increasing (or decreasing) the effective contact rate β by 10% increases (or decreases) the basic reproduction number, $\mathfrak{R}_0$, by 10%. On the other hand, the negative sign of S.I of the basic reproduction number to the model parameters implies that an increase (or decrease) in the value of each of the parameters in this case leads to a corresponding decrease (or increase) in the basic reproduction number of the disease. For example, $\Upsilon_{\alpha}^{\mathfrak{R}_0} = -0.5$ means that increasing (or decreasing) the coefficient α, by 10%, decreases (or increases) the $\mathfrak{R}_0$ by 5%. In Table 2, we present the sensitivity indices of all model parameters to $\mathfrak{R}_0$. The parameters are arranged from the most sensitive to the least sensitive.

Hence, with sensitivity analysis, one can get insight into the appropriate intervention strategies to prevent and control the spread of the abstention behavior of the registration on the electoral lists described by model (1).

3. The Optimal Control Problem

3.1. Problem Statement. Any country seeking to improve its image and political position is in an urgent need of improving many democratic indicators including increasing the registration rate on the electoral lists. To achieve this objective, it must develop some optimal strategies for an awareness program that helps officials of that country to increase the rate of citizens registered on the electoral lists

Table 2

Parameter	Description	Value	Sensitivity index
β	The effective contact rate	0.3	+1.00
α	The registration rate of potential electors	0.04	−0.50
γ	The registration rate of individuals who were abstainers	0.08	−0.67
μ	The natural death rate	0.04	−0.17

with an optimal effort, which are always time, money, and human resources.

So, our objective in this proposed strategy of control is to minimize the number of abstainers $A^t(t)$ and maximize the number of registrants $R(t)$ during the time interval $[t_0, t_f]$ and also to minimize the cost spent in an awareness program.

In the model (1), we include two controls $u_1(t)$ and $u_2(t)$ for $t \in [t_0, t_f]$: the control u_1 represents the awareness campaign effort and administrative and legal facilities effort to motivate the potential electors to register on the electoral lists. The control u_2 represents efforts, i.e., dialogues, debates, and media coverage that are made to influence the abstainers to participate in the electoral process. So, the controlled mathematical system is given by the following system of differential equations:

$$\begin{cases} \dot{P}(t) = \Lambda - \beta \dfrac{A(t)P(t)}{N} - (\alpha+\mu)P(t) - u_1(t)P(t), \\ \dot{A}(t) = \beta \dfrac{A(t)P(t)}{N} - (\gamma+\mu)A(t) - u_2(t)A(t), \\ \dot{R}(t) = \alpha P(t) + \gamma A(t) - \mu R(t) + u_1(t)P(t) + u_2(t)A(t), \end{cases} \tag{32}$$

where $P_0 \geq 0$, $R_0 \geq 0$, $A_0^t \geq 0$, $U_0 \geq 0$, $V_0^f \geq 0$, and $V_0^a \geq 0$ are the given initial states.

Then, the problem is to minimize the objective functional.

$$J(u_1, u_2, u_3) = A(t_f) - R(t_f) + \int_{t_0}^{t_f} \left(A(s) - R(s) + \frac{M}{2} u_1^2(s) + \frac{N}{2} u_2^2(s) \right) \mathrm{d}s, \tag{33}$$

where the parameters M and N are the strictly positive cost coefficients; they are selected to weigh the relative importance of u_1 and u_2 at time t; t_f is the final time.

In other words, we seek the optimal controls u_1 and u_2 such that

$$J(u_1^*, u_2^*) = \min_{(u_1,u_2)\in U_{\mathrm{ad}}^3} J(u_1, u_2), \tag{34}$$

where U_{ad} is the set of admissible controls defined by

$$U_{\mathrm{ad}} = \left\{ u_i(t)\colon 0 \leq u_i \leq 1, \quad \text{for } i = 1, 2, \text{ and } t \in [t_0, t_f] \right\}. \tag{35}$$

3.2. Existence of Optimal Controls. The existence of the optimal controls can be obtained using a result by Fleming and Rishel [20] (see Corollary 4.1).

