Research Article

On the Multiplicity of a Proportionally Modular Numerical Semigroup

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A proportionally modular numerical semigroup is the set $S(a, b, c)$ of nonnegative integer solutions to a Diophantine inequality of the form $ax \bmod b \leq cx$, where a, b , and c are positive integers. A formula for the multiplicity of $S(a, b, c)$, that is, $m(S(a, b, c)) = \lfloor kb/a \rfloor$ for some positive integer k , is given by A. Moscardiello. In this paper, we give a new proof of the formula and determine a better bound for k . Furthermore, we obtain $k = 1$ for various cases and a formula for the number of the triples (a, b, c) such that $k \neq 1$ when the number $a - c$ is fixed.

1. Introduction and Preliminaries

Given two integers m and n with $n \neq 0$, we denote by $m \bmod n$ the remainder of the division of m by n . A proportionally modular Diophantine inequality is an expression of the form $ax \bmod b \leq cx$, where a, b , and c are positive integers. In [1, 2], it is shown that the set $S(a, b, c)$ of nonnegative integer solutions of the former inequality is a numerical semigroup; that is, it is a subset of the set $\mathbb{N}$ of the nonnegative integers that is closed under addition, contains 0, and whose complement in $\mathbb{N}$ is finite. This type of numerical semigroups is known as proportionally modular numerical semigroups. If S is a numerical semigroup and d is a positive integer, then we denote by S/d the set $\{x \in \mathbb{N} \mid dx \in S\}$. It is clear that S/d is a numerical semigroup containing S . We say that S/d is the quotient of S by d . Given two relatively prime positive integers a_1 and a_2 , we denote by $\langle a_1, a_2 \rangle$ the numerical semigroup generated by them, that is, $\langle a_1, a_2 \rangle = \{s_1 a_1 + s_2 a_2 \mid s_1, s_2 \in \mathbb{N}\}$. From Theorem 6.18 in [1], we know that $\langle a_1, a_2 \rangle/d$ is a proportionally modular numerical semigroup, and every proportionally modular numerical semigroup can be represented in this form. Let S

be a numerical semigroup. We say that S has a Toms decomposition if there exist $q_1, \dots, q_n, m_1, \dots, m_n$ and L such that (1) $\gcd(q_i, m_i) = \gcd(L, q_i) = \gcd(L, m_i) = 1$ for all $i \in \{1, \dots, n\}$ and (2) $S = (1/L) \cap_{i=1}^n \langle q_i, m_i \rangle$. Let a, b , and c be positive integers. We say that the numerical semigroup $\langle a, b \rangle/c$ is a Toms block if $\gcd(a, b) = \gcd(a, c) = \gcd(b, c) = 1$. Observe that $(1/L) \cap_{i=1}^n \langle q_i, m_i \rangle = \cap_{i=1}^n (\langle q_i, m_i \rangle/L)$. So, a numerical semigroup admits a Toms decomposition if and only if it can be expressed as an intersection of finitely many Toms blocks with the same denominator. In [1, 3], it is shown that numerical semigroups having a Toms decomposition are precisely those numerical semigroups that can be expressed as a finite intersection of proportionally modular numerical semigroups.

As usual, for a rational number r , $\lceil r \rceil$ denotes the least integer not smaller than r and $\lfloor r \rfloor$ denotes the greatest integer not bigger than r . If S is a numerical semigroup, then the least positive integer that belongs to S is an important invariant of S called the multiplicity of S , and we denote it by $m(S)$ (see, for example, [1]). Some partial results on the invariant of a proportionally modular numerical semigroup

are given in [4, 5]. In particular, a formula, that is, $m(S(a, b, c)) = \lceil kb/a \rceil$ where k is some positive integer, is given by Moscariello in [6].

The contents of this paper are organized as follows. In Section 2, we give a clearer proof on the formula given in [6] and the bound of k . In Section 3, we obtain $k = 1$ for various cases and a formula for the number of the triples (a, b, c) such that $k \neq 1$ when the number $a - c$ is fixed. In Section 4, we describe an algorithm that allows us to calculate the multiplicity of $S(a, b, c)$ and apply the algorithm to some questions.

