Research Article

Hamilton-Connected Mycielski Graphs

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Jarnicki, Myrvold, Saltzman, and Wagon conjectured that if $G$ is Hamilton-connected and not $K_2$, then its Mycielski graph $\mu(G)$ is Hamilton-connected. In this paper, we confirm that the conjecture is true for three families of graphs: the graphs $G$ with $\delta(G) > |V(G)|/2$, generalized Petersen graphs $GP(n, 2)$ and $GP(n, 3)$, and the cubes $G^3$. In addition, if $G$ is pancyclic, then $\mu(G)$ is pancyclic.

1. Introduction

All graphs considered in this paper are simple and finite. For notations and terminologies not defined here, we refer to Bondy and Murty [1]. A spanning cycle (path) of a graph is called Hamilton cycle (Hamilton path). A graph which contains a Hamiltonian path between every two vertices of $G$ is called Hamilton-connected (HC). Mycielski [2] proved that the chromatic numbers of triangle-free graphs can be arbitrarily large by introducing a graph transformation as follows. For a graph $G$ on vertices $V = \{v_1, v_2, \ldots, v_n\}$, its Mycielski graph, denoted by $\mu(G)$, is the graph on vertices $X \cup Y \cup \{z\} = \{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\} \cup \{z\}$ with edges $zy_i$ for all $i$ and edges $x_ix_j$, $y_ix_j$, and $x_iy_j$ for all edges $v_iv_j$ in $G$. In recent years, a number of papers are devoted to various properties of Mycielski graphs, such as Hamilton-connectivity, Hamiltonicity [3–7], total chromatic number [8, 9], circular chromatic number [10–14], and connectivity [15, 16]. Fisher et al. [4] obtained the following results.

Theorem 1 (see Fisher et al. [4]). The following results hold for a graph $G$:

1. If $G$ is Hamiltonian, then $\mu(G)$ is Hamiltonian
2. If $G$ is not connected, then $\mu(G)$ is not Hamiltonian
3. If $G$ has at least two pendant vertices, then $\mu(G)$ is not Hamiltonian

Cheng, Wang, and Liu studied Hamiltonicity and Hamilton-connectedness in Mycielski graphs of bipartite graphs.

Theorem 2 (see Cheng et al. [3]). For a bipartite graph $G$, the following are true:

1. If $\mu(G)$ is Hamiltonian, then $G$ is balanced
2. If $\mu(G)$ is Hamiltonian, then $G$ has a Hamilton path

In 2017, Jarnicki et al. [17] established the following results for $\mu(G)$ being Hamilton-connected or not.

Theorem 3 (see Jarnicki et al. [17]). The following results hold for a graph $G$:

1. If $G$ is an odd cycle, then $\mu(G)$ is Hamilton-connected
2. If $G$ is a Hamilton-connected graph with order odd, then $\mu(G)$ is Hamilton-connected
3. If $G$ is an even cycle, then $\mu(G)$ is not Hamilton-connected

They posed the following conjecture.

Conjecture 1 (see Jarnicki et al. [17]). If $G$ is Hamilton-connected and not $K_2$, then $\mu(G)$ is Hamilton-connected.

In this paper, we confirm that the conjecture is true for three families of graphs: the graphs $G$ with $\delta(G) > |V(G)|/2$,
generalized Petersen graphs $GP(n, 2)$ and $GP(n, 3)$, and the cubes $G^3$. In addition, if $G$ is pan cyclic, then $\mu(G)$ is pan cyclic.

2. Mycielski Factor

Let $G$ be a connected graph of order $n$ even, and $v_1 \in V(G)$. We call a connected spanning subgraph of $G$ to be a Mycielski factor starting at $v_1$ if it consists of an even number of odd cycles $C_1, \ldots, C_k$ (possibly $s = 0$) and an even cycle $C_{2s+1}$ with the chord (possibly empty), joined by $2s$ edges $e_1, \ldots, e_{2s}$, where $e_i = v_i v_{i+1}$ for each $i \in \{1, \ldots, 2s\}$ such that $v_i v_{i+1} \in V(C_i)$ for each $i \in \{1, \ldots, 2s + 1\}$, and the chord joins $v_{2s+1}^1$ and a vertex at distance even on $C_{2s+1}$.

