Research Article

Periodic Solutions with Prescribed Minimal Period for $2n$ th-Order Nonlinear Discrete Systems

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In this paper, by using the critical point theory, some new results of the existence of at least two nontrivial periodic solutions with prescribed minimal period to a class of $2n$ th-order nonlinear discrete system are obtained. The main approach used in our paper is variational technique and the linking theorem. The problem is to solve the existence of periodic solutions with prescribed minimal period of $2n$ th-order discrete systems.

1. Introduction

In the following and in the sequel, we denote by $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{R}$ the sets of all natural numbers, integers, and real numbers, respectively. The symbol $*$ is defined by the transpose of a vector. Let $[\cdot]$ be the greatest integer function. For any integers a and b with $a \leq b$, we let $\mathbb{Z}(a, b)$ denote the discrete interval $\{a, a+1, \dots, b\}$ and $\mathbb{Z}(a) = \{a, a+1, \dots\}$.

Now, we concern with the following $2n$ th-order nonlinear discrete system:

$$\Delta^{2n} x_{k-n} = (-1)^n g(k, x_k), \quad n \in \mathbb{Z}(3), k \in \mathbb{Z}, \quad (1)$$

where Δ is the forward difference operator defined by $\Delta x_k = x_{k+1} - x_k$, $\Delta^0 x_k = x_k$, and $\Delta^j x_k = \Delta^{j-1}(\Delta x_k)$ for $j \geq 2$, $g(t, x) \in C^1(\mathbb{R}^2, \mathbb{R})$, and $g(t+M, x) = g(t, x)$ for a given integer $M \geq 3$.

We may think of (1) as a discrete analogue of the following $2n$ th-order differential equation:

$$\frac{d^{2n} x(t)}{dt^{2n}} = (-1)^n g(t, x(t)), \quad t \in \mathbb{R}. \quad (2)$$

Equations similar in structure to (2) have been studied by many authors [1–6]. The difficulty of this paper comparing with system (2) is that there are few known techniques for

studying the existence of periodic solutions with minimal period of (1).

Denote

$$\omega = \frac{2\pi}{M}. \quad (3)$$

Throughout this paper, we suppose that there is a function $G(t, x) \in C^2(\mathbb{R}^2, \mathbb{R})$ with $G(t+M, x) = G(t, x)$, $G(-t, -x) = G(t, x)$, $G(t, x) \geq 0$, and

$$\frac{\partial G(t, x)}{\partial x} = g(t, x), \quad \forall t \in \mathbb{R}. \quad (4)$$

Bin [7] in 2013 considered the second-order discrete Hamiltonian systems:

$$\Delta x_{k-1} + W'(k, x_k) = 0, \quad x_{k+M} = x_k. \quad (5)$$

By using Morse theory, some new results concerning the existence of nontrivial periodic solution are obtained.

Using the critical point method, Liu et al. [8] studied the following forward and backward difference equation:

$$\Delta^n (r_{k-n} \Delta^n x_{k-n}) = (-1)^n f(k, x_{k+1}, x_k, x_{k-1}), \quad n \in \mathbb{Z}(3), k \in \mathbb{Z}. \quad (6)$$

Some new criteria for the existence and multiplicity of periodic and subharmonic solutions are established.

In 2016, Leng [9] established some new criteria for the existence and multiplicity of periodic and subharmonic solutions to the $2n$ th-order difference equation with ϕ_c -Laplacian

$$\Delta^n(r_{k-n}\phi_c(\Delta^n x_{k-n})) = (-1)^n f(k, x_{k+1}, x_k, x_{k-1}), \quad k \in \mathbb{Z}, \quad (7)$$

$$\Delta^n(r_{k-n}\phi_p(\Delta^n x_{k-n})) + q_n \phi_p(x_k) = f(k, x_{k+n}, \dots, x_k, \dots, x_{k-n}), \quad n \in \mathbb{Z}(3), k \in \mathbb{Z}, \quad (8)$$

has infinitely many homoclinic orbits.

Lin and Zhou [11] concerned with the existence and multiplicity of periodic and subharmonic solutions of the following $2n$ th-order difference equation

$$\Delta^n(r_{k-n}\phi(\Delta^n x_{k-n})) = (-1)^n f(k, x_{k+n}, \dots, x_k, \dots, x_{k-n}), \quad k \in \mathbb{Z}. \quad (9)$$

$$\sum_{i=0}^n r_i (X_{k-i} + X_{k+i}) + f(k, X_{k+\Gamma}, \dots, X_k, \dots, X_{k-\Gamma}) = 0, \quad n \in \mathbb{N}, k \in \mathbb{Z}, \quad (10)$$

by making use of the critical point theory. Some existence criteria are established. Some results are generalized.