2. A Formula for $m(S(a, b, c))$

As the inequality $ax \bmod b \leq cx$ has the same set of integer solutions as $(a \bmod b)x \bmod b \leq cx$, we do not lose generality by supposing that $a < b$. If $c \geq a$, then $S(a, b, c) = \mathbb{N}$; thus, we may also suppose that $c < a$. In this paper, unless otherwise stated, we suppose that a, b , and c are positive integers satisfying $c < a < b$.

Lemma 1 (Proposition 1 [7]). $S(a, b, c) = S(b + c - a, b, c)$.

The following lemma has as a consequence that we may also suppose that $a \leq (b + c)/2$.

Remark 1

- (1) Under the above notations, we have $m(S(a, b, c)) < b$. In fact, for any positive integer x , if $cx \geq b - 1$, then $x \in S(a, b, c)$. It follows that $m(S(a, b, c)) \leq (b - 1)/c \leq b - 1 < b$.
- (2) Let a and b be positive integers such that $a < b$. Then, for each integer k there exists exactly one integer α_k such that $kb \leq a\alpha_k < kb + a$. Moreover, $\alpha_k = \lceil kb/a \rceil$. Indeed, since $0 \leq (-kb) \bmod a < a$, we have $kb \leq kb + (-kb) \bmod a < kb + a$. Therefore, $kb \leq a\lceil kb/a \rceil < kb + a$. Uniqueness follows from the fact that the set $\{kb, kb + 1, \dots, kb + a - 1\}$ consists of a consecutive integers and therefore contains exactly one multiple of a .

As a consequence of the uniqueness of α_k in Remark 1 (2), we obtain the following result.

Lemma 2. Let a and b be positive integers such that $a < b$, and let k be an integer. Then, $a\lceil kb/a \rceil$ is the least multiple of a in the set $\{kb, kb + 1, \dots, kb + b - 1\}$.

Remark 2. For an integer x , if $kb \leq ax < (k + 1)b$, then $\lfloor ax/b \rfloor = k$. Thus, $ax \bmod b = ax - kb$.

Now, we are ready to prove the result given in [6] concerning the form of the multiplicity of $S(a, b, c)$.

Theorem 1 (Proposition 3 in [6]). Under the above hypothesis and notations, we have $m(S(a, b, c)) \in \{\lceil kb/a \rceil \mid k \in \{1, 2, \dots, a - 1\}\}$.

Proof. Let $m = m(S(a, b, c))$, then $m < b$ by Remark 1 (1). Thus, there exists $k \in \{0, 1, \dots, a - 1\}$ such that $kb \leq am < (k + 1)b$, i.e., $m \geq \lceil kb/a \rceil$. If $m > \lceil kb/a \rceil$, then $m - 1 \geq \lceil kb/a \rceil$, and we have $kb \leq a(m - 1) < (k + 1)b$. By Remark 2, we have $a(m - 1) \bmod b = a(m - 1) - kb = am - kb - a = am \bmod b - a$. Since $m \in S(a, b, c)$, we have $am \bmod b \leq cm$. Thus $a(m - 1) \bmod b = am \bmod b - a \leq cm - a < cm - c = c(m - 1)$. Consequently, $m - 1 \in S(a, b, c)$. From the fact that m is the multiplicity of $S(a, b, c)$, we have $m - 1 = 0$ and so $m = 1$. Thus, $a = a \bmod b \leq c$ which contradicts that $c < a$. Hence, $m = \lceil kb/a \rceil$.

Finally, if $k = 0$, then $m = \lceil 0b/a \rceil = 0$ which contradicts the definition of m . Therefore, k must not be 0, and this completes the proof. $\square$

From the preceding theorem, it follows that $m(S(a, b, c)) = \lceil \xi b/a \rceil$, where $\xi = \min\{k \in \{1, 2, \dots, a - 1\} \mid \lceil kb/a \rceil \in S(a, b, c)\}$. The following three lemmas will allow us to reformulate this fact.

Lemma 3. Let $k \in \{1, 2, \dots, a - 1\}$. Then, $\lceil kb/a \rceil \in S(a, b, c)$ if and only if $(-kb) \bmod a \leq kbc/(a - c)$.

