Research Article

Numerical Solution of Nonlinear Stochastic Itô–Volterra Integral Equations Driven by Fractional Brownian Motion Using Block Pulse Functions

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This paper presents a valid numerical method to solve nonlinear stochastic Itô–Volterra integral equations (SIVIEs) driven by fractional Brownian motion (FBM) with Hurst parameter $H \in ((1/2), 1)$. On the basis of FBM and block pulse functions (BPFs), a new stochastic operational matrix is proposed. The nonlinear stochastic integral equation is converted into a nonlinear algebraic equation by this method. Furthermore, error analysis is given by the pathwise approach. Finally, two numerical examples exhibit the validity and accuracy of the approach.

1. Introduction

Stochastic equations have been widely used in engineering, economic management, biological sciences, finance, etc. A lot of problems in these areas are modeled by stochastic Volterra integral equations. However, the vast majority of

these equations do not have exact solutions, so it is meaningful to discuss numerical solutions of the equations [1, 2, 3, 4, 5, 6, 7].

In the present work, we use BPFs to solve the following nonlinear SIVIEs driven by FBM:

$$x(u) = x_0(u) + \int_0^u g_1(s, u) \nu(x(s)) ds + \int_0^u g_2(s, u) \kappa(x(s)) dB_s^H, \quad u \in [0, 1], \quad (1)$$

where $g_1(s, u)$ and $g_2(s, u)$ are kernel functions, for $s \in [0, u]$. $x(u)$ is an unknown stochastic process, B_s^H is FBM with $H \in ((1/2), 1)$, and the stochastic integral is pathwise Riemann–Stieltjes (R–S) integral. They are all defined on the same probability space $(\Omega, \mathcal{F}, P)$. ν and κ are bounded analytic functions and satisfy Lipschitz conditions.

Although FBM is not semimartingale or Markov process, FBM can describe more phenomena than Brownian motion. This is also the main reason why FBM can be used in the Internet, traffic, weather, and so on. In the past two decades, many scholars have extensively researched the stochastic

equations driven by FBM [8–16]. According to [15], we only discuss the case where $H \in ((1/2), 1)$, and the stochastic integral in equation (1) is the pathwise R–S integral. In [17], Zähle used the fractional integration by parts formula to express the pathwise R–S integral in terms of fractional derivative operators. In [18], Nualart and Răşcanu used the pathwise R–S integral to study stochastic differential equations in regard to FBM. Pei and Xu [19] solved the stochastic equations about FBM with $H > (1/2)$ and standard Brownian motion by pathwise approach and stochastic average. Moreover, Mishura and Shevchenko studied the Euler

approximate solutions of stochastic differential equations in regard to FBM in [10]. Hu et al. [20] also gave the Euler scheme of stochastic differential equation with respect to FBM.

Since FBM is not martingale, the classical Itô formula is invalid. Hashemi and Khodabin [8] used Hat functions to solve nonlinear SIVIEs driven by FBM, but the error analysis is insufficient. On the contrary, Maleknejad et al. [3, 4], Sang et al. [6], and Ezzati et al. [12] studied numerical solution of stochastic equation using BPFs and gave the error analysis. In this paper, we employ BPFs to solve equation (1) and a high-precision numerical solution is acquired. In [10], authors prove that the rate of convergence for Euler approximations of solutions of pathwise SDEs driven by FBM with $H > (1/2)$ can be estimated by $O(\delta^{2H-1})$ (δ is the diameter of partition). Compared with [10], the convergence rate of the approximate solution obtained by this method may be faster when $H \rightarrow 1$. In addition, the calculation also is simpler.

This article's structure is as follows. In Section 2, the definitions and related properties of FBM and BPFs are given. In Section 3, the effective numerical method is used to solve equation (1). The error analysis is acquired in Section 4. In Section 5, the validity and accuracy of this approach are validated by two examples. Section 6 is the conclusion.

2. Preliminaries

FBM and BPFs have been extensively studied by many scholars. For more contents, please refer to [3, 4, 14, 16].

2.1. Fractional Brownian Motion. FBM is a stochastic process $(B_u^H, u \geq 0)$ with $H \in ((1/2), 1)$ on $(\Omega, \mathcal{F}, P)$, and B_u^H is a continuous Gaussian process that has zero mean and the following covariance function:

$$\text{Cov}(B_s^H, B_u^H) = \frac{1}{2} \{s^{2H} + u^{2H} - |u - s|^{2H}\}. \quad (2)$$

It satisfies the following properties:

- (i) $B_0^H = 0$ and $\mathbb{E}(B_u^H) = 0, u \geq 0$
- (ii) B_u^H has homogeneous increments
- (iii) $\mathbb{E}(B_u^H)^2 = u^{2H}, u \geq 0$
- (iv) B_u^H has continuous trajectories

It is obvious that B_u^H is a standard Brownian motion when $H = (1/2)$ in [4, 6].