In 1978, Rabinowitz [13] proposed a conjecture that the Hamiltonian system has a nonconstant periodic solution with prescribed minimal period under some given conditions. From then on, there has been much progress [14–16] on Rabinowitz's conjecture under various conditions. Ambrosetti and Mancini [14] assumed that the dual functional is bounded from below and the Hamiltonian system has a minimum to which correspond a solution with given minimal period. Ekeland and Hofer [16] proved that if the Hamiltonian system is flat near an equilibrium and superquadratic near infinity, it has a periodic solution with minimal period. The estimate of number of periodic solutions was established in [15]. In contrast to differential equations, the research on periodic solutions with prescribed minimal period of higher order discrete systems is fresh and there are very few literature (see [7–12, 17–29]) on it. Comparing this paper with references [8–11], the advantages and differences of this paper are that the existence of periodic solutions with prescribed minimal period of (1) is obtained in this paper; however, only periodic solutions are obtained in the references [8–11]. Given integer $T \geq 2$, Long [23] considered the following T -cycle discrete Hamiltonian systems:

$$J\Delta x_k = \nabla H(x, Lx_k). \quad (11)$$

By making use of minimax theory and geometrical index theory, some results on the existence and multiplicity of subharmonic solutions with prescribed minimal period to the abovementioned discrete Hamiltonian systems are

using the linking theorem in combination with variational technique.

By establishing a new proper variational framework and using the critical point theory, He [10] established some new existence criteria to guarantee that the $2n$ th-order nonlinear difference equation containing both advance and retardation with p -Laplacian

By using the critical point theory, some new sufficient conditions are obtained and some previous results are generalized.

Xia [12] in 2018 considered the existence of periodic solutions for a higher order nonlinear difference equation

obtained. Yu et al. [27] in 2004 obtained some sufficient conditions on the existence of subharmonic solutions with prescribed minimal period for the second-order difference equation by using variational methods. Therefore, there is still spacious room to explore the periodic solutions with prescribed minimal period of higher order discrete systems. Motivated by the papers [9, 12], for a given integer s with $s > 1$, the aim of this paper is to obtain some new results for the existence of at least two nontrivial periodic solutions with minimal period sM to a $2n$ th-order discrete system by using critical point method.

Here, we give the existence results of at least two nontrivial periodic solutions with minimal period sM as follows.

Theorem 1. Suppose that $G(t, x)$ satisfies the following assumptions:

(G₁) There are constants $\delta_1 > 0$ and $\rho \in (0, ((4 \sin^2(\pi/sM))^n/2))$ such that

$$G(t, x) \leq \rho x^2, \quad \forall t \in \mathbb{Z}, |x| \leq \delta_1. \quad (12)$$

(G₂) There are constants $\tau > 0$ and $\gamma \in ((4^n/2), +\infty)$ such that

$$G(t, x) \geq \gamma x^2 - \tau, \quad \forall (t, x) \in \mathbb{R}^2. \quad (13)$$

(G₃) There are three constants $\kappa > 0$ and $C > D > 0$ such that

$$\begin{aligned} \left(\frac{\partial^2 G(t, x)}{\partial x^2} \xi, \xi \right) &\leq C\xi^2, \quad \forall (t, x) \in \mathbb{R}^2, \xi \in \mathbb{R}, \\ \left(\frac{\partial^2 G(t, x)}{\partial x^2} \xi, \xi \right) &\geq D\xi^2, \quad \forall |x| \leq \kappa, s \in \mathbb{R}, \xi \in \mathbb{R}. \end{aligned} \quad (14)$$

(G₄) If ω is a rational number, x is a solution of (1) with a minimal period ωM , and $g(t, x)$ also has a minimal period ωM , then ω must be an integer.

(G₅) Denote by m_s the least prime factor of s :

$$\begin{aligned} \left(4 \sin^2 \frac{\omega m_s}{2s} \right)^n &> C, \\ \left(4 \sin^2 \frac{\omega}{2s} \right)^n &< D, \\ \sum_{k=1}^{sM} g^2(k, 0) &< \frac{4\pi\kappa^2 \left[\left(4 \sin^2 (\omega m_s / 2s) \right)^n - C \right] \left[D - \left(4 \sin^2 (\omega / 2s) \right)^n \right]}{\omega}. \end{aligned} \quad (15)$$

Then, (1) possesses at least two nontrivial periodic solutions with minimal period sM .

then (1) has at least two nontrivial periodic solutions with minimal period sM .