Theorem 6. *Consider the control problem with system (32). There exists an optimal control $(u_1^*, u_2^*) \in U_{ad}^2$ such that*

$$J(u_1^*, u_2^*) = \min_{(u_1,u_2)\in U_{\mathrm{ad}}^2} J(u_1, u_2). \tag{36}$$

If the following conditions are met:

(1) The set of controls and the corresponding state variables is nonempty.

(2) The control set U_{ad} is convex and closed.

(3) The right-hand side of the state system is bounded by a linear function in the state and control variables.

(4) The integrand $L(P, A, R, u_1, u_2)$ of the objective functional is convex on U_{ad} and there exist constants $c_1, c_2 > 0$ and $\beta > 1$ such that:

$$L(P, A, R, u_1, u_2) \geq -c_1 + c_2\left(|u_1|^2 + |u_2|^2\right)^{\beta/2}. \tag{37}$$

Proof □

Condition 1. To prove that the set of controls and the corresponding state variables is nonempty, we will use a simplified version of an existence result ([21], Theorem 7.1.1).Let $\dot{P} = F_P(t; P, A, R)$, $\dot{A} = F_A(t; P, A, R)$ and $\dot{R} = F_R(t; P, A, R)$ where F_P, F_A, and F_R form the right-hand side of the system of equations (32). Let $u_i(t) = c_i$ for $i = 1, 2$ for some constants and since all parameters are constants and P, A, and R are continuous, then F_P, F_A, and F_R are also continuous. Additionally, the partial derivatives $\partial F_P/\partial P$, $\partial F_P/\partial A$, $\partial F_P/\partial R$, $\partial F_A/\partial A$, $\partial F_A/\partial A$, $\partial F_A/\partial R$, and $\partial F_R/\partial P$, $\partial F_R/\partial A$, and $\partial F_R/\partial R$ are all continuous. Therefore, there exists a unique solution (P, A, R) that satisfies the initial conditions. Therefore, the set of controls and the corresponding state variables is nonempty and condition 1 is satisfied.

Condition 2. By definition, U_{ad} is closed. Take any controls $u, v \in U_{\mathrm{ad}}$ and $\lambda \in [0, 1]$. Then $0 \leq \lambda u + (1-\lambda)v$

Additionally, we observe that $\lambda u \leq \lambda$ and $(1-\lambda)v \leq (1-\lambda)$, then $\lambda u + (1-\lambda)v \leq \lambda + (1-\lambda) = 1$

Hence,

$$0 \leq \lambda u + (1-\lambda)v \leq 1, \quad \text{for all } u, v \in U_{\mathrm{ad}} \text{ and } \lambda \in [0, 1]. \tag{38}$$

Therefore, U_{ad} is convex and condition 2 is satisfied.

Condition 3. From the system of differential equations (32), we have

$$\frac{dN}{dt} \leq \Lambda - \mu N. \tag{39}$$

Then,

$$\limsup_{t\longrightarrow\infty} N(t) \leq \frac{\Lambda}{\mu}. \tag{40}$$

Therefore, all solutions of the model (32) are bounded. So, there exist positive constants B_1, B_2, and B_3 such that $\forall t \in [t_0, t_f]$:

$$\begin{aligned} P(t) &\leq B_1, \\ A(t) &\leq B_2, \\ R(t) &\leq B_3. \end{aligned} \tag{41}$$

We consider,

$$\begin{cases} F_P = \dot{P}(t) \leq \Lambda, \\ F_A = \dot{A}(t) \leq \beta P(t) - u_2(t)A(t), \\ F_R = \dot{R}(t) \leq \alpha P(t) + \gamma A(t) + u_1(t)B_1 + u_2(t)B_2. \end{cases} \tag{42}$$

