Research Article

The Truncated Theta-EM Method for Nonlinear and Nonautonomous Hybrid Stochastic Differential Delay Equations with Poisson Jumps

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In this paper, we study a class of nonlinear and nonautonomous hybrid stochastic differential delay equations with Poisson jumps (HSDDEwPJs). The convergence rate of the truncated theta-EM numerical solutions to HSDDEwPJs is investigated under given conditions. An example is shown to support our theory.

1. Introduction

Stochastic differential equations have been widely used in many fields and have attracted many scholars [1–3]. Sometimes, an emergency may occur in the system, and it is necessary to consider the influence of the emergency. For example, the surprising outbreak of COVID-19 has a huge impact on the world economy, especially on the stock market. Therefore, stochastic differential equations with jumps considering continuous and discontinuous random effects have been investigated to analyze these situations [4–7]. In practical applications, the parameters and forms in the stochastic systems will change when certain emergencies occur. In this case, we could use stochastic differential equations with Markovian switching to describe them [8]. In this paper, we will take the Markovian switching and jumps into consideration; i.e., we shall study hybrid stochastic differential delay equations with Poisson jumps (HSDDEwPJs).

Numerical methods have been extensively studied, due to the fact that many true solutions of plenty of stochastic differential equations could not be obtained. For example, the explicit Euler–Maruyama (EM) schemes are well known for approximating the true solutions [9]. However, when the coefficients grow superlinearly, Hutzenhaler et al. in [10] proved that, for all $p \in [1, \infty)$, the p th moment of the EM

approximations diverges to infinity. Therefore, many implicit methods have been proposed to approximate the solutions of the stochastic differential equations with nonlinear growing coefficients [11–13]. In addition, considering that the amount of calculations of the explicit schemes is less, some modified EM methods have been used to approximate the nonlinear stochastic differential equations [14–16]. In particular, the truncated EM method was initialized by Mao in [17] with both the drift and diffusion coefficients growing superlinearly. The convergence rate of the truncated EM method was given in [18]. Subsequently, there have been many papers discussing the truncated EM method for stochastic differential equations with superlinear coefficients [19–25]. In addition, there are many papers which consider the stability of the systems [26–30]. The truncated EM scheme for time-changed nonautonomous stochastic differential equations was shown in [31]. In [32], it was extended to the truncated theta-EM scheme on the basis of truncated EM scheme, and the strong convergence rate of the truncated theta-EM scheme for stochastic delay differential equations under local Lipschitz condition was investigated. The truncated theta-EM method will become the EM method when $\theta = 0$ and degenerate to the backward EM method when $\theta = 1$. Additionally, there are a few results on the numerical solutions for HSDDEwPJs. The convergence

of EM approximation solution to the true solution in probability under some weaker conditions was proved in [33]. The EM approximate solutions converge to the true solutions for stochastic differential delay equations with Poisson jumps and Markovian switching under local Lipschitz condition [34]. The convergence of EM method for stochastic differential delay equations with Poisson jumps and Markovian switching in the sense of L^1 -norm under one non-Lipschitz condition was discussed in [35]. The strong convergence between the true solutions and the numerical solutions to stochastic differential delay equations with Poisson jumps and Markovian switching was studied when the drift and diffusion coefficients are Taylor approximations [36]. To the best of our knowledge, there are few papers concerning the numerical solutions of the nonlinear and nonautonomous HSDDEwPJs. Thus, in this paper, we will give the strong convergence rate of the truncated theta-EM method for nonlinear and nonautonomous HSDDEwPJs.

This paper is organized as follows. We will introduce some necessary notations in Section 2. The rate of convergence in $\mathcal{L}^2$ sense will be discussed in Section 3. Finally, in Section 4, we will give an example to illustrate that our main result could cover a large class of nonlinear and nonautonomous HSDDEwPJs.

2. Mathematical Preliminaries

Throughout this paper, unless otherwise specified, we will use the following notations. If A is a vector or matrix, its transpose is denoted by A^T . $\forall x \in \mathbb{R}^n$, let $|x|$ denote its Euclidean norm. If A is a matrix, its trace norm is denoted by $|A| = \sqrt{\text{trace}(A^T A)}$. $A \leq 0$ and $A < 0$ mean that A is nonpositive and negative definite, respectively. If a, b are real numbers, then $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$. Let $[a]$ be the largest integer which does not exceed a . Let $\mathbb{R}_+ = [0, +\infty)$ and $\tau > 0$. Let $C([- \tau, 0]; \mathbb{R}^n)$ be the family of continuous functions ν from $[- \tau, 0]$ to $\mathbb{R}^n$ with the norm $\|\nu\| = \sup_{-\tau \leq \theta \leq 0} |\nu(\theta)|$. If H is a set, let $\mathbb{I}_H$ denote its indicator function which means $\mathbb{I}_H(\omega) = 1$ if $\omega \in H$ and $\mathbb{I}_H(\omega) = 0$ if $\omega \notin H$. Let C be a generic positive real constant which could be different in different cases.

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t \geq 0}$ satisfying the usual conditions (i.e., it is increasing and right continuous while $\mathcal{F}_0$ contains all $\mathbb{P}$ -null sets). In addition, let $\mathbb{E}$ denote the probability expectation with respect to $\mathbb{P}$. For $p > 0$, let $L^p_{\mathcal{F}_0}([- \tau, 0]; \mathbb{R}^n)$ denote the family of all $\mathcal{F}_0$ -measurable and $C([- \tau, 0]; \mathbb{R}^n)$ -value random variables ξ such that $\mathbb{E}\|\xi\|^p < \infty$. Let $B(t) = (B_1(t), \dots, B_m(t))^T$ be an m -dimensional Brownian motion defined on the probability space. Let $N(t)$ denote a scalar Poisson process with the compensated Poisson process $\tilde{N}(t) = N(t) - \lambda t$, where the parameter $\lambda > 0$ is the jump intensity. Moreover, we assume that $B(\cdot)$ and $N(\cdot)$ are independent in this paper.

Let $r(t)$ ($t \geq 0$) be a right-continuous Markov chain on the probability space taking values in a finite state space $\mathbb{S} = \{1, 2, \dots, N\}$ with the generator $\Gamma = (\gamma_{ij})_{N \times N}$ given by

$$\mathbb{P}\{r(t + \Delta) = j | r(t) = i\} = \begin{cases} \gamma_{ij}\Delta + o(\Delta), & \text{if } i \neq j, \\ 1 + \gamma_{ii}\Delta + o(\Delta), & \text{if } i = j, \end{cases} \quad (1)$$

where $\Delta > 0$ and γ_{ij} is the transition rate from i to j with $\gamma_{ij} > 0$ if $i \neq j$, while $\gamma_{ii} = -\sum_{j \neq i} \gamma_{ij}$. We suppose that the Markov chain $r(\cdot)$ is independent of $B(\cdot)$ and $N(\cdot)$. As we know in [37], almost every sample path of $r(t)$ is a right-continuous step function with a finite number of simple jumps in any finite subinterval of $\mathbb{R}_+$. Thus, there exists a sequence of stopping times $0 = \tau_0 < \tau_1 < \tau_2 < \dots < \tau_k \rightarrow \infty$, almost surely such that

$$r(t) = \sum_{k=0}^{\infty} r(\tau_k) \mathbb{I}_{[\tau_k, \tau_{k+1})}(t). \quad (2)$$

Hence, $r(t)$ is constant on each interval $[\tau_k, \tau_{k+1})$,

$$r(t) = r(\tau_k), \quad \forall t \in [\tau_k, \tau_{k+1}), k = 0, 1, 2, \dots \quad (3)$$

In this paper, we consider the nonlinear and nonautonomous hybrid stochastic differential delay equations with Poisson jumps of the form

$$\begin{aligned} dx(t) = & f(t, x(t), x(t - \tau), r(t))dt \\ & + g(t, x(t), x(t - \tau), r(t))dB(t) \\ & + h(t, x(t), x(t - \tau), r(t))dN(t), \quad t \in [0, T], \end{aligned} \quad (4)$$

with the initial data

$$x_0 = \xi \in L^p_{\mathcal{F}_0}([- \tau, 0]; \mathbb{R}^n), \quad r(0) = r_0 \in \mathbb{S}. \quad (5)$$

Here, $f: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^n$, $g: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^{n \times m}$, $h: \mathbb{R}_+ \times \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{S} \rightarrow \mathbb{R}^n$. They are all Borel-measurable functions.

To estimate the truncated theta-EM method for (4), we need the following lemma ([8], Theorem 1.44).

Lemma 1. *Given $\Delta > 0$, let $r_k^\Delta = r(k\Delta)$ for $k \geq 0$. Then, $\{r_k^\Delta, k = 0, 1, 2, \dots\}$ is a discrete Markov chain with the one-step transition probability matrix*

$$\mathbb{P}(\Delta) = (\mathbb{P}_{ij}(\Delta))_{N \times N} = e^{\Delta \Gamma}. \quad (6)$$

Then, we impose the standard hypothesis on the initial data.