Proof. Our conclusion can be obtained from the following deduction:

$$\begin{aligned} \lceil \frac{kb}{a} \rceil \in S(a, b, c) &\Leftrightarrow a\lceil \frac{kb}{a} \rceil \bmod b \leq c\lceil \frac{kb}{a} \rceil, \\ &\Leftrightarrow a\lceil \frac{kb}{a} \rceil \bmod b = a\lceil \frac{kb}{a} \rceil - kb \leq c\lceil \frac{kb}{a} \rceil \\ &\Leftrightarrow (a - c)\lceil \frac{kb}{a} \rceil \leq kb \\ &\Leftrightarrow (a - c)\frac{kb + (-kb) \bmod a}{a} \leq kb \\ &\Leftrightarrow (-kb) \bmod a \leq \frac{kbc}{a - c}. \end{aligned} \quad (1)$$

$\square$

Lemma 4. Under the above notations, we have $\xi \leq \lfloor a(1 - (1/b))/c \rfloor$.

Proof. According to Remark 1 (1), we know that $m(S(a, b, c)) \leq (b - 1)/c$. It follows that $\lceil \xi b/a \rceil \leq (b - 1)/c$ and so $(\xi b/a) \leq (b - 1)/c$. Thus, $\xi \leq (a(1 - (1/b)))/c$. Since ξ is a positive integer, we have $\xi \leq \lfloor a(1 - (1/b))/c \rfloor$. $\square$

Lemma 5. $\lfloor a(1 - (1/b))/c \rfloor \leq a - 1$, where the equality holds if and only if $c = 1$.

Proof. Observing that $\lfloor a(1 - (1/b))/c \rfloor \leq a(1 - (1/b))/c < a$, we have $\lfloor a(1 - (1/b))/c \rfloor \leq a - 1$. If $c = 1$, then $\lfloor a(1 - (1/b))/c \rfloor = \lfloor a(1 - (1/b)) \rfloor = \lfloor a - (a/b) \rfloor = a - 1$. Conversely, if the equality holds, $(a - 1) \leq (a(1 - (1/b)))/c$. Thus, $c \leq (a - (a/b))/(a - 1) = 1 + (b - a)/(b(a - 1))$. Since c is a

positive integer, we have $c \leq \lfloor 1 + (b-a)/(b(a-1)) \rfloor = 1$ and so $c = 1$. $\square$

Therefore, we can reformulate the fact in Theorem 1 as follows.

Corollary 1. *Under the above notations, we have $m(S(a, b, c)) = \lceil \xi b/a \rceil$, where $\xi = \min\{k \in \{1, 2, \dots, \lfloor a(1 - (1/b)/c) \rfloor\} \mid (-kb) \bmod a \leq kbc/(a-c)\}$.*

3. The Cases of $m(S(a, b, c)) = \lceil b/a \rceil$ and $m(S(a, b, c)) \neq \lceil b/a \rceil$

In this section, we study the cases of $k = 1$ and the number of the triples (a, b, c) such that $k \neq 1$ if the number $a - c$ is fixed.

As a consequence of Corollary 1 and the proof of Lemma 3, we have the following result.

Theorem 2. *$m(S(a, b, c)) = \lceil b/a \rceil$ if and only if $(-b) \bmod a \leq bc/(a-c)$ if and only if $\lceil b/a \rceil \leq b/(a-c)$.*

Corollary 2. *If a, b , and c satisfy one of the following conditions:*

- (1) $a - c \leq (ab/(a+b-1))$,
- (2) $a - c \leq c + 1$,

then $m(S(a, b, c)) = \lceil b/a \rceil$.

Proof

- (1) If $a - c \leq ab/(a+b-1)$, then $bc \geq (a-1)(a-c)$. Thus, $bc/(a-c) \geq a-1 \geq (-b) \bmod a$. By Theorem 2, we have $m(S(a, b, c)) = \lceil b/a \rceil$.
- (2) If $a - c \leq c + 1$, then $bc/(a-c) \geq bc/(c+1)$. Since $a < b$, we have $b \geq a + 1$. Thus, $bc/(c+1) \geq ((a+1)c)/(c+1) = a - (a-c)/(c+1) \geq a-1 \geq (-b) \bmod a$. It follows from Theorem 2 that $m(S(a, b, c)) = \lceil b/a \rceil$. $\square$

Remark 3. Since c is a positive integer, $c + 1 \geq 2$. If $a - c = 1$ or 2, then $m(S(a, b, c)) = \lceil b/a \rceil$.

