Research Article

Hamilton-Connected Mycielski Graphs

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Jarnicki, Myrvold, Saltzman, and Wagon conjectured that if $G$ is Hamilton-connected and not $K_2$, then its Mycielski graph $\mu(G)$ is Hamilton-connected. In this paper, we confirm that the conjecture is true for three families of graphs: the graphs $G$ with $\delta(G) > |V(G)|/2$, generalized Petersen graphs $GP(n, 2)$ and $GP(n, 3)$, and the cubes $G^3$. In addition, if $G$ is pancyclic, then $\mu(G)$ is pancyclic.

1. Introduction

All graphs considered in this paper are simple and finite. For notations and terminologies not defined here, we refer to Bondy and Murty [1]. A spanning cycle (path) of a graph is called Hamilton cycle (Hamilton path). A graph which contains a Hamiltonian path between every two vertices of $G$ is called Hamilton-connected (HC). Mycielski [2] proved that the chromatic numbers of triangle-free graphs can be arbitrarily large by introducing a graph transformation as follows. For a graph $G$ on vertices $V = \{v_1, v_2, \ldots, v_n\}$, its Mycielski graph, denoted by $\mu(G)$, is the graph on vertices $X \cup Y \cup \{z\} = \{x_1, x_2, \ldots, x_n\} \cup \{y_1, y_2, \ldots, y_n\} \cup \{z\}$ with edges $zy_i$ for all $i$ and edges $x_iy_j$, $y_ix_j$, and $x_iy_j$ for all edges $v_iv_j$ in $G$. In recent years, a number of papers are devoted to various properties of Mycielski graphs, such as Hamilton-connectivity, Hamiltonicity [3–7], total chromatic number [8, 9], circular chromatic number [10–14], and connectivity [15, 16]. Fisher et al. [4] obtained the following results.

Theorem 1 (see Fisher et al. [4]). The following results hold for a graph $G$:

(1) If $G$ is Hamiltonian, then $\mu(G)$ is Hamiltonian

(2) If $G$ is not connected, then $\mu(G)$ is not Hamiltonian

(3) If $G$ has at least two pendant vertices, then $\mu(G)$ is not Hamiltonian

Cheng, Wang, and Liu studied Hamiltonicity and Hamilton-connectedness in Mycielski graphs of bipartite graphs.

Theorem 2 (see Cheng et al. [3]). For a bipartite graph $G$, the following are true:

(1) If $\mu(G)$ is Hamiltonian, then $G$ is balanced

(2) If $\mu(G)$ is Hamiltonian, then $G$ has a Hamilton path

In 2017, Jarnicki et al. [17] established the following results for $\mu(G)$ being Hamilton-connected or not.

Theorem 3 (see Jarnicki et al. [17]). The following results hold for a graph $G$:

(1) If $G$ is an odd cycle, then $\mu(G)$ is Hamilton-connected

(2) If $G$ is a Hamilton-connected graph with order odd, then $\mu(G)$ is Hamilton-connected

(3) If $G$ is an even cycle, then $\mu(G)$ is not Hamilton-connected

They posed the following conjecture.

Conjecture 1 (see Jarnicki et al. [17]). If $G$ is Hamilton-connected and not $K_2$, then $\mu(G)$ is Hamilton-connected.

In this paper, we confirm that the conjecture is true for three families of graphs: the graphs $G$ with $\delta(G) > |V(G)|/2$, \ldots
generalized Petersen graphs $GP(n, 2)$ and $GP(n, 3)$, and the cubes $G^3$. In addition, if $G$ is pancyclic, then $\mu(G)$ is pancyclic.

2. Mycielski Factor

Let $G$ be a connected graph of order $n$ even, and $v_1 \in V(G)$. We call a connected spanning subgraph of $G$ to be a Mycielski factor starting at $v_1$ if it consists of an even number of odd cycles $C_1, \ldots, C_s$ (possibly $s = 0$) and an even cycle $C_{2s+1}$ with the chord (possibly empty), joined by $2s$ edges $e_1, \ldots, e_{2s}$, where $e_i = v_i v_{i+1}$ for each $i \in \{1, \ldots, 2s\}$ such that $v_i v_j \in V(C_i)$ for each $i \in \{1, \ldots, 2s + 1\}$, and the chord joins $v_{2s+1}$ and a vertex at distance even on $C_{2s+1}$.

**Lemma 1.** Assume that a graph $G$ is Hamilton-connected. If, for any $v \in V(G)$, there exists a Mycielski factor starting at $v$, then $\mu(G)$ is Hamilton-connected.