Assumption 1. There exist constants $K_1 > 0$ and $\gamma \in (0, 1]$ such that

$$|\xi(t) - \xi(s)| \leq K_1 |t - s|^\gamma, \quad -\tau \leq t, s \leq 0. \quad (7)$$

Since γ_{ij} is independent of x , the paths of r could be generated before approximating x . The discrete Markovian chain $\{r_k^\Delta, k = 0, 1, 2, \dots\}$ could be generated as follows: Compute the one-step transition probability matrix $\mathbb{P}(\Delta)$. Let $r_0^\Delta = i_0$, and generate a random number ξ_1 which is uniformly distributed in $[0, 1]$. Define

$$r_1^\Delta = \begin{cases} i_1, & \text{if } i_1 \in \mathbb{S} - \{N\} \text{ such that } \sum_{j=1}^{i_1-1} \mathbb{P}_{i_0,j}(\Delta) \leq \xi_1 < \sum_{j=1}^{i_1} \mathbb{P}_{i_0,j}(\Delta), \\ N, & \text{if } \sum_{j=1}^{N-1} \mathbb{P}_{i_0,j}(\Delta) \leq \xi_1, \end{cases} \quad (8)$$

where we set $\sum_{j=1}^0 \mathbb{P}_{i_0,j}(\Delta) = 0$ as usual. Then, we generate a new random number ξ_2 independently which is uniformly distributed in $[0, 1]$ as well. Define

$$r_2^\Delta = \begin{cases} i_2, & \text{if } i_2 \in \mathbb{S} - \{N\} \text{ such that } \sum_{j=1}^{i_2-1} \mathbb{P}_{r_1^\Delta,j}(\Delta) \leq \xi_2 < \sum_{j=1}^{i_2} \mathbb{P}_{r_1^\Delta,j}(\Delta), \\ N, & \text{if } \sum_{j=1}^{N-1} \mathbb{P}_{r_1^\Delta,j}(\Delta) \leq \xi_2. \end{cases} \quad (9)$$

Repeating this procedure, we could obtain a trajectory of $\{r_k^\Delta, k = 1, 2, \dots\}$. The procedure could be applied independently to get more trajectories. After generating the discrete Markov chain $\{r_k^\Delta, k = 0, 1, 2, \dots\}$, we can now define the truncated theta-EM approximate solution for HSDDEwPJs (4) with the initial data (5).

In order to define the truncated theta-EM scheme, we first choose a strictly decreasing function $\varphi: (0, 1] \rightarrow (0, \infty)$ such that

$$\lim_{\Delta \rightarrow 0} \varphi(\Delta) = \infty, \quad K_{\varphi(\Delta)} \Delta^{(1/4)} \leq 1, \forall \Delta \in (0, 1], \quad (10)$$

where $K_{\varphi(\Delta)}$ is a function that depends on $\varphi(\Delta)$. For example, we could choose

$$\begin{aligned} \varphi(\Delta) &= \Delta^{-\varepsilon}, \\ K_{\varphi(\Delta)} &= \varphi(\Delta), \end{aligned} \quad (11)$$

for some $\varepsilon \in (0, (1/8)]$.

For a given step size $\Delta \in (0, 1]$, we give the definition of the truncated mapping

$$\pi_\Delta(x) = (|x| \wedge \varphi(\Delta)) \frac{x}{|x|}, \quad (12)$$

where we let $(x/|x|) = 0$ when $x = 0$. The truncated functions are defined as

$$\begin{aligned} f_\Delta(t, x, y, i) &= f(t, \pi_\Delta(x), \pi_\Delta(y), i) = f\left(t, (|x| \wedge \varphi(\Delta)) \frac{x}{|x|}, (|y| \wedge \varphi(\Delta)) \frac{y}{|y|}, i\right), \\ g_\Delta(t, x, y, i) &= g(t, \pi_\Delta(x), \pi_\Delta(y), i) = g\left(t, (|x| \wedge \varphi(\Delta)) \frac{x}{|x|}, (|y| \wedge \varphi(\Delta)) \frac{y}{|y|}, i\right). \end{aligned} \quad (13)$$

Now we give the definition of the discrete-time truncated theta-EM scheme to approximate the true solution of (4). Assume that there exist two positive integers M and M' such that $\Delta = \tau/M = T/M'$. Hence, Δ will become sufficiently small when we choose M sufficiently large. Define $t_k = k\Delta$ for $k = -M, -M+1, \dots, 0, 1, \dots, M'-1$. Set $X_\Delta(t_k) = \xi(t_k)$ for $k = -M, -M+1, \dots, 0$ and then form

$$\begin{aligned} X_\Delta(t_{k+1}) &= X_\Delta(t_k) + \theta f_\Delta(t_{k+1}, X_\Delta(t_{k+1}), X_\Delta(t_{k+1-M}), r_{k+1}^\Delta) \Delta \\ &\quad + (1-\theta) f_\Delta(t_k, X_\Delta(t_k), X_\Delta(t_{k-M}), r_k^\Delta) \Delta \\ &\quad + g_\Delta(t_k, X_\Delta(t_k), X_\Delta(t_{k-M}), r_k^\Delta) \Delta B_k \\ &\quad + h(t_k, X_\Delta(t_k), X_\Delta(t_{k-M}), r_k^\Delta) \Delta N_k, \end{aligned} \quad (14)$$

for $k = 0, 1, 2, \dots, M'-1$, where $\Delta B_k = B(t_{k+1}) - B(t_k)$, $\Delta N_k = N(t_{k+1}) - N(t_k)$. To form the continuous-time scheme, define

$$\begin{aligned} \mu(t) &= \sum_{k=0}^{M'-1} t_k \mathbb{I}_{[t_k, t_{k+1})}(t), \\ \bar{r}(t) &= \sum_{k=0}^{M'-1} r_k^\Delta \mathbb{I}_{[t_k, t_{k+1})}(t). \end{aligned} \quad (15)$$

It is well known that there exist two kinds of the continuous-time step approximations. The first one is that

$$\bar{x}_\Delta(t) = \sum_{k=0}^{M'-1} X_\Delta(t_k) \mathbb{I}_{[t_k, t_{k+1})}(t). \quad (16)$$

The other one is that

$$\begin{aligned}
x_\Delta(t) &= \xi(0) - \theta f_\Delta(0, \xi(0), \xi(-\tau), r_0)\Delta \\
&\quad + \theta f_\Delta(t, x_\Delta(t), x_\Delta(t-\tau), r(t))\Delta \\
&\quad + \int_0^t f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))ds \\
&\quad + \int_0^t g_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))dB(s) \\
&\quad + \int_0^t h(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))dN(s).
\end{aligned} \tag{17}$$

Then, we could observe that $X_\Delta(t_k) = \bar{x}_\Delta(t_k) = x_\Delta(t_k)$. Namely, they coincide at t_k . For simplicity, we write

$$\begin{aligned}
y_\Delta(t) &= x_\Delta(t) - \theta f_\Delta(t, x_\Delta(t), x_\Delta(t-\tau), r(t))\Delta, \\
\bar{y}_\Delta(t) &= \bar{x}_\Delta(t) - \theta f_\Delta(\mu(t), \bar{x}_\Delta(t), \bar{x}_\Delta(t-\tau), \bar{r}(t))\Delta.
\end{aligned} \tag{18}$$

3. Convergence Rate

To obtain the rate of convergence for the truncated theta-EM method for (4) in $\mathcal{L}^2$ sense, we need to impose the following assumptions on the coefficients.

Assumption 2. For any $R \geq 1$, there exists a constant $L_R > 0$ such that

$$\begin{aligned}
&|f(t, x, y, i) - f(t, \bar{x}, \bar{y}, i)| \vee |g(t, x, y, i) - g(t, \bar{x}, \bar{y}, i)| \\
&\leq L_R(|x - \bar{x}| + |y - \bar{y}|),
\end{aligned} \tag{19}$$

for all $t \in [0, T]$, $i \in \mathbb{S}$, and any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$ with $|x| \vee |y| \vee |\bar{x}| \vee |\bar{y}| \leq R$.

Assumption 3. There exists a constant $K_2 > 0$ such that

$$|h(t, x, y, i) - h(t, \bar{x}, \bar{y}, i)| \leq K_2(|x - \bar{x}| + |y - \bar{y}|), \tag{20}$$

for all $t \in [0, T]$, any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, and $i \in \mathbb{S}$.

From Assumption 3, we could derive that there exists a constant $\bar{K}_2 > 0$ such that

$$|h(t, x, y, i)| \leq \bar{K}_2(1 + |x| + |y|), \tag{21}$$

where $\bar{K}_2 = 3K_2 + \sup_{t \in [0, T], i \in \mathbb{S}} |h(t, 0, 0, i)|$.

Before presenting the next assumption, we need more notations. Let $\mathcal{U}$ be the family of continuous functions $U: \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}_+$ such that, for each $b > 0$, there exists a positive constant κ_b satisfying

$$U(x, y) \leq \kappa_b |x - y|^2, \tag{22}$$

for any $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq b$.