Lemma 1. Assume that a graph $G$ is Hamilton-connected. If, for any $v \in V(G)$, there exists a Mycielski factor starting at $v$, then $\mu(G)$ is Hamilton-connected.

Proof. As in the assumption, let $G$ be HC. Trivially, $G$ has a Hamilton cycle. By Theorem 3 (2), $\mu(G)$ is HC if the order of $G$ is odd. So, it remains to tackle the case when the order is even. Let $V(G) = \{v_1, \ldots, v_{2n}\}$, where $n \geq 2$. Recall that $V(\mu(G)) = X \cup Y \cup \{z\}$. Take any two vertices $A, B \in V(\mu(G))$. We consider five cases in terms of the location of $A$ and $B$ in $X, Y$, and $\{z\}$.

Case 1. $A \in X$ and $B \in X$.

Without loss of generality, let $A = x_1$ and $B = x_{2n-1}$. Since $G$ is HC, there exists a Hamilton path $P$ connecting $v_1$ and $v_{2n}$ in $G$. We shall find a Hamilton path of $\mu(G)$ depending on $P$ as follows. Zigzag up from $x_1$, until $y_{2n}$ is reached. Then, jump via $z$ to $y_1$, and zigzag right until $x_2n$ is reached. Formally, it is

$$x_1 - y_2 - x_3 - \cdots - x_{2n} - z - y_1 - x_2 - \cdots - x_{2n-2}$$

as shown in Figure 1.

Case 2. $A \in Y$ and $B \in Y$.

Without loss of generality, let $A = y_1$ and $B = y_{2n}$. Since $G$ is HC, there exists a Hamilton path $P$ connecting $v_1$ and $v_{2n}$ in $G$. Thus, there exists a neighbor, say $y_{2n}$, of $y_1$. Zigzag up from $y_{2n}$ to $y_1$ and then back to $x_2$ and zigzag up to $x_{2n-1}$ and then up to $x_{2n}$, and zigzag left to $y_1$ and then up to $z$ and $y_{2n}$, as shown in Figure 2. Formally,

$$y_1 - x_2 - y_3 - \cdots - y_{2n} - x_1 - y_2 - \cdots - y_{2n-2} - z - y_{2n}$$

If $s \geq 1$, for every integer $i \in \{1, \ldots, 2s\}$, label the vertices of $C_i$ in the clockwise order $u_i, u_i+1, \ldots, u_{2k+1}$. One can find a Hamilton path $P_i$ of $\mu(C_i)$ as follows:

$$x_1, y_2, x_3, y_4, \ldots, x_{2k+1}, y_1, x_2, y_3, x_4, \ldots, y_{2k+1},$$

where $u_i = v_i$ and $u_{2k+1} = v_i$.

Case 3. $A \in X$ and $B \in Y$.

Without loss of generality, let $A = x_1$. Since $G$ is HC, $G$ has a Hamilton cycle $C$. Label the vertices of $C$ as $v_1 v_2, \ldots, v_{2n} v_1$. We are able to find a Hamilton path joining $A$ and $B$: zigzag from $y_1$ to $x_2$, and then go to $x_1$, and then zigzag right to $y_2$, and finish at $z$, as shown in Figure 3. Formally, it is

$$y_1 - x_2 - y_3 - \cdots - x_{2n} - x_1 - y_2 - \cdots - y_{2n} - z$$

where $w_i = v_i$ and $w_{2n} = v_i$.

Thus, $(\cup_{i=1}^{2s} P_i) \cup (\cup_{1 \leq i \leq 2s}) \cup \{y_{2n} - z\}$ is a Hamilton path of $\mu(G)$ from joining $x_1$ and $z$.

3. Hamiltonian Connectedness

Theorem 4. Assume that $G$ is a Hamilton-connected graph of order $n \geq 3$. If $\delta(G) \geq (n/2) + 1$, then $\mu(G)$ is Hamilton-connected.

Proof. Let $v$ be a vertex of $G$. We consider a Hamilton cycle $C$ of $G$. Let $u$ be a neighbor of $v$ on $C$. Since $d(u) \geq (n/2) + 1$, it has a neighbor at distance even on $C$. By Lemma 1, $\mu(G)$ is HC.