Corollary 1. Suppose that $G(t, x)$ satisfies (G₁) – (G₄) and

$$\begin{aligned} \left(4 \sin^2 \frac{\omega}{2} \right)^n &> C, \\ \left(4 \sin^2 \frac{\omega}{2s} \right)^n &< D. \end{aligned} \quad (16)$$

If

$$\sum_{k=1}^{sM} g^2(k, 0) < \frac{4\pi\kappa^2 \left[\left(4 \sin^2 (\omega/2) \right)^n - C \right] D}{\omega}, \quad (17)$$

then there is $S > 0$ such that for any prime integer $s > S$, (1) possesses at least two nontrivial periodic solutions with minimal period sM .

Theorem 2. Suppose that $G(t, x)$ satisfies (G₃) – (G₅) and the following assumptions:

$$\begin{aligned} (G_6) \quad &\lim_{|x| \rightarrow +\infty} (G(t, x)/x^2) = 0 \text{ uniformly for } t \in \mathbb{R}. \\ (G_7) \quad &\text{There are constants } \delta_2 > 0 \text{ and } \theta > 2 \text{ such that} \\ &0 < \theta G(t, x) \leq xg(t, x), \quad \forall t \in \mathbb{Z}, |x| \leq \delta_2. \end{aligned} \quad (18)$$

Then, (1) has at least two nontrivial periodic solutions with minimal period sM .

Theorem 3. Suppose that $G(t, x)$ satisfies (G₁) – (G₄) and the following assumptions. If

$$\begin{aligned} g(k, 0) &= 0, \quad \forall k \in \mathbb{Z}, \\ \left(4 \sin^2 \frac{\omega m_s}{2s} \right)^n &> C, \\ \left(4 \sin^2 \frac{\omega}{2s} \right)^n &< D, \end{aligned} \quad (19)$$

Corollary 2. Suppose that $G(t, x)$ satisfies (G₁) – (G₄) and the following assumptions. If

$$\begin{aligned} g(k, 0) &= 0, \quad \forall k \in \mathbb{Z}, \\ \left(4 \sin^2 \frac{\omega}{2} \right)^n &> C, \\ \left(4 \sin^2 \frac{\omega}{2s} \right)^n &< D. \end{aligned} \quad (20)$$

then there is $S > 0$ such that for any prime integer $s > S$, (1) possesses at least two nontrivial periodic solutions with minimal period sM .

The remainder of this paper is organized as follows. In Section 2, we build the variational functional and gather some basic notations that are necessary in the proofs of our main theorems. In Section 3, we state some useful lemmas. In Section 4, the main results will be proved.

Regarding the basis for variational methods, we refer the reader to [30]. Regarding the basic knowledge of integral inequalities and extended hypergeometric functions, the reader is referred to [31].

2. Preliminaries

In this section, we shall establish the variational framework associated with (1) and gather some basic notations that are necessary in the proofs of our main theorems.

Let the vector space $\mathbb{B}$ be defined by

$$\mathbb{B} = \{x | x_{k+sM} = x_k\}, \quad (21)$$

and for any $x \in \mathbb{B}$, define the inner product

$$(x, y) = \sum_{i=1}^{sM} x_i y_i, \quad (22)$$

and the norm

$$\|x\| = \left(\sum_{i=1}^{sM} x_i^2 \right)^{1/2}. \quad (23)$$

For any $x \in \mathbb{B}$, let I be the functional defined by

$$I(x) = \frac{1}{2} \sum_{k=1}^{sM} (\Delta^n x_{k-1})^2 - \sum_{k=1}^{sM} G(k, x_k). \quad (24)$$

Then, I is continuously differentiable and

$$\frac{\partial I}{\partial u_k} = (-1)^n \Delta^{2n} x_{k-n} - g(k, x_k), \quad \forall k \in \mathbb{Z}(1, sM). \quad (25)$$

Thus, x is a critical point of $I(x)$ on $\mathbb{B}$ if and only if

$$\Delta^{2n} x_{k-n} = (-1)^n g(k, x_k), \quad \forall k \in \mathbb{Z}(1, sM). \quad (26)$$

Therefore, we reduce the problem of finding sM -periodic solutions of (1) to that of seeking critical points of the functional I on $\mathbb{B}$.