Assumption 4. There exist constants $K_3 > 0, q > 2$, and $U \in \mathcal{U}$, such that

$$\begin{aligned}
&(x - \bar{x})^T (f(t, x, y, i) - f(t, \bar{x}, \bar{y}, i)) + \frac{q-1}{2} |g(t, x, y, i) - g(t, \bar{x}, \bar{y}, i)|^2 \\
&\leq K_3(|x - \bar{x}|^2 + |y - \bar{y}|^2) - U(x, \bar{x}) + U(y, \bar{y}),
\end{aligned} \tag{23}$$

for all $t \in [0, T]$, any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$, and $i \in \mathbb{S}$.

Assumption 5. There exist constants $K_4 > 0, p > q > 2$ such that

$$x^T f(t, x, y, i) + \frac{p-1}{2} |g(t, x, y, i)|^2 \leq K_4(1 + |x|^2 + |y|^2), \tag{24}$$

for all $t \in [0, T]$, any $x, y \in \mathbb{R}^n$, and $i \in \mathbb{S}$.

Assumption 6. There exist constants $\bar{L}_R > 0, \vartheta \in (0, 1]$ and $\sigma \in (0, 1]$ such that

$$\begin{aligned}
&|f(t_1, x, y, i) - f(t_2, x, y, i)| \leq \bar{L}_R |t_1 - t_2|^\vartheta, \\
&|g(t_1, x, y, i) - g(t_2, x, y, i)| \leq \bar{L}_R |t_1 - t_2|^\sigma,
\end{aligned} \tag{25}$$

for all $t_1, t_2 \in [0, T]$, $i \in \mathbb{S}$, and any $x, y \in \mathbb{R}^n$ with $|x| \vee |y| \leq R$.

Assumption 7. There exist constants $K_5 > 0, \eta \in (0, 1]$ such that

$$|h(t_1, x, y, i) - h(t_2, x, y, i)| \leq K_5(1 + |x| + |y|)|t_1 - t_2|^\eta, \tag{26}$$

for all $t_1, t_2 \in [0, T]$, any $x, y \in \mathbb{R}^n$, and $i \in \mathbb{S}$.

The boundedness of the p -moment of the true solution is shown in the following lemma which could be proved by using the standard method.

Lemma 2. Let Assumptions 2, 3, and 5 hold. Then, for any given initial data (5), there exists a unique solution $x(t)$ to (4) on $t \geq -\tau$. Moreover,

$$\sup_{-\tau \leq t \leq T} \mathbb{E}|x(t)|^q < \infty, \quad \forall T > 0. \tag{27}$$

Furthermore, we could obtain the next two lemmas in an analogous way to the proof of Lemmas 2.2 and 2.3 in [32].

Lemma 3. Let Assumption 2 holds. For any $\Delta \in (0, 1]$ with $K_{\varphi(\Delta)} \geq 1$, we have

$$\begin{aligned} |f_{\Delta}(t, x, y, i) - f_{\Delta}(t, \bar{x}, \bar{y}, i)| \vee |g_{\Delta}(t, x, y, i) - g_{\Delta}(t, \bar{x}, \bar{y}, i)| &\leq 2K_{\varphi(\Delta)}(|x - \bar{x}| + |y - \bar{y}|), \\ (x - \bar{x})^T (f_{\Delta}(t, x, y, i) - f_{\Delta}(t, \bar{x}, \bar{y}, i)) &\leq 5K_{\varphi(\Delta)}^2(|x - \bar{x}|^2 + |y - \bar{y}|^2), \end{aligned} \quad (28)$$

for all $t \in [0, T]$, $i \in \mathbb{S}$, and any $x, y, \bar{x}, \bar{y} \in \mathbb{R}^n$.

From Lemma 3, we derive that

$$|f_{\Delta}(t, x, y, i)| \vee |g_{\Delta}(t, x, y, i)| \leq 2K_{\varphi(\Delta)}(|x| + |y|) + \sup_{t \in [0, T], i \in \mathbb{S}} [|f(t, 0, 0, i)| + |g(t, 0, 0, i)|]. \quad (29)$$

In addition, by the monotone operator theory in [38], $5K_{\varphi(\Delta)}^2 \theta \Delta < 1$ needs to be satisfied to ensure the existence and uniqueness of $X_{\Delta}(t_{k+1})$ for given $X_{\Delta}(t_k)$. We get from (10) that $K_{\varphi(\Delta)}^2 \Delta^{(1/2)} \leq 1$; thus, we need $\theta^2 \Delta < (1/25)$. Moreover, to get the main result, $\theta \Delta^{(3/4)} < (1/8)$ should be satisfied in the proof of Lemma 6. Denote $\Delta^* = (1/25\theta^2) \wedge (1/16\theta^{4/3})$. Let $\Delta \in (0, \Delta^*)$ and $\theta \in (0, 1]$ in the rest of this paper.

Lemma 4. Let Assumption 5 hold. Then, for any $\Delta \in (0, \Delta^*)$ and $x, y \in \mathbb{R}^n$, we derive that

$$x^T f_{\Delta}(t, x, y, i) + \frac{p-1}{2} |g_{\Delta}(t, x, y, i)|^2 \leq \bar{K}_4 (1 + |x|^2 + |y|^2), \quad (30)$$

where $\bar{K}_4 = 3K_4([1/\varphi(1)] \vee 1)$.

Before the next lemma, define

$$\kappa(t) = \lfloor \frac{t}{\Delta} \rfloor \Delta, \quad \forall -\tau \leq t \leq T. \quad (31)$$

Lemma 5. Let Assumptions 2 and 3 hold. For any $\Delta \in (0, \Delta^*)$ and $t \in [0, T]$, we obtain that

$$\begin{aligned} \mathbb{E}(|y_{\Delta}(t) - \bar{y}_{\Delta}(t)|^{\bar{p}} | \mathcal{F}_{\kappa(t)}) &\leq C \left(\Delta^{(\bar{p}/2)} K_{\varphi(\Delta)}^{\bar{p}} + \Delta \right) \left(1 + |\bar{x}_{\Delta}(s)|^{\bar{p}} + |\bar{x}_{\Delta}(s - \tau)|^{\bar{p}} \right), \quad \bar{p} \geq 2, \\ \mathbb{E}(|y_{\Delta}(t) - \bar{y}_{\Delta}(t)|^{\bar{p}} | \mathcal{F}_{\kappa(t)}) &\leq C \Delta^{(\bar{p}/2)} K_{\varphi(\Delta)}^{\bar{p}} \left(1 + |\bar{x}_{\Delta}(s)|^{\bar{p}} + |\bar{x}_{\Delta}(s - \tau)|^{\bar{p}} \right), \quad 0 < \bar{p} < 2. \end{aligned} \quad (32)$$

Proof. Fix any $\bar{p} \geq 2$. By (17), for any $\Delta \in (0, \Delta^*)$ and $t \in [0, T]$, we get that

$$\begin{aligned} y_{\Delta}(t) &= y_{\Delta}(0) + \int_0^t f_{\Delta}(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s - \tau), \bar{r}(s)) ds \\ &\quad + \int_0^t g_{\Delta}(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s - \tau), \bar{r}(s)) dB(s) \\ &\quad + \int_0^t h(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s - \tau), \bar{r}(s)) dN(s), \end{aligned} \quad (33)$$

where $y_{\Delta}(0) = \xi(0) - \theta f_{\Delta}(0, \xi(0), \xi(-\tau), r_0) \Delta$.

For any $t \in [0, T]$, there always exists an integer $k \geq 0$ such that $t_k \leq t < t_{k+1}$. By Hölder's inequality and BDG's inequality, we obtain that

$$\begin{aligned}
& \mathbb{E} \left(|y_{\Delta}(t) - \bar{y}_{\Delta}(t)|^{\bar{p}} \middle| \mathcal{F}_{\kappa(t)} \right) \\
& \leq C \mathbb{E} \left(\left| \int_{\kappa(t)}^t f_{\Delta}(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s)) ds \right|^{\bar{p}} \middle| \mathcal{F}_{\kappa(t)} \right) \\
& \quad + C \mathbb{E} \left(\left| \int_{\kappa(t)}^t g_{\Delta}(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s)) dB(s) \right|^{\bar{p}} \middle| \mathcal{F}_{\kappa(t)} \right) \\
& \quad + C \mathbb{E} \left(\left| \int_{\kappa(t)}^t h(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s)) dN(s) \right|^{\bar{p}} \middle| \mathcal{F}_{\kappa(t)} \right) \\
& \leq C \Delta^{\bar{p}-1} \mathbb{E} \left(\int_{\kappa(t)}^t |f_{\Delta}(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))|^{\bar{p}} ds \middle| \mathcal{F}_{\kappa(t)} \right) \\
& \quad + C \Delta^{(\bar{p}/2)-1} \mathbb{E} \left(\int_{\kappa(t)}^t |g_{\Delta}(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))|^{\bar{p}} ds \middle| \mathcal{F}_{\kappa(t)} \right) \\
& \quad + C \mathbb{E} \left(\left| \int_{\kappa(t)}^t h(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s)) dN(s) \right|^{\bar{p}} \middle| \mathcal{F}_{\kappa(t)} \right) \\
& \leq C \Delta^{(\bar{p}/2)} K_{\varphi(\Delta)}^{\bar{p}} \left(1 + |\bar{x}_{\Delta}(s)|^{\bar{p}} + |\bar{x}_{\Delta}(s-\tau)|^{\bar{p}} \right) \\
& \quad + C \mathbb{E} \left(\left| \int_{\kappa(t)}^t h(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s)) dN(s) \right|^{\bar{p}} \middle| \mathcal{F}_{\kappa(t)} \right).
\end{aligned} \tag{34}$$

