Research Article

Global Exponential Stability of a Discrete-Time Rayleigh System with Delays

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This paper deals with the problem of global exponential stability for a discrete-time Rayleigh system with delays. By using the mathematical induction method, some sufficient conditions are proposed for the global exponential stability of the discrete-time Rayleigh system. Finally, a numerical example is given to illustrate the effectiveness and application of the obtained results.

1. Introduction

In recent years, Rayleigh system has been widely investigated, due to its broad applications in physics, neural networks, electronics, engineering technique and mechanics, and other fields. The numerical solutions of Rayleigh equation for the scattering of electromagnetic waves from rough dielectric films on perfectly conducting substrates were studied by Madrazo and Maradudin [1]. In 2014, Kudryashov and Sinelshchikov [2] considered the analytical solutions of the Rayleigh equation for empty and gas-filled bubble. Russell [3] studied period bounds for a generalized Rayleigh equation. The small oscillations of the periodic Rayleigh equation were obtained in [4]. Omari and Villari [5] considered a certain planar system with applications to Lienard and Rayleigh equations. In [6], the authors dealt with the existence of asymptotically stable periodic solutions of a Rayleigh-type equation.

On the other hand, time delays, including constant, discrete-time, and distributed, are often encountered in many areas: physical, biological, engineering, and economic systems due to the finite switching speed of amplifiers in electronic networks or to the finite signal diffuse time in biological networks and so on (see [7–12]). For the dynamic behavior analysis of delayed dynamic systems, different types of time delays have been taken into account by using a variety of methods that include linear matrix inequality

(LMI) approach, M-matrix theory, topological degree theory, and Lyapunov functional method, see e.g., [13–16]. The stability problem of the Rayleigh system with delays is an important issue from both theoretical and practical points of view, which has been extensively studied by many authors. Cao and Wan [17] considered stability and synchronization of an inertial BAM neural network with time delays. Hu et al. [18] studied pinning synchronization of coupled inertial delayed neural networks. Wang et al. [19, 20] studied synchronization transitions on small-world neuronal networks and scale-free neuronal networks.

To the best of our knowledge, there are few results involving a discrete-time Rayleigh system with delays till now. In this paper, we will fill in the research gaps for the Rayleigh system. First, we derive a discrete-time Rayleigh system by the difference method. After that, some sufficient conditions are obtained for global exponential stability of a discrete-time Rayleigh system with delays by using the mathematical induction method. Our research enriches and develops the research content and research scope of the Rayleigh equation.

Remark 1.1. We list the advantage of this paper comparing with the existing related results as follows:

- (1) We derive a new discrete-time Rayleigh system with delays by using the difference method.

- (2) For convenience of research, we introduce a proper variable substitution to change the second-order differential system into a two-dimensional system.
- (3) We use the mathematical induction method for obtaining global exponential stability condition in this paper which is easier than the traditional Lyapunov functional method and linear matrix inequality method.

The following sections are organized as follows: in Section 2, problem formulation and some preliminaries are given. Section 3 gives global exponential stability for the considered system. In Section 4, a numerical example is given to show the feasibility of our results. Finally, Section 5 concludes the paper.

Notations: let $\mathbb{R}^n$ be the n -dimensional Euclidean space and $\mathbb{R}^{n \times m}$ be the $n \times m$ real matrix. I_n is the $n \times n$ identity matrix. The superscript T is the transpose and $\text{diag}\{\dots\}$ stands for a diagonal matrix. For real symmetric matrices X and Y , the notation $X \geq Y$ (respectively, $X > Y$) means that the matrix $X - Y$ is positive semidefinite (respectively, positive definite). Denote by $\mathbb{R}_{\geq}^{n \times m}$ and $\mathbb{R}_{>}^{n \times m}$ the sets of all $m \times n$ nonnegative and positive matrices, respectively. Denote

$$[a, b]_{\mathbb{Z}} = \{a, a+1, \dots, b-1, b\} \text{ for } a, b \in \mathbb{Z} \text{ and } a \leq b. \quad (1)$$

$$\begin{cases} x'(t) = -\eta x(t) + y(t), \\ y'(t) = (\eta - b)y(t) - \eta(\eta - b)x(t) - g(x(t), x(t - \tau)) + \omega, \quad t > 0. \end{cases} \quad (4)$$