Denote

$$L = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 & -1 \\ -1 & 2 & -1 & \cdots & 0 & 0 \\ 0 & -1 & 2 & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & 2 & -1 \\ -1 & 0 & 0 & \cdots & -1 & 2 \end{pmatrix}_{sM \times sM}. \quad (27)$$

It is clear that the eigenvalues of L are

$$\lambda_i = 4 \sin^2 \frac{i\pi}{sM}, \quad i = 0, 1, 2, \dots, sM - 1. \quad (28)$$

Furthermore, L is positively semidefinite and all of eigenvalues of L are positive except for 0, and

$$\min\{\lambda_1, \lambda_2, \dots, \lambda_{sM-1}\} = 4 \sin^2 \frac{\pi}{sM}, \quad (29)$$

$$\max\{\lambda_1, \lambda_2, \dots, \lambda_{sM-1}\} \leq 4.$$

Obviously, 0 is an eigenvalue of L and $(1, 1, \dots, 1)^*$ is an eigenvector associated to 0. Let $A = \{(c, c, \dots, c)^* \in \mathbb{B} | c \in \mathbb{R}\}$, then A is an invariant subspace of $\mathbb{B}$. Denote B by $\mathbb{B} = A \oplus B$.

The eigenvectors of L corresponding to λ_i are defined by

$$\begin{aligned} \alpha_i &= \left(\cos \frac{2i\pi}{sM}, \cos \frac{2i\pi \cdot 2}{sM}, \dots, \cos \frac{2i\pi \cdot sM}{sM} \right)^*, \quad i = 1, 2, \dots, \left[\frac{sM-1}{2} \right], \\ \beta_i &= \left(\sin \frac{2i\pi}{sM}, \sin \frac{2i\pi \cdot 2}{sM}, \dots, \sin \frac{2i\pi \cdot sM}{sM} \right)^*, \quad i = 1, 2, \dots, \left[\frac{sM-1}{2} \right]. \end{aligned} \quad (30)$$

Set

$$\begin{aligned} Y &= \text{span}\{\alpha_i\}, \quad i = 1, 2, \dots, \left[\frac{sM-1}{2} \right], \\ Z &= \text{span}\{\beta_i\}, \quad i = 1, 2, \dots, \left[\frac{sM-1}{2} \right]. \end{aligned} \quad (31)$$

If sM is odd, then $\mathbb{B} = A \oplus Y \oplus Z$. For any $x \in \mathbb{B}$ and $i \in \mathbb{Z}$,

$$x_i = p + (-1)^i q + \sum_{k=1}^{[(sM-1)/2]} \left(p_k \cos \frac{\omega k}{s} i + q_k \sin \frac{\omega k}{s} i \right), \quad (32)$$

where p, q, p_k , and q_k are constants.

If sM is even, then 4 is the eigenvalue of L . Let η denote the eigenvector corresponding to 4, and $W = \text{span}\{\eta\}$. We have $\mathbb{B} = A \oplus Y \oplus Z \oplus W$. For any $x \in \mathbb{B}$ and $i \in \mathbb{Z}$,

$$x_i = p + \sum_{k=1}^{[(sM-1)/2]} \left(p_k \cos \frac{\omega k}{s} i + q_k \sin \frac{\omega k}{s} i \right), \quad (33)$$

where p, p_k , and q_k are constants.

Suppose that $\mathbb{B}$ is a real Banach space and $I \in C^1(\mathbb{B}, \mathbb{R})$. As usual, I is said to satisfy the Palais-Smale condition if every sequence $\{x^{(j)}\} \subset \mathbb{B}$, such that $\{I(x^{(j)})\}$ is bounded and $I'(x^{(j)}) \rightarrow 0$, ($j \rightarrow \infty$), has a convergent subsequence. The sequence $\{x^{(j)}\}$ is called a Palais-Smale sequence.

3. Some Lemmas

To apply critical point theory to study the existence of periodic solutions with minimal period sM of (1), some lemmas should be stated in this section which will be used in proofs of our main results.

Below, we denote by $B_r(x)$ the open ball centered at $x \in \mathbb{B}$ with radius $r > 0$, $\bar{B}_r(x)$ as its closure, and $\partial B_r(x)$ as its boundary.

Lemma 1 (linking theorem [30]). *Let $\mathbb{B}$ be a real Banach space, $\mathbb{B} = \mathbb{B}_1 \oplus \mathbb{B}_2$, where $\mathbb{B}_1$ is finite dimensional. Suppose that $I \in C^1(\mathbb{B}, \mathbb{R})$ satisfies the Palais–Smale condition and the following:*

(I₁) *There are positive constants c and r such that $I|_{\partial B_r(0) \cap \mathbb{B}_2} \geq c$*

(I₂) *There is $\mu \in \partial B_1(0) \cap \mathbb{B}_2$ and a positive constant $\hat{c} \geq r$ such that $I|_{\partial \Omega} \leq 0$, where $\Omega = (\bar{B}_{\hat{c}}(0) \cap \mathbb{B}_1) \oplus \{t\mu | 0 < t < \hat{c}\}$*

Then, I possesses a critical value $c_0 \geq c$, where

$$c_0 = \inf_{d \in Y} \sup_{x \in \Omega} I(d(x)), \quad (34)$$

and $Y = \{d \in C(\bar{\Omega}, \mathbb{B}) | d|_{\partial \Omega} = i d\}$, where $i d$ denotes the identity operator.