By the characteristic function's argument in [39], for any $\Delta \in (0, \Delta^*)$, we have

$$\mathbb{E} |N(t) - N(\kappa(t))|^{\bar{p}} \leq c_0 \Delta, \tag{35}$$

where c_0 is a constant independent of Δ . Then, we get from Assumption 3 that

$$\begin{aligned}
& \mathbb{E} \left(\left| \int_{\kappa(t)}^t h(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s)) dN(s) \right|^{\bar{p}} \middle| \mathcal{F}_{\kappa(t)} \right) \\
& \leq \mathbb{E} \left(\left| h(\kappa(t), x_{\Delta}(\kappa(t)), x_{\Delta}(\kappa(t)-\tau), r_{[t/\Delta]}^{\Delta}) (N(t) - N(\kappa(t))) \right|^{\bar{p}} \middle| \mathcal{F}_{\kappa(t)} \right) \\
& \leq \left| h(\kappa(t), x_{\Delta}(\kappa(t)), x_{\Delta}(\kappa(t)-\tau), r_{[t/\Delta]}^{\Delta}) \right|^{\bar{p}} \mathbb{E} |N(t) - N(\kappa(t))|^{\bar{p}} \\
& \leq C \Delta \left(1 + |\bar{x}_{\Delta}(s)|^{\bar{p}} + |\bar{x}_{\Delta}(s-\tau)|^{\bar{p}} \right).
\end{aligned} \tag{36}$$

Thus,

$$\begin{aligned}
& \mathbb{E} \left(|y_{\Delta}(t) - \bar{y}_{\Delta}(t)|^{\bar{p}} \middle| \mathcal{F}_{\kappa(t)} \right) \\
& \leq C \left(\Delta^{(\bar{p}/2)} K_{\varphi(\Delta)}^{\bar{p}} + \Delta \right) \left(1 + |\bar{x}_{\Delta}(s)|^{\bar{p}} + |\bar{x}_{\Delta}(s-\tau)|^{\bar{p}} \right).
\end{aligned} \tag{37}$$

For $0 < \bar{p} < 2$, the use of Jensen's inequality yields that

$$\begin{aligned}
& \mathbb{E} \left(|y_{\Delta}(t) - \bar{y}_{\Delta}(t)|^{\bar{p}} \middle| \mathcal{F}_{\kappa(t)} \right) \\
& \leq \left(\mathbb{E} |y_{\Delta}(t) - \bar{y}_{\Delta}(t)|^2 \middle| \mathcal{F}_{\kappa(t)} \right)^{(\bar{p}/2)} \\
& \leq C \Delta^{(\bar{p}/2)} K_{\varphi(\Delta)}^{\bar{p}} \left(1 + |\bar{x}_{\Delta}(s)|^{\bar{p}} + |\bar{x}_{\Delta}(s-\tau)|^{\bar{p}} \right).
\end{aligned} \tag{38}$$

The proof is completed. $\square$

Lemma 6. Let Assumptions 2, 3, and 5 hold; then, we have

$$\sup_{0 < \Delta < \Delta^*} \sup_{0 \leq t \leq T} \mathbb{E}|x_\Delta(t)|^p \leq C, \quad \forall T > 0. \quad (39)$$

where $y_\Delta(0) = \xi(0) - \theta f_\Delta(0, \xi(0), \xi(-\tau), r_0)\Delta$.
Using Itô's formula, we derive that

Proof. By (17), we get that

$$\begin{aligned} y_\Delta(t) &= y_\Delta(0) + \int_0^t f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))ds \\ &\quad + \int_0^t g_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))dB(s) \\ &\quad + \int_0^t h(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))dN(s), \end{aligned} \quad (40)$$

$$\begin{aligned} \mathbb{E}|y_\Delta(t)|^p &\leq \mathbb{E}|y_\Delta(0)|^p + \mathbb{E} \int_0^t p|y_\Delta(s)|^{p-2} [y_\Delta^T(s) f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)) \\ &\quad + \frac{p-1}{2} |g_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2] ds \\ &\quad + \lambda \mathbb{E} \int_0^t (|x_\Delta(s) - h(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^p - |x_\Delta(s)|^p) ds \\ &\leq \mathbb{E}|y_\Delta(0)|^p + \mathbb{E} \int_0^t p|y_\Delta(s)|^{p-2} [\bar{x}_\Delta^T(s) f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)) + \frac{p-1}{2} |g_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2] ds \\ &\quad + \mathbb{E} \int_0^t p|y_\Delta(s)|^{p-2} (y_\Delta(s) - \bar{x}_\Delta(s))^T f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)) ds \\ &\quad + \lambda \mathbb{E} \int_0^t (|x_\Delta(s) - h(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^p - |x_\Delta(s)|^p) ds =: \mathbb{E}|y_\Delta(0)|^p + A_1 + A_2 + A_3. \end{aligned} \quad (41)$$

With the help of (10), Lemma 4, and Young's inequality, we could obtain

$$\begin{aligned} A_1 &\leq p\bar{K}_4 \mathbb{E} \int_0^t |y_\Delta(s)|^{p-2} (1 + |\bar{x}_\Delta(s)|^2 + |\bar{x}_\Delta(s-\tau)|^2) ds \\ &\leq p\bar{K}_4 \mathbb{E} \int_0^t |x_\Delta(s) - \theta f_\Delta(s, x_\Delta(s), x_\Delta(s-\tau), r(s))\Delta|^{p-2} \\ &\quad \cdot (1 + |\bar{x}_\Delta(s)|^2 + |\bar{x}_\Delta(s-\tau)|^2) ds \\ &\leq C \mathbb{E} \int_0^t \left(1 + |x_\Delta(s)|^p + K_{\varphi(\Delta)}^p \Delta^p (|x_\Delta(s)|^p + |x_\Delta(s-\tau)|^p) \right) \\ &\quad + |\bar{x}_\Delta(s)|^p + |\bar{x}_\Delta(s-\tau)|^p ds \\ &\leq C \left(1 + \int_0^t \sup_{0 \leq l \leq s} \mathbb{E}|x_\Delta(l)|^p ds \right). \end{aligned} \quad (42)$$

Moreover,

$$\begin{aligned}
 A_2 &\leq C \mathbb{E} \int_0^t |\bar{y}_\Delta(s)|^{p-2} |y_\Delta(s) - \bar{x}_\Delta(s)| \|f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))\| ds \\
 &\quad + C \mathbb{E} \int_0^t |y_\Delta(s) - \bar{y}_\Delta(s)|^{p-2} |y_\Delta(s) - \bar{x}_\Delta(s)| \|f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))\| ds \\
 &=: A_{21} + A_{22}.
 \end{aligned} \tag{43}$$

By (10), Young's inequality, and Lemmas 3 and 5, we could show that

$$\begin{aligned}
 A_{21} &\leq C \int_0^t \mathbb{E} \left[|\bar{y}_\Delta(s)|^{p-2} |f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))| \right. \\
 &\quad \cdot \mathbb{E} \left(|y_\Delta(s) - \bar{y}_\Delta(s) - \theta f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)) \Delta| | \mathcal{F}_{\kappa(s)} \right) \Big] ds \\
 &\leq C \int_0^t \mathbb{E} \left[|\bar{y}_\Delta(s)|^{p-2} |f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))| \right. \\
 &\quad \cdot \mathbb{E} \left(|y_\Delta(s) - \bar{y}_\Delta(s)| | \mathcal{F}_{\kappa(s)} \right) \Big] ds \\
 &\quad + C \int_0^t \mathbb{E} \left[|\bar{y}_\Delta(s)|^{p-2} |f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2 \Delta \right] ds \\
 &\leq C \int_0^t \Delta^{(1/2)} K_{\varphi(\Delta)}^2 \mathbb{E} \left[|x_\Delta(s) - \theta f_\Delta(s, x_\Delta(s), x_\Delta(s-\tau), r(s)) \Delta|^{p-2} \right. \\
 &\quad \cdot (1 + |\bar{x}_\Delta(s)|^2 + |\bar{x}_\Delta(s-\tau)|^2) \Big] ds \\
 &\quad + C \int_0^t \Delta \mathbb{E} |x_\Delta(s) - \theta f_\Delta(s, x_\Delta(s), x_\Delta(s-\tau), r(s)) \Delta|^{p-2} ds \\
 &\quad + C \int_0^t \Delta K_{\varphi(\Delta)}^2 \mathbb{E} \left[|x_\Delta(s) - \theta f_\Delta(s, x_\Delta(s), x_\Delta(s-\tau), r(s)) \Delta|^{p-2} \right. \\
 &\quad \cdot (1 + |\bar{x}_\Delta(s)|^2 + |\bar{x}_\Delta(s-\tau)|^2) \Big] ds \\
 &\leq C \left(1 + \int_0^t \sup_{0 \leq l \leq s} \mathbb{E} |x_\Delta(l)|^p ds \right).
 \end{aligned} \tag{44}$$