By (3), initial conditions of system (4) are

$$\begin{aligned} x(s) &= \phi(s), \\ y(s) &= \psi(s) + \eta\phi(s) = \tilde{\phi}(s), \quad s \in [-\tau, 0]. \end{aligned} \quad (5)$$

$$\begin{cases} x'_i(t) = -\eta_i x_i(t) + y_i(t), \\ y'_i(t) = (\eta_i - b_i)y_i(t) - \eta_i(\eta_i - b_i)x_i(t) - g_i(x_i(t), x_i(t - \tau)) + \omega_i, \end{cases} \quad (6)$$

with initial conditions

$$\begin{aligned} x_i(s) &= \phi_i(s), \\ y_i(s) &= \psi_i(s) + \eta_i\phi_i(s) = \tilde{\phi}_i(s), \quad s \in [-\tau, 0]. \end{aligned} \quad (7)$$

In this paper, we need the following assumption. H_1 : there exist constants $l_{1i}, l_{2i} > 0$ such that

$$|g_i(x_2, y_2) - g_i(x_1, y_1)| \leq l_{1i}|x_2 - x_1| + l_{2i}|y_2 - y_1|, \quad \forall x_1, y_1, x_2, y_2 \in \mathbb{R}, i = 1, \dots, m. \quad (8)$$

A matrix $M \in \mathbb{R}^{n \times n}$ is called M -matrix, if all off-diagonal elements of M are nonnegative. Let $\lambda(M) = \{z \in \mathbb{C} : \det(z - I_n M)\} = 0$. The spectral abscissa of M is defined by $s(M) = \max\{\text{Re}\lambda : \lambda \in \lambda(M)\}$. The spectral radius of M is defined by $\rho(M) = \max\{|\lambda| : \lambda \in \sigma(M)\}$.

2. Preliminaries and Discrete-Time Rayleigh System

Consider the following continuous-time Rayleigh system with delays:

$$x''(t) + bx'(t) + g(x(t), x(t - \tau)) = \omega, \quad t > 0, \quad (2)$$

with initial conditions

$$\begin{aligned} x(s) &= \phi(s), \\ x'(s) &= \psi(s), \quad s \in [-\tau, 0], \end{aligned} \quad (3)$$

where $x(t) = (x_1(t), \dots, x_m(t))^T$, $\omega = (\omega_1, \dots, \omega_m)^T$, ω_i is a constant for $i = 1, \dots, m$, $g(x(t), x(t - \tau)) = (g_1(x_1(t), x_1(t - \tau)), \dots, g_m(x_m(t), x_m(t - \tau)))^T \in C(\mathbb{R}^m \times \mathbb{R}^m, \mathbb{R}^m)$, $b = \text{diag}\{b_1, \dots, b_m\}$ is a constant matrix, and $\phi, \psi \in C(-\tau, 0], \mathbb{R}^m$, $\tau > 0$ is a constant.

Let $y(t) = x'(t) + \eta x(t)$, where $\eta = \text{diag}\{\eta_1, \dots, \eta_m\}$, $\eta_i > 0$, $i = 1, \dots, m$. Then, system (2) is rewritten as

For $i = 1, \dots, m$, system (4) and initial conditions (5) can be rewritten as follows:

Remark 2.1. System (2) is a high-order system which is difficult to study it directly. Hence, we first change system (2) into a two-dimensional system, and then some mathematic methods can be easily used to study the above two-dimensional system (a first-order system).

Remark 2.2. In this paper, based on the semidiscrete method, discrete-time formulation for the Rayleigh system is going to be given by analysis and approximation techniques. The benefits of this approach are to ensure that the discrete

system and original system have the same nature of the solution. In [21], Chen, Zhao, and Fu obtained discrete analogue of high-order periodic Cohen–Grossberg neural networks with delay by the semidiscrete method. Since Rayleigh system is a second-order system, it is harder to discretize the Rayleigh system by the semidiscrete method, and hence, we develop mathematical analysis skills for overcoming the above difficulty. Furthermore, some recent results about the discrete-time system with delays can be considered, such as robust H^∞ sliding mode controller design of a class of time-delayed discrete conic-type nonlinear systems, see [22], and sliding mode controller design for conic-type nonlinear semi-Markovian jumping systems of time-delayed Chua's circuit, see [23, 24].