Set

$$\tilde{\mathbb{B}} = \{x \in \mathbb{B} | x_{-k} = x_k, \forall k \in \mathbb{Z}\}. \quad (35)$$

We have $\tilde{\mathbb{B}} = \mathbb{Z}$, then

$$x_k = \sum_{i=1}^{[(sM-1)/2]} q_i \sin \frac{\omega i}{s} k, \quad \forall k \in \mathbb{Z}. \quad (36)$$

Lemma 2. *Suppose that $G(t, x)$ satisfies $(G_1) - (G_5)$. Then,*

$$I(x) = \frac{1}{2} \sum_{k=1}^{sM} (\Delta^n x_{k-1})^2 - \sum_{k=1}^{sM} G(k, x_k), \quad (37)$$

is bounded from above in $\mathbb{B}$.

Proof. For any $x \in \mathbb{B}$, by (G_2) , we have

$$\begin{aligned} I(x) &= \frac{1}{2} \sum_{k=1}^{sM} (\Delta^n x_{k-1}, \Delta^n x_{k-1}) - \sum_{k=1}^{sM} G(k, x_k) \\ &= \frac{1}{2} \sum_{k=1}^{sM} (\Delta^n x_k, \Delta^n x_k) - \sum_{k=1}^{sM} G(k, x_k) \\ &= \frac{1}{2} y^* L y - \sum_{k=1}^{sM} G(k, x_k) \\ &\leq \frac{1}{2} \times 4 \|y\|^2 - \sum_{k=1}^{sM} (\varrho x^2 - \tau) \\ &= 2 \|y\|^2 - \varrho \|x\|^2 + sM\tau, \end{aligned} \quad (38)$$

where $y = (\Delta^{n-1} x_1, \Delta^{n-1} x_2, \dots, \Delta^{n-1} x_{sM})^*$. Since

$$\|y\|^2 = \sum_{k=1}^{sM} (\Delta^{n-2} x_{k+1} - \Delta^{n-2} x_k)^2 \leq 4 \sum_{k=1}^{sM} (\Delta^{n-2} x_k)^2 \leq \dots \leq 4^{n-1} \|x\|^2, \quad (39)$$

then

$$I(x) \leq \frac{1}{2} (4^n - 2\varrho) \|x\|^2 + sM\tau \leq sM\tau. \quad (40)$$

The proof is finished.

Lemma 3. *Suppose that $G(t, x)$ satisfies $(G_1) - (G_5)$. Then,*

$$I(x) = \frac{1}{2} \sum_{k=1}^{sM} (\Delta^n x_{k-1})^2 - \sum_{k=1}^{sM} G(k, x_k), \quad (41)$$

satisfies the Palais–Smale condition.

Proof. Assume that $\{I(x^{(j)})\}$ is a bounded sequence from the lower bound. Then, there is a positive constant $c_1 > 0$ such that

$$-c_1 \leq I(x^{(j)}), \quad \forall j \in \mathbb{N}. \quad (42)$$

The proof of Lemma 2 implies that

$$-c_1 \leq I(x^{(j)}) \leq (4^n - 2\varrho) \|x^{(j)}\|^2 + sM\tau, \quad \forall j \in \mathbb{N}. \quad (43)$$

Therefore,

$$\|x^{(j)}\|^2 \leq \frac{c_1 + sM\tau}{2\varrho - 4^n}. \quad (44)$$

It comes from $\varrho > (4^n/2)$ that we can find a positive constant c_2 such that for any $j \in \mathbb{N}$, $\|x^{(j)}\| \leq c_2$. As a consequence of this, we know that the sequence $\{x^{(j)}\}_{j \in \mathbb{N}}$ is a bounded in the finite-dimensional space $\mathbb{B}$. Thus, it has a convergent subsequence. The Palais–Smale condition is verified.

Lemma 4. *Suppose that x is a critical point of $I(x)$ on $\tilde{\mathbb{B}}$. Then, x is a critical point of $I(x)$ on $\mathbb{B}$.*

The proof of Lemma 4 is similar to the proof of Lemma 2.2 in [12]. For simplicity, we omit its proof.