In the same way, with (10) and Lemmas 3 and 5, we have

$$A_{22} \leq C \left(1 + \int_0^t \sup_{0 \leq l \leq s} \mathbb{E} |x_\Delta(l)|^p ds \right). \tag{45}$$

By inserting (44) and (45) into (43), it follows that

$$A_2 \leq C \left(1 + \int_0^t \sup_{0 \leq l \leq s} \mathbb{E} |x_\Delta(l)|^p ds \right). \tag{46}$$

From Assumption 3, one can see that

$$\begin{aligned}
 A_3 &\leq C \mathbb{E} \int_0^t |1 + |x_\Delta(s)|^p + |\bar{x}_\Delta(s)|^p + |\bar{x}_\Delta(s-\tau)|^p| ds \\
 &\leq C \left(1 + \int_0^t \sup_{0 \leq l \leq s} \mathbb{E} |x_\Delta(l)|^p ds \right).
 \end{aligned} \tag{47}$$

Combining (41), (42), (46), and (47) together, we obtain

$$\sup_{0 \leq l \leq t} \mathbb{E} |y_\Delta(l)|^p \leq C \left(1 + \int_0^t \sup_{0 \leq l \leq s} \mathbb{E} |x_\Delta(l)|^p ds \right). \tag{48}$$

By the inequality $|u - v|^p \geq 2^{1-p} |u|^p - |v|^p$ and (29), we get

$$\begin{aligned}
|y_\Delta(t)|^p &\geq 2^{1-p}|x_\Delta(t)|^p - \theta^p \Delta^p |f_\Delta(t, x_\Delta(t), x_\Delta(t-\tau), r(t))|^p \\
&\geq 2^{1-p}|x_\Delta(t)|^p - 2^{2p-2} \theta^p \Delta^p K_{\varphi(\Delta)}^p (|x_\Delta(t)|^p + |x_\Delta(t-\tau)|^p) \\
&\quad - 2^{p-1} \theta^p \Delta^p \sup_{t \in [0, T], i \in \mathbb{S}} [|f(t, 0, 0, i)| + |g(t, 0, 0, i)|] \\
&= \left(2^{1-p} - 2^{2p-2} \theta^p \Delta^p K_{\varphi(\Delta)}^p\right) |x_\Delta(t)|^p - 2^{2p-2} \theta^p \Delta^p K_{\varphi(\Delta)}^p |x_\Delta(t-\tau)|^p \\
&\quad - 2^{p-1} \theta^p \Delta^p \sup_{t \in [0, T], i \in \mathbb{S}} [|f(t, 0, 0, i)| + |g(t, 0, 0, i)|].
\end{aligned} \tag{49}$$

Note that $\Delta < (1/16\theta^{4/3})$; hence,

$$\begin{aligned}
\sup_{0 \leq l \leq t} \mathbb{E}|x_\Delta(l)|^p &\leq C \left(1 + \int_0^t \sup_{0 \leq l \leq s} \mathbb{E}|y_\Delta(l)|^p ds\right) \\
&\leq C \left(1 + \int_0^t \sup_{0 \leq l \leq s} \mathbb{E}|x_\Delta(l)|^p ds\right).
\end{aligned} \tag{50}$$

Applying Gronwall's inequality gives the desired result. $\square$

Lemma 7. Let Assumptions 2, 3, and 5 hold. For any $\Delta \in (0, \Delta^*)$ and $t \in [0, T]$, we derive that

$$\mathbb{E}|y_\Delta(t) - \bar{y}_\Delta(t)|^p \leq C \left(\Delta^{(p/2)} K_{\varphi(\Delta)}^p + \Delta \right), \quad p \geq 2, \tag{51}$$

$$\mathbb{E}|y_\Delta(t) - \bar{y}_\Delta(t)|^p \leq C \Delta^{(p/2)} K_{\varphi(\Delta)}^p, \quad 0 < p < 2, \tag{52}$$

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p \leq C \left(\Delta^{(p/2)} K_{\varphi(\Delta)}^p + \Delta \right), \quad p \geq 2, \tag{53}$$

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p \leq C \Delta^{(p/2)} K_{\varphi(\Delta)}^p, \quad 0 < p < 2. \tag{54}$$

Hence,

$$\begin{aligned}
\lim_{\Delta \rightarrow 0} \mathbb{E}|y_\Delta(t) - \bar{y}_\Delta(t)|^p \\
= \lim_{\Delta \rightarrow 0} \mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p = 0, \quad \forall p > 0.
\end{aligned} \tag{55}$$

Proof. By Lemmas 5 and 6, we could obtain (51) and (52). Then, the use of a similar technique in Lemma 6 gives the following when $p \geq 2$:

$$\mathbb{E}|x_\Delta(t) - \bar{x}_\Delta(t)|^p \leq C \mathbb{E}|y_\Delta(t) - \bar{y}_\Delta(t)|^p \leq C \left(\Delta^{(p/2)} K_{\varphi(\Delta)}^p + \Delta \right). \tag{56}$$

We could get (54) by applying Jensen's inequality. $\square$

By Lemmas 2 and 6 and Chebyshev's inequality, we could get the following lemma right away.

Lemma 8. Let Assumptions 2, 3, and 5 hold. For any number $R \geq \|\xi\|$, define the stopping time

$$\begin{aligned}
\tau_R &= \inf\{t \geq 0: |x(t)| \geq R\}, \\
\tau_{\Delta, R} &= \inf\{t \geq 0: |x_\Delta(t)| \geq R\}.
\end{aligned} \tag{57}$$

Then, we obtain that

$$\begin{aligned}
\mathbb{P}(\tau_R \leq T) &\leq \frac{C}{R^2}, \\
\mathbb{P}(\tau_{\Delta, R} \leq T) &\leq \frac{C}{R^2}.
\end{aligned} \tag{58}$$

Lemma 9. Let Assumptions 1–7 hold. Let $\Delta \in (0, \Delta^*)$ be sufficiently small such that $\varphi(\Delta) \geq R \vee \|\xi\|$. Then, we get

$$\mathbb{E} \left| x(T \wedge \rho_{\Delta, R}) - x_\Delta(T \wedge \rho_{\Delta, R}) \right|^2 \leq C \left((K_{\varphi(\Delta)}^4 \Delta) \vee \Delta^{2(\gamma \wedge \vartheta \wedge \sigma \wedge \eta)} \right), \tag{59}$$

where $\rho_{\Delta, R} := \tau_R \wedge \tau_{\Delta, R}$.

Proof. Let $e_\Delta(t) = x(t) - y_\Delta(t)$ for $t \in [0, T]$ and $\Delta \in (0, \Delta^*)$. For simplicity, we rewrite $\rho_{\Delta, R} = \rho$. Recalling the definition of f_Δ and g_Δ , we have

$$\begin{aligned}
f_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)) &= f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)), \\
g_\Delta(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)) &= g(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s)),
\end{aligned} \tag{60}$$

for any $0 \leq s \leq t \wedge \rho$. By Itô's formula, we could show that

$$\begin{aligned}
& \mathbb{E}|e_{\Delta}(t \wedge \rho)|^2 \\
& \leq \mathbb{E}(\theta^2 \Delta^2 |f_{\Delta}(0, \xi(0), \xi(-\tau), r_0)|^2) \\
& \quad + \mathbb{E} \int_0^{t \wedge \rho} [2e_{\Delta}^T(s) (f(s, x(s), x(s-\tau), r(s)) - f_{\Delta}(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))) \\
& \quad + |g(s, x(s), x(s-\tau), r(s)) - g_{\Delta}(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))|^2] ds \\
& \quad + \lambda \mathbb{E} \int_0^{t \wedge \rho} (|e_{\Delta}(s) + h(s, x(s), x(s-\tau), r(s)) \\
& \quad - h(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))|^2 - |e_{\Delta}(s)|^2) ds \\
& \leq CK_{\varphi(\Delta)}^2 \Delta^2 + \mathbb{E} \int_0^{t \wedge \rho} [2(x(s) - \bar{x}_{\Delta}(s))^T \\
& \quad \cdot (f(s, x(s), x(s-\tau), r(s)) - f(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))) \\
& \quad + |g(s, x(s), x(s-\tau), r(s)) - g(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))|^2] ds \\
& \quad + \mathbb{E} \int_0^{t \wedge \rho} 2(\bar{x}_{\Delta}(s) - x_{\Delta}(s))^T \\
& \quad \cdot (f(s, x(s), x(s-\tau), r(s)) - f(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))) ds \\
& \quad + \mathbb{E} \int_0^{t \wedge \rho} 2\theta \Delta f_{\Delta}^T(s, x_{\Delta}(s), x_{\Delta}(s-\tau), r(s)) \\
& \quad \cdot (f(s, x(s), x(s-\tau), r(s)) - f(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))) ds \\
& \quad + \lambda \mathbb{E} \int_0^{t \wedge \rho} (|e_{\Delta}(s) + h(s, x(s), x(s-\tau), r(s)) \\
& \quad - h(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))|^2 - |e_{\Delta}(s)|^2) ds \\
& = CK_{\varphi(\Delta)}^2 \Delta^2 + B_1 + B_2 + B_3 + B_4.
\end{aligned} \tag{61}$$