**Proof.** As in the assumption, let $G$ be HC. Trivially, $G$ has a Hamilton cycle. By Theorem 3 (2), $\mu(G)$ is HC if the order of $G$ is odd. So, it remains to tackle the case when the order is even. Let $V(G) = \{v_1, \ldots, v_{2n}\}$, where $n \geq 2$. Recall that $V(\mu(G)) = X \cup Y \cup \{z\}$. Take any two vertices $A, B \in V(\mu(G))$. We consider five cases in terms of the location of $A$ and $B$ in $X, Y$, and $\{z\}$.

*Case 1.* $A \in X$ and $B \in X$.

Without loss of generality, let $A = x_1$ and $B = x_{2n}$. Since $G$ is HC, there exists a Hamilton path $P$ connecting $v_1$ and $v_{2n}$ in $G$. We shall find a Hamilton path of $\mu(G)$ depending on $P$ as follows. Zigzag up from $x_1$ until $y_{2n}$ is reached. Then, jump via $z$ to $y_1$, and zigzag right until $x_{2n}$ is reached. Formally, it is

$$x_1 - y_2 - x_3 - \cdots - y_{2n} - z - y_1 - x_2 - \cdots - x_{2n},$$

(1)

as shown in Figure 1.

*Case 2.* $A \in Y$ and $B \in Y$.

Without loss of generality, let $A = y_1$ and $B = y_{2n}$. Since $G$ is HC, there exists a Hamilton path $P$ connecting $v_1$ and $v_{2n}$ in $G$. Thus, there exists a neighbor, say $v_2$, of $v_1$. Zigzag up from $y_1$ to $y_2$, and then back to $y_1$, zigzag up to $x_{2n-1}$ and then up to $x_{2n}$, and zigzag left to $y_2$, and then up to $z$ and $y_{2n}$, as shown in Figure 2. Formally,

$$y_1 - x_2 - x_1 - y_2 - \cdots - x_{2n-1} - x_2 - y_{2n-1} - x_2 - y_2 - \cdots - y_3 - z - y_{2n}.$$ (2)

If $s \geq 1$, for every integer $i \in \{1, \ldots, 2s\}$, label the vertices of $C_i$ in the clockwise order $u_i, u_{i+1}, \ldots, u_{2k+1}$. One can find a Hamilton path $P_i$ of $\mu(C_i)$ as follows:

$$x_1 y_2 x_3 y_4, \ldots, x_{2k+1}, y_1 y_2 y_3 y_4, \ldots, y_{2k+1},$$ (3)

where $u_i = v_i$ and $u_{2k+1} = v'_i$.

Let $C_{2s+1} = w_1 w_2 \cdots w_{2s}$ be an even cycle with chord $w_3 w_4$ in $H$. One can find a Hamilton path $P_{2s+1}$ of $\mu(C_i)$ as follows:

$$x_1 y_2 x_3 y_4, \ldots, y_{2i}, x_{2i} y_{2i+1} x_{2i+2}, \ldots, y_{2i}, y_1 y_2 y_3 y_4, \ldots, y_{2i-1},$$ (4)

where $w_i = v_i$ and $w_{3i} = v'_i$.

Thus, $(\cup_{i=1}^{s+1} P_i) \cup \{v_i v'_i : 1 \leq i \leq 2s\} \cup \{y_{2n-1} z\}$ is a Hamilton path of $\mu(G)$ from joining $x_1$ and $z$.

3. Hamiltonian Connectedness

**Theorem 4.** Assume that $G$ is a Hamilton-connected graph of order $n \geq 3$. If $\delta(G) \geq (n/2) + 1$, then $\mu(G)$ is Hamilton-connected.

**Proof.** Let $v$ be a vertex of $G$. We consider a Hamilton cycle $C$ of $G$. Let $u$ be a neighbor of $v$ on $C$. Since $d(u) \geq (n/2) + 1$, it has a neighbor at distance even on $C$. By Lemma 1, $\mu(G)$ is HC.

The $k$th power of a graph $G$, denoted by $G^k$, is a graph with the same vertex set as $G$ in which two vertices are adjacent if and only if their distance in $G$ is at most $k$. Thus,
Subcase 2.1: \( \min\{n_v, n_w\} = n_v = 2 \).
Subcase 2.1.1: \( \min\{n_v, n_w\} = n_w = 1 \).

Since \( n_v + n_w = n \) is an even number at least 3, \( n_w \) is an odd number at least 3. By Theorem 5, let \( C_{uvw} \) be a Hamilton cycle of \( T^3_w \) containing \( uuw \). It is easy to see that \( C_{uvw} + vw + vv'w \) is a Mycielski factor of \( T^3 \) starting at \( v \).