So, we can rewrite system (32) in matrix form as

$$F(t; P, A, R) \leq \overline{\Lambda} + AX(t) - BU(t), \tag{43}$$

where

$$\begin{aligned} F(t; P, A, R) &= \begin{bmatrix} F_P & F_A & F_R \end{bmatrix}^T, \\ \overline{\Lambda} &= \begin{bmatrix} \Lambda & 0 & 0 \end{bmatrix}^T, \\ X(t) &= \begin{bmatrix} P & A & R \end{bmatrix}^T, \\ U(t) &= \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T, \\ A &= \begin{bmatrix} 0 & 0 & 0 \\ \beta & 0 & 0 \\ \alpha & \gamma & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} P & 0 \\ 0 & A \\ P & A \end{bmatrix}. \end{aligned} \tag{44}$$

It gives a linear function of control vector and state variable vector. Therefore, we can write

$$\begin{aligned} \|F(t; P, A, R)\| &\leq \|\overline{\Lambda}\| + \|A\|\|X(t)\| + \|B\|\|U(t)\| \\ &\leq \varphi + \phi(\|X(t)\| + \|U(t)\|), \end{aligned} \tag{45}$$

where $\varphi = \|\overline{\Lambda}\|$ and $\phi = \max(\|A\|, \|B\|)$.

Hence, we see the right-hand side is bounded by a sum of state and control vectors. Therefore, condition 3 is satisfied.

Condition 4: The integrand in the objective functional (33) is convex on U_{ad}. It rests to show that there exist constants $c_1, c_2 > 0$ and $\beta > 1$ such that the integrand $L(P, A, R, u_1, u_2)$ of the objective functional satisfies

$$\begin{aligned} L(P, A, R, u_1, u_2) &= A(t) - R(t) + \frac{M}{2}u_1^2 + \frac{N}{2}u_2^2 \\ &\geq -c_1 + c_2\left(|u_1|^2 + |u_2|^2\right)^{\beta/2}. \end{aligned} \tag{46}$$

The state variables being bounded, let $c_1 = 2\sup_{t\in[t_0,t_f]}(A, R)$, $c_2 = \inf(M/2, N/2)$, and $\beta = 2$, then it follows that

$$L(P, A, R, u_1, u_2) \geq -c_1 + c_2\left(|u_1|^2 + |u_2|^2\right)^{\beta/2}. \tag{47}$$

3.3. Characterization of the Optimal Controls. In this section, we apply the Pontryagin's maximum principle [9]. The key idea is introducing the adjoint function to attach the system of differential equations to the objective functional resulting in the formation of a function called the Hamiltonian. This principle converts the problem of finding the control to optimize the objective functional subject to the state of differential equations with initial condition to find the control to optimize Hamiltonian pointwise (with respect to the control).

Now, we have the Hamiltonian H in time t, defined by

$$H(t) = A(t) - R(t) + \frac{M}{2}u_1^2(t) + \frac{N}{2}u_2^2(t) + \sum_{i=1}^{3}\lambda_i f_i, \tag{48}$$

where f_i is the right side of the system of differential equations (32) of the i^{th} state variable.

Theorem 7. *Given an optimal control $u^* = (u_1^*, u_2^*) \in U_{ad}^2$ and solutions P^*, A^*, and R^* of the corresponding state system (32), there exist adjoint functions λ_1, λ_2, and λ_3 satisfying*

$$\begin{cases} \dot{\lambda}_1 = \lambda_1\left\{\beta\frac{A(t)}{N} + \alpha + \mu + u_1(t)\right\} - \lambda_2\beta\frac{A(t)}{N} - \lambda_3\{\alpha + u_1(t)\}, \\ \dot{\lambda}_2 = -1 + \lambda_1\beta\frac{P(t)}{N} - \lambda_2\left\{\beta\frac{P(t)}{N} - \gamma - \mu - u_2(t)\right\} - \lambda_3\{\gamma + u_2(t)\}, \\ \dot{\lambda}_3 = 1 + \lambda_3\mu, \end{cases} \tag{49}$$

with the transversality conditions at time t_f

$$\begin{aligned} \lambda_1(t_f) &= 0, \\ \lambda_2(t_f) &= 1, \\ \lambda_3(t_f) &= -1. \end{aligned} \tag{50}$$