Let $\bar{q} \in (2, q)$, so

$$\begin{aligned}
& \frac{1}{2} |g(s, x(s), x(s-\tau), r(s)) - g(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))|^2 \\
& \leq \frac{\bar{q}-1}{2} |g(s, x(s), x(s-\tau), r(s)) - g(s, \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), r(s))|^2 \\
& \quad + \frac{\bar{q}-1}{2(\bar{q}-2)} |g(s, \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), r(s)) - g(\mu(s), \bar{x}_{\Delta}(s), \bar{x}_{\Delta}(s-\tau), \bar{r}(s))|^2.
\end{aligned} \tag{62}$$

Then, we have

$$\begin{aligned}
B_1 &\leq 2\mathbb{E} \int_0^{t\wedge\rho} \left[(x(s) - \bar{x}_\Delta(s))^T \right. \\
&\quad \cdot (f(s, x(s), x(s-\tau), r(s)) - f(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s))) \\
&\quad \left. + \frac{\bar{q}-1}{2} |g(s, x(s), x(s-\tau), r(s)) - g(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s))|^2 \right] ds \\
&\quad + 2\mathbb{E} \int_0^{t\wedge\rho} (x(s) - \bar{x}_\Delta(s))^T \cdot (f(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)) - f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))) ds \\
&\quad + \mathbb{E} \int_0^{t\wedge\rho} \frac{\bar{q}-1}{\bar{q}-2} |g(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)) - g(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2 ds \\
&=: B_{11} + B_{12} + B_{13}.
\end{aligned} \tag{63}$$

By Assumptions 1 and 4 and Lemma 7, it follows that

We derive from Assumption 6 that

$$\begin{aligned}
B_{11} &\leq 2\mathbb{E} \int_0^{t\wedge\rho} K_3 (|x(s) - \bar{x}_\Delta(s)|^2 + |x(s-\tau) - \bar{x}_\Delta(s-\tau)|^2) ds \\
&\quad + 2\mathbb{E} \int_0^{t\wedge\rho} (-U(x(s), \bar{x}_\Delta(s)) + U(x(s-\tau), \bar{x}_\Delta(s-\tau))) ds \\
&\leq 4K_3 \mathbb{E} \int_0^{t\wedge\rho} (|x(s) - x_\Delta(s)|^2 + |x_\Delta(s) - \bar{x}_\Delta(s)|^2 \\
&\quad + |x(s-\tau) - x_\Delta(s-\tau)|^2 + |x_\Delta(s-\tau) - \bar{x}_\Delta(s-\tau)|^2) ds \\
&\quad + 2 \int_{-\tau}^0 U(\xi(s), \xi(\kappa(s))) ds \\
&\leq C \mathbb{E} \int_0^{t\wedge\rho} |x(s) - x_\Delta(s)|^2 ds + C \mathbb{E} \int_0^T K_{\varphi(\Delta)}^2 \Delta ds \\
&\quad + C \int_{-\tau}^0 |\xi(s) - \xi(\kappa(s))|^2 ds \\
&\leq C \left(\int_0^t \mathbb{E} |x(s\wedge\rho) - x_\Delta(s\wedge\rho)|^2 ds + K_{\varphi(\Delta)}^2 \Delta + \Delta^{2\gamma} \right).
\end{aligned} \tag{64}$$

$$\begin{aligned}
B_{12} &\leq \mathbb{E} \int_0^{t\wedge\rho} (|x(s) - \bar{x}_\Delta(s)|^2 + |f(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)) - f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2) ds \\
&\leq 2\mathbb{E} \int_0^{t\wedge\rho} (|x(s) - x_\Delta(s)|^2 + |x_\Delta(s) - \bar{x}_\Delta(s)|^2) ds \\
&\quad + 2\mathbb{E} \int_0^{t\wedge\rho} (|f(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)) - f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s))|^2 \\
&\quad + |f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)) - f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2) ds \\
&\leq C \int_0^t \mathbb{E} |x(s\wedge\rho) - x_\Delta(s\wedge\rho)|^2 ds + CK_{\varphi(\Delta)}^2 \Delta + C\Delta^{2\gamma} \\
&\quad + C \mathbb{E} \int_0^{t\wedge\rho} |f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)) - f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2 ds.
\end{aligned} \tag{65}$$

In addition, let j denote the integer part of T/Δ . Thus,

$$\begin{aligned}
& \mathbb{E} \int_0^{t \wedge \rho} |f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)) - f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2 ds \\
&= \sum_{k=0}^j \mathbb{E} \int_{t_k}^{t_{k+1}} |f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)) - f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2 \mathbb{I}_{[0, t \wedge \rho]}(s) ds \\
&\leq 2 \sum_{k=0}^j \mathbb{E} \int_{t_k}^{t_{k+1}} \left(|f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s))|^2 + |f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2 \right) \mathbb{I}_{[0, t \wedge \rho]}(s) \mathbb{I}_{\{r(s) \neq \bar{r}(t_k)\}} ds \\
&\leq C \sum_{k=0}^j \mathbb{E} \int_{t_k}^{t_{k+1}} \mathbb{E} \left(\mathbb{E} \left[(1 + K_{\varphi(\Delta)}^2 (|\bar{x}_\Delta(s)|^2 + |\bar{x}_\Delta(s-\tau)|^2)) \cdot \mathbb{I}_{\{r(s) \neq \bar{r}(t_k)\}} |r(t_k)| \right] \right) ds,
\end{aligned} \tag{66}$$

where we have utilized the fact that $\bar{x}_\Delta(s)$ and $\bar{x}_\Delta(s-\tau)$ are conditionally independent of $\mathbb{I}_{\{r(s) \neq \bar{r}(t_k)\}}$ with reference to the σ -algebra generated by $r(t_k)$. The application of the Markov property gives

From Lemma 6, it follows that

$$\begin{aligned}
\mathbb{E} \left(\mathbb{I}_{\{r(s) \neq \bar{r}(t_k)\}} |r(t_k)| \right) &= \sum_{i \in \mathbb{S}} \mathbb{I}_{\{r(t_k)=i\}} \mathbb{P}(r(s) \neq i | r(t_k) = i) \\
&\leq \max_{0 \leq i \leq N} (-\gamma_{ii} \Delta + o(\Delta)) \sum_{i \in \mathbb{S}} \mathbb{I}_{\{r(t_k)=i\}} \\
&\leq C \Delta + o(\Delta).
\end{aligned} \tag{67}$$

$$\begin{aligned}
& \mathbb{E} \int_0^{t \wedge \rho} |f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)) - f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2 ds \\
&\leq (C \Delta + o(\Delta)) \sum_{k=0}^j \int_{t_k}^{t_{k+1}} (1 + K_{\varphi(\Delta)}^2 \mathbb{E} |\bar{x}_\Delta(s)|^2 + K_{\varphi(\Delta)}^2 \mathbb{E} |\bar{x}_\Delta(s-\tau)|^2) ds \\
&\leq K_{\varphi(\Delta)}^2 (C \Delta + o(\Delta)).
\end{aligned} \tag{68}$$

Inserting (68) into (65), we get

$$B_{12} \leq C \left(\int_0^t \mathbb{E} |x(s \wedge \rho) - x_\Delta(s \wedge \rho)|^2 ds + K_{\varphi(\Delta)}^2 \Delta + \Delta^{2\theta} + o(\Delta) \right). \tag{69}$$

Similar to B_{12} , we obtain

$$B_{13} \leq C (K_{\varphi(\Delta)}^2 \Delta + \Delta^{2\sigma} + o(\Delta)). \tag{70}$$

Combining (63), (64), (69), and (70) together gives

$$\begin{aligned}
B_1 &\leq C \left(\int_0^t \mathbb{E} |x(s \wedge \rho) - x_\Delta(s \wedge \rho)|^2 ds + K_{\varphi(\Delta)}^2 \Delta \right. \\
&\quad \left. + \Delta^{2\gamma} + \Delta^{2\theta} + \Delta^{2\sigma} + o(\Delta) \right).
\end{aligned} \tag{71}$$