Theorem 2.1. *Under assumption (H_1) , system (6) exhibits unique equilibrium, if*

$$0 < \tilde{\kappa} < 1, \quad (9)$$

where

$$\tilde{\kappa} = \max \left\{ \kappa^2 e^{2\eta_i \kappa}, \kappa^2 e^{2|b_i - \eta_i| \kappa} (\eta_i |\eta_i - b_i| + l_{1i} + l_{2i})^2 \right\}. \quad (10)$$

Proof. Let $\mathcal{B} = C([0, \kappa], \mathbb{R})$, $\kappa > 0$ is a constant. Let $T(x_i(t), y_i(t)) = (T_1(x_i(t), y_i(t)), tT_2^n(x_i(t), y_i(t)))^\top$ on $\mathcal{B} \times \mathcal{B}$, where

$$\begin{aligned} T_1(x_i(t), y_i(t)) &= x_i(0)e^{-\eta_i t} + \int_0^t y_i(s)e^{\eta_i(s-t)} ds, \\ T_2(x_i(t), y_i(t)) &= y_i(0)e^{-(b_i - \eta_i)t} + \int_0^t [-\eta_i(\eta_i - b_i)x_i(s) - g_i(x_i(s), x_i(s - \tau)) + \omega_i]e^{(b_i - \eta_i)(s-t)} ds. \end{aligned} \quad (11)$$

Obviously, $T(x_i(t), y_i(t)) \in \mathcal{B} \times \mathcal{B}$. We will show that $T: \mathcal{B} \times \mathcal{B} \rightarrow \mathcal{B} \times \mathcal{B}$ is a contraction mapping. In fact,

from assumption (H_1) and (9), for any $(x_i(t), y_i(t)), (\bar{x}_i(t), \bar{y}_i(t)) \in \mathcal{B} \times \mathcal{B}$, we have

$$\begin{aligned} |T(x_i(t), y_i(t)) - T(\bar{x}_i(t), \bar{y}_i(t))|^2 &= \left| \int_0^t (y_i(s) - \bar{y}_i(s))e^{\eta_i(s-t)} ds \right|^2 \\ &\quad + \left| \int_0^t [-\eta_i(\eta_i - b_i)(x_i(s) - \bar{x}_i(s)) - (g_i(x_i(s), x_i(s - \tau)) - g_i(\bar{x}_i(s), \bar{x}_i(s - \tau)))]e^{(b_i - \eta_i)(s-t)} ds \right|^2 \\ &\leq \kappa^2 e^{2\eta_i \kappa} \|y_i - \bar{y}_i\|^2 + \kappa^2 e^{2|b_i - \eta_i| \kappa} (\eta_i |\eta_i - b_i| + l_{1i} + l_{2i})^2 \|x_i - \bar{x}_i\|^2 \\ &\leq \tilde{\kappa} (\|x_i - \bar{x}_i\|^2 + \|y_i - \bar{y}_i\|^2). \end{aligned} \quad (12)$$

Thus,

$$\|T(x_i, y_i) - T(\bar{x}_i, \bar{y}_i)\|^2 \leq \tilde{\kappa} (\|x_i - \bar{x}_i\|^2 + \|y_i - \bar{y}_i\|^2). \quad (13)$$

It yields that T is contractive on set $\mathcal{B} \times \mathcal{B}$. Therefore, T possesses a unique fixed point $z^* = (\phi^*, \psi^*) \in \mathcal{B} \times \mathcal{B}$ such

that $Tz^* = z^*$; it follows that z^* is the unique solution of (6). Since solution $z^*(t)$ of (6) is continuous for $t \in [0, \kappa]$, $z^*(t)$ can be extended to infinity.