Let

$$\Theta_\omega = -\frac{s}{2[(4 \sin^2(\omega\omega/2s))^n - C]} \sum_{k=1}^{sM} g^2(k, 0). \quad (45)$$

Lemma 5. *Suppose that $G(t, x)$ satisfies $(G_1) - (G_5)$ and $I(x) < \Theta_{m_s}$. If x is a critical point of $I(x)$ on $\tilde{\mathbb{B}}$, then x has a minimal period sM .*

Proof. Suppose, for the sake of contradiction, that x exists a minimal period (sM/ω) . In view of the condition (G_4) , we have $\omega \geq m_s$.

Similarly, x_i can be written in the form of

$$x_k = \sum_{i=1}^{[(sM-\omega)/2\phi]} q_i \sin \frac{\omega\omega i}{s} k, \quad \forall x \in \tilde{\mathbb{B}}. \quad (46)$$

Thus,

$$I(x) = \frac{1}{2}y^*Ly - \sum_{k=1}^{sM} G(k, x_k) \geq 2 \sin^2 \frac{\omega\omega}{2s} \|y\|^2 - \sum_{k=1}^{sM} G(k, x_k), \quad (47)$$

where $y = (\Delta x_1, \Delta x_2, \dots, \Delta x_{sM})^*$. It is obvious that

$$\|y\|^2 = \sum_{k=1}^{sM} (\Delta^{n-2} x_{k+1} - \Delta^{n-2} x_k)^2 \geq 4 \sin^2 \frac{\omega\omega}{2s} \sum_{k=1}^{sM} (\Delta^{n-2} x_k)^2 \geq \dots \geq \left(4 \sin^2 \frac{\omega\omega}{2s}\right)^{n-1} \|x\|^2. \quad (48)$$

Therefore, by (G_5) ,

$$\begin{aligned} I(x) &\geq \frac{1}{2} \left(4 \sin^2 \frac{\omega\omega}{2s}\right)^n \|x\|^2 - \left(\sum_{k=1}^{sM} g^2(k, 0)\right)^{1/2} \|x\| - \frac{C}{2} \|x\|^2 \\ &\geq -\frac{1}{2 \left[\left(4 \sin^2 (\omega\omega/2s)\right)^n - C \right]} \sum_{k=1}^{sM} g^2(k, 0) \\ &\geq -\frac{s}{2 \left[\left(4 \sin^2 (\omega\omega/2s)\right)^n - C \right]} \sum_{k=1}^{sM} g^2(k, 0) \\ &= \Theta_{\omega} \geq \Theta_{m_s}. \end{aligned} \quad (49)$$

That is, $I(x) \geq \Theta_{m_s}$ which is a contradiction with the condition $I(x) < \Theta_{m_s}$, and the proof of Lemma 5 is now complete.

4. Proofs of the Main Theorems

In this section, the proofs of Theorems 1–3 and Corollaries 1 and 2 are given by using the critical point theory.

Proof. of Theorem 1. In view of Lemma 2, $I(x)$ is bounded from above in $\mathbb{B}$.

Set

$$\bar{m} = \sup_{x \in \mathbb{B}} I(x). \quad (50)$$

Thus, there is a sequence $\{x^{(i)}\}$ on $\mathbb{B}$ such that

$$\|y\|^2 = \sum_{k=1}^{sM} (\Delta^{n-2} x_{k+1} - \Delta^{n-2} x_k)^2 \geq 4 \sin^2 \frac{\pi}{sM} \sum_{k=1}^{sM} (\Delta^{n-2} x_k)^2 \geq \dots \geq \left(4 \sin^2 \frac{\pi}{sM}\right)^{n-1} \|x\|^2. \quad (55)$$

Therefore,

$$I(x) \geq \left[\frac{(4 \sin^2 (\pi/sM))^n}{2} - \rho \right] \|x\|^2. \quad (56)$$

Take $c = [(4 \sin^2 (\pi/sM))^n/2 - \rho]\delta_1^2$. Thus, for any $x \in \partial B_{\delta_1}(0) \cap B$,

$$I(x) \geq c. \quad (57)$$

Consequently,

$$\bar{m} = \lim_{i \rightarrow \infty} I(x^{(i)}). \quad (51)$$

In addition, by the proof of Lemma 2, for any $x \in \mathbb{B}$,

$$I(x) \leq \frac{1}{2} (4^n - 2Q) \|x\|^2 + sM\tau \leq sM\tau. \quad (52)$$

Therefore, $\lim_{\|x\| \rightarrow +\infty} I(x) = -\infty$. This means that $\{x^{(i)}\}$ is bounded. Consequently, $\{x^{(i)}\}$ has a convergent subsequence. We define it as $\{x^{(i_k)}\}$. Denote

$$\bar{x} = \lim_{k \rightarrow +\infty} x^{(i_k)}. \quad (53)$$

On account of the continuity of $I(x)$ in x , there must be a point $\bar{x} \in \mathbb{B}$, $I(\bar{x}) = \bar{m}$. Obviously, $\bar{x} \in \mathbb{B}$ is a critical point of $I(x)$.