Furthermore, for $t \in [t_0, t_f]$, the optimal controls $u_1^*(t)$ and $u_2^*(t)$ are given by

$$u_1^*(t) = \min\left(1, \max\left(0, \frac{1}{M} P(t)(\lambda_1(t) - \lambda_3(t))\right)\right). \quad (51)$$

$$u_2^*(t) = \min\left(1, \max\left(0, \frac{1}{N} A(t)(\lambda_3(t) - \lambda_2(t))\right)\right). \quad (52)$$

Proof. The Hamiltonian in time t is given by

$$\begin{aligned} H = {} & A(t) - R(t) + \frac{M}{2}u_1^2(t) + \frac{N}{2}u_2^2(t) + \lambda_1 \\ & \cdot \left\{\Lambda - \beta\frac{A(t)P(t)}{N} - (\alpha+\mu)P(t) - u_1(t)P(t)\right\} \\ & + \lambda_2\left\{\beta\frac{A(t)P(t)}{N} - (\gamma+\mu)A(t) - u_2(t)A(t)\right\} \\ & + \lambda_3\{\alpha P(t) + \gamma A(t) - \mu R(t) + u_1(t)P(t) + u_2(t)A(t)\}. \end{aligned} \quad (53)$$

For $t \in [t_0, t_f]$, the adjoint equations and transversality conditions can be obtained by using Pontryagin's maximum principle given in [22] such that

$$\begin{cases} \dot{\lambda}_1 = -\dfrac{\partial H}{\partial P}, & \lambda_1(t_f) = 0, \\ \dot{\lambda}_2 = -\dfrac{\partial H}{\partial A}, & \lambda_2(t_f) = 1, \\ \dot{\lambda}_3 = -\dfrac{\partial H}{\partial R}, & \lambda_3(t_f) = -1. \end{cases} \quad (54)$$

For $t \in [t_0, t_f]$, the optimal controls $u_1^*(t)$ and $u_2^*(t)$ can be solved from the optimality condition

$$\frac{\partial H}{\partial u_1} = 0, \qquad \frac{\partial H}{\partial u_2} = 0, \quad (55)$$

that is,

$$\begin{aligned} \frac{\partial H}{\partial u_1} &= Mu_1(t) + \lambda_1(t)\{-P(t)\} + \lambda_3(t)\{P(t)\} = 0, \\ \frac{\partial H}{\partial u_2} &= Nu_2(t) + \lambda_2(t)\{-A(t)\} + \lambda_3(t)\{A(t)\} = 0. \end{aligned} \quad (56)$$

$$\begin{aligned} u_1(t) &= \frac{1}{M}P(t)(\lambda_1(t) - \lambda_3(t)), \\ u_2(t) &= \frac{1}{N}A(t)(\lambda_3(t) - \lambda_2(t)). \end{aligned} \quad (57)$$

By the bounds in U_{ad} of the controls, it is easy to obtain $u_1^*(t)$ and $u_2^*(t)$ in the form of ((51) and (52)). □

3.4. Numerical Simulation. In this section, we present the results obtained by numerically solving the optimality system. In our control problem, we have initial conditions for the state variables and terminal conditions for the adjoints. That is, the optimality system is a two-point boundary value problem with separated boundary conditions at times step $i = t_0$ and $i = t_f$. We solve the optimality system by an iterative method with forward solving of the state system followed by backward solving of the adjoint system. We start with an initial guess for the controls at the first iteration and then before the next iteration, we update the controls by using the characterization. We continue until convergence of successive iterates is achieved. A code is written and compiled in Matlab using the following data: $\Lambda = 3 \times 10^6$, $\alpha = 0.293$, $\beta = 0.700$, $\gamma = 0.020$, $\mu = 0.054$, and the initial values $P_0 = 10502796$, $A_0 = 9041311$, and $R_0 = 8020655$.