We reformulate system (6) with the following approximations:

$$\begin{cases} x'_i(t) = -\eta_i x_i(t) + y_i\left(\left[\frac{t}{h}\right]h\right), \\ y'_i(t) = (\eta_i - b_i)y_i(t) - \eta_i(\eta_i - b_i)x_i\left(\left[\frac{t}{h}\right]h\right) - g_i\left(x_i\left(\left[\frac{t}{h}\right]h\right), x_i\left(\left[\frac{t}{h}\right]h - \left[\frac{\tau}{h}\right]h\right)\right) + \omega_i, \end{cases} \quad (14)$$

where $t \in [nh, (n+1)h]$, $n \in \mathbb{Z}_0^+ = \{k: k \geq 0, k \in \mathbb{Z}\}$, h is a fixed positive real number denoting a uniform discretionary step-size, and $[e]$ denotes the integer part of the real number

e . Obviously, for $t \in [nh, (n+1)h]$, then $[t/h] = n$. Let the notation $[\tau/h] = k \in \mathbb{Z}_0^+$, $x_i(nh) = x_i(n)$ and $y_i(nh) = y_i(n)$. We rewrite (14) as

$$\begin{cases} x_i'(t) = -\eta_i x_i(t) + y_i(t), \\ y_i'(t) = (\eta_i - b_i)y_i(t) - \eta_i(\eta_i - b_i)x_i(n) - g_i(x_i(n), x_i(n-k)) + \omega_i. \end{cases} \quad (15)$$

Integrating (15) over $[nh, t]$, $t < (n+1)h$, we have

$$\begin{cases} x_i(t) = e^{\eta_i(nh-t)} x_i(n) + \frac{1 - e^{\eta_i(nh-t)}}{\eta_i} y_i(n) \\ y_i(t) = e^{(b_i - \eta_i)(nh-t)} y_i(n) + \frac{1 - e^{(b_i - \eta_i)(nh-t)}}{b_i - \eta_i} [-\eta_i(\eta_i - b_i)x_i(n) - g_i(x_i(n), x_i(n-k)) + \omega_i]. \end{cases} \quad (16)$$

Letting $t \rightarrow (n+1)h$ in (16), we have

$$\begin{cases} x_i(n+1) = e^{-\eta_i h} x_i(n) + \frac{1 - e^{-\eta_i h}}{\eta_i} y_i(n), \\ y_i(n+1) = e^{-(b_i - \eta_i)h} y_i(n) + \frac{1 - e^{-(b_i - \eta_i)h}}{b_i - \eta_i} [-\eta_i(\eta_i - b_i)x_i(n) - g_i(x_i(n), x_i(n-k)) + \omega_i]. \end{cases} \quad (17)$$

Assume that

$$h > 0, b_i > \eta_i > 0 \text{ and } \frac{1 - e^{-\eta_i h}}{\eta_i} = h + O(h^2), \frac{1 - e^{-(b_i - \eta_i)h}}{b_i - \eta_i} = h + O(h^2) \text{ for sufficiently small } h. \quad (18)$$

It follows that the discrete-time system (17) converges to the continuous-time system (6) when $h \rightarrow 0^+$. System (18) is supplemented with the initial conditions:

$$\begin{aligned} x_i(s) &= \phi_i(s), \\ y_i(s) &= \psi_i(s) + \eta_i \phi_i(s) = \tilde{\phi}_i(s), \quad s \in [-\tau, 0]_{\mathbb{Z}}. \end{aligned} \quad (19)$$

Let $(x^*, y^*)^T$ is said to be an equilibrium of (17), then

$$\begin{cases} x_i^* = (1 - \eta_i)x_i^* + y_i^*, \\ y_i^* = (1 + \eta_i - b_i)y_i^* - \eta_i(\eta_i - b_i)x_i^* - g_i(x_i^*, x_i^*) + \omega_i. \end{cases} \quad (20)$$

Obviously, the continuous-time system (6) and the discrete-time analogue (17) have the same equilibrium $(x^*, y^*)^T$.

For $i = 1, \dots, m$, if $(x_i^*, y_i^*)^T$ is an equilibrium of (17), then system (17) can be written as the following equations:

$$\begin{cases} x_i^* = e^{\eta_i(nh-t)} x_i^* + \frac{1 - e^{\eta_i(nh-t)}}{\eta_i} y_i^*, \\ y_i^* = e^{(b_i - \eta_i)(nh-t)} y_i^* + \frac{1 - e^{(b_i - \eta_i)(nh-t)}}{b_i - \eta_i} [-\eta_i(\eta_i - b_i)x_i^* - g_i(x_i^*, x_i^*) + \omega_i]. \end{cases} \quad (21)$$