Subcase 2.1.2: \( \min\{n_v, n_w\} = n_v = 2 \).

If \( n_w = 2 \), then \( n = 4 \). Trivially, \( T^3_w = K_4 \) has a Mycielski factor starting at \( v \).

If \( n_w \neq 2 \), then \( n_w \) is an even number at least 4. By Theorem 5, let \( P_{vw} \) be a Hamilton path of \( T^3_w \). It can be seen that \( P_{vw} + vw + wv' + vv'w \) is a Mycielski factor starting at \( v \).

Subcase 2.2: \( \min\{n_v, n_w\} = n_w = 2 \).

If \( T \equiv K_{1, w-1} \), then \( T^3 \equiv K_w \) has a Mycielski factor starting at \( v \). Next, we assume that \( T \not\equiv K_{1, w-1} \). We can choose a neighbor \( w \) of \( v \) such that \( n_w \geq 2 \). Combining with our assumption that \( \min\{n_v, n_w\} = n_w \leq 2 \), we have \( n_w = 2 \). By Theorem 5, let \( P_{vv} \) be a Hamilton path of \( T^3_v \). It can be checked that \( P_{vv} + vv + wv' + vv'w \) is a Mycielski factor starting at \( v \).

In 1969, Watkins [19] introduced the notion of the generalized Petersen graph \( GP(n, k) \), \( 1 \leq k \leq n \), as follows. The vertex set is \( \{i, j | 1 \leq i \leq n\} \), and the edge set is \( \{ui, vj | i \equiv j \mod n\} \), where the subscript arithmetic performs modulo \( n \) using the residues \( 1, \ldots, n \). In 1971, Frucht et al. [20] showed that \( GP(n, k) \) is vertex-transitive if and only if \( k^2 \equiv \pm 1 \mod n \) or \( (n, k) = (10, 2) \). Next, we consider the Hamilton-connectedness of the Mycielski graph of the generalized Petersen graphs \( GP(n, 2) \) and \( GP(n, 3) \). 

Theorem 7 (see Alspach and Liu [21]). The generalized Petersen graph \( GP(n, 2) \) with \( n \geq 6 \) is Hamilton-connected if and only if \( n \equiv 1, 2, 3 \mod 6 \).

Theorem 8. If \( GP(n, 2) \) is Hamilton-connected for \( n \geq 6 \), then \( \mu(GP(n, 2)) \) is Hamilton-connected.

Proof. In view of Lemma 1, it suffices to show that \( GP(n, 2) \) has a Mycielski factor starting at any \( v \in V(GP(n, 2)) \). We consider two cases:

Case 1: \( n \equiv 1 \) or \( 3 \mod 6 \).

Since \( n \) is odd, \( GP(n, 2) \) is vertex-transitive, and we may assume that \( v = u_1 \), without loss of generality. Let \( C_1 \) and \( C_2 \) be the outer cycle and inner cycle of \( GP(n, 2) \). Let \( v \) be a vertex of \( GP(n, 2) \). It is clear that \( C_1 \cup C_2 + u_1v_2 \) is a Mycielski factor of \( GP(n, 2) \) starting at \( v \).

Case 2: \( n \equiv 2 \mod 6 \).

By the symmetry, it suffices to tackle two possibilities according to the location of \( v \) in \( GP(n, 2) \): \( v \) lies on the outer cycle or inner cycle of \( GP(n, 2) \). Without loss of generality, let \( v = u_t \) or \( v = v_1 \).

First, for the case when \( n = 8 \), we can find a Mycielski factor \( F_n \) of \( GP(n, 2) \) as follows:
By the symmetry, it suffices to tackle two possibilities according to the location of \( v \) in \( GP(n, 3) \): \( v \) lies on the outer cycle or inner cycle of \( GP(n, 3) \). Without loss of generality, let \( v = u_1 \) or \( v = v_1 \).

