Research Article

Projected Adaptive Cubic Regularization Algorithm with Derivative-Free Filter Technique for Box Constrained Optimization

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An adaptive projected affine scaling algorithm of cubic regularization method using a filter technique for solving box constrained optimization without derivatives is put forward in the passage. The affine scaling interior-point cubic model is based on the quadratic probabilistic interpolation approach on the objective function. The new iterations are obtained by the solutions of the projected adaptive cubic regularization algorithm with filter technique. We prove the convergence of the proposed algorithm under some assumptions. Finally, experiments results showed that the presented algorithm is effective in detail.

1. Introduction

In this passage, we investigate the following optimization:

$$\begin{aligned} \min & f(y) \\ \text{s.t. } & y = (y_1, y_2, \dots, y_n) \in R^n \\ & l_i \leq y_i \leq u_i \\ & i = 1, 2, \dots, n, \end{aligned} \quad (1)$$

where $f: R^n \rightarrow R$ is smooth, but its derivative information is unavailable or unreliable. For each i , l_i is not greater than u_i , and there exists some i_k such that l_k is less than u_k strictly. Define the feasible set and the strict interior by Ω , $\text{int}(\Omega)$, respectively.

The structure of this passage is as shown below. The relevant information is reviewed and our algorithm for problem (1) is introduced in Section 2. The corresponding analysis are investigated in Sections 3 and 4, respectively. Then, the numerical experiments are listed in details in Section 5.

2. Literature Review

Optimization (1) forms an ordinary but important class in various circumstances. There are lots of researchers using different tools to solve it ([1–4]). In order to complements the algorithm library for solving this kind of optimization, we extent the cubic regularization algorithm (shorted by ARC), which is presented by Cartis et al. in [5, 6], with filter technique to solve (1) by projection measure based on probabilistic models.

The numerical results of classical ARC showed excellent performance. In [7], Cartis et al. extended ARC for unconstrained optimizations to finite-difference version. In [8], the authors extended the bound of [5, 6] to nonlinear problems with convex constraints. Gould et al., in [9], updated the parameter in the large-scale least-squares nonlinear optimization. Huang and Zhu, in [3], presented an affine scaling ARC algorithm for derivative-free box constrained minimization using backtracking line search technique to obtain the step size. In the framework of ARC, the authors did not use box constraints' information; thus,

we turn to an affine scaling technique in [10], which usually combine with the methods of trust region [10, 11]. However, for various reasons, there are many examples in computational science and engineering; their (at least some) derivatives were unavailable or unreliable. However, they may still be desirable to get the optimizations. This situation led researchers to look for the way to solve (1). Recently, there are many papers proposing various different techniques for the optimization without derivatives. Algorithms for bound constrained optimization with global convergence results using the trust region derivative-free methods were presented in [12, 13]. Powell, in [14, 15], proposed the NEWUOA algorithm, which employs a quadratic polynomial interpolation for the unconstrained or box constrained optimization with excellent experiment performance. In [16], Conn et al. gave a derivative-free version of the trust region method. The corresponding model was constructed by using the polynomial interpolation techniques. Li and Zhu, in [17], proposed an affine trust region method to solve (1) by using a backtracking relevant condition and a filter technique.

Motivated by these ideas, we will try our best to introduce the ARC method using filter technique to solve (1) without derivatives under certain assumptions in this paper. In absence of derivatives, we use probabilistic polynomials' interpolation to construct models of f . The presented algorithm has two merits. The first merit is that to solve the subproblem which is only needed once at every point under some conditions, but the ARC algorithms need to solve repeatedly in [5, 6], in which solving a subproblem is computationally more expensive than using line search to get the trial step. The other merit is that we take a backing line search technique to make the new iteration strictly feasible.

In this paper, unless otherwise noted, we write $\|\cdot\| = \|\cdot\|_2$ for brevity.

2.1. Notations. There are many constants and notations displayed in the next sections. For convenience, we collect the following:

- κ_g is bound on $\nabla f(y)$
- κ_h is bound on $\nabla^2 f(y)$
- κ_{ng} is bound on $\nabla m(y) \triangleq g(y)$
- κ_H is bound on $\nabla^2 m(y) \triangleq H(y)$
- D affines scaling matrix for f
- $\bar{D}$ affines scaling matrix for m
- κ_D is bound on $D, \bar{D}$
- $\bar{\beta}_i$ are constants related to m
- κ is projected gradient for f
- $\bar{\kappa}$ is projected gradient for m
- α_k is the step size at y_k
- $\alpha_{k,l}$ is the l th trial step size at y_k

3. Development of the Algorithm

At first, we review the elements of our presented algorithm.

Following the notations proposed by Cartis et al., in [5, 6], we give the ARC subproblem as follows:

$$\min \psi_k(d) = \nabla f(y_k)^T d + \frac{1}{2} d^T \nabla^2 f(y_k) d + \frac{1}{3} \sigma_k \|d\|^3, \quad (2)$$

where σ_k is a dynamic positive parameter.

Definition 1. A mapping $P_C: R^n \rightarrow C$ is called the projection onto C if

$$P_C(y) = \arg \min_{x \in C} \|x - y\|, \quad (3)$$

where $C \neq \emptyset$ and C is closed. $P_C(y) = \max\{l, \min\{y, u\}\}$ since Ω in (1) is a box. $\kappa(y) = \|y - P_\Omega(y - \nabla f(y))\|$ is a well-known criticality measure.

Definition 2 (see [18, 19]). Function m is said fully quadratic of f in $B(y_k, \delta_k)$ if, for all $y \in B(y_k, \delta_k)$, there exist some positive constants $\kappa_{ef}, \kappa_{eg}, \kappa_{eh}$, such that

$$|f(y) - m(y)| \leq \kappa_{ef} \delta_k^3, \quad (4)$$

$$\|\nabla f(y) - \nabla m(y)\| \leq \kappa_{eg} \delta_k^2, \quad (5)$$

$$\|\nabla^2 f(y) - \nabla^2 m(y)\| \leq \kappa_{eh} \delta_k. \quad (6)$$

Next, random models M_k will be considered; $m_k = M_k(\omega_k)$ is denoted for their realizations. X_k and Δ_k will be used for random quantities, while their realizations are denoted by $x_k = X_k(\omega_k)$ and $\delta_k = \Delta_k(\omega_k)$, respectively.

Definition 3 (see [18, 19]). A sequence $\{M_k\}$ is said (p) -probabilistically fully quadratic for sequence $\{B(X_k, \Delta_k)\}$ if $S_k = \{M_k \text{ is a full quadratic on sequence } \{B(X_k, \Delta_k)\}\}$ satisfy $P(S_k | F_{k-1}^M) \geq p$, in which $F_{k-1}^M = \sigma(M_0, \dots, M_{k-1})$ is generated by $M_0, \dots, M_{k-1}$. The random models are probabilistically fully quadratic if $p \geq (1/2)$.

Now, we expand (2) to the derivative-free version to solve (1):

$$\min \varphi_k(d) = g_k^T d + \frac{d^T H_k d}{2} + \frac{\sigma_k \|d\|^3}{3} \quad (7)$$

$$\text{s.t. } l \leq x_k + d \leq u,$$

where $g_k = \nabla m(y_k)$ and $H_k = \nabla^2 m(y_k)$. And, $g_k \neq \nabla f(y_k)$ and $H_k \neq \nabla^2 f(y_k)$ in the derivative-free case, so $\bar{\kappa}(y) \stackrel{\text{def}}{=} \|y - P_\Omega(y - \nabla m(y))\| \neq \kappa(y)$.

Filter technique was proposed firstly by Fletcher and Leyffer for constrained nonlinear optimization. The idea of using filter is to interpret the system of (1) as a biobjective optimization with two goals: minimizing $f(y)$ and minimizing $\|h(y)\|$, where $h(y) = \bar{D}(y)g(y)$, $\bar{D}(y) = \text{diag}\{\bar{v}(y)\}$, and

$$[\bar{v}(y)]_j = \begin{cases} 1, & \text{if } l_j = -\infty \text{ and } u_j = +\infty, \\ \min\{y_j - l_j + \mu \max\{0, -[\nabla m(y)]_j\}, & \\ u_j - y_j + \mu \max\{0, [\nabla m(y)]_j\}, & \text{otherwise.} \end{cases} \quad (8)$$

Then, we also definite $D(y)$ is the derivative version of $\bar{D}(y)$.

In order to ensure f or $\|h\|$ decreasing sufficiently, we give such a strategy:

$$(1 - \eta_h)\|h(y_k)\| \geq \|h(y_k(\alpha_{k,l}))\|, \quad (9)$$

or

$$f(y_k) - \eta_f \alpha_{k,l} \|h(y_k)\|^2 \geq f(y_k(\alpha_{k,l})), \quad (10)$$

where $y_k(\alpha_{k,l}) = y_k + \alpha_{k,l} d_k$ and $\eta_h, \eta_f \in (0, 1)$ are small positive constants.