Let $u(n) = x(n) - x^*$, $v(n) = y(n) - y^*$, $n \in [0, \infty)_{\mathbb{Z}}$, where $x^* = (x_1^*, \dots, x_m^*)^T$ and $y^* = (y_1^*, \dots, y_m^*)^T$. In view of (17) and (21), we have

$$\begin{cases} u_i(n+1) = e^{-\eta_i h} u_i(n) + \frac{1 - e^{-\eta_i h}}{\eta_i} v_i(n), \\ v_i(n+1) = e^{-(b_i - \eta_i)h} v_i(n) + \frac{1 - e^{-(b_i - \eta_i)h}}{b_i - \eta_i} [-\eta_i(\eta_i - b_i)u_i(n) - \tilde{g}_i(u_i(n), u_i(n - \tau))], \end{cases} \quad (22)$$

where $\tilde{g}_i(u_i(n), u_i(n - \tau)) = g_i(u_i(n) + x^*, u_i(n - \tau) + x_i^*) - g(x_i^*, x_i^*)$. Clearly, $(0, 0)$ is an equilibrium of system (22). $\tilde{g}_i$ satisfies assumption (H_1) .

Let $C((-\tau, 0]_{\mathbb{Z}}, \mathbb{R}^m)$ is a linear space with $\sup_{s \in (-\tau, 0]_{\mathbb{Z}}} \phi(s) < \infty$. Define the norm $\|\cdot\|$ on $\mathbb{R}^m \times \mathbb{R}^m$ by $\|(f, g)\| = (\|f\|_2^2 + \|g\|_2^2)^{1/2}$ for any $f, g \in \mathbb{R}^m$. Define the norm $\|\cdot\|_r$ on $C((-\tau, 0]_{\mathbb{Z}}, \mathbb{R}^m) \times C((-\tau, 0]_{\mathbb{Z}}, \mathbb{R}^m)$ by

$$\|(\phi, \tilde{\phi})\|_r = \max \left\{ \sup_{s \in (-\tau, 0]_{\mathbb{Z}}} \|\phi(s)\|_2, \sup_{s \in (-\tau, 0]_{\mathbb{Z}}} \|\tilde{\phi}(s)\|_2 \right\} \quad (23)$$

for any $\phi, \tilde{\phi} \in C((-\tau, 0]_{\mathbb{Z}}, \mathbb{R}^m)$, where $\|\cdot\|_2$ is the Euclidean norm on $\mathbb{R}^m$.

Definition 2.1 (see [25]). The zero equilibrium of system (21) is said to be globally exponentially stable, if there exist constants $M > 0$ and $\gamma > 0$ such that every solution of system (21) $(u(n), v(n))^T$ satisfies

$$\|(u(n), v(n))\| \leq M e^{-\gamma n} \|(\phi, \tilde{\phi})\|_r, \quad \forall n \in [0, \infty)_{\mathbb{Z}}, \quad (24)$$

where $\phi, \tilde{\phi} \in C((-\tau, 0]_{\mathbb{Z}}, \mathbb{R}^m)$.

Definition 2.2. A real matrix $A = (a_{ij})_{n \times n}$ is an M -matrix if $a_{ij} \leq 0$ for $i \neq j$, $i, j = 1, \dots, n$ and all successive principal minors of A are positive.

Lemma 2.1 (see [26]). Let $A_0 \in \mathbb{R}^{n \times n}$ be a M -matrix and $B_0, C_0, D_0 \in \mathbb{R}^{n \times n}$. Then, the following statements (i)–(iii) are equivalent:

- (i) $A_0 x + B_0 y < 0$ and $C_0 x + D_0 y < y$ for $x, y \in \mathbb{R}_+^n$
- (ii) $\rho(D_0) < 1$ and $s(A_0 + B_0(I_n - D_0)^{-1}C_0) < 0$
- (iii) $s(A_0) < 0$ and $\rho(C_0(-A_0)^{-1}B_0 + D_0) < 1$

3. Global Exponential Stability

In this section we obtain sufficient conditions under which the zero equilibrium of system (22) is globally exponentially stable, that is, system (17) has a unique equilibrium which is globally exponentially stable.