For any $x \in B$, $\|x\| \leq \delta_1$, from the condition (G_1) ,

$$\begin{aligned} I(x) &= \frac{1}{2}y^*Ly - \sum_{k=1}^{sM} G(k, x_k) \\ &\geq \frac{1}{2}y^*Ly - \rho \sum_{k=1}^{sM} x_k^2 \\ &\geq 2 \sin^2 \frac{\pi}{sM} \|y\|^2 - \rho \|x\|^2, \end{aligned} \quad (54)$$

where $y = (\Delta x_1, \Delta x_2, \dots, \Delta x_{sM})^*$. It is easy to see that

$$\bar{m} = \sup_{x \in \mathbb{B}} I(x) \geq c > 0. \quad (58)$$

Furthermore, there are positive constants $c > 0$ and $\delta_1 > 0$ such that for any $x \in \partial B_{\delta_1}(0) \cap B$,

$$I(x) \geq c. \quad (59)$$

For any $x \in A$, note that $\sum_{k=1}^{sM} (\Delta^n x_{k-1}, \Delta^n x_{k-1}) = 0$, we have

$$I(x) = \frac{1}{2} \sum_{k=1}^{sM} (\Delta^n x_{k-1}, \Delta^n x_{k-1}) - \sum_{k=1}^{sM} G(k, x_k) = - \sum_{k=1}^{sM} G(k, x_k) \leq 0. \quad (60)$$

Thus, $\bar{x} \notin A$ and the critical point $\bar{x}$ of $I(x)$ corresponding to the critical value $\bar{m}$ is a nontrivial periodic solution of (1) with period sM .

Choose $\mu \in \partial B_1(0) \cap B$. Denote

$$x = t\mu + \nu, \quad \forall \nu \in B, t \in \mathbb{R}. \quad (61)$$

We have

$$\begin{aligned} I(x) &= \frac{1}{2} \sum_{k=1}^{sM} (\Delta^n x_{k-1}, \Delta^n x_{k-1}) - \sum_{k=1}^{sM} G(k, x_k) \\ &\leq \frac{t^2}{2} \sum_{k=1}^{sM} (\Delta^n \mu_{k-1}, \Delta^n \mu_{k-1}) - \sum_{k=1}^{sM} G(k, t\mu_k + \nu_k) \\ &\leq \frac{t^2}{2} z^* Lz - \sum_{k=1}^{sM} [\varrho(t\mu_k + \nu_k)^2 - \tau] \\ &\leq 2t^2 \|z\|^2 - \varrho \sum_{k=1}^{sM} (t\mu_k + \nu_k)^2 + sM\tau \\ &= 2t^2 \|z\|^2 - \varrho t^2 - \varrho \|\nu\|^2 + sM\tau, \end{aligned} \quad (62)$$

where $z = (\Delta\mu_1, \Delta\mu_2, \dots, \Delta\mu_{sM})^*$. Since

$$\|z\|^2 = \sum_{k=1}^{sM} (\Delta^{n-2} \mu_{k+1} - \Delta^{n-2} \mu_k)^2 \leq 4 \sum_{k=1}^{sM} (\Delta^{n-2} \mu_k)^2 \leq \dots \leq 4^{n-1}, \quad (63)$$

then

$$I(x) \leq \left(\frac{4^n}{2} - \varrho \right) t^2 - \varrho \|\nu\|^2 + sM\tau \leq -\varrho \|\nu\|^2 + sM\tau. \quad (64)$$

Accordingly, there exists a positive number $\varsigma > \delta_1$ such that

$$I(x) \leq 0, \quad \forall x \in \partial\Omega, \quad (65)$$

where $\Omega = (\bar{B}_\varsigma(0) \cap \mathbb{B}_1) \oplus \{t\mu | 0 < t < \varsigma\}$. Applying the linking theorem, $I(x)$ has a critical value $c_0 \geq c > 0$, where

$$c_0 = \inf_{d \in Y} \sup_{x \in \Omega} I(d(x)), \quad (66)$$

and $Y = \{d \in C(\bar{\Omega}, \mathbb{B}) | d|_{\partial\Omega} = i \cdot d\}$.