Example 1

- (1) Let $a = 13, b = 70$, and $c = 4$. Then, $S(a, b, c) = \{0, 6, 7, 11, 12, 13, 14, 15\} \cup \{x \in \mathbb{N} \mid x \geq 17\}$. As $(a-c)(a+b-1) = 9 \times 82 = 738 \leq 910 = 13 \times 70 = ab$, it follows that $m(S(a, b, c)) = \lceil b/a \rceil = \lceil 70/13 \rceil = 6$.
- (2) Let $(a, b, c) = (5, 42, 2)$. Then, $S(a, b, c) = \{0, 9, 10, 11, 12, 13, 14\} \cup \{x \in \mathbb{N} \mid x \geq 17\}$. Since $a - c = 3 = c + 1$, we have that $m(S(a, b, c)) = \lceil b/a \rceil = \lceil 42/5 \rceil = 9$.

Theorem 3. *Let $a - c = d$. Then, the number N_d of the triples (a, b, c) such that $m(S(a, b, c)) \neq \lceil b/a \rceil$ is given by*

$$N_d = \frac{1}{2} \sum_{i=1}^{d-2} \left\lfloor \frac{d-1}{i} \right\rfloor [d-1-i+(d-1) \bmod i]. \quad (2)$$

Proof. If $a - c = d$, then $a \geq d + 1$. By Corollary 2 (2), if $m(S(a, b, c)) \neq \lceil b/a \rceil$, then $a < 2d - 1$. Thus, $a \in \{d+1, d+2, \dots, 2d-2\}$. Let $a = d+i$ (i.e., $c = i$) for some $i \in \{1, 2, \dots, d-2\}$ and n_i be the number of b such that $m(S(a, b, c)) \neq \lceil b/a \rceil$. By Theorem 2, we know that $m(S(d+i, b, i)) = \lceil b/a \rceil$ if and only if $\lceil b/(d+i) \rceil \leq b/d$, that is, $m(S(d+i, b, i)) \neq \lceil b/a \rceil$ if and only if $j(d+i) < b < (j+1)d$ for some positive integer j (notice that $b > d+i$). For a positive integer j , we obtain the number $d-1-ij$ of b satisfying this condition between $j(d+i)$ and $(j+1)d$. Since $d-1-ij$ is a natural number, we have $j \leq \lfloor (d-1)/i \rfloor$. Thus, $n_i = \sum_{j=1}^{\lfloor (d-1)/i \rfloor} (d-1-ij)$. Therefore,

$$\begin{aligned} N_d &= \sum_{i=1}^{d-2} n_i, \\ &= \sum_{i=1}^{d-2} \sum_{j=1}^{\lfloor (d-1)/i \rfloor} (d-1-ij) \\ &= (d-1) \sum_{i=1}^{d-2} \left\lfloor \frac{d-1}{i} \right\rfloor - \sum_{i=1}^{d-2} \sum_{j=1}^{\lfloor (d-1)/i \rfloor} ij \\ &= (d-1) \sum_{i=1}^{d-2} \left\lfloor \frac{d-1}{i} \right\rfloor - \sum_{i=1}^{d-2} i \frac{\lfloor (d-1)/i \rfloor (\lfloor (d-1)/i \rfloor + 1)}{2} \\ &= \sum_{i=1}^{d-2} \left\lfloor \frac{d-1}{i} \right\rfloor \left[d-1 - \frac{1}{2} i \left(\left\lfloor \frac{d-1}{i} \right\rfloor + 1 \right) \right] \\ &= \frac{1}{2} \sum_{i=1}^{d-2} \left\lfloor \frac{d-1}{i} \right\rfloor [d-1-i+(d-1) \bmod i]. \end{aligned} \quad (3)$$

This completes our proof. $\square$

Remark 4. By the formula in Theorem 3, we obtain that $N_3 = 1$ and the triple $(a, b, a-3)$ satisfying $m(S(a, b, a-3)) \neq \lceil b/a \rceil$ is exactly $(4, 5, 1)$. Moreover, $N_4 = 4, N_5 = 9, N_6 = 17$, and $N_7 = 27$. This gives the integer sequence A078567 (see [8]) 1, 4, 9, 17, 27, 41, 57, 77, ... In fact, we know from [8] that $a_n = \sum_{i=1}^{n-1} \sum_{j=1}^{\lfloor (n-1)/i \rfloor} (n-ij)$ is one of the formulations of the integer sequence A078567. If $i \mid n$, then $\lfloor n/i \rfloor = n/i = \lfloor (n-1)/i \rfloor + 1$ and $n - i \lfloor n/i \rfloor = n - i(n/i) = 0$. Otherwise, $\lfloor n/i \rfloor = \lfloor (n-1)/i \rfloor$. Thus, $a_n = \sum_{i=1}^{n-1} \sum_{j=1}^{\lfloor n/i \rfloor} (n-ij)$, which is consistent with the formula of N_d from the second equality in the proof of Theorem 3.