Theorem 3.1. Assume that assumption (H_1) holds. System (17) has a unique equilibrium which is globally exponentially stable, if there exist $\tilde{u}, \tilde{v} \in \mathbb{R}_+^m$ and a positive constant γ such that

$$\Gamma_1 \tilde{u} + \Gamma_2 \tilde{v} \leq 0, \quad (25)$$

$$\Gamma_3 \tilde{u} + \Gamma_4 \tilde{v} \leq 0, \quad (26)$$

where

$$\Gamma_1 = \Xi_1 - e^{-\gamma} I_n, \quad \Xi_1 = \text{diag}\{e^{-\eta_1 h}, \dots, e^{-\eta_m h}\}, \quad (27)$$

$$\Gamma_2 = \text{diag}\left\{\frac{1 - e^{-\eta_1 h}}{\eta_1}, \dots, \frac{1 - e^{-\eta_m h}}{\eta_m}\right\},$$

$$\Gamma_3 = \text{diag}\left\{\frac{K(1 - e^{-(b_1 - \eta_1)h})[\eta_1(b_1 - \eta_1) + l_{11} + l_{21}e^{\gamma\tau}]}{b_1 - \eta_1}, \dots, \frac{K(1 - e^{-(b_m - \eta_m)h})[\eta_m(b_m - \eta_m) + l_{1m} + l_{2m}e^{\gamma\tau}]}{b_m - \eta_m}\right\}, \quad (28)$$

$$\Gamma_4 = \Xi_2 - e^{-\gamma} I_n, \quad \Xi_2 = \text{diag}\{e^{-(b_1 - \eta_1)h}, \dots, e^{-(b_m - \eta_m)h}\}.$$

Proof. Choose $K > 0$ such that

$$K\tilde{u} > (1, \dots, 1)^T, \quad K\tilde{v} > (1, \dots, 1)^T. \quad (29)$$

For each initial functions $\phi, \tilde{\phi} \in C((-\tau, 0]_{\mathbb{Z}}, \mathbb{R}^m)$ of system (22), let

$$\hat{u}(n) = K\|(\phi, \tilde{\phi})\|_r e^{-\gamma n} \tilde{u}, \quad \forall n \in [-\tau, \infty)_{\mathbb{Z}}, \quad (30)$$

$$\hat{v}(n) = K\|(\phi, \tilde{\phi})\|_r e^{-\gamma n} \tilde{v}, \quad \forall n \in [-\tau, \infty)_{\mathbb{Z}}. \quad (31)$$

Applying mathematical induction method, we claim that

$$u(n) \leq \hat{u}(n), \quad v(n) \leq \hat{v}(n), \quad \forall n \in [-\tau, \infty)_{\mathbb{Z}}. \quad (32)$$

In view of the definition of $\|\cdot\|_r$ and (29), we have

$$u(p) \leq \hat{u}(p), \quad v(p) \leq \hat{v}(p), \quad \forall p \in [-\tau, 0]_{\mathbb{Z}}. \quad (33)$$

Assume that (32) holds for $n \leq p$ ($p \geq 0$). When $n = p + 1$, for each $i = 1, \dots, m$, by the first equation of (22), we have

$$\begin{aligned}
|u_i(p+1)| &= \left| e^{-\eta_i h} u_i(p) + \frac{1 - e^{-\eta_i h}}{\eta_i} v_i(p) \right| \\
&\leq e^{-\eta_i h} K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma p} \tilde{u}_i + \frac{1 - e^{-\eta_i h}}{\eta_i} K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma p} \tilde{v}_i \\
&\leq K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma p} \left(e^{-\eta_i h} \tilde{u}_i + \frac{1 - e^{-\eta_i h}}{\eta_i} \tilde{v}_i \right).
\end{aligned} \tag{34}$$

Inequality (34) is equivalent to

$$u(p+1) \leq K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma p} (\Xi_1 \tilde{u} + \Gamma_2 \tilde{v}). \tag{35}$$

From (25) and (34), we have

$$u(p+1) \leq K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma(p+1)} \tilde{u} = \hat{u}(p+1). \tag{36}$$