The $k$th power of a graph $G$, denoted by $G^k$, is a graph with the same vertex set as $G$ in which two vertices are adjacent if and only if their distance in $G$ is at most $k$. Thus,
$G^3 = G$. We need the following result due to Karaganis [18].

**Theorem 5** (see Karaganis [18]). **The cube** $G^3$ of every connected graph $G$ of order $n \geq 3$ is Hamilton-connected.

**Theorem 6.** For any connected graph $G$ of order $n \geq 3$, $\mu(G^3)$ is Hamilton-connected.

**Proof.** By Theorem 5, $G^3$ is HC for $G$. Since $\mu(H^3)$ is a spanning subgraph of $\mu(G^3)$ for any spanning graph $H$ of $G$, to show $\mu(G^3)$ is HC, it suffices to show that $\mu(T^3)$ is HC for any tree $T$ of order $n \geq 3$. Since $T^3$ is HC, by Theorem 3 (1), we may assume that $n$ is even. By Lemma 1, it remains to show that $T^3$ has a Mycielski factor starting from each vertex $v \in V(T)$.

Let $w$ be a neighbor of $v$ in $T$, and let $T_v$ and $T_w$ be the components of $T - vw$ containing $v$ and $w$, respectively. Let $n_v$ and $n_w$ be the order of $T_v$ and $T_w$, respectively. Let $v'$ be a neighbor of $v$ in $T_v$ and let $w'$ be a neighbor of $w$ in $T_w$.

- **Case 1:** both $n_v$ and $n_w$ are at least 3.
  - **Subcase 1.1:** both $n_v$ and $n_w$ are odd.
    - By Theorem 5, both $T_v^3$ and $T_w^3$ are HC. Let $C_v$ and $C_w$ be Hamilton cycles of $T_v$ and $T_w$, respectively. One can see that $C_v \cup C_w + v'w'$ is a Mycielski factor of $T^3$ starting at $v$.
  - **Subcase 1.2:** both $n_v$ and $n_w$ are even.
    - By the induction hypothesis, $T_v^3$ has a Hamilton path $P_{v_v'}$ joining $v$ and $v'$, and $T_w^3$ has a Hamilton path $P_{w_w'}$ joining $w$ and $w'$. One can see that $P_{v_v'} \cup P_{w_w'} + vw + v'w' + v'w'w$ is a Mycielski factor of $T^3$ starting at $v$.

- **Case 2:** $\min\{n_v, n_w\} \leq 2$.

  **Subcase 2.1:** $\min\{n_v, n_w\} = n_v = 2$.
  - Since $n_v + n_w = n$ is an even number at least 3, $n_w$ is an odd number at least 3. By Theorem 5, let $C_{vw'}$ be a Hamilton cycle of $T_v^3$ containing $vw$. It is easy to see that $C_{vw'} + vw + vw'$ is a Mycielski factor of $T_v^3$ starting at $v$.
  - **Subcase 2.1.1:** $\min\{n_v, n_w\} = n_w = 2$.
    - If $n_w = 2$, then $n = 4$. Trivially, $T^3 = K_4$ has a Mycielski factor starting at $v$.
    - If $n_w \neq 2$, then $n_w$ is an even number at least 4. By Theorem 5, let $P_{vw'}$ be a Hamilton path of $T^3_w$. It can be seen that $P_{vw'} + vw + w' + v'w$ is a Mycielski factor starting at $v$.

  **Subcase 2.2:** $\min\{n_v, n_w\} = n_w \leq 2$.

  - If $T \equiv K_{1,n-1}$, then $T^3 \equiv K_n$ has a Mycielski factor starting at $v$. Next, we assume that $T \neq K_{1,n-1}$. We can choose a neighbor $w$ of $v$ such that $n_w \geq 2$. Combining with our assumption that $\min\{n_v, n_w\} = n_w \leq 2$, we have $n_w = 2$. By Theorem 5, let $P_{vv'}$ be a Hamilton path of $T^3_v$. It can be checked that $P_{vv'} + vw + wv' + v'w + v'w$ is a Mycielski factor starting at $v$.