By Lemmas 3 and 7, we could show that

$$\begin{aligned}
B_2 &\leq 4 \mathbb{E} \int_0^{t \wedge \rho} K_{\varphi(\Delta)} |x_\Delta(s) - \bar{x}_\Delta(s)| \cdot (|x(s) - \bar{x}_\Delta(s)| + |x(s-\tau) - \bar{x}_\Delta(s-\tau)|) ds \\
&\leq 2 \mathbb{E} \int_0^{t \wedge \rho} K_{\varphi(\Delta)}^2 |x_\Delta(s) - \bar{x}_\Delta(s)|^2 ds \\
&\quad + 2 \mathbb{E} \int_0^{t \wedge \rho} (|x(s) - \bar{x}_\Delta(s)| + |x(s-\tau) - \bar{x}_\Delta(s-\tau)|) ds \\
&\leq C \left(\int_0^t \mathbb{E} |x(s \wedge \rho) - x_\Delta(s \wedge \rho)|^2 ds + K_{\varphi(\Delta)}^4 \Delta + K_{\varphi(\Delta)}^2 \Delta + \Delta^{2\gamma} + \Delta^{2\theta} + o(\Delta) \right),
\end{aligned} \tag{72}$$

where the same techniques used in the proofs of B_{11} and B_{12} have been applied. Similarly,

$$\begin{aligned} B_3 &\leq \mathbb{E} \int_0^{t \wedge \rho} \theta^2 \Delta^2 |f_\Delta(s, x_\Delta(s), x_\Delta(s-\tau), r(s))|^2 ds \\ &\quad + \mathbb{E} \int_0^{t \wedge \rho} |f(s, x(s), x(s-\tau), r(s)) - f(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2 ds \\ &\leq C(K_{\varphi(\Delta)}^2 \Delta^2 + K_{\varphi(\Delta)}^2 \Delta + \Delta^{2\gamma} + \Delta^{2\theta} + o(\Delta)). \end{aligned} \quad (73)$$

By Assumptions 3 and 7, we obtain that

$$\begin{aligned} B_4 &\leq \lambda \mathbb{E} \int_0^{t \wedge \rho} (|e_\Delta(s)|^2 + 2|h(s, x(s), x(s-\tau), r(s)) - h(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2) ds \\ &\leq 2\lambda \mathbb{E} \int_0^{t \wedge \rho} (|x(s) - x_\Delta(s)|^2 + \theta^2 \Delta^2 |f_\Delta(s, x_\Delta(s), x_\Delta(s-\tau), r(s))|^2) ds \\ &\quad + 2\lambda \mathbb{E} \int_0^{t \wedge \rho} |h(s, x(s), x(s-\tau), r(s)) - h(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s))|^2 ds \\ &\quad + 2\lambda \mathbb{E} \int_0^{t \wedge \rho} |h(s, \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)) - h(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s))|^2 ds \\ &\quad + 2\lambda \mathbb{E} \int_0^{t \wedge \rho} |h(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), r(s)) - h(\mu(s), \bar{x}_\Delta(s), \bar{x}_\Delta(s-\tau), \bar{r}(s))|^2 ds \\ &\leq C \left(\int_0^t \mathbb{E} |x(s \wedge \rho) - x_\Delta(s \wedge \rho)|^2 ds + K_{\varphi(\Delta)}^2 \Delta^2 + K_{\varphi(\Delta)}^2 \Delta + \Delta^{2\gamma} + \Delta^{2\eta} + o(\Delta) \right), \end{aligned} \quad (74)$$

where the skills of estimating B_1 have been used. Inserting (71)–(74) into (61), one can see that

$$\begin{aligned} \mathbb{E} |e_\Delta(t \wedge \rho)|^2 &\leq C \left(\int_0^t \mathbb{E} |x(s \wedge \rho) - x_\Delta(s \wedge \rho)|^2 ds + K_{\varphi(\Delta)}^4 \Delta \right. \\ &\quad \left. + \Delta^{2\gamma} + \Delta^{2\eta} + \Delta^{2\theta} + \Delta^{2\sigma} + o(\Delta) \right). \end{aligned} \quad (75)$$

Thus,

$$\begin{aligned} \mathbb{E} |x(t \wedge \rho) - x_\Delta(t \wedge \rho)|^2 &\leq \mathbb{E} |e_\Delta(t \wedge \rho)|^2 + \mathbb{E} |\theta \Delta f_\Delta(t, x_\Delta(t), x_\Delta(t-\tau), r(t))|^2 \\ &\leq C \left(\int_0^t \mathbb{E} |x(s \wedge \rho) - x_\Delta(s \wedge \rho)|^2 ds + ((K_{\varphi(\Delta)}^4 \Delta) \vee \Delta^{2(\gamma \wedge \theta \wedge \sigma \wedge \eta)}) \right). \end{aligned} \quad (76)$$

The use of Gronwall's inequality yields that

$$\mathbb{E} |x(T \wedge \rho) - x_\Delta(T \wedge \rho)|^2 \leq C((K_{\varphi(\Delta)}^4 \Delta) \vee \Delta^{2(\gamma \wedge \theta \wedge \sigma \wedge \eta)}). \quad (77)$$

The proof is completed. $\square$

Let us now give the rate of convergence in $\mathcal{L}^2$ sense.

Theorem 1. *Let Assumptions 1–7 hold. For any sufficiently small $\Delta \in (0, \Delta^*)$, we assume there exists a positive constant c^* such that*

$$\varphi(\Delta) \geq c^* ((K_{\varphi(\Delta)}^4 \Delta) \vee \Delta^{2(\gamma \wedge \theta \wedge \sigma \wedge \eta)})^{(-p/2(p-2))}. \quad (78)$$

Then, for any $T > 0$, we have

$$\mathbb{E} |x(T) - x_\Delta(T)|^2 \leq C((K_{\varphi(\Delta)}^4 \Delta) \vee \Delta^{2(\gamma \wedge \theta \wedge \sigma \wedge \eta)}), \quad (79)$$

$$\mathbb{E} |x(T) - \bar{x}_\Delta(T)|^2 \leq C((K_{\varphi(\Delta)}^4 \Delta) \vee \Delta^{2(\gamma \wedge \theta \wedge \sigma \wedge \eta)}). \quad (80)$$

Proof. Let $\delta > 0$ be arbitrary. By Young's inequality, we obtain

$$\begin{aligned} u^2 v &= (\delta u^p)^{(2/p)} \left(\frac{v^{p/(p-2)}}{\delta^{2/(p-2)}} \right)^{(p-2)/p} \\ &\leq \frac{2\delta}{p} u^p + \frac{p-2}{p\delta^{2/(p-2)}} v^{p/(p-2)}, \quad \forall u, v > 0. \end{aligned} \quad (81)$$

Therefore,

$$\begin{aligned}
& \mathbb{E}|x(T) - x_\Delta(T)|^2 \\
&= \mathbb{E}\left(|x(T) - x_\Delta(T)|^2 \mathbb{1}_{\{\rho > T\}}\right) + \mathbb{E}\left(|x(T) - x_\Delta(T)|^2 \mathbb{1}_{\{\rho \leq T\}}\right) \\
&\leq \mathbb{E}|x(T \wedge \rho) - x_\Delta(T \wedge \rho)|^2 + \frac{2\delta}{p} \mathbb{E}|x(T) - x_\Delta(T)|^p \\
&\quad + \frac{p-2}{p\delta^{2/(p-2)}} \mathbb{P}\{\rho \leq T\}.
\end{aligned} \tag{82}$$

By Lemmas 2 and 6, we get

$$\mathbb{E}|x(T) - x_\Delta(T)|^p \leq C. \tag{83}$$

From Lemma 8, it follows that

$$\mathbb{P}\{\rho \leq T\} \leq \mathbb{P}\{\tau_R \leq T\} + \mathbb{P}\{\tau_{\Delta,R} \leq T\} \leq \frac{C}{R^2}. \tag{84}$$

We can choose $\delta = (K_{\varphi(\Delta)}^4 \Delta) \vee \Delta^{2(\gamma \wedge \vartheta \wedge \sigma \wedge \eta)}$ and $R = c^* ((K_{\varphi(\Delta)}^4 \Delta) \vee \Delta^{2(\gamma \wedge \vartheta \wedge \sigma \wedge \eta)})^{(-p/2(p-2))}$ such that

$$\begin{aligned}
\mathbb{E}|x(T) - x_\Delta(T)|^2 &\leq \mathbb{E}|x(T \wedge \rho) - x_\Delta(T \wedge \rho)|^2 \\
&\quad + C((K_{\varphi(\Delta)}^4 \Delta) \vee \Delta^{2(\gamma \wedge \vartheta \wedge \sigma \wedge \eta)}).
\end{aligned} \tag{85}$$

With condition (78), we obtain

$$\varphi(\Delta) \geq c^* ((K_{\varphi(\Delta)}^4 \Delta) \vee \Delta^{2(\gamma \wedge \vartheta \wedge \sigma \wedge \eta)})^{(-p/2(p-2))} = R. \tag{86}$$

From Lemma 9 and (85), we derive that

$$\mathbb{E}|x(T) - x_\Delta(T)|^2 \leq C((K_{\varphi(\Delta)}^4 \Delta) \vee \Delta^{2(\gamma \wedge \vartheta \wedge \sigma \wedge \eta)}). \tag{87}$$

Then, combining Lemma 7 and (79) gives (80). We complete the proof. $\square$

4. Example

An example is presented to illustrate our theory in this section.