On the other hand, assume that (32) holds for $n \leq p$ ($p \geq 0$). When $n = p+1$, for each $i = 1, \dots, m$, by the second equation of (21) and assumption (H_1) , we have

$$\begin{aligned}
|v_i(p+1)| &= \left| e^{-(b_i - \eta_i)h} v_i(p) + \frac{1 - e^{-(b_i - \eta_i)h}}{b_i - \eta_i} [-\eta_i(\eta_i - b_i)u_i(p) - \tilde{g}_i(u_i(p), u_i(p - \tau))] \right| \\
&\leq e^{-(b_i - \eta_i)h} K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma p} \tilde{v}_i + \eta_i \left(1 - e^{-(b_i - \eta_i)h} \right) K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma p} \tilde{u}_i \\
&\quad + \frac{\left(1 - e^{-(b_i - \eta_i)h} \right) l_{1i}}{b_i - \eta_i} K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma p} \tilde{u}_i \\
&\quad + \frac{\left(1 - e^{-(b_i - \eta_i)h} \right) l_{2i}}{b_i - \eta_i} K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma(p-\tau)} \tilde{u}_i.
\end{aligned} \tag{37}$$

Inequality (37) is equivalent to

$$v(p+1) \leq K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma p} (\Gamma_3 \tilde{u} + \Xi_2 \tilde{v}). \tag{38}$$

From (25) and (38), we have

$$v(p+1) \leq K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma(p+1)} \tilde{v} = \hat{v}(p+1). \tag{39}$$

It follows (36) and (39) that (32) holds. By (30)–(32), we have

$$\begin{aligned}
\|(u(n), v(n))\| &= \left(\|u(n)\|_2^2 + \|v(n)\|_2^2 \right)^{1/2} \\
&\leq \left(\|u(n)\|_2^2 + \|v(n)\|_2^2 \right)^{1/2} \\
&= K \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma n} \left(\|u(n)\|_2^2 + \|v(n)\|_2^2 \right)^{1/2}, \\
n &\in [0, \infty)_{\mathbb{Z}}.
\end{aligned} \tag{40}$$

Let $\tilde{K} = K (\|u(n)\|_2^2 + \|v(n)\|_2^2)^{1/2}$, then

$$\|(u(n), v(n))\| \leq \tilde{K} \|(\phi, \tilde{\phi})\|_{\tau} e^{-\gamma n}, \quad n \in [0, \infty)_{\mathbb{Z}}. \tag{41}$$

Since $\phi, \tilde{\phi} \in C((-\tau, 0]_{\mathbb{Z}}, \mathbb{R}^m)$ are arbitrary, which implies that the zero equilibrium of system (22) is globally exponentially stable, system (17) has unique equilibrium which is globally exponentially stable.

From Lemma 2.1 and Theorem 3.1, we have the following theorem: $\square$

Theorem 3.2. Suppose that assumption (H_1) holds. System (17) has unique equilibrium which is globally exponentially stable, if one of the following statements (i)–(iii) holds:

- (i) $\Gamma_1 \tilde{u} + \Gamma_2 \tilde{v} \leq 0$ and $\Gamma_3 \tilde{u} + (\Gamma_4 + I_n) \tilde{v} \leq \tilde{v}$ for $\tilde{u}, \tilde{v} \in \mathbb{R}_{>}^m$
- (ii) $\rho(\Gamma_4 + I_n), 1$ and $s(\Gamma_1 - \Gamma_2 \Gamma_4^{-1} \Gamma_3) < 0$
- (iii) $s(\Gamma_1) < 0$ and $\rho(\Gamma_4 + I_n - \Gamma_3 \Gamma_1^{-1} \Gamma_2) < 1$

Remark 3.1. Theorems 3.1 and 3.2 give the delay-dependent global exponential stability criteria for the equilibrium of system (17). We will give delay independent global exponential stability criteria for the equilibrium of system (17) in the future work.

Remark 3.2. In [27], the authors studied the exponential stability for the discrete-time bidirectional associative memory neural networks with time-varying delays by using the Lyapunov functional method. Raja and Marshal Anthoni [28] dealt with the problem of stability analysis for a class of discrete-time BAM neural networks with time-varying delays by employing the Lyapunov functional and linear matrix inequality (LMI) approach. Gao and Cui [29] considered global robust exponential stability of discrete-time interval BAM neural networks with time-varying delays by employing the Lyapunov functional and LMI approach. We find that the existing results for discrete-time neural networks are based on Lyapunov functional and LMI approach

FIGURE 1: State trajectories $((x_i(k), y_i(k))^T$ ($i = 1, 2$) of system (42). FIGURE 2: State trajectories $((x_i(k), y_i(k))^T$ ($i = 1, 2$) of system (42).

which are complicated than the mathematical induction approach belonging to this paper. Furthermore, the conditions in the present paper are easier to verify than the conditions in [27–29].