In this paper, the stochastic integral of equation (1) is the stochastic pathwise R-S integral. For any $\alpha \in (1 - H, (1/2))$, the stochastic process $f = \{f_u, u \in [0, T]\}, T < \infty$, belongs to the space $W_0^{\alpha,1}(0, T)$ that is the space of measurable functions f on $[0, T]$ such that

$$\|f\|_{\alpha,1} := \int_0^u \frac{|f(s)|}{s^\alpha} ds + \int_0^u \int_0^s \frac{|f(s) - f(y)|}{(s - y)^{\alpha+1}} dy ds < \infty, \quad (3)$$

and then the integral exists (for the details, see [18]). And, the pathwise integral satisfies the following:

$$\left| \int_0^u f(s) dB_s^H \right| \leq \Lambda \left(\int_0^u \frac{|f(s)|}{s^\alpha} ds + \int_0^u \int_0^s \frac{|f(s) - f(y)|}{(s - y)^{\alpha+1}} dy ds \right), \quad (4)$$

where

$$\Lambda = \frac{1}{\Gamma(1 - \alpha)} \sup_{0 < s < u < T} \left| (D_{u-}^{1-\alpha} B_{u-})(s) \right| \quad (5)$$

is a random variable. It is well known that Λ has finite moments of all orders (see [19]).

2.2. Block Pulse Functions. The definition of BPFs is as follows:

$$\varphi_i(u) = \begin{cases} 1, & (i-1)h \leq u < h, i = 1, 2, \dots, m, \\ 0, & \text{otherwise,} \end{cases} \quad (6)$$

where m is a positive integer and $h = (1/m)$ and $u \in [0, 1)$.

The BPFs have the following basic properties:

(i) Disjointness:

$$\varphi_i(u) \varphi_j(u) = \delta_{ij} \varphi_i(u), \quad i, j = 1, 2, \dots, m, \quad (7)$$

where δ_{ij} is Kronecker delta.

(ii) Orthogonality:

$$\int_0^1 \varphi_i(u) \varphi_j(u) du = h \delta_{ij}, \quad i, j = 1, 2, \dots, m. \quad (8)$$

(iii) Completeness: if the function f is square integrable, then

$$\int_0^1 f^2(u) du = \lim_{m \rightarrow \infty} \sum_{n=1}^m f_n^2 \|\varphi_n(u)\|^2, \quad (9)$$

where $f_n = (1/h) \int_0^1 f(u) \varphi_n(u) du$ and the above equation is called Parseval's identity.

The vector form of BPFs is as follows:

$$\Phi(u) = (\varphi_1(u), \varphi_2(u), \dots, \varphi_m(u))^T, \quad (10)$$

$$\Phi(u) \Phi^T(u) = \begin{pmatrix} \varphi_1(u) & 0 & \cdots & 0 \\ 0 & \varphi_2(u) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \varphi_m(u) \end{pmatrix}_{m \times m}, \quad (11)$$

$$\Phi^T(u) \Phi(u) = 1,$$

$$\Phi(u) \Phi^T(u) F = \mathbf{D}_F \Phi(u),$$

where $F = (f_1, f_2, \dots, f_m)^T$ and $\mathbf{D}_F$ denotes a diagonal matrix whose diagonal entries are the constant vector F .

2.3. Function Approximation. The square integrable function $x(u)$ on $[0, 1)$ can be expressed as

$$x(u) \approx x_m(u) = \sum_{i=1}^m x_i \varphi_i(u) = X^T \Phi(u) = \Phi^T(u) X, \quad (12)$$

where

$$X = (x_1, x_2, \dots, x_m)^T. \quad (13)$$

Let $g(s, u) \in L^2([0, 1] \times [0, 1])$. It can also be approximated as

$$g(s, u) = \Phi^T(s) \mathbf{G} \Psi(u), \quad (14)$$

where $\Phi(s)$ and $\Psi(s)$ are m_1 and m_2 dimensional BPFs vectors, respectively, $\mathbf{G} = (g_{ij})_{m_1 \times m_2}$,

$$g_{ij} = \frac{1}{h_1 h_2} \int_0^1 \int_0^1 g(s, u) \varphi_i(s) \psi_j(u) ds du, \quad (15)$$

and $h_1 = (1/m_1)$ and $h_2 = (1/m_2)$.

2.4. Related Lemmas

Lemma 1 (see [4]). Suppose that $\Phi(u)$ is defined in (10); then, we have

$$\int_0^u \Phi(s) ds \simeq \mathbf{P} \Phi(u), \quad (16)$$

where

$$\mathbf{P} = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{m \times m}. \quad (17)$$

Lemma 2 (see [12]). The integral of $\Phi(u)$ with respect to FBM has the following approximation:

$$\int_0^u \Phi(s) dB_s^H \simeq \mathbf{P}_H \Phi(u), \quad (18)$$

where $\mathbf{P}_H$ is the stochastic operational matrix with

$$\mathbf{P}_H = \begin{pmatrix} B_{(h/2)}^H & B_h^H & B_h^H & \cdots & B_h^H \\ 0 & B_{(3h/2)}^H - B_h^H & B_{2h}^H - B_h^H & \cdots & B_{2h}^H - B_h^H \\ 0 & 0 & B_{(5h/2)}^H - B_{2h}^H & \cdots & B_{3h}^H - B_{2h}^H \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B_{((2m-1)h/2)}^H - B_{(m-1)h}^H \end{pmatrix}_{m \times m}. \quad (19)$$

3. Numerical Method

We solve equation (1) by BPFs in this section. To solve this integral equation, a lemma is given to deal with nonlinear analytic functions in the first place.