In 1969, Watkins [19] introduced the notion of the generalized Petersen graph $GP(n, k)$, $1 \leq k \leq n$, as follows. The vertex set is $\{u_i, v_i : 1 \leq i \leq n\}$, and the edge set is $\{u_iu_{i+1}, v_iv_{i+k}, u_iv_i\}$, where the subscript arithmetic performs modulo $n$. The Petersen graph $GP(5, 2)$ is vertex-transitive if and only if $k^2 \equiv \pm 1 \pmod{n}$ or $(n, k) = (10, 2)$. Next, we consider the Hamilton-connectedness of the generalized Petersen graph $GP(n, 2)$ and $GP(n, 3)$.

**Theorem 7** (see Alspach and Liu [21]). **The generalized Petersen graph** $GP(n, 2)$ with $n \geq 6$ is Hamilton-connected if and only if $n \equiv 1, 2, 3 \pmod{6}$.

**Theorem 8.** If $GP(n, 2)$ is Hamilton-connected for $n \geq 6$, then $\mu(GP(n, 2))$ is Hamilton-connected.

**Proof.** In view of Lemma 1, it suffices to show that $GP(n, 2)$ has a Mycielski factor starting at any $v \in V(GP(n, 2))$. We consider two cases:

- **Case 1:** $n \equiv 1$ or $3 \pmod{6}$.
  - Since $n$ is odd, $GP(n, 2)$ is vertex-transitive, and we may assume that $v = u_1$, without loss of generality. Let $C_1$ and $C_2$ be the outer cycle and inner cycle of $GP(n, 2)$. Let $v$ be a vertex of $GP(n, 2)$. It is clear that $C_1 \cup C_2 + u_i, v_j$ is a Mycielski factor of $GP(n, 2)$ starting at $v$.

- **Case 2:** $n \equiv 2 \pmod{6}$.
  - By the symmetry, it suffices to tackle two possibilities according to the location of $v$ in $GP(n, 2)$: $v$ lies on the outer cycle or inner cycle of $GP(n, 2)$. Without loss of generality, let $v = u_1$ or $v = v_1$. First, for the case when $n = 8$, we can find a Mycielski factor $F_n$ of $GP(n, 2)$ as follows:
By the symmetry, it suffices to tackle two possibilities according to the location of \( v \) in \( GP(n, 3) \): \( v \) lies on the outer cycle or inner cycle of \( GP(n, 3) \). Without loss of generality, let \( v = u_1 \) or \( v = v_1 \).

First, for the case when \( n = 9 \), we can find a Mycielski factor \( F_n \) of \( GP(n, 3) \) starting at \( v \) as follows:

\[
F_n = \begin{cases} 
C_9 + u_1v_1 & \text{if } v = u_1, \\
C_9 + u_1v_2 & \text{if } v = v_1,
\end{cases}
\]

(7)

where

\[
C_9 = \begin{cases} 
u_1v_1v_2u_1v_3v_4v_5u_2v_6v_7u_3v_8v_1 & \text{if } v = u_1, \\
v_1v_2v_3v_4v_5v_6v_7v_8v_1 & \text{if } v = v_1.
\end{cases}
\]

(8)

For \( n = 14 \), by inserting 12 new vertices to \( C_9 \) of \( F_8 \), we get \( C_{14} \) as illustrated in Figures 5–7 for the case that \( v \) lies in the outer cycle and for the case that \( v \) lies in the inner cycle as illustrated in Figures 6, 8, and 9. For the case when \( n \geq 20 \), by inserting 12 new vertices to \( F_{n-6} \) with type A insertion, we obtain a Mycielski factor \( F_n \) of \( GP(n, 2) \) starting at \( v \).

Theorem 9 (see Alspach and Liu [21]). The generalized Petersen graph \( GP(n, 3) \) with \( n \geq 6 \) is Hamilton-connected if and only if \( n \) is odd.

Theorem 10. If \( GP(n, 3) \) is Hamilton-connected, then \( \mu(GP(n, 3)) \) is Hamilton-connected.

Proof. Since \( GP(n, 3) \) is Hamilton-connected, by Theorem 9, \( n \) is an odd number at least 7. Let \( v \) be a vertex of \( GP(n, 3) \). In view of Lemma 1, it suffices to show that \( GP(n, 3) \) has a Mycielski factor starting at \( v \). We consider two cases:

Case 1: \( n \equiv 1 \) or \( 5 \) (mod 6).