4. Numerical Examples

In this section, a numerical example is given to show the effectiveness of theoretical results.

For $i = 1, 2$, consider the following system of model (17):

$$\begin{cases} x_i(n+1) = e^{-\eta_i h} x_i(n) + \frac{1 - e^{-\eta_i h}}{\eta_i} y_i(n), \\ y_i(n+1) = e^{-(b_i - \eta_i)h} y_i(n) + \frac{1 - e^{-(b_i - \eta_i)h}}{b_i - \eta_i} [-\eta_i(\eta_i - b_i)x_i(n) - g_i(x_i(n), x_i(n-k)) + h_i]. \end{cases} \quad (42)$$

Let $\eta_i = 0.5$, $b_i = 1$, $h_i = 0.01$ and

$$g_i(x, y) = \tanh(x) + \tanh(y), \quad \forall x, y \in \mathbb{R}. \quad (43)$$

Obviously, $g_i(x, y)$ satisfies assumption (H_1) with $l_{1i} = l_{2i} = 1$. In view of the discussion in [30], one can easily obtain an equilibrium $(x^*, y^*)^T$ of (42) as follows:

$$\begin{aligned} x^* &= (0.01362, 0.00532)^T, \\ y^* &= (0.02429, 0.00716)^T. \end{aligned} \quad (44)$$

Choosing $\gamma = 0.1$ and $\tau = 1$, by simple calculation, we have

$$\begin{aligned} \Xi_1 &= \text{diag}\{e^{-\eta_1 h}, e^{-\eta_2 h}\} = \text{diag}\{e^{-0.0005}, e^{-0.0005}\}, \\ \Gamma_1 &= \Xi_1 - e^{-\gamma} I_2 = \text{diag}\{-0.105, -0.105\}, \\ \Xi_2 &= \text{diag}\{e^{-(b_1 - \eta_1)h}, e^{-(b_2 - \eta_2)h}\} = \text{diag}\{e^{-0.0005}, e^{-0.0005}\}, \\ \Gamma_4 &= \Xi_2 - e^{-\gamma} I_2 = \text{diag}\{-0.105, -0.105\}, \\ \Gamma_3 &= \text{diag}\left\{ \frac{K(1 - e^{-(b_1 - \eta_1)h})[\eta_1(b_1 - \eta_1) + l_{11} + l_{21}e^{\gamma\tau}]}{b_1 - \eta_1}, \dots, \frac{K(1 - e^{-(b_2 - \eta_2)h})[\eta_2(b_2 - \eta_2) + l_{12} + l_{22}e^{\gamma\tau}]}{b_2 - \eta_2} \right\} \\ &\doteq \text{diag}\{0.0023K, 0.0023K\}. \end{aligned} \quad (45)$$

Choose $K = 1.1$, $\tilde{u} = (1, 1)^T$, $\tilde{v} = (1, 1)^T$, then

$$\Gamma_1 \tilde{u} + \Gamma_2 \tilde{v} = (-1.1025, -1.1025)^T < 0, \quad (46)$$

$$\Gamma_3 \tilde{u} + \Gamma_4 \tilde{v} = (-1.1025, -1.1025)^T < 0. \quad (47)$$

Thus, the conditions of Theorem 3.1 are satisfied. Figures 1 and 2 show the theoretical results presented in Theorem 3.1 under the different initial functions.

5. Conclusions and Discussion

In this paper, we obtain some results for global exponential stability for a class of discrete-time Rayleigh system with delays by using the M -matrix analysis method and mathematical induction method. Some sufficient conditions is proposed for the global exponential stability of the discrete-time Rayleigh system. Finally, a

numerical test has been given at the end of this paper to illustrate the effectiveness and feasibility of the proposed criterion.

The methods of this article can also be used to study other types of dynamic system, especially for genetic regulatory neural networks [31, 32], singular dynamic systems [33, 34], and fuzzy dynamic system [35–37]. We will study different types of neural networks by using the mathematical induction method and M -matrix analysis method.

Data Availability

The data used to support the findings of this study are available from the author upon request.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Authors' Contributions

The author contributed equally to the writing of this paper. The author read and approved the final manuscript.

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