If (9) or (10) holds, the current filter $\mathcal{F}$ accepts $y_k(\alpha_{k,l})$. The filter remains unchanged. Otherwise, neither (9) nor (10) holds; the current filter rejects $y_k(\alpha_{k,l})$; the filter can be augmented as

$$\mathcal{F}_{k+1} = \left\{ (\|h\|, f) \in R^n : (1 - \eta_h)\|h(y_k)\| \geq \|h\| \text{ and } f(y_k) - \eta_f \alpha_k \|h(y_k)\|^2 \geq f \right\} \cup \mathcal{F}_k. \quad (11)$$

At the beginning, initialize $\mathcal{F}_0 = \{(\|h\|, f) : \|h\| \geq \|h\|_0\}$.

Now, we present a projected adaptive cubic regularization algorithm with filter technique (shorted by PFARC) for the solutions of derivative-free box constraint optimization:

3.1. Initialization Step. An initial point $y_0 \in \text{int}(\Omega) = \{y : l < y < u\}$. Given a positive parameter $\sigma_0 > 0$, a radius $1 \geq \delta_0 > 0$, and constants $\gamma_1, \gamma_2, \gamma, \lambda_1, \lambda_2$, they satisfy that

$$\begin{aligned} 0 < \gamma_1 \leq \gamma_2 < 1, \\ 0 < \gamma < 1, \\ 0 < \lambda_1 \leq \lambda_2 < 1. \end{aligned} \quad (12)$$

Initialize the filter $\mathcal{F}_0$. Choose $\varepsilon \in (0, 1)$, $\eta_f, \eta_h \in (0, 1)$, and $\gamma_3 \in (0, 1)$. Set $k = 0$.

3.2. Main Step

Step 1: set a interpolation points set X_k such that $y_k \in X_k \subseteq B(y_k, \delta_k)$; then, apply Algorithms 6.1 and 6.3 in [20] to construct the fully quadratic model $m_k = m(y_k)$ on $B(y_k, \delta_k)$. Set

$$\begin{aligned} g_k &= \nabla m_k, \\ H_k &= \nabla^2 m_k, \\ \bar{\kappa}_k &= \bar{\kappa}(y_k), \\ h(y_k) &\stackrel{\text{def}}{=} \bar{D}_k g_k. \end{aligned} \quad (13)$$

Step 2: if $\bar{\kappa}_k = 0$, stop at y_k .

Step 3: solve (7) and obtain d_k .

Step 4: line search.

Step 4.1: let $l = 0$ and $\alpha_{k,0} = 1$.

Step 4.2: set $y_k(\alpha_{k,l}) = y_k + \alpha_{k,l} d_k$.

Step 4.3: if $\mathcal{F}$ accepts $y_k(\alpha_{k,l})$, go to Step 4.4, else go to Step 4.6.

Step 4.4: if $(1 - \eta_h)\|h(y_k)\| \geq \|h(y_k(\alpha_{k,l}))\|$, $y_k + \alpha_{k,l} d_k \in \Omega$ holds, and accept $\alpha_{k,l}$ and go to Step 4.7, else go to Step 4.5.

Step 4.5: if $f(y_k) - \eta_f \alpha_{k,l} \|h(y_k)\|^2 \geq f(y_k(\alpha_{k,l}))$, $y_k + \alpha_{k,l} d_k \in \Omega$ holds, and accept $\alpha_{k,l}$ and go to Step 4.7, else go to Step 4.6.

Step 4.6: set $\alpha_{k,l+1} \in [\lambda_1 \alpha_{k,l}, \lambda_2 \alpha_{k,l}]$; let $l = l + 1$; then, go to Step 4.2.

Step 4.7: set $\alpha_k = \alpha_{k,l}$ and

$$s_k = \begin{cases} \alpha_k d_k, & \text{if } y_k + \alpha_k d_k \in \text{int}(\Omega), \\ \theta_k \alpha_k d_k, & \text{if otherwise,} \end{cases} \quad (14)$$

where $\theta_k \in (\theta_0, 1]$, for some $0 < \theta_0 < 1$ and $\theta_k - 1 = O(\|d_k\|)$.

Step 5: let $\tilde{\delta}_k = \delta_k$. If $\tilde{\delta}_k > \min(\kappa_k, \|s_k\|)$, set $\tilde{\delta}_k = \gamma_3 \tilde{\delta}_k$, and apply Algorithm 6.1 and 6.3 in [20] to construct fully quadratic model $\tilde{m}_k$ in $B(y_k, \tilde{\delta}_k)$. Set $\tilde{g}_k = \nabla \tilde{m}_k$ and $\tilde{H}_k = \nabla^2 \tilde{m}_k$. Go to Step 3 to compute $\tilde{d}_k$. Until $\tilde{\delta}_k \leq \min(\bar{\kappa}_k, \|\tilde{s}_k\|)$. Update $m_k = \tilde{m}_k$, $g_k = \nabla \tilde{m}_k$, $H_k = \nabla^2 \tilde{m}_k$, $\bar{\kappa}_k = \bar{\kappa}_k$, and $s_k = \tilde{s}_k$, and compute $\varphi_k(s_k)$.

Step 6: set $y_{k+1} = y_k + s_k$. Compute $f_{k+1}, m_{k+1}, g_{k+1}$, and H_{k+1} .

Step 7: compute ratio $\rho_k = (f(y_k) - f(y_{k+1})) / -\varphi_k(s_k)$. Set

$$\sigma_{k+1} = \begin{cases} \gamma^{-1} \sigma_k, & \text{if } \rho_k < \gamma_1, \\ \sigma_k, & \text{if } \gamma_2 \geq \rho_k \geq \gamma_1, \\ \gamma \sigma_k, & \text{if } \rho_k > \gamma_2. \end{cases} \quad (15)$$

Step 8: if neither (9) nor (10) holds, augment the filter set, else the filter does not change.

Step 9: set $x_0^{k+1} = y_{k+1}$ and $\delta_{k+1} = \delta_k$. Then, the sample set $X_{k+1} = \{x_0^{k+1}, \dots, x_q^{k+1}\}$ is constructed in $B(y_{k+1}, \delta_{k+1})$. Compute g_{k+1}, H_{k+1} . Set $k = k + 1$.

Remark 1. Now, we describe a concrete implementation in Step 3 using a Cauchy step which is constructed by an affine scaled gradient [21]. The search direction d_k satisfies $\varphi_k(d_k) \leq \varphi_k(d_k^C)$, where

$$\begin{aligned} d_k^C &= -\zeta_k^C \bar{D}_k^2 g_k, \\ \zeta_k^C &= \arg \min_{\zeta \in \mathbb{R}_+} \varphi_k(-\zeta \bar{D}_k^2 g_k), \\ \varphi_k(d) &= g_k^T d + \frac{1}{2} d^T H_k d + \frac{1}{3} \sigma_k \|d\|^3. \end{aligned} \quad (16)$$

As we all know that the computation of Cauchy condition usually is much cheaper than the computation in Step 3, we will give a relationship between the critical measure $\bar{\kappa}$ and the affine gradient. Therefore, we can replace the predicted decrease by the Cauchy step at each criticality measure satisfying this relationship.

Remark 2. Similar to Lemma 5.1 in [16], we can prove that Step 5 will terminate in a finite number if $\kappa_k \neq 0$.

Remark 3. In Step 4, the scalar α_k with $y_k + \alpha_k d_k \in \Omega$ denotes that

$$\alpha_k^{\text{def}} = \min \left\{ \max \left\{ \frac{u_j - [y_k]_j}{[d_k]_j}, \frac{l_j - [y_k]_j}{[d_k]_j} \right\}, j = 1, 2, \dots, n \right\}, \quad (17)$$

in which $u_j, l_j, [y_k]_j$, and $[d_k]_j$ represent the j th elements of u, l, y_k , and d_k , respectively.

4. Global Convergence Analysis

Next, we will present some properties of the algorithm PFARC which will be referenced in analysis of global and local convergence. Denote the indices set of the augmented filter by $\mathcal{A}$. First, we make some assumptions as following.

Assumption 1. The iterates y_k remain in $\mathcal{L}(y_0)$, where $\mathcal{L}(y_0)$ is the level set of f , $\mathcal{L}(y_0) = \{y \in \mathbb{R}^n | f(y_0) \geq f(y), y \in \Omega\}$.

Assumption 2. Let $f: \mathbb{R}^n \rightarrow \mathbb{R}$ be twice continuously differentiable and bounded from below, and its gradient is Lipschitz continuous with Lipschitz constants L_g .

Assumption 3. Assume that $\|D(y)\| \leq \kappa_D$, $\|\bar{D}(y)\| \leq \kappa_D$, $\|\nabla f(y)\| \leq \kappa_g$, and $\|\nabla^2 f(y)\| \leq \kappa_h$, for all $y \in \mathcal{L}(y_0)$, respectively, where κ_D, κ_g , and κ_h are positive constants.

Let d_k be a solution to (7); the following equations,

$$(H_k + \sigma_k \|d_k\| I) d_k = -g_k, \quad (18)$$

are the first-order necessary conditions of (7).