The proposed control strategy in this work helps to improve an important political indicator of a democratic election that is the rate of registration on the electoral lists:

Case 1: we use only the optimal control u_1^*.

In this strategy, we focus the effort of the awareness campaign on potential electors in order to sensitize them of the importance of political and electoral participation and to protect them from the negative impact of abstainers.

From these figures, we observe that the number of registered individuals on the electoral lists has increased from 1.8765×10^7 to 2.0913×10^7 at the end of this awareness campaign (Figure 4(a)). Also, the number of abstainers has changed, such that the number of abstainers has decreased from 1.4632×10^7 to 1.3011×10^7 at the end of the campaign (Figure 4(b)). Consequently, we observe that the rate of registration abstinence on the electoral lists has decreased by 11% (Figure 5(a)). On the other hand, in (Figure 5(b)), we show that the registration rate on the lists is increased by 11.42%.

These changes are important but not sufficient; it is for this reason that we must also target by this strategy of control the abstainers.

Case 2: we combine the optimal control u_1^* and u_2^*.

In this strategy, the two optimal controls u_1^* and u_2^* are activated at the same time in order to improve the numerical results of case 1. The optimal control u_2^* represents the effort to convince the abstainers to change their opinion and reduce their negative effect on the potential electors.

From Figure 6(a), we can see that the number of registered individuals increases significantly from 1.8765×10^7 to 2.7204×10^7. Also, Figure 6(b) demonstrates that the number of abstainers decreased from 1.4632×10^7 to 5.8388×10^6 at the end of this political campaign.

Therefore, we observe that the rate of registration abstinence on the electoral lists has decreased by 105.25% (Figure 7(a)) and the registration rate on the lists is increased by 44.82% (Figure 7(b)).

Finally, we conclude that the proposed strategy becomes more effective when we combined two optimal controls u_1^* and u_2^*.