Since \( n \) is odd, \( GP(n, 3) \) is vertex-transitive; by the symmetry, we may assume that \( v = u_1 \), without loss of generality. Let \( C_1 \) and \( C_2 \) be the outer cycle and inner cycle of \( GP(n, 3) \). It is clear that \( C_1 \cup C_2 + u_2v_2 \) is a Mycielski factor of \( GP(n, 3) \) starting at \( v \).

Case 2: \( n \equiv 3 \) (mod 6).

4. Pancyclicity

In this section, we show that if a graph \( G \) is pancyclic, then \( \mu(G) \) is also pancyclic.

Theorem 11. If \( G \) is pancyclic, then \( \mu(G) \) is pancyclic.

Proof. Let \( G \) be a pancyclic graph of order \( n \). Since \( \mu(G) \) contains \( G \) as its subgraph, \( \mu(G) \) contains a cycle of length \( l \) for each \( l \in \{3, \ldots, n\} \).

Now, we find a cycle of length \( n + 1 \) in \( \mu(G) \). Take a cycle \( C \) of length \( n - 2 \) in \( G \). Without loss of generality, let \( P = v_1v_2, \ldots, v_{n-2} \) be a path resulting from \( C \) deleting an edge. It can be seen that \( x_1x_2, x_2x_3, \ldots, x_{n-2}z \) is a cycle of length \( n + 1 \), as illustrated in Figure 13. In a similar way, one can find a cycle of length \( n + 2 \) in \( \mu(G) \) in terms of a cycle of length \( n - 1 \) in \( G \).

Next, we will find a cycle of length \( n + k \) in \( \mu(G) \) for each \( k \in \{3, \ldots, n + 1\} \). Take a Hamilton cycle \( C \). Without loss of
generality, let $C = v_1v_2, \ldots, v_nv_1$ in $G$. We consider two cases according to the parity of $k$:

**Case 1:** $k$ is odd.

One can find a cycle of length $n+k$ in $\mu(G)$, as shown in Figure 14. Formally, it is

$$x_1 - y_2 - x_3 - \cdots - y_{k-1} - z - y_1 - \cdots - x_{k-1} - x_k - \cdots - x_n - x_1.$$  \hspace{1cm} (10)

**Case 2:** $k$ is even.

Zigzag up from $x_1$ to $x_{k-1}$ and left to $x_{k-2}$, then zigzag left to $y_1$, $z$, $y_{k-1}$, and $x_k$, and go right to $x_n$ and back to $x_1$, as shown in Figure 15. Formally, it is

$$x_1 - \cdots - x_{k-2} - x_{k-1} - y_{k-2} - \cdots - y_1 - z - y_{k-1} - x_k - \cdots - x_n - x_1.$$  \hspace{1cm} (11)
Figure 10: \( F_9 \) from \( u_1 \) to \( v_1 \) in \( GP(9, 3) \).

Figure 11: Type B insertion.

Figure 12: \( F_{15} \) obtained from \( F_9 \) by type B insertion in \( GP(9, 3) \).

Figure 13: Finding a cycle \( C_{n+1} \) in \( \mu(G) \) from a cycle \( C_{n-2} \) in \( G \).

Figure 14: A cycle of length \( n+k \) in \( \mu(G) \) if \( k \) is odd.

Figure 15: A cycle of length \( n+k \) in \( \mu(G) \) if \( k \) is even. \( \Box \)
5. Conclusion

In this paper, we introduce the notion of the Mycielski factor of a graph. If a graph $G$ has a Mycielski factor starting at $v$ for any $v \in V(G)$, then $\mu(G)$ is Hamilton-connected. Applying this result, we are able to show that if a graph $G$ belongs to three (well-defined) families of graphs, then $\mu(G)$ is Hamilton-connected. However, the full conjecture of Jarnicki, Myrvold, Saltzman, and Wagon is not yet solved. We also prove that if $G$ is pancyclic, then $\mu(G)$ is pancyclic.

One of the reviewers proposed the following two interesting problems.

Zhong et al. [7] showed that the line graph of the generalized Petersen graph $GP(n,k)$ is always Hamilton-connected. Is it easy to show that the Mycielski graph of $L(GP(n,k))$ is Hamilton-connected?

It is known that the line graph of a Hamilton-connected graph $G$ is also Hamilton-connected. Is $\mu(L(G))$ Hamilton-connected if $L(G)$ is Hamilton-connected? [22].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References