Consequently, from Assumptions 1–3, (5), and the choice of δ_k in Step 5, we can get that

$$\begin{aligned} \kappa_{lf} &\leq f(y) \leq \kappa_{uf}, \\ \|H_k\| &\leq \kappa_h + \kappa_{eh} \delta_0^{\text{def}} = \kappa_H, \\ \|g_k\| &\leq \|\nabla f(y_k)\| + \|\nabla f(y_k) - g_k\| \leq \kappa_g + \kappa_{eg} \delta_0^{2\text{def}} = \kappa_{ng}. \end{aligned} \quad (19)$$

Let $\bar{\kappa}$ denote any criticality measure for the derivative-free model such that

$$\bar{\beta}_1 \bar{\kappa}(y) \leq \|h(y)\| \stackrel{\text{def}}{=} \|\bar{D}(y) \nabla m(y)\| \quad (20)$$

holds on Ω , for some $\bar{\beta}_1 > 0$. Certainly, $\bar{\beta}_1 \bar{\kappa}_k \leq \kappa_D \kappa_{ag}$ and a natural choice for $\bar{\kappa}$ satisfying (20) is $\bar{\kappa} = \|h(y)\|/\bar{\beta}_1$. And, there exist an $\beta_1 > 0$ such that $\beta_1 \kappa(y) = \|D(y) \nabla f(y)\|$. First, we show that $\|h(y)\|$ is a criticality measure, which can be found in [21].

Lemma 1. The function $\|h(y_k)\|$ defined in (20) is a critical measure.

Lemma 2. Let d_k satisfy the Cauchy decrease condition in Remark 1. If $\bar{\kappa}_k \neq 0$, there is a constant $\bar{\beta}_2 > 0$ which only depends on $\bar{\beta}_1, \kappa_D$, and κ_H such that the following holds:

$$-\varphi_k(d_k) \geq \bar{\beta}_2 \bar{\kappa}_k \min \left\{ 1, \sqrt{\frac{\bar{\kappa}_k}{\sigma_k}}, \bar{\kappa}_k \right\}. \quad (21)$$

Proof. Set $d_k = -\bar{D}_k^2 g_k$ and $h(y_k) = \bar{D}_k g_k$. First, we derive an upper bound to $\varphi_k(d_k^C) = \varphi_k(\zeta_k^C d_k) = \min\{\varphi_k(\zeta d_k); \zeta \geq 0, \zeta d_k \in \Omega\}$; then, apply $\varphi_k(d_k) \leq \varphi_k(d_k^C)$. It is easy to see that $g_k^T d_k = -\|h(y_k)\|^2 \leq -\bar{\beta}_1^2 \bar{\kappa}_k^2 < 0$, since, by (20), the maximum step-size ζ_1 arrived at the lower bounds to $\Omega - y_k$ is

$$\begin{aligned} \zeta_1 &= \min \left\{ \frac{[y_k - l]_j}{[d_k]_j}; [d_k]_j \leq 0 \right\} \\ &= \min \left\{ \frac{[y_k - l]_j}{[\bar{D}_k]_{jj} [h(y_k)]_j}; [g_k]_j > 0, [y_k]_j > l_j \right\} \\ &\geq \min \left\{ \frac{[y_k - l]_j}{\min\{[y_k - l]_j, [u - y_k]_j + \mu [g_k]_j\} \|h(y_k)\|}; [g_k]_j > 0, [y_k]_j > l_j \right\} \\ &\geq \frac{1}{\|h(y_k)\|}. \end{aligned} \quad (22)$$

The step-size ζ_2 arrived at the upper bounds to $\Omega - y_k$ is calculated in the same way:

$$\zeta_2 = \min \left\{ \frac{[u - y_k]_j}{[d_k]_j}; [d_k]_j \geq 0 \right\} \geq \frac{1}{\|h(y_k)\|}. \quad (23)$$

Let $a_k = (1/3)\sigma_k\|d_k\|^3$, $b_k = (1/2)d_k^T H_k d_k$, and $c_k = g_k^T d_k$. Noting that $a_k > 0$ and $c_k < 0$, the function $\varphi_k(\zeta d_k) = \zeta g_k^T d_k + (\zeta^2/2)d_k^T H_k d_k + (\zeta^3/3)\sigma_k\|d_k\|^3 \stackrel{\text{def}}{=} a_k \zeta^3 + b_k \zeta^2 + c_k \zeta$, $\zeta \geq 0$ attains its global minimum at $\zeta = \zeta_3$, where

$$\begin{aligned} \zeta_3 &= \frac{-b_k + \sqrt{b_k^2 - 3a_k c_k}}{3a_k} \\ &= \frac{-c_k}{\sqrt{b_k^2 - 3a_k c_k} + b_k} \\ &\geq \frac{1}{\sqrt{(1/4)\kappa_D^4 \kappa_H + \sigma_k \kappa_D^3 \|h(x_k)\|} + (1/2)\kappa_D^2 \kappa_H} \\ &\geq \frac{1}{2} \min \left\{ \frac{1}{\kappa_D^2 \kappa_H}, \frac{1}{\kappa_D \sqrt{\sigma_k} \|h(x_k)\|} \right\}. \end{aligned} \quad (24)$$

We have $\zeta_k^C = \min\{\zeta_1, \zeta_2, \zeta_3\}$. If $\zeta_k^C < \zeta_3$, then

$$\zeta_k^C \left(\sqrt{b_k^2 - 3a_k c_k} + b_k \right) < -c_k, \quad (25)$$

and furthermore,

$$a_k < \frac{c_k^2 + 2b_k c_k \zeta_k^C}{-3c_k (\zeta_k^C)^2} = \frac{-c_k - 2b_k \zeta_k^C}{3(\zeta_k^C)^2}. \quad (26)$$

Then, if H_k is positive semidefinite, that is, $b_k \geq 0$, we have that

$$\begin{aligned} 2\zeta_k^C b_k &< -c_k, \\ \zeta_k^C \sqrt{-3a_k c_k} &< -c_k. \end{aligned} \quad (27)$$

That is,

$$b_k < -\frac{c_k}{2\zeta_k^C},$$

$$a_k < \frac{c_k}{3(\zeta_k^C)^2},$$

$$\begin{aligned} \varphi_k(\zeta_k^C d_k) &= a_k (\zeta_k^C)^3 + b_k (\zeta_k^C)^2 + c_k \zeta_k^C < \frac{c_k}{3(\zeta_k^C)^2} (\zeta_k^C)^3 \\ &\quad - \frac{c_k}{2\zeta_k^C} (\zeta_k^C)^2 + c_k \zeta_k^C = -\frac{5}{6} \zeta_k^C \|h(x_k)\|^2. \end{aligned} \quad (28)$$

Otherwise, $b_k < 0$, we can get that $(-3a_k c_k \zeta_k^C / 2 \max\{\sqrt{-3a_k c_k}, -2b_k\}) < -c_k$, from $-3a_k c_k \zeta_k^C / (\sqrt{b_k^2 - 3a_k c_k} - b_k) < -c_k$, which is equivalent to (25).

First, assume that $\max\{\sqrt{-3a_k c_k}, -2b_k\} = \sqrt{-3a_k c_k}$. Therefore, from (25), we have

$$\zeta_k^C \left(\sqrt{5b_k^2 + b_k} \right) < -c_k. \quad (29)$$

Noting that $-b_k < 0$, we can also obtain

$$-b_k < \frac{-c_k}{(\sqrt{5} - 1)\zeta_k^C}. \quad (30)$$

By combining $b_k < 0$, (26), and (30), we can get that

$$\begin{aligned} \varphi_k(\zeta_k^C) &= a_k (\zeta_k^C)^3 + b_k (\zeta_k^C)^2 + c_k \zeta_k^C < \frac{2}{3} c_k \zeta_k^C \\ &\quad - \frac{2}{3(\sqrt{5} - 1)} c_k \zeta_k^C < -\frac{1}{6} \zeta_k^C \|h(x_k)\|^2. \end{aligned} \quad (31)$$

Now, assume that $\max\{\sqrt{-3a_k c_k}, -2b_k\} = -2b_k$; then,

$$b_k < -\frac{c_k + 3a_k (\zeta_k^C)^2}{2\zeta_k^C},$$

$$\begin{aligned} \varphi_k(\zeta_k^C d_k) &= a_k (\zeta_k^C)^3 + b_k (\zeta_k^C)^2 + c_k \zeta_k^C \\ &< a_k (\zeta_k^C)^3 - \frac{c_k + 3a_k (\zeta_k^C)^2}{2\zeta_k^C} (\zeta_k^C)^2 + c_k \zeta_k^C \\ &< -\frac{1}{2} \zeta_k^C \|h(x_k)\|^2. \end{aligned} \quad (32)$$

On the contrary, $\zeta_k^C = \zeta_3 = (-b_k + \sqrt{b_k^2 - 3a_k c_k})/3a_k$, and we obtain that

$$a_k (\zeta_k^C)^3 + \frac{2}{3} b_k (\zeta_k^C)^2 + \frac{1}{3} c_k \zeta_k^C = 0,$$

$$\varphi_k(\zeta_k^C d_k) = a_k (\zeta_k^C)^3 + b_k (\zeta_k^C)^2 + c_k \zeta_k^C = \frac{1}{3} b_k (\zeta_k^C)^2 + \frac{2}{3} c_k \zeta_k^C. \quad (33)$$

If $b_k < 0$,

$$\varphi_k(\zeta_k^C d_k) < \frac{2}{3} c_k \zeta_k^C = -\frac{2}{3} \zeta_k^C \|h(x_k)\|^2. \quad (34)$$

If $b_k \geq 0$, we can have that $b_k < -(c_k/\zeta_k^C)$ from $\zeta_k^C = \zeta_3 = (-b_k + \sqrt{b_k^2 - 3a_k c_k})/3a_k$, and then,

$$\varphi_k(\zeta_k^C d_k) = \frac{1}{3} b_k (\zeta_k^C)^2 + \frac{2}{3} c_k \zeta_k^C < \frac{1}{3} c_k \zeta_k^C = -\frac{1}{3} \zeta_k^C \|h(x_k)\|^2. \quad (35)$$

Then, combining $\varphi_k(d_k) \leq \varphi_k(d_k^C) = \varphi_k(\zeta_k^C d_k)$ and (20) and (22)–(35), $-\varphi_k(d_k) \geq \bar{\beta}_2 \bar{\kappa}_2^2$ holds, where $\bar{\beta}_2 > 0$ is a constant which only depends on $\bar{\beta}_1$, κ_D , and κ_H . The proof is completed.