Similar to the Proof of Theorem 1 in [22], we can prove that (1) has at least two sM -periodic nontrivial solutions and so we omit it. By Lemma 5, it suffices to prove that

$$I(x) < \Theta_{m_s}, \quad \forall x \in \tilde{\mathbb{B}}. \quad (67)$$

From (G_3) ,

$$G(k, x) = g(k, 0)x + \frac{1}{2} \times \frac{\partial^2 G(k, \varphi x)}{\partial x^2} x^2 \geq g(k, 0)x + \frac{D}{2} x^2, \quad \forall |x| \leq \kappa. \quad (68)$$

Thus,

$$\begin{aligned} I(x) &= \frac{1}{2} \sum_{k=1}^{sM} (\Delta^2 x_k, \Delta^2 x_k) - \sum_{k=1}^{sM} G(k, x_k) \\ &\leq \frac{1}{2} \sum_{k=1}^{sM} (\Delta^2 x_k, \Delta^2 x_k) - \frac{D}{2} \sum_{k=1}^{sM} x_k^2 - \sum_{k=1}^{sM} g(k, 0)x_k, \quad \forall |x| \leq \kappa. \end{aligned} \quad (69)$$

Choose

$$x_k = \kappa \sin \frac{\omega k}{s}. \quad (70)$$

It comes from $g(-k, 0) = g(k, 0)$ and $g(k+M, 0) = g(k, 0)$ that

$$g(k, 0) = \sum_{i=1}^{[(M-1)/2]} p_i \sin \frac{2i\pi}{M} k = \sum_{i=1}^{[(M-1)/2]} p_i \sin \frac{2i\pi}{sM} sk, \quad (71)$$

where p_i is a constant. In addition,

$$\sum_{k=1}^{sM} g(k, 0)x_k = \sum_{i=1}^{[(M-1)/2]} \kappa a_i \sum_{k=1}^{sM} \sin \frac{2i\pi}{sM} sk \cdot \sin \frac{2\pi}{sM} k = 0. \quad (72)$$

Consequently,

$$I(x) \leq 2 \left[\left(4 \sin^2 \frac{\omega}{2s} \right)^n - D \right] \|x\|^2. \quad (73)$$

It is easy to see that

$$\|x\| = \kappa \left(\frac{s\pi}{\omega} \right)^{1/2}. \quad (74)$$

Thus,

$$I(x) = \frac{2 \left[\left(4 \sin^2 (\omega/2s) \right)^n - D \right] \kappa^2 s\pi}{\omega} < \Theta_{m_s}. \quad (75)$$

The Proof of Theorem 1 is completed.

Proof. of Corollary 1. Since s is a prime integer and $s > 0$, it is easy to see that $m_s = s$. Therefore,

$$\sum_{k=1}^{sM} g^2(k, 0) < \frac{4\pi\kappa^2 \left[\left(4 \sin^2 (\omega/2) \right)^n - C \right] \left[D - \left(4 \sin^2 (\omega/2s) \right)^n \right]}{\omega}. \quad (76)$$

In virtue of Theorem 1, the conclusion of Corollary 1 is obtained. The proof of Corollary 1 is fulfilled.

Proof of Theorem 2. In fact, it is evident that condition (G_6) implies (G_1) and condition (G_7) implies (G_2) . As a result of Theorem 1, Theorem 2 holds. The result of Theorem 2 is achieved.

Remark 1. Similar to the proof of Theorem 1, we can also prove that Theorem 3 is right. For simplicity, we omit its proof. Thanks to Theorem 3, the conclusion of Corollary 2 is obviously accomplished.

Remark 2. A real example is given by using the main results of this paper.

For $n = 2$ and $s = 4$, assume that

$$\Delta^4 x_{k-2} = \frac{1}{6} x_k^2 \sin\left(\frac{2k\pi}{5} x_k\right), \quad k \in \mathbb{Z}. \quad (77)$$

We have

$$\begin{aligned} \omega &= \frac{2\pi}{5}, \\ m_s &= 2, \\ M &= 5, \\ g(t, x) &= \frac{1}{6} x^2 \sin\left(\frac{2t\pi}{5} x\right). \end{aligned} \quad (78)$$

It is easy to verify that all the assumptions of Theorem 1 are satisfied. Consequently, (77) possesses at least two nontrivial periodic solutions with minimal period (20).

Data Availability

This paper is purely theoretical, so there are no supporting data.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally and significantly to writing this manuscript. All authors read and approved the final manuscript.

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