4. The Algorithm and Some Applications

In this section, our aim is to give an algorithm to compute the multiplicity of a proportionally modular numerical

OUTPUT: $m(S(a, b, c))$.
INSTRUCTIONS:
 (1) $a := a \bmod b$.
 (2) If $a > (b + c)/2$, then $a := b + c - a$.
 (3) If $a \leq c$, then return 1.
 (4) If $a|b$, then return b/a .
 (5) $k := 1$.
 (6) while $(-kb) \bmod a > kbc/(a - c)$ do $k := k + 1$.
 (7) return $\lceil kb/a \rceil$.

ALGORITHM 1: **INPUT:** positive integers a, b , and c .

semigroup and apply this algorithm to solve some questions on numerical semigroups.

Lemma 6 (Proposition 2 (2) in [6]). *If $a|b$, then $m(a, b, c) = b/a$.*

Remark 5. The observations made at the beginning of Section 2 and the fact $m(\mathbb{N}) = 1$ justify the steps (1), (2), and (3). Step (4) is a consequence of Lemma 6. Finally, steps (5), (6), and (7) are consequences of Corollary 1. Also by Corollary 1, we know that the Algorithm 1 stops after a finite number of steps.

At last, we apply the given algorithm to three questions on numerical semigroups. Given two relatively prime positive integers a_1 and a_2 , we know from Lemma 18 in [5] that $\langle a_1, a_2 \rangle = \{x \in \mathbb{N} | a_2^{-1}a_2x \bmod a_1a_2 \leq x\}$, where $a_2^{-1} \in \{i \in \mathbb{N} | 0 \leq i < a_1, ia_2 \equiv 1 \pmod{a_1}\}$. Given two relatively prime positive integers a_1 and a_2 and a positive integer d , it is an open problem (see [1]) to find a formula for the smallest multiple of d that belongs to $\langle a_1, a_2 \rangle$.

Remark 6. Consider the quotient $\langle a_1, a_2 \rangle/d = \{x \in \mathbb{N} | dx \in \langle a_1, a_2 \rangle\} = \{x \in \mathbb{N} | a_2^{-1}a_2dx \bmod a_1a_2 \leq dx\}$ whose multiplicity is $m(\langle a_1, a_2 \rangle/d) = m(S(a_2^{-1}a_2d, a_1a_2, d))$, and therefore it can be calculated by applying Algorithm 1.

Given a nonempty subset A of $\mathbb{Q}_0^+$ (here, $\mathbb{Q}_0^+$ is the set of nonnegative rational numbers), we denote by $\langle A \rangle$ the submonoid of $(\mathbb{Q}_0^+, +)$ generated by A , that is, $\langle A \rangle = \{\lambda_1a_1 + \dots + \lambda_na_n | n \in \mathbb{N}_+, \lambda_1, \dots, \lambda_n \in \mathbb{N} \text{ and } a_1, \dots, a_n \in A\}$. Clearly, $S(A) = \langle A \rangle \cap \mathbb{N}$ is a submonoid of $\mathbb{N}$. Let p and q be two positive rational numbers with $p < q$. We use the notations $[p, q] = \{x \in \mathbb{Q} | p \leq x \leq q\}$ and $]p, q[= \{x \in \mathbb{Q} | p < x < q\}$. It is known that $S([p, q])$ and $S(]p, q[)$ are proportionally modular numerical semigroups. If S is a numerical semigroup, then the greatest integer that does not belong to S is called the Frobenius number of S and denoted here by $F(S)$. In [9], a relationship between $F(S(a, b, 1))$ and $m(S([b/a, b/(a-1)])$ is given by the following three lemmas.