The following lemma from [5] gives a useful bound on the step d_k . $\square$

Lemma 3. *If d_k is a solution in Step 3, then*

$$\|d_k\| \leq \frac{3}{\sigma_k} \max \left\{ \kappa_H, \sqrt{\frac{\bar{\beta}_1 \sigma_k \bar{\kappa}_k}{\kappa_D}} \right\}. \quad (36)$$

Assumption 4. Suppose that $\|\nabla_d \varphi_k(d_k)\| \leq \kappa_\theta \min\{1, \|d_k\|\} \cdot \|g_k\|$, where $\kappa_\theta \in (0, 1)$.

Lemma 4. Let Assumptions 1–4 hold and m_k be fully quadratic. If

$$\sqrt{\sigma_k \bar{\kappa}_k} \geq \frac{9\bar{\beta}_1}{(1-\gamma_1)\bar{\beta}_2 \kappa_D} \max\{(L_g + \kappa_{eg}\delta_0) + \kappa_H, 1, \bar{\kappa}_k\} \stackrel{\text{def}}{=} \kappa_{HB}, \quad (37)$$

then $\rho_k \geq \gamma_1$ and $\sigma_k \geq \sigma_{k+1}$.

Proof. From (37), we will get that $\bar{\kappa}_k \neq 0$ because, otherwise, the algorithm would have terminated; then, $\bar{\kappa}_k = 0$ will conflict with (37). However,

$$\rho_k - \gamma_1 = \frac{f(y_k + s_k) - f(y_k) - \varphi_k(s_k) + (1 - \gamma_1)\varphi_k(s_k)}{\varphi_k(s_k)}. \quad (38)$$

Firstly, using a Taylor expansion, for some $\xi_k \in (0, 1)$, the first term of the numerator in the fraction above becomes that

$$\begin{aligned} & f(y_k + s_k) - f(y_k) - \varphi_k(s_k) \\ &= f(y_k) + \nabla f(y_k + \xi_k s_k)^T s_k - f(y_k) - g_k^T s_k - \frac{1}{2} s_k^T H_k s_k - \frac{\sigma_k}{3} \|s_k\|^3 \\ &\leq (\|\nabla f(y_k + \xi_k s_k) - \nabla f(y_k)\| + \|\nabla f(y_k) - g_k\|) \cdot \|s_k\| + \frac{1}{2} \kappa_H \|s_k\|^2 \\ &\leq \theta_k \alpha_k \max\{(L_g + \kappa_{eg}\delta_0) + \kappa_H, 1, \bar{\kappa}_k\} \cdot \frac{9\bar{\beta}_1 \bar{\kappa}_k}{\kappa_D \sigma_k}, \end{aligned} \quad (39)$$

where Assumption 2, Lemma 3, and (37) are used.

Next, we consider the second term of the numerator in the fraction:

$$\begin{aligned} (1 - \gamma_1)\varphi_k(s_k) &\leq -(1 - \gamma_1)\bar{\beta}_2 \theta_k \alpha_k \bar{\kappa}_k \min \left\{ 1, \sqrt{\frac{\bar{\kappa}_k}{\sigma_k}}, \bar{\kappa}_k \right\} \\ &\stackrel{(37)}{\leq} -(1 - \gamma_1)\bar{\beta}_2 \theta_k \alpha_k \bar{\kappa}_k^2 \cdot \sqrt{\frac{1}{\sigma_k \bar{\kappa}_k}}. \end{aligned} \quad (40)$$

However, the denominator $\varphi_k(s_k) < 0$ is from Lemma 2. Thus, following with (37)–(40), we get that

$$\begin{aligned} \rho_k - \gamma_1 &\geq \frac{\theta_k \alpha_k \max\{L_g + \kappa_{eg}\delta_0 + \kappa_H, 1, \bar{\kappa}_k\} (9\bar{\beta}_1 \bar{\kappa}_k / \kappa_D \sigma_k) - \bar{\beta}_2 (1 - \gamma_1) \alpha_k \theta_k \bar{\kappa}_k^2 \cdot \sqrt{1/\sigma_k \bar{\kappa}_k}}{\varphi_k(s_k)} \\ &= \frac{\theta_k \alpha_k \bar{\kappa}_k \cdot [9\bar{\beta}_1 \max\{L_g + \kappa_{eg}\delta_0 + \kappa_H, 1, \bar{\kappa}_k\} - \bar{\beta}_2 (1 - \gamma_1) \kappa_D \sqrt{\sigma_k \bar{\kappa}_k}]}{\kappa_D \sigma_k \varphi_k(s_k)} \\ &\stackrel{(37)}{\geq} 0. \end{aligned} \quad (41)$$

Thus, $\rho_k \geq \gamma_1$ and $\sigma_k \geq \sigma_{k+1}$ are from Step 7. $\square$

Assumption 5. Let $\sigma_k \geq \sigma_{\min} > 0$; then, $\sigma_k \rightarrow \infty$ as $\alpha_k \rightarrow 0$.

Assumption 5, which is reasonable when we use Armijo backing line search or Wolfe condition (ii) for

global convergence, is important to promote global convergence.

If $y_k + \alpha_k d_k$ satisfies Armijo condition, $f(y_k + \alpha_k d_k) \leq f(y_k) + c_1 \alpha_k \nabla f(y_k)^T d_k$, where c_1 is in $(0, 1)$, we assume that $\alpha_k = \alpha_{k,l}$, where l is the first integer such that $\alpha_{k,l}$ satisfies Armijo condition, and we can get that

$$f\left(y_k + \frac{\alpha_k}{\lambda_1} d_k\right) > f(y_k) + \frac{\alpha_k}{\lambda_1} \nabla f(y_k)^T d_k. \quad (42)$$

Then, we can get that

$$\begin{aligned} & \frac{\alpha_k}{\lambda_1} \left(\nabla f\left(y_k + \xi_k \frac{\alpha_k}{\lambda_1} d_k\right) - \nabla f(y_k) \right)^T d_k + \frac{\alpha_k}{\lambda_1} (1 - c_1) \\ & \cdot (\nabla f(y_k) - g_k)^T d_k + \frac{\alpha_k}{\lambda_1} c_1 g_k^T d_k > 0, \end{aligned} \quad (43)$$

where $\xi_k \in (0, 1)$. From Assumption 2 and (5), then

$$\left(\frac{\alpha_k}{\lambda_1}\right)^2 L_g \|d_k\|^2 + \frac{\alpha_k^3}{\lambda_1} (1 - c_1) \kappa_{eg} \|d_k\|^3 + \frac{\alpha_k}{\lambda_1} c_1 g_k^T d_k > 0. \quad (44)$$

Dividing (44) by (α_k/λ_1) and applying Lemma 2,

$$\frac{\alpha_k}{\lambda_1} L_g \|d_k\|^2 - \alpha_k^2 (c_1 - 1) \kappa_{eg} \|d_k\|^3 > c_1 \bar{\beta}_2 \bar{\kappa}_k \min \left\{ 1, \sqrt{\frac{\bar{\kappa}_k}{\sigma_k}}, \bar{\kappa}_k \right\}. \quad (45)$$

If $\bar{\kappa}_k > \varepsilon$, using Assumption 3 and the fact that $\|d_k\|$ is bounded, $\|d_k\| \leq 3 \max \left\{ (\kappa_H/\sigma_k), \sqrt{\bar{\beta}_1 \bar{\kappa}_k / \kappa_D \sigma_k} \right\} \leq 3 \max \left\{ (\kappa_H/\sigma_{\min}), \sqrt{\kappa_{ng}/\sigma_{\min}} \right\}$, we can infer that $\sigma_k \rightarrow \infty$ as $\alpha_k \rightarrow 0$ from (45).