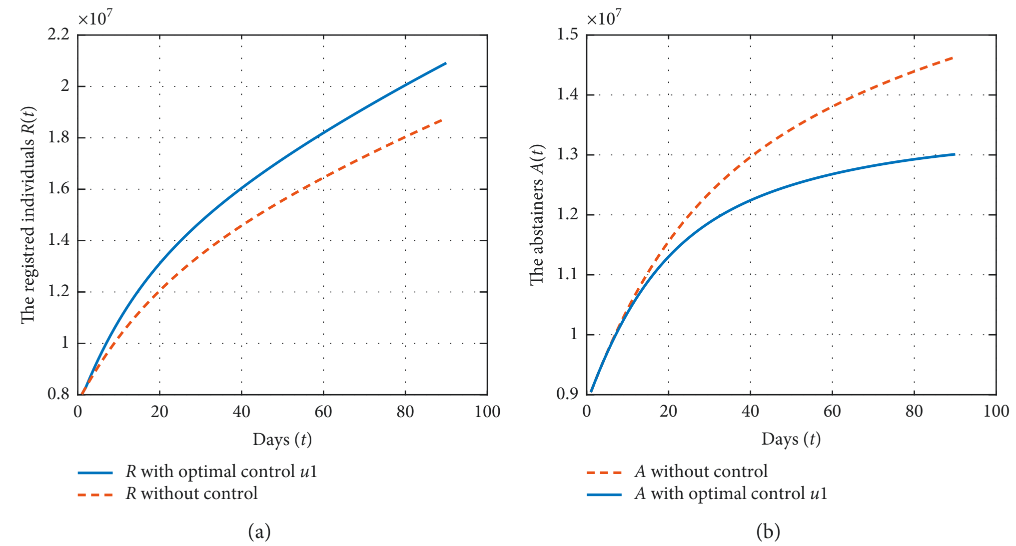


Figure 4

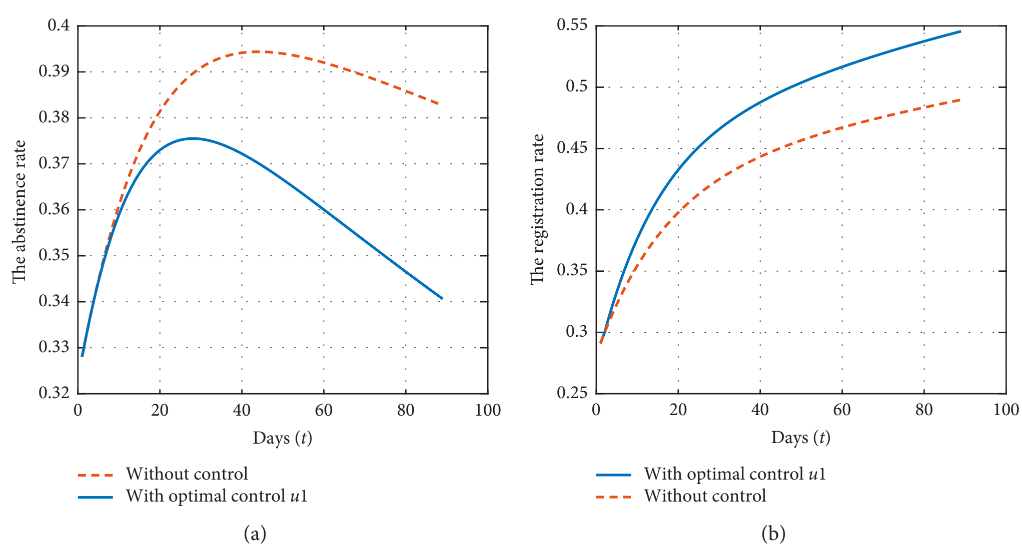


Figure 5

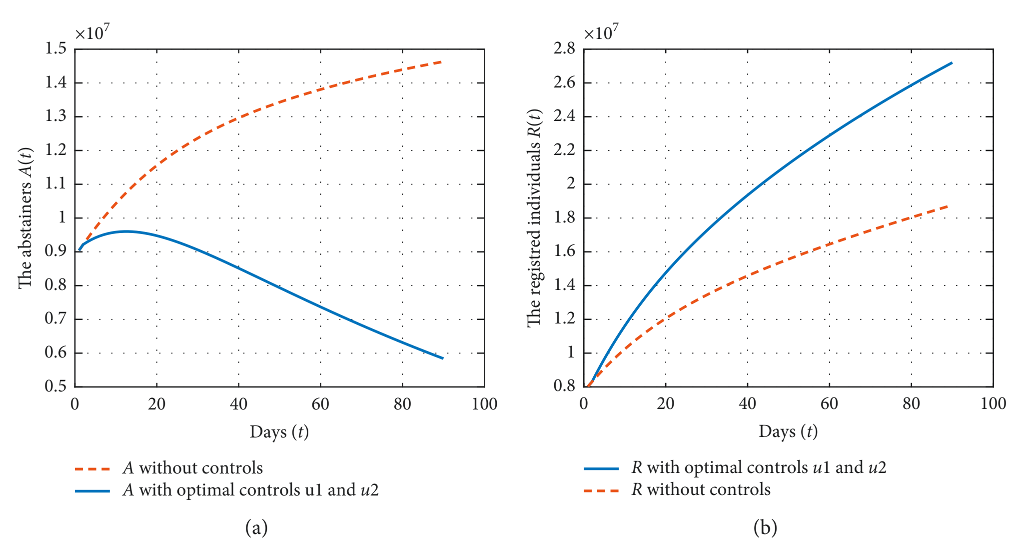


Figure 6

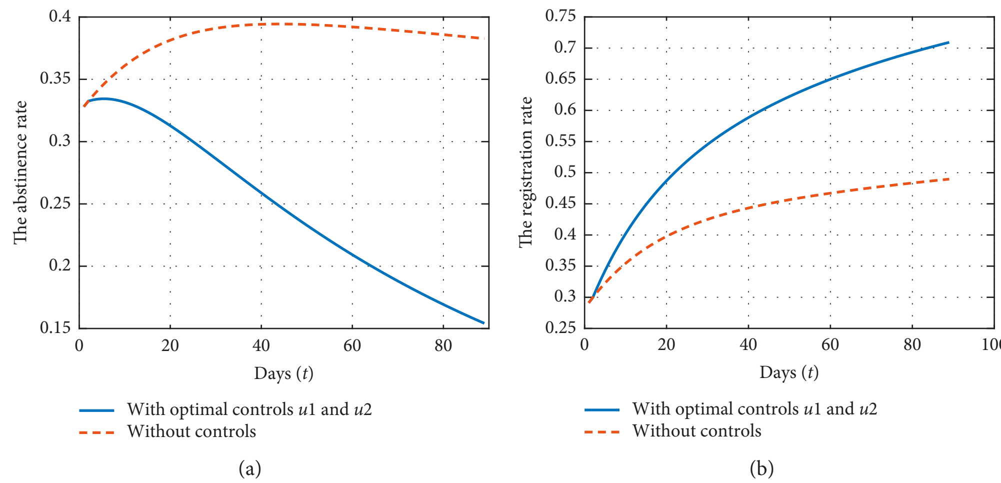


Figure 7

4. Conclusion

In this work, we formulated a mathematical model that describes the dynamics of citizens who have the right to register on the electoral lists and the negative influence of abstainers, who abstain from registration on the electoral lists and the electoral process, on potential electors. By using Routh–Hurwitz criteria and constructing Lyapunov functions, the local stability and the global stability of abstaining-free equilibrium and abstaining equilibrium are obtained. We also studied the sensitivity analysis of the model parameters to know the parameters that have a high impact on the reproduction number $\mathfrak{R}_0$. In addition, we proposed an optimal strategy for an awareness program that helps politicians and officials to increase the rate of citizens registered on the electoral lists with an optimal effort. In this work, we introduced two controls: the first control represented the awareness campaign effort and administrative and legal facilities effort to motivate the potential electors to register on the electoral lists. The second control measured the efforts directed to dialogues, debates, and media coverage that are made to make an impact on the abstainers to participate in the electoral process. Pontryagin's maximum principle is used to characterize the optimal controls and the optimality system is solved by an iterative method.

The numerical simulation was carried out using Matlab. We proposed an algorithm based on the forward and backward difference approximation and we show that the optimal strategy becomes more effective when we combined two optimal controls.

Data Availability

The disciplinary data used to support the findings of this study have been deposited in the Network Repository (http://www.networkrepository.com).

Conflicts of Interest

The authors declare that there is no conflict of interest regarding the publication of this paper.

References

[1] S. Ansolabehere, "The introduction of voter registration and its effect on turnout," *Political Analysis*, vol. 14, no. 1, pp. 83–100, 2005.

[2] S. Ansolabehere, E. Hersh, A. Gerber, and D. Doherty, *Voter Registration List Quality Pilot Studies: Report on Methodology*, Harvard University and Yale University, New Haven, CT, USA, 2010.

[3] M. Henn and N. Foard, "Social differentiation in young people's political participation: the impact of social and educational factors on youth political engagement in Britain," *Journal of Youth Studies*, vol. 17, pp. 360–380, 2014.