First, for the case when \( n = 9 \), we can find a Mycielski factor \( F_n \) of \( GP(n, 3) \) starting at \( v \) as follows:

\[
F_9 = \begin{cases} 
C_9 + u_1u_7 & \text{if } v = u_1, \\
C_9 + u_1u_2 & \text{if } v = v_1,
\end{cases}
\]

where

\[
C_9 = u_1v_1v_2u_3v_4u_5v_3v_5u_6v_4v_7u_7u_8v_8u_6v_9u_3u_4u_5u_6v_4v_3u_2u_1
\]

For \( n = 14 \), by inserting 12 new vertices to \( C_8 \) of \( F_8 \), we get \( C_{14} \) as illustrated in Figures 5–7 for the case that \( v \) lies in the outer cycle and for the case that \( v \) lies in the inner cycle as illustrated in Figures 6, 8, and 9. For the case when \( n \geq 20 \), by inserting 12 new vertices to \( F_{n-6} \) with type A insertion, we obtain a Mycielski factor \( F_n \) of \( GP(n, 2) \) starting at \( v \).

\[\text{Theorem 9} \text{ (see Alspach and Liu [21]). The generalized Petersen graph } GP(n, 3) \text{ with } n \geq 6 \text{ is Hamilton-connected if and only if } n \text{ is odd.}\]

\[\text{Theorem 10. If } GP(n, 3) \text{ is Hamilton-connected, then } \mu(GP(n, 3)) \text{ is Hamilton-connected.}\]

\[\text{Proof. Since } GP(n, 3) \text{ is Hamilton-connected, by Theorem 9, } n \text{ is an odd number at least 7. Let } v \text{ be a vertex of } GP(n, 3). \text{ In view of Lemma 1, it suffices to show that } GP(n, 3) \text{ has a Mycielski factor starting at } v. \text{ We consider two cases:}\]

Case 1: \( n \equiv 1 \) or \( 5 \) (mod 6).

Since \( n \) is odd, \( GP(n, 3) \) is vertex-transitive; by the symmetry, we may assume that \( v = u_1 \), without loss of generality. Let \( C_1 \) and \( C_2 \) be the outer cycle and inner cycle of \( GP(n, 3) \). It is clear that \( C_1 \cup C_2 + u_2v_2 \) is a Mycielski factor of \( GP(n, 3) \) starting at \( v \).

Case 2: \( n \equiv 3 \) (mod 6).

Next, we will find a cycle of length \( n + k \) in \( \mu(G) \) for each \( k \in \{3, \ldots, n + 1\} \). Take a Hamilton cycle \( C \). Without loss of
generality, let $C = v_1 v_2, \ldots, v_n v_1$ in $G$. We consider two cases according to the parity of $k$:

Case 1: $k$ is odd.

One can find a cycle of length $n+k$ in $\mu(G)$, as shown in Figure 14. Formally, it is

$$x_1 - y_2 - x_3 - \cdots - y_{k-1} - z - y_1 - \cdots - x_{k-1} - x_k - \cdots - x_n - x_1. \quad (10)$$

Case 2: $k$ is even.

Zigzag up from $x_1$ to $x_{k-1}$ and left to $x_{k-2}$, then zigzag left to $y_1$, $z$, $y_{k-1}$, and $x_k$, and go right to $x_n$ and back to $x_1$, as shown in Figure 15. Formally, it is

$$x_1 - \cdots - x_{k-2} - x_{k-1} - y_{k-2} - \cdots - y_1 - z - y_{k-1} - x_k - \cdots - x_n - x_1. \quad (11)$$
Figure 10: $F_9$ from $u_1$ to $v_1$ in $GP(9, 3)$.

Figure 11: Type $B$ insertion.

Figure 12: $F_{15}$ obtained from $F_9$ by type $B$ insertion in $GP(9, 3)$.

Figure 13: Finding a cycle $C_{n+1}$ in $\mu(G)$ from a cycle $C_{n-2}$ in $G$.

Figure 14: A cycle of length $n + k$ in $\mu(G)$ if $k$ is odd.

Figure 15: A cycle of length $n + k$ in $\mu(G)$ if $k$ is even. □
5. Conclusion

In this paper, we introduce the notion of the Mycielski factor of a graph. If a graph $G$ has a Mycielski factor starting at $v$ for any $v \in V(G)$, then $\mu(G)$ is Hamilton-connected. Applying this result, we are able to show that if a graph $G$ belongs to three (well-defined) families of graphs, then $\mu(G)$ is Hamilton-connected. However, the full conjecture of Jar-nicki, Myrvold, Saltzman, and Wagon is not yet solved. We also prove that if $G$ is pancyclic, then $\mu(G)$ is pancyclic.

One of the reviewers proposed the following two interesting problems.

Zhong et al. [7] showed that the line graph of the generalized Petersen graph $GP(n,k)$ is always Hamilton-connected. Is it easy to show that the Mycielski graph of $L(GP(n,k))$ is Hamilton-connected?

It is known that the line graph of a Hamilton-connected graph $G$ is also Hamilton-connected. Is $\mu(L(G))$ Hamilton-connected if $L(G)$ is Hamilton-connected? [22].

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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