Then, we show that Assumption 5 can be also achieved when $y_k(\alpha_k) = y_k + \alpha_k d_k$ satisfies the Wolfe condition (ii):

$$\nabla f(y_k + \alpha_k d_k)^T d_k \geq c_2 \nabla f(y_k)^T d_k, \quad \text{in which } c_2 \in (0, 1). \quad (46)$$

We rewrite it as

$$\begin{aligned} & (\nabla f(y_k + \alpha_k d_k) - \nabla f(y_k))^T d_k + (1 - c_2) \\ & \cdot (\nabla f(y_k) - g_k)^T d_k + (1 - c_2) g_k^T d_k > 0. \end{aligned} \quad (47)$$

From Assumption 2 and (5) again, we have that

$$L_g \alpha_k \|d_k\|^2 + (1 - c_2) \alpha_k^2 \kappa_{eg} \kappa_D^2 \|d_k\|^3 + (1 - c_2) g_k^T d_k > 0. \quad (48)$$

Using Lemma 2 and Assumption 3, the following inequality,

$$\begin{aligned} & L_g \alpha_k \|d_k\|^2 + (1 - c_2) \alpha_k^2 \kappa_{eg} \kappa_D^2 \|d_k\|^3 > (1 - c_2) \bar{\beta}_2 \bar{\kappa}_k \\ & \cdot \min \left\{ 1, \sqrt{\frac{\bar{\kappa}_k}{\sigma_k}}, \bar{\kappa}_k \right\}, \end{aligned} \quad (49)$$

holds. Consequently, $\lim_{\alpha_k \rightarrow 0} \sigma_k = \infty$ if $\bar{\kappa}_k > \varepsilon$.

Assumption 6. Assume that $\text{sign}([g(y)]_j) = \text{sign}([\nabla f(y)]_j) \forall y \in B(y_*, \delta)$, when $[\nabla f(y_*)]_j = 0, j = 1, 2, \dots, n$.

Lemma 5. Let Assumptions 1–6 hold. If $|\mathcal{A}| < \infty$ and after many steps with a KKT points y_k of (1), algorithm PFARC does not stop; then, $\lim_{k \rightarrow \infty} \bar{\kappa}_k = 0$.

Proof. Choose an integer K that satisfy that the filter is not augmented in iteration for all iteration $k \geq K$. Step 4 in algorithm PFARC gives that either (9) or (10) holds. If (9) holds for $k \geq K$, then $\|h(y_{k+1})\| \leq (1 - \eta_h) \|h(y_k)\| \leq \dots \leq (1 - \eta_h)^{k-K} \|h(y_{K+1})\|$ since η_h is in $(0, 1)$. Recall that $\bar{\beta}_1 \bar{\kappa}_k \leq \|h(y_k)\|$, we infer that $\lim_{k \rightarrow \infty} \bar{\kappa}_k = 0$ holds. If (9) does not hold for certain k sufficiently large, algorithm DFFARC implies that, for each k which is greater than K ,

$$f(y_{k+1}) \leq f(y_k) - \eta_f \alpha_k \|h(y_k)\|^2. \quad (50)$$

Hence, for all $i = 1, 2, \dots$,

$$\begin{aligned} f(y_{K+i}) &= \sum_{k=K}^{K+i-1} (f(y_{k+1}) - f(y_k)) + f(y_K) \leq \\ &= \sum_{k=K}^{K+i-1} \eta_f \alpha_k \|h(y_k)\|^2 + f(y_K). \end{aligned} \quad (51)$$

To have a contradiction, we let $\bar{\kappa}_k > \varepsilon$. Then, $\|h(y_k)\| \geq \bar{\beta}_1 \bar{\kappa}_k \geq \bar{\beta}_1 \varepsilon$ and $f(y_K) - f(y_{K+i}) \geq \sum_{k=K}^{K+i-1} \eta_f \alpha_k \|h(y_k)\|^2 \geq \sum_{k=K}^{K+i-1} \eta_f \alpha_k \bar{\beta}_1^2 \varepsilon^2$. Since $f(y_{K+i})$ is bounded from below, the right-hand side series is bounded as $i \rightarrow \infty$; then, $\lim_{k \rightarrow \infty} \alpha_k = 0$.

Let α_k be given in Step 4 with $y_k + \alpha_k d_k \in \Omega$. It is easy to see that

$$\forall y \in \Omega, D(y) \nabla f(y) = 0 \Leftrightarrow [\nabla f(y)]_j \begin{cases} \geq 0, & \text{if } [y]_j = l_j, \\ \leq 0, & \text{if } [y]_j = u_j, \\ = 0, & \text{if } l_j \leq [y]_j \leq u_j, \end{cases} \quad (52)$$

which is the KKT-conditions of (1). From the continuity of f , Assumption 6, and (5), we can get that $[g(y_k)]_j$ has the same sign when y_k is close to y ; then, $\alpha_k \rightarrow 0$ as $k \rightarrow \infty$.

Furthermore, if α_k is obtained from the reduced conditions in Step 4, we have that $\sigma_k \rightarrow \infty$ from Assumption 5. Recall that $\bar{\kappa}_k > \varepsilon$, we can have that, for sufficiently large k , $\sqrt{\sigma_k \bar{\kappa}_k} > \sqrt{\sigma_k \varepsilon} > \kappa_{HB}$; then, $\sigma_k \geq \sigma_{k+1}$ from Lemma 4. According to the updating rule in σ_k , if $\sigma_k < \sigma_{k+1} = \gamma^{-1} \sigma_k$, then $\sigma_k \leq (\kappa_{HB}^2/\varepsilon) < (\kappa_{HB}^2/\gamma \varepsilon)$ from Lemma 4. By induction, $\sigma_0 \leq \max\{\sigma_0, (\kappa_{HB}^2/\gamma \varepsilon)\}$. Suppose the claim that $\sigma_k \leq \max\{\sigma_0, (\kappa_{HB}^2/\gamma \varepsilon)\}$ holds. It is obvious that $\sigma_{k+1} \leq \max\{\sigma_0, (\kappa_{HB}^2/\gamma \varepsilon)\}$ if $\sigma_k \geq \sigma_{k+1}$. If $\sigma_k < \sigma_{k+1}$, then $\sigma_k \leq (\kappa_{HB}^2/\varepsilon)$; consequently, $\sigma_{k+1} \leq \max\{\sigma_0, (\kappa_{HB}^2/\gamma \varepsilon)\}$ since $\sigma_{k+1} = \gamma^{-1} \sigma_k \leq (\kappa_{HB}^2/\gamma \varepsilon)$. The claim $\sigma_k \leq \sigma_{\max} = \max\{\sigma_0, (\kappa_{HB}^2/\gamma \varepsilon)\}$, which contradicts to $\sigma_k \rightarrow \infty$. Hence, $\alpha_k \rightarrow 0$, and our hypothesis $\bar{\kappa}_k > \varepsilon$ is impossible. Thus, the conclusion holds. $\square$

Lemma 6. Let Assumptions 1–6 hold and $|\mathcal{A}| = \infty$. If algorithm PFARC does not stop after finite steps at a KKT points of y_k in (1), then

$$\lim_{k \rightarrow \infty, k \in \mathcal{A}} \bar{\kappa}_k = 0. \quad (53)$$

Proof. To get a contradiction, we suppose that, for all large k_i , $\bar{\kappa}_{k_i} \geq \varepsilon$, where $\{k_i\}$ is a subsequence of $\mathcal{A}$ and $\varepsilon > 0$ is a constant. By the definition of k_i , the pair of $(\|h(y_{k_i})\|, f(y_{k_i}))$ is augmented to $\mathcal{F}$. Thus, there exists an infinite subsequence $\{k_{i_j}\} \subseteq \{k_i\}$ such that $\bar{\kappa}_{k_{i_j}} \rightarrow 0$ from the compactness theorem, which contradicts to $\bar{\kappa}_{k_i} \geq \varepsilon$. Hence, (53) holds. $\square$

Lemma 7. *Let Assumptions 1–6 hold, and sequence $\{Q_k\}$ is probabilistically $(\kappa_{eh}, \kappa_{eg}, \kappa_{ef})$ fully quadratic. $\{Y_k\}$ is a sequence of random iterates generated by algorithm PFARC. If any limit point y_* of $\{y_k\}$ satisfies the strict complementarity, then $\liminf_{k \rightarrow \infty} \|D(Y_k) \nabla f(Y_k)\| = 0$ almost surely.*

Proof. Combining the affine scaling technique with the proof in Theorem 4.2 in [18], it is easy to derive the conclusion similar to Theorem 4.8 in [19]. We have omitted this section for reasons of length. $\square$

Theorem 1. *Let Assumptions 1–5 hold, and algorithm PFARC does not stop after finite steps at a KKT points of y_k in (1); thus,*

$$\lim_{k \rightarrow \infty} \bar{\kappa}_k = 0. \quad (54)$$

Proof. From Lemmas 5 and 6, we just think about this one case in which there are k_1, k_2 with $k_1 \in \mathcal{A}$ and $k_2 \notin \mathcal{A}$, and k_1, k_2 are larger than integer K . If (54) does not hold, there is a subsequence $\{y_{k_i}\}$ such that $\bar{\kappa}_{k_i} \geq 2\varepsilon$ and $k_i \notin \mathcal{A}$. From Lemma 6, we also have, for every k_i , that $\bar{\kappa}_{l_i} < \varepsilon$; the iterate y_{l_i}

is the first iterate after y_{k_i} such that $(\|h(y_{l_i})\|, f(y_{l_i}))$ is augmented to $\mathcal{F}$, that is, $l_i \in \mathcal{A}$. Thus,

$$\bar{\kappa}_k \geq \varepsilon, \text{ with } k_i \leq k < l_i \text{ for all } i. \quad (55)$$

Obviously, $\mathcal{K} = \{k \in \mathbb{N} | k_i \leq k < l_i\}$ is infinite, where k_i and l_i are defined as above.