[4] J. R. Neiheisel and B. C. Burden, "The impact of election day registration on voter turnout and election outcomes," *American Politics Research*, vol. 40, pp. 636–664, 2012.

[5] J. S. Rosenberg, *Expanding Democracy: Voter Registration around the World*, Brennan Center for Justice at New York University School of Law, New York, NY, USA, 2009.

[6] T. Wang, *Voter Registration in the Middle East and North Africa: Select Case Studies*, National Democratic Institute for International Affairs (NDI), Washington, DC, USA, 2015.

[7] E. A. Bennion and D. W. Nickerson, "I will register and vote if you teach me how: a field experiment testing voter registration in college classrooms," *PS: Political Science & Politics*, vol. 49, no. 4, pp. 867–871, 2016.

[8] C. Braconnier, J.-Y. Dormagen, and V. Pons, "Voter registration costs and disenfranchisement: experimental evidence from France," *American Political Science Review*, vol. 111, no. 3, pp. 584–604, 2017.

[9] L. S. Pontryagin, V. G. Boltyanskii, R. V. Gamkrelidze, and E. F. Mishchenko, *The Mathematical Theory of Optimal Processes*, Wiley, New York, NY, USA, 1962.

[10] C. J. Snelling, "Young people and electoral registration in the UK: examining local activities to maximise youth registration," *Parliamentary Affairs*, vol. 69, no. 3, pp. 663–685, 2016.

[11] O. Balatif, A. Labzai, and M. Rachik, "A discrete mathematical modeling and optimal control of the electoral behavior with regard to a political party," *Discrete Dynamics in Nature and Society Research Article*, vol. 2018, Article ID 9649014, 14 pages, 2018.

[12] K. Calderon, C. Orbe, A. Panjwani, D. M. Romero, C. Kribs-Zaleta, and K. Ros-Soto, *An Epidemiological Approach to the*

Spread of Political Third Parties, Arizona State University, Tempe, AZ, USA, 2005, http://mtbi.asu.edu/Sum.

[13] Q. J. A. Khan, "Hopf bifurcation in multiparty political systems with time delay in switching," *Applied Mathematics Letters*, vol. 13, no. 7, pp. 43–52, 2000.

[14] I. Petersen, "Stability of equilibria in multi-party political systems," *Mathematical Social Sciences*, vol. 21, no. 1, pp. 81–93, 1991.

[15] M. Bani-Yaghoub, R. Gautam, Z. Shuai, P. van den Driessche, and R. Ivanek, "Reproduction numbers for infections with free-living pathogens growing in the environment," *Journal of Biological Dynamics*, vol. 6, no. 2, pp. 923–940, 2012.

[16] P. V. Driessche and J. Watmough, "Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission," *Mathematical Biosciences*, vol. 180, pp. 29–48, 2002.

[17] J. P. LaSalle, "The stability of dynamical systems," *Regional Conference Series in Applied Mathematics*, Vol. 25, SIAM, Philadelphia, PA, USA, 1976.

[18] O. D. Makinde and K. O. Okosun, "Impact of chemo-therapy on optimal control of malaria disease with infected immigrants," *BioSystems*, vol. 104, pp. 32–41, 2011.

[19] C. Nakul, J. M. Cushing, and J. M. Hyman, "Bifurcation analysis of a mathematical model for malaria transmission," *SIAM Journal on Applied Mathematics*, vol. 67, no. 1, pp. 24–45, 2006.

[20] W. H. Fleming and R. W. Rishel, *Deterministic and Stochastic Optimal Control*, Springer, New York, NY, USA, 1975.

[21] W. E. Boyce and R. C. DiPrima, *Elementary Differential Equations and Boundary Value Problems*, John Wiley and Sons, New York, NY, USA, 2009.

[22] D. W. Nickerson, "Do voter registration drives increase participation? For whom and when?," *Journal of Politics*, vol. 77, no. 1, pp. 88–101, 2015.