Lemma 7 (Proposition 1 in [9])

$$(1) S([b/a, b/(a-c)]) = S(a, b, c)$$

(2) *Conversely, if a_1, b_1, a_2 , and b_2 are positive integers such that $(b_1/a_1) < (b_2/a_2)$, then $S([(b_1/a_1), (b_2/a_2)]) = S(a_1b_2, b_1b_2, a_1b_2 - a_2b_1)$*

Lemma 8 (Theorem 9 in [9]). $S([b/a, b/(a-1)]) = S([(2b^2+1)/2ab], ((2b^2-1)/2b(a-1)))$.

Lemma 9 (Theorem 18 in [9]). *Let $S = S([b/a, b/(a-1)])$ and $T = S([b/a, b/(a-1)])$. Then, $F(S) = b - m(T)$.*

By Lemma 7 (1), we have $S(a, b, 1) = S([b/a, b/(a-1)])$. Moreover, Lemmas 8 and 9 assert that $S([b/a, b/(a-1)]) = S(4ab^3 - 2ab, 4b^4 - 1, 4b^3 - 4ab + 2b)$. Therefore, $F(S(a, b, 1)) = b - m(S(4ab^3 - 2ab, 4b^4 - 1, 4b^3 - 4ab + 2b))$. Next, we generalize the result to $S(a, b, c)$.

Theorem 4. *Let $c|a$ and $c|b$. Then, $F(S(a, b, c)) = (b/c) - m(S(4ab^3 - 2abc, 4b^4 - c^2, 4b^3c - 4abc + 2bc^2))$.*

Proof. Let $a = ca_1$ and $b = cb_1$. Then, $ax \bmod b \leq cx$ if and only if $a_1x \bmod b_1 \leq x$. It follows that $S(a, b, c) = S(a_1, b_1, 1) = S((a/c), (b/c), 1)$. Therefore, $F(S(a, b, c)) = (b/c) - m(S(4ab^3 - 2abc, 4b^4 - c^2, 4b^3c - 4abc + 2bc^2))$. $\square$

Remark 7. From Theorem 4, we can apply Algorithm 1 to compute $F(S(a, b, c))$ if c is a common divisor of a and b .

Let S be a numerical semigroup minimally generated by $\{n_1, n_2\}$. Sylvester proved in [10] that $F(S) = n_1n_2 - n_1 - n_2$. If S is minimally generated by $\{n_1, n_2, \dots, n_p\}$, with $p > 2$, then it is still an open problem to find a formula for $F(S)$. Next, we apply our algorithm to compute the Frobenius number of a numerical semigroup minimally generated by $\{n_1, n_2, n_3\}$.

Let n_1, n_2 , and n_3 be three positive integers such that $\gcd(n_1, n_2, n_3) = 1$ and $n_1 > n_2 > n_3$, denoted by S the numerical semigroup $S = \langle n_1, n_2, n_3 \rangle$. In the study of $F(S)$, we can assume that $\gcd(n_i, n_j) = 1$, for $i \neq j$ with $i, j \in \{1, 2, 3\}$. This is due to the following result.

Lemma 10 (Theorem 3 in [11]). *Let $n_1, n_2, n_3 \in \mathbb{N}_+$ and $d_{ij} = \gcd(n_i, n_j)$ for every $\{i, j\} \subset \{1, 2, 3\}$ with $i \neq j$. Define*

$m_1, m_2, m_3 \in \mathbb{N}_+$ such that $n_i = m_i d_{ij} d_{ik}$ for every $\{i, j, k\} = \{1, 2, 3\}$. Then,

$$F(\langle n_1, n_2, n_3 \rangle) = d_{12} d_{23} d_{31} F(\langle m_1, m_2, m_3 \rangle). \quad (4)$$

Remark 8. Denote by L_i the positive integer $\min\{x \in \mathbb{N}_+ | x n_i \in \langle n_j, n_k \rangle\}$, with $\{i, j, k\} = \{1, 2, 3\}$. In Theorem 3.4 in [12], it is proved that

$$F(S) = L_1 n_1 + \max\{(L_2 n_2 n_3^{-1} \bmod n_1) n_3, (L_3 n_3 n_2^{-1} \bmod n_1) n_2\} - n_1 - n_2 - n_3. \quad (5)$$

Observe that, by applying Algorithm 1, we obtain L_1, L_2 , and L_3 . Thus, we have $F(\langle n_1, n_2, n_3 \rangle)$ by the previous formula.

Data Availability

The data used to support this study are available upon request to the corresponding author.

Conflicts of Interest

The author declares that there are no conflicts of interest.

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