For every $k = k_i, \dots, l_i - 1 \notin \mathcal{A}$, we have that either (9) or (10) holds. If (9) holds, then $(1 - \eta_h)\|h(y_k)\| \geq \|h(y_{k+1})\|$. Since $0 < \eta_h < 1$ and $\|h(y_k)\| \leq \kappa_D \kappa_{ng}$ and $\|h(y_{k+1})\| \rightarrow 0$, then $\bar{\kappa}_{k+1} \rightarrow 0$ which contradicts to (55). Then, algorithm PFARC implies that (10) holds. Thus, $f(y_k) - \eta_f \alpha_k \bar{\beta}_1 \varepsilon^2 > f(y_k) - \eta_f \alpha_k \|h(y_k)\|^2 \geq f(y_{k+1})$.

Moreover, $\{f(y_k)\}$ is monotonically decreasing and $f(y_{k_{i+1}}) \geq f(y_{l_i})$. Consequently,

$$\begin{aligned} f(y_{k_i}) - \eta_f \alpha_{k_i} \bar{\beta}_1 \varepsilon^2 &> f(y_{k_i}) - \eta_f \alpha_{k_i} \|h(y_{k_i})\|^2 \\ &> f(y_{k_{i+1}}) \geq f(y_{l_i}), \quad \forall i. \end{aligned} \quad (56)$$

This ensures that $f(y_{k_{(i+1)}}) \geq f(y_{l_i})$, $\forall K \in \mathbb{N}$, with some $i \geq K$, because otherwise the inequality above will give that

$$\begin{aligned} f(y_{k_i}) - \eta_f \alpha_{k_i} \bar{\beta}_1 \varepsilon^2 &> f(y_{k_{i+1}}) \leq f(y_{k_i}) - \eta_f \alpha_{k_i} \|h(y_{k_i})\|^2 \\ &\geq f(y_{l_i}) > f(y_{k_{(i+1)}}), \quad \forall i. \end{aligned} \quad (57)$$

Similar to proof of Lemma 5, we can to prove that $\alpha_{k \in \mathcal{K}_1} \rightarrow 0$, where $\mathcal{K}_1 \subseteq \mathbb{N}$. Furthermore, from Assumption 5, we can get $\sigma_k \rightarrow \infty$ as $\alpha_k \rightarrow 0$. Recall that $\bar{\beta}_1 \bar{\kappa}_k \leq \kappa_D \kappa_{ng}$ and $\|d_k\| \leq 3\sqrt{\bar{\beta}_1 \bar{\kappa}_k / \kappa_D \sigma_k}$; thus, $\|d_k\| \rightarrow 0$, and then, $\theta_k \rightarrow 1$, where θ_k is defined in Step 4, so $s_k = \alpha_k d_k$. However,

$$\begin{aligned} -\bar{\varphi}_k(s_k) &= -\bar{\varphi}(\alpha_k d_k) \geq \alpha_k \bar{\beta}_2 \bar{\kappa}_k \min \left\{ 1, \sqrt{\frac{\bar{\kappa}_k}{\sigma_k}}, \bar{\kappa}_k \right\} \\ &\geq \alpha_k \bar{\beta}_2 \varepsilon \cdot \frac{\kappa_D^2}{3\bar{\beta}_1^2} \|d_k\| = \bar{\beta}_2 \varepsilon \cdot \frac{\kappa_D^2}{3\bar{\beta}_1^2} \|s_k\|, \\ f(y_k + s_k) - f(y_k) - \varphi_k(s_k) &= \nabla f(y_k + \xi_k s_k)^T s_k - g_k^T s_k - \frac{1}{2} s_k^T H_k s_k - \frac{1}{3} \sigma_k \|s_k\|^3 \\ &\leq [\nabla f(y_k + \xi_k d_k) - \nabla f(y_k)]^T s_k + [\nabla f(y_k) - g_k]^T s_k + \kappa_H \|s_k\|^2 \\ &\leq (L_g + \kappa_{eg} \|s_k\| + \kappa_H) \|s_k\|^2. \end{aligned} \quad (58)$$

Thus, combining (58) and inequalities above, we can have for sufficiently large k that

$$\rho_k = 1 - \frac{f(y_k + s_k) - f(y_k) - \varphi_k(s_k)}{-\varphi_k(s_k)} \geq 1 - \frac{3\bar{\beta}_1^2(L_g + \kappa_{eg}\|s_k\| + \kappa_H)\|s_k\|^2}{\bar{\beta}_2\kappa_D^2\epsilon\|s_k\|} \geq \gamma_2. \quad (59)$$

Then,

$$\begin{aligned} f(y_k) - f(y_{k+1}) &= f(y_k) - f(y_k + s_k) \\ &\geq \eta_2[-\varphi_k(s_k)] \geq \frac{\eta_2\bar{\beta}_2\kappa_D^2\epsilon}{3\bar{\beta}_1}\|s_k\|. \end{aligned} \quad (60)$$

Consequently,

$$\|y_{k_i} - y_{l_i}\| \leq \sum_{k=k_i}^{l_i-1} \|s_k\| \leq \frac{3\bar{\beta}_1}{\eta_2\bar{\beta}_2\kappa_D^2\epsilon} [f(y_{k_i}) - f(y_{l_i})]. \quad (61)$$

Since $f(y_k)$ is convergent, we can get $\|y_{k_i} - y_{l_i}\| \rightarrow 0$ from (61). However, $\bar{\kappa}$ is uniform continuous on $\mathcal{L}(x_0)$. Thus, $\bar{\kappa}_{k_i} - \bar{\kappa}_{l_i} \rightarrow 0$ holds, which conflicts to the fact that $|\bar{\kappa}_{k_i} - \bar{\kappa}_{l_i}| \geq \epsilon$. Therefore, our assumption is wrong and $\alpha_k \rightarrow 0$.

Combining (57) with $\alpha_{k_i} \rightarrow 0$, we get that $f(y_{k_i}) \rightarrow -\infty$, which conflicts to $\{f(y_k)\}$ is bounded below. Thus, there is a subsequence $\{i_j\} \subset \{i\}$ such that

$$f(y_{k_{(i_j+1)}}) \geq f(y_{l_{i_j}}). \quad (62)$$

Since $k_{(i_j+1)}$ is not in $\mathcal{A}$ and $\mathcal{F}_{l_{i_j}} \subseteq \mathcal{F}_{k_{(i_j+1)}}$, it follows from (7) and (62) that

$$(1 - \eta_h)\|h(y_{l_{i_j}})\| \geq \|h(y_{k_{(i_j+1)}})\|. \quad (63)$$

For all j , $l_{i_j} \in \{l_i\} \subseteq \mathcal{A}$, Lemma 6 implies $\lim_{j \rightarrow \infty} \|h(y_{l_{i_j}})\| = 0$. Consequently, from (63), we get that $\lim_{j \rightarrow \infty} \|h(y_{k_{(i_j+1)}})\| = 0$; it contradicts to that $\|h(y_{k_{(i_j+1)}})\| \geq \bar{\beta}_1\bar{\kappa}_{(i_j+1)} \geq 2\bar{\beta}_1\epsilon$ since $k_{(i_j+1)} \notin \mathcal{A}$. Thus, the claim holds. $\square$

5. Properties of the Local Convergence

We can know that any limit point y_* of (1) is a KKT point from Theorem 1. In the following section, we consider the local convergence of algorithm PFARC, where the following assumptions are required.

Assumption 7. Assume that $\lim_{k \rightarrow \infty} \|(\nabla^2 f(y_k) - H_k)d_k\| / \|d_k\| = 0$ as $\lim_{k \rightarrow \infty} \bar{\kappa}_k = 0$.

Assumption 8. Assume that y_* satisfies

$$\exists \kappa_c > 0 \text{ such that } d^T \nabla^2 f(y_*)d \geq 2\kappa_c \|d\|^2, \quad \forall d \neq 0. \quad (64)$$

Noting that $D(y_*)^2 \nabla f(y_*) = 0$ is equivalent to $D(y_*) \nabla f(y_*) = 0$, then $\|D_k^2 \nabla f(y_k)\| \rightarrow 0$ is equivalent to

the fact $\|D_k \nabla f(y_k)\| \rightarrow 0$. Let $h_1(y) = \bar{D}(y)^2 g(y)$. From (5), the fact $\|h_1(y_k)\| \rightarrow 0$ is equivalent to the fact $\|h(y_k)\| \rightarrow 0$, as $k \rightarrow \infty$. To illustrate the properties of the local convergence, we substitute $h(y_k)$ for $h_1(y_k)$ in Step 4.4. That is, we accept the trial step $\alpha_{k,l}$ if

$$\|h_1(y_k(\alpha_{k,l}))\| \leq (1 - \eta_h)\|h_1(y_k)\| \text{ with } y_k + \alpha_{k,l}d_k \in \Omega \quad (65)$$

holds in Step 4.4 in the following sections.