Example 1. Consider the following nonlinear and nonautonomous scalar hybrid stochastic differential delay equation with Poisson jumps:

$$\begin{aligned}
dx(t) &= f(t, x(t), x(t-\tau), r(t))dt \\
&\quad + g(t, x(t), x(t-\tau), r(t))dB(t) \\
&\quad + h(t, x(t), x(t-\tau), r(t))dN(t),
\end{aligned} \tag{88}$$

with the initial value x_0 which satisfies Assumption 1. $B(t)$ is a scalar Brownian motion, and $N(t)$ is a scalar Poisson process with intensity $\lambda = 0.4$. Furthermore, $r(t)$ is a Markovian chain defined on the state space $\mathbb{S} = \{1, 2\}$ with generator

$$\Gamma = \begin{pmatrix} -2 & 2 \\ 1 & -1 \end{pmatrix}. \tag{89}$$

Moreover, for any $t \in [0, 2]$ and $x, y \in \mathbb{R}$, let

$$\begin{aligned}
f(t, x, y, i) &= \begin{cases} -5x^3 + 3x \sin t + \frac{1}{8}(t(2-t))^{(1/3)}|y|^{(5/4)}, & \text{if } i = 1, \\ -20x^5 + 6x \cos t + \frac{1}{8}(t(2-t))^{(1/4)}|y|^{(5/4)}, & \text{if } i = 2, \end{cases} \\
g(t, x, y, i) &= \begin{cases} \frac{1}{2}(t(2-t))^{(1/3)}|x|^{(3/2)}, & \text{if } i = 1, \\ (t(2-t))^{(1/4)}|x|^{(5/2)}, & \text{if } i = 2, \end{cases} \\
h(t, x, y, i) &= \begin{cases} (t(2-t))^{(1/5)}x, & \text{if } i = 1, \\ (t(2-t))^{(1/5)}y, & \text{if } i = 2. \end{cases}
\end{aligned} \tag{90}$$

Obviously, Assumptions 2 and 3 hold with $L_R = 20R^4$, $K_2 = 1$. Moreover, when $i = 1$,

$$\begin{aligned}
& (x - \bar{x})^T (f(t, x, y, 1) - f(t, \bar{x}, \bar{y}, 1)) \\
& \leq (x - \bar{x})^2 - 5(x^2 + x\bar{x} + \bar{x}^2) + 3(x - \bar{x})^2 + \frac{1}{8}(x - \bar{x})(|y|^{(5/4)} - |\bar{y}|^{(5/4)}) \\
& \leq -\frac{5}{2}(x - \bar{x})^2(x^2 + \bar{x}^2) + 4(x - \bar{x})^2 + \frac{25}{256}|y - \bar{y}|^2(|y|^{(1/4)} - |\bar{y}|^{(1/4)})^2 \\
& \leq -\frac{5}{2}(x - \bar{x})^2(|x|^2 + |\bar{x}|^2) + 4|x - \bar{x}|^2 + \frac{25}{64}|y - \bar{y}|^2 + \frac{25}{128}|y - \bar{y}|^2(|y|^2 + |\bar{y}|^2).
\end{aligned} \tag{91}$$

Let $q = 3$. In a similar way, we have

$$\begin{aligned}
& \frac{q-1}{2}|g(t, x, y, 1) - g(t, \bar{x}, \bar{y}, 1)|^2 \\
& \leq \frac{1}{2}||x|^{(3/2)} - |\bar{x}|^{(3/2)}|^2 \\
& \leq \frac{9}{8}|x - \bar{x}|^2(|x|^{(1/2)} + |\bar{x}|^{(1/2)})^2 \\
& \leq \frac{9}{2}|x - \bar{x}|^2 + \frac{9}{4}|x - \bar{x}|^2(|x|^2 + |\bar{x}|^2).
\end{aligned} \tag{92}$$

$$\begin{aligned}
& (x - \bar{x})^T (f(t, x, y, 1) - f(t, \bar{x}, \bar{y}, 1)) \\
& + \frac{q-1}{2}|g(t, x, y, 1) - g(t, \bar{x}, \bar{y}, 1)|^2 \\
& \leq 9(|x - \bar{x}|^2 + |y - \bar{y}|^2) - \frac{1}{4}|x - \bar{x}|^2(|x|^2 + |\bar{x}|^2) \\
& + \frac{1}{4}|y - \bar{y}|^2(|y|^2 + |\bar{y}|^2).
\end{aligned} \tag{93}$$

Similarly, for $i = 2$, we have

Thus,

$$\begin{aligned}
& (x - \bar{x})^T (f(t, x, y, 2) - f(t, \bar{x}, \bar{y}, 2)) \\
& \leq (x - \bar{x})^2(-20(x^4 + x^3\bar{x} + x^2\bar{x}^2 + x\bar{x}^3 + \bar{x}^4)) + 6(x - \bar{x})^2 \\
& + \frac{1}{8}(x - \bar{x})(|y|^{(5/4)} - |\bar{y}|^{(5/4)}) \\
& \leq -10(x - \bar{x})^2(|x|^4 + |\bar{x}|^4) + 7|x - \bar{x}|^2 + \frac{25}{256}|y - \bar{y}|^2(|y|^{(1/4)} + |\bar{y}|^{(1/4)})^2 \\
& \leq -10(x - \bar{x})^2(|x|^4 + |\bar{x}|^4) + 7|x - \bar{x}|^2 + \frac{25}{64}|y - \bar{y}|^2 + \frac{25}{128}|y - \bar{y}|^2(|y|^4 + |\bar{y}|^4), \\
& \cdot \frac{q-1}{2}|g(t, x, y, 2) - g(t, \bar{x}, \bar{y}, 2)|^2 \\
& \leq \frac{1}{2}||x|^{(5/2)} - |\bar{x}|^{(5/2)}|^2 \\
& \leq \frac{25}{8}|x - \bar{x}|^2(|x|^{(1/2)} + |\bar{x}|^{(1/2)})^2 \\
& \leq \frac{25}{2}|x - \bar{x}|^2 + \frac{25}{4}|x - \bar{x}|^2(|x|^4 + |\bar{x}|^4).
\end{aligned} \tag{94}$$

Therefore,

$$\begin{aligned}
& (x - \bar{x})^T (f(t, x, y, 2) - f(t, \bar{x}, \bar{y}, 2)) + \frac{q-1}{2}|g(t, x, y, 2) - g(t, \bar{x}, \bar{y}, 2)|^2 \\
& \leq 20(|x - \bar{x}|^2 + |y - \bar{y}|^2) - \frac{75}{4}|x - \bar{x}|^2(|x|^4 + |\bar{x}|^4) + \frac{75}{4}|y - \bar{y}|^2(|y|^4 + |\bar{y}|^4).
\end{aligned} \tag{95}$$

Hence, Assumption 4 is satisfied for any $i \in \mathbb{S}$. For Assumption 5, by letting $p = 4$, we obtain that

$$\begin{aligned} & x^T f(t, x, y, 1) + \frac{p-1}{2} |g(t, x, y, 1)|^2 \\ & \leq -5x^4 + 3x^2 + \frac{1}{8} x|y|^{(5/4)} + \frac{3}{8} |x|^3 \leq C(1 + |x|^2 + |y|^2), \\ & \cdot x^T f(t, x, y, 2) + \frac{p-1}{2} |g(t, x, y, 2)|^2 \\ & \leq -20x^6 + 6x^2 + \frac{1}{8} x|y|^{(5/4)} + \frac{3}{2} |x|^5 \leq C(1 + |x|^2 + |y|^2). \end{aligned} \quad (96)$$

Thus, Assumption 5 is satisfied. Then, it is easy to see that Assumptions 6 and 7 hold with $\vartheta = \sigma = (1/4)$, $\eta = (1/5)$.

For any $\varepsilon \in (0, (1/4)]$, define $\varphi(\Delta) = \sqrt{[4]} (1/20) \Delta^{-(\varepsilon/4)}$. Therefore, when $K_{\varphi(\Delta)} = 20(\varphi(\Delta))^4$, we have $K_{\varphi(\Delta)} \Delta^{(1/4)} = \Delta^{(1/4)-\varepsilon} \leq 1$ and $\lim_{\Delta \rightarrow 0} \varphi(\Delta) = \infty$. Choose $p = 4$, $\varepsilon = (33/136)$, $c^* = \sqrt{[4]} (1/20)$; then, we derive that

$$\sqrt[4]{\frac{1}{20}} (K_{\varphi(\Delta)} \Delta)^{(-p/2(p-2))} = \sqrt[4]{\frac{1}{20}} \Delta^{4\varepsilon-1} \leq \sqrt[4]{\frac{1}{20}} \Delta^{-(\varepsilon/4)} = \varphi(\Delta). \quad (97)$$

By Theorem 1, we could get that

$$\mathbb{E}|x(T) - x_{\Delta}(T)|^2 \leq C\Delta^{(1-4\varepsilon)\wedge 2\gamma}, \quad (98)$$

where $\gamma \in (0, 1]$ is defined in Assumption 1. Thus, the convergence rate of the truncated theta-EM method for (88) is $(1-4\varepsilon)/2\wedge\gamma$. This example shows that our main result could cover a large class of nonlinear and nonautonomous HSDDEwPJs.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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