Theorem 2. Assume that Assumptions 1–8 hold and algorithm PFARC does not stop finite step at a KKT points y_k of (1); then, $\|d_k\| \rightarrow 0$, $\alpha_k \equiv 1$ for large k and all iterations eventually satisfy $\rho_k > \gamma_2$, σ_k is bounded from above as $k \rightarrow \infty$.

Proof. Theorem 1 gives that $\lim_{k \rightarrow \infty} \bar{\kappa}_k \rightarrow 0$ from the assumptions in Theorem 2. Consequently, Assumption 7 implies that $\lim_{k \rightarrow \infty} \|(\nabla^2 f(y_k) - H_k)d_k\| / \|d_k\| = 0$.

Thus, for large k ,

$$d_k^T (\nabla^2 f(y_k) - H_k)d_k \leq \kappa_c \|d_k\|^2, \quad (66)$$

in which κ_c appears in Assumption 8. And, we also have that $d_k^T \nabla^2 f(y_k)d_k \geq 2\kappa_c \|d_k\|^2$ for large k from Assumptions 2 and 8. Then, we can infer that $2\kappa_c \|d_k\|^2 \leq d_k^T [\nabla^2 f(y_k) H_k] d_k + d_k^T H_k d_k \leq \kappa_c \|d_k\|^2 + d_k^T H_k d_k$. As a consequence, $d_k^T H_k d_k \geq \kappa_c \|d_k\|^2$. So, H_k is positive definite. And,

$$\begin{aligned} \kappa_c \|d_k\|^2 &\leq d_k^T H_k d_k \leq d_k^T [H_k + \sigma_k \|d_k\| I] d_k \\ &\stackrel{(18)}{=} -g_k^T d_k \leq \|g_k\| \cdot \|d_k\|. \end{aligned} \quad (67)$$

Since $d_k \neq 0$, the first and last terms above give

$$\|d_k\| \leq \frac{\|g_k\|}{\kappa_c}. \quad (68)$$

Otherwise, $-\varphi_k(d_k) = 0$, from Lemma 2. This, however, contradicts $-\varphi_k(d_k) > 0$ since $\bar{\kappa}_k \neq 0$. Recall that

$$\|\bar{D}_k g_k\| = 0 \Leftrightarrow [g(y)]_j \begin{cases} \geq 0, & \text{if } [y]_j = l_i, \\ \leq 0, & \text{if } [y]_j = u_i, \\ = 0, & \text{if } l_j \leq [y]_j \leq u_j, \end{cases} \quad (69)$$

and we can get $\|g_k\| \rightarrow 0$, from $\bar{\kappa}_k \rightarrow 0$. Thus, (68) gives that $\|d_k\| \rightarrow 0$. Furthermore, $\theta_k \rightarrow 1$, in which θ_k appears in Step 4. Then, $s_k = \alpha_k d_k$ and $\hat{s}_k = \alpha_k \bar{d}_k$ for large k .

Assumption 6 provides that $\text{sign}[g_k]_j$ and $\text{sign}[\nabla f(y_k)]_j$ is the same for large k if $\nabla[f(y_*)]_j = 0$. On the other side, (5) gives that $\text{sign}[g_k]_j$ and $\text{sign}[\nabla f(x_k)]_j$ are also the same for large k if $[\nabla f(y_*)]_j \neq 0$ for some j . Thus, $D_k = \bar{D}_k$ for large k .

Now let us think about σ_k . From Step 7, $\rho_k = 1 - (f(y_k + s_k) - f(y_k) - \varphi_k(s_k)) / -\varphi_k(s_k)$.

However, using a Taylor expansion, for some $\xi_k \in (0, 1)$,

$$\begin{aligned}
& f(y_k + s_k) - f(y_k) - \varphi_k(s_k) \\
&= \nabla f(y_k)^T s_k + \frac{1}{2} s_k^T \nabla^2 f(y_k + \xi_k s_k) s_k - g_k^T s_k - \frac{1}{2} s_k^T H_k s_k - \frac{1}{3} \sigma_k \|s_k\|^3 \\
&\leq (\nabla f(y_k) - g_k)^T s_k + \frac{1}{2} s_k^T \nabla^2 f(y_k + \xi_k s_k) s_k - \frac{1}{2} s_k^T H_k s_k \\
&\leq \kappa_D \kappa_{eg} \bar{\kappa}_k \|s_k\|^2 + \frac{1}{2} \kappa_D^2 \|s_k\|^2 \cdot \|\nabla^2 f(y_k + \xi_k s_k) - \nabla^2 f(x_k)\| + \frac{1}{2} \|s_k\|^2 \frac{\|\nabla^2 f(y_k) - H_k\| \|s_k\|}{\|s_k\|}, \tag{70}
\end{aligned}$$

$$\begin{aligned}
\varphi_k(s_k) &= g_k^T s_k + \frac{1}{2} s_k^T H_k s_k + \frac{1}{3} \sigma_k \|s_k\|^3 \\
&\stackrel{(18)}{=} -[s_k^T (H_k + \sigma_k \|s_k\| I) s_k] + \frac{1}{2} s_k^T H_k s_k + \frac{1}{3} \sigma_k \|s_k\|^3 \\
&= -\frac{1}{2} s_k^T H_k s_k - \frac{2}{3} \sigma_k \|s_k\|^3 \leq -\frac{1}{2} s_k^T H_k s_k \leq -\frac{\kappa_c}{2} \|s_k\|^2.
\end{aligned}$$

Together with the inequalities above, for large k ,

$$\rho_k \geq 1 - \frac{\kappa_D \kappa_{eg} \bar{\kappa}_k + (1/2) \kappa_D^2 \|\nabla^2 f(y_k + \xi_k s_k) - \nabla^2 f(y_k)\| + (1/2) (\|\nabla^2 f(y_k) H_k s_k\| / \|s_k\|)}{(\kappa_c/2)} > \gamma_2, \tag{71}$$

where the last inequality is from $\bar{\kappa}_k \rightarrow 0$ and Assumptions 2 and 7. Recall the updating rule in σ_k in Step 7, and we get that

$$\sigma_k > \sigma_{k+1} = \gamma \sigma_k, 0 < \gamma < 1, k \text{ sufficiently large.} \tag{72}$$

Thus, σ_k is bounded from above and $\sigma_k \|d_k\| \rightarrow 0$. Clearly, $\|h_1(y_k)\| = \|\bar{D}_k^2 g_k\| \neq 0$. Set $\varepsilon_k = (1 - \eta_h)/(2\delta_1) \|h_1(y_k)\|$, where δ_1 is sufficiently small such that

$$\begin{aligned}
\|h_1(y_k) + \nabla h_1(y_k) d_k\| &= \|\bar{D}_k^2 g_k + (\bar{D}_k^2 H_k + \bar{D}_k \text{diag}\{g_k\} \bar{J}_k) d_k\| \\
&\stackrel{(18)}{\leq} \sigma_k \kappa_D^2 \|d_k\|^2 + \|\bar{D}_k g_k\| \cdot \|d_k\| \\
&\leq (\kappa_D^2 \sigma_k \|d_k\| + \beta_1 \bar{\kappa}_k) \|d_k\| \leq \varepsilon_k \|d_k\|, \tag{73}
\end{aligned}$$

whenever $\|d_k\| \leq \delta_1$. Let $\omega_k = ((1 - \eta_h)/2\delta_2) \|h_1(y_k)\|$, where δ_2 is sufficiently small such that

$$\|h_1(y_k + d_k) - h_1(y_k) - \nabla h_1(y_k) d_k\| \leq \omega_k \|d_k\|, \tag{74}$$

whenever $\|d_k\| \leq \delta_2$. Such δ_2 exists by Lemma 1.2 in [22].

Then, it follows that

$$\begin{aligned}
\|h_1(y_k + d_k)\| &\leq \|h_1(y_k + d_k) - h_1(y_k) - \nabla h_1(y_k) d_k\| \\
&\quad + \|h_1(y_k) + \nabla h_1(y_k) d_k\| \\
&\leq \omega_k \|d_k\| + \varepsilon_k \|d_k\| \\
&\leq \frac{(1 - \eta_h)}{\delta} \|h_1(y_k)\| \cdot \|d_k\| \\
&\leq (1 - \eta_h) \|h_1(y_k)\|, \tag{75}
\end{aligned}$$

whenever $\|d_k\| \leq \delta$, where $\delta = \min\{\delta_1, \delta_2\}$.

Let α_k be given by $y_k + \alpha_k d_k \in \Omega$ in Step 4. Similar to the proof of Lemma 5, we can also get that $\alpha_k \rightarrow 0$. Therefore, $\alpha_k \equiv 1$ for large k . $\square$

Corollary 1. Under the conditions of Theorem 2, then

$$\frac{\|D_{k+1}^2 \nabla f(y_{k+1})\|}{\|D_k^2 \nabla f(y_k)\|} \rightarrow 0, \quad \text{as } k \rightarrow \infty, \tag{76}$$

$$\frac{\|y_{k+1} - y_*\|}{\|y_k - y_*\|} \rightarrow 0, \quad \text{as } k \rightarrow \infty. \tag{77}$$

Proof. Recalling (5) and $D_k = \bar{D}_k$ for large k , we obtain the following inequalities:

$$\begin{aligned}
\|D_{k+1} \nabla f(y_{k+1})\| &\leq \|D_{k+1} \nabla f(y_{k+1}) - h(y_{k+1})\| \\
&\quad + \|h(y_{k+1})\| \leq (\kappa_D \kappa_{eg} \bar{\kappa}_{k+1} + \bar{\beta}_1) \cdot \bar{\kappa}_{k+1}, \\
\|D_k \nabla f(y_k)\| &\geq \|h(y_k)\| - \|h(y_k) - D_k \nabla f(y_k)\| \\
&\geq (\bar{\beta}_1 - \kappa_D \kappa_{eg} \bar{\kappa}_k) \cdot \bar{\kappa}_k. \tag{78}
\end{aligned}$$

Hence,

$$\frac{\|D_{k+1} \nabla f(y_{k+1})\|}{\|D_k \nabla f(y_k)\|} \leq \frac{(\kappa_D \kappa_{eg} \bar{\kappa}_{k+1} + \bar{\beta}_1) \cdot \bar{\kappa}_{k+1}}{(\bar{\beta}_1 - \kappa_D \kappa_{eg} \bar{\kappa}_k) \cdot \bar{\kappa}_k} \rightarrow 0, \tag{79}$$

as $k \rightarrow \infty$. The limit in (79) is from (75). In fact, (75) gives that $\|h_1(y_{k+1})\| = o(\|h_1(y_k)\|)$ since $\|d_k\| \rightarrow 0$. Thus, $\|h(y_{k+1})\| = o(\|h(y_k)\|)$. Furthermore, $\bar{\kappa}_{k+1} = o(\bar{\kappa}_k)$. Noting

that $\|D_k^2 \nabla f(y_k)\| \rightarrow 0$ is equivalently to $\|D_k \nabla f(y_k)\| \rightarrow 0$, we have that (76) holds.

Since y_* is the limit point of $\{y_k\}$, there exist positive scales $\beta_3, \beta_4, \beta_5, \beta_6$, such that

$$\begin{aligned} \beta_3 \|y_{k+1} - y_*\| &\leq \|D_{k+1}^{-2} \nabla f(y_{k+1})\| \leq \beta_4 \|y_k - y_*\|, \\ \beta_5 \|y_{k+1} - y_*\| &\leq \|D_k^{-2} \nabla f(y_k)\| \leq \beta_6 \|y_k - y_*\|. \end{aligned} \quad (80)$$

Consequently, (77) holds from the inequalities above and (76). $\square$

6. Numerical Analysis

The main task of this section is to test the algorithm PFARC constructed in this paper by solving actual test problems. We use MATLAB 2014a to program the algorithm, solve all the test problems in Table 1, and record the corresponding results. The initial parameters given by the algorithm PFARC are as follows:

$$\begin{aligned} \sigma_0 &= 1, \\ \delta_0 &= 0.5, \\ \gamma_1 &= 0.3, \\ \gamma_2 &= 0.7, \\ \gamma &= 0.6, \\ \lambda_1 &= 0.1, \\ \lambda_2 &= 0.7, \\ \eta_f &= 0.8, \\ \eta_h &= 0.8, \\ \gamma_3 &= 0.5. \end{aligned} \quad (81)$$

In this paper, the termination precision of PFARC is set to $\varepsilon = 10^{-5}$.

In the following table, every test problem comes from [23], where n represents the dimension of the problems and l and u represent the lower bound and upper bound of the problems, respectively.

Next, we will use PFARC and BOBYQA to solve the same problems in Table 1, respectively, and the results are recorded in Table 2, where INTR_1 and INTR_2 represent the number of iterations in PFARC and BOBYQA, respectively.

From the comparison results in Table 2, we can easily see that our algorithm PFARC can use a few iterations to get the optimal solution when solving the test problem given in Table 1, while BOBYQA algorithm needs a large number of iterations, which shows that PFARC has advantages in calculating the given test problems in Table 1.

In order to further test our algorithm, we will use PFARC given by this paper to calculate the problems given in [3], and the comparison results are recorded in Table 3. In Table 3, n represents the dimension of the problems, INTR_3 represents the number of iterations in PFARC, and INTR_4 represents the number of iterations for the algorithm given in [3]. In addition, the questions in Table 3 are from [24].

TABLE 1: The information of test problems.

Problems	n	l	u
HS05	2	$[-1.5, -3]$	$[4, 3]$
HS18	2	$[2, 0]$	$[50, 50]$
HS19	2	$[13, 0]$	$[100, 100]$
HS21	2	$[2, -50]$	$[50, 50]$
HS23	3	$[-50, -50]$	$[50, 50]$
HS30	3	$[1, -10, -10]$	$[10, 10, 10]$
HS31	3	$[-10, 1, -10]$	$[10, 10, 1]$
HS34	3	$[0, 0, 0]$	$[100, 100, 100]$
HS36	3	$[0, 0, 0]$	$[20, 11, 42]$
HS37	3	$[0, 0, 0]$	$[42, 42, 42]$
HS38	4	$[-10, -10, -10, -10]$	$[10, 10, 10, 10]$
HS41	4	$[0, 0, 0, 0]$	$[1, 1, 1, 2]$
HS45	5	$[0, 0, 0, 0, 0]$	$[1, 2, 3, 4, 5]$
HS53	5	$[-10, \dots, -10]$	$[10, \dots, 10]$
HS59	2	$[0, 0]$	$[75, 65]$
HS60	3	$[-10, -10, -10]$	$[10, 10, 10]$
HS62	3	$[0, 0, 0]$	$[1, 1, 1]$
HS65	3	$[-5, -5, -5]$	$[5, 5, 5]$
HS66	3	$[0, 0, 0]$	$[100, 100, 1]$
HS68	4	$[0.0001, 0, 0, 0]$	$[100, 100, 2, 2]$
HS71	4	$[1, 1, 1, 1]$	$[5, 5, 5, 5]$
HS80	5	$[-2.3, -2.3, -3.2, -3.2, -3.2]$	$[2.3, 2.3, 3.2, 3.2, 3.2]$
HS83	5	$[78, 33, 27, 27, 27]$	$[102, 45, 45, 45, 45]$
HS110	10	$[2.001, \dots, 2.001]$	$[9.999, \dots, 9.999]$

TABLE 2: The calculation results.

Problems	n	INTR_1	INTR_2
HS05	2	4	145
HS18	2	2	76
HS21	2	22	467
HS23	2	2	68
HS25	3	2	40
HS30	3	3	85
HS31	3	3	99
HS34	3	4	148
HS36	3	4	171
HS37	3	5	270
HS38	4	17	385
HS41	4	2	158
HS45	5	2	189
HS53	5	4	168
HS59	2	10	387
HS60	3	3	271
HS62	3	3	233
HS65	3	3	152
HS66	3	7	311
HS68	4	6	298
HS71	4	8	176
HS80	5	3	73
HS83	5	14	692
HS110	10	14	771

From the results in Table 3, it can be seen that the same problem is solved by the ARC method using the interpolation technology without derivative. The number of iterations required by PFARC is less than the number of iterations in [3]. The import reason is that filter technique

TABLE 3: The calculation results.

Problems	n	INTR ₃	INTR ₄
S201	2	2	4
S203	2	5	7
S206	2	3	5
S207	2	6	7
S212	3	5	9
S213	3	12	27
S240	3	1	3
S256	4	10	16
S261	4	12	12
S283	10	10	34

implemented in PFARC has two goals: minimization of f and $\|h\|$, but a line search technique in [3] only performed the minimization of f . Therefore, our algorithm PFARC is more effective for solving bounded constrained derivative-free optimization problems.

In this paper, we propose a new algorithm PFARC for solving box constrained optimization problems without derivatives, which further complements the algorithm library for solving this kind of optimization. Since box constrained optimization is a classical problem, the proposed algorithm is an evolution of the existing algorithms, which further improved the existing ones. Box constrained optimization can be used in many fields, such as finance, science and technology, and military industry. Because the derivative information observed often is unavailable or unreliable, the derivative-free algorithm we proposed is closer to the reality. In the future, we can use the idea of this algorithm PFARC to solve other optimization problems, which is what we are going to do later.

Data Availability

The data used to support the findings of the study can be obtained from W. Hock, K. Schittkowski, Test examples for nonlinear programming codes, Lecture Notes in Economics and Mathematics System, 187. Springer (1981), and K. Schittkowski, MORE test examples for nonlinear programming codes, Lecture Notes in Economics and Mathematics System, 187. Springer (1981).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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