Lemma 3 (see [6, 7]). Let $v(u) = \sum a_j u^j$ and $\kappa(u) = \sum b_j u^j$ be the analytic functions and $j \in \mathbb{Z}^+$; then,

$$\begin{aligned} v(x_m(u)) &= v^T(X) \Phi(u), \\ \kappa(x_m(u)) &= \kappa^T(X) \Phi(u), \end{aligned} \quad (20)$$

where

$$\begin{aligned} v^T(X) &= (v(x_1), v(x_2), \dots, v(x_m)), \\ \kappa^T(X) &= (\kappa(x_1), \kappa(x_2), \dots, \kappa(x_m)). \end{aligned} \quad (21)$$

On the basis of (12) and (14), $x(u)$, $x_0(u)$, $g_1(s, u)$, and $g_2(s, u)$ have the following approximate forms, respectively:

$$\begin{aligned} x(u) &\simeq X^T \Phi(u) = \Phi^T(u) X, \\ x_0(u) &\simeq C^T \Phi(u) = \Phi^T(u) C, \end{aligned}$$

$$\begin{aligned} g_1(s, u) &\simeq \Phi^T(s) \mathbf{G}_1 \Psi(u) = \Psi^T(u) \mathbf{G}_1^T \Phi(s), \\ g_2(s, u) &\simeq \Phi^T(s) \mathbf{G}_2 \Psi(u) = \Psi^T(u) \mathbf{G}_2^T \Phi(s), \end{aligned} \quad (22)$$

where X and C are BPFs coefficients vectors and $\mathbf{G}_1$ and $\mathbf{G}_2$ are BPFs coefficients matrices similar to (15). For convenience, we put $m_1 = m_2 = m$. Substituting above equations into equation (1), we obtain

$$\begin{aligned} X^T \Phi(u) &\simeq C^T \Phi(u) + v^T(X) \int_0^u \Phi(s) \Phi^T(s) ds \mathbf{G}_1 \Phi(u) \\ &\quad + \kappa^T(X) \int_0^u \Phi(s) \Phi^T(s) dB_s^H \mathbf{G}_2 \Phi(u). \end{aligned} \quad (23)$$

Applying the operational matrixes $\mathbf{P}$ and $\mathbf{P}_H$ for BPFs derived in (16) and (18), equation (23) can be rewritten as

$$X^T \Phi(u) \simeq C^T \Phi(u) + v^T(X) \mathbf{M}_1 \Phi(u) + \kappa^T(X) \mathbf{M}_2 \Phi(u), \quad (24)$$

where $\mathbf{M}_1$ and $\mathbf{M}_2$ are $m \times m$ matrixes similar to [12]; see appendix, for details,

$$\begin{aligned}
\mathbf{M}_1 &= \frac{h}{2} \begin{pmatrix} g_{11}^1 & 2g_{12}^1 & 2g_{13}^1 & \cdots & 2g_{1m}^1 \\ 0 & g_{22}^1 & 2g_{23}^1 & \cdots & 2g_{2m}^1 \\ 0 & 0 & g_{33}^1 & \cdots & 2g_{3m}^1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g_{mm}^1 \end{pmatrix}_{m \times m}, \\
\mathbf{M}_2 &= \begin{pmatrix} g_{11}^2 B_{(h/2)}^H & g_{12}^2 B_h^H & g_{13}^2 B_h^H & \cdots & g_{1m}^2 B_h^H \\ 0 & g_{22}^2 (B_{(3h/2)}^H - B_h^H) & g_{23}^2 (B_{2h}^H - B_h^H) & \cdots & g_{2m}^2 (B_{2h}^H - B_h^H) \\ 0 & 0 & g_{33}^2 (B_{(5h/2)}^H - B_{2h}^H) & \cdots & g_{3m}^2 (B_{3h}^H - B_{2h}^H) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g_{mm}^2 (B_{((2m-1)h/2)}^H - B_{(m-1)h}^H) \end{pmatrix}_{m \times m},
\end{aligned} \tag{25}$$

with g_{ij}^1 and g_{ij}^2 are the same as (15).

Then,

$$X^T \simeq C^T + \nu^T(X) \mathbf{M}_1 + \kappa^T(X) \mathbf{M}_2. \tag{26}$$

4. Error Analysis

The error analysis of the presented method is provided in this section. Firstly, two important lemmas are shown.

Lemma 4 (see [3]). Let $e(s) = f(s) - f_m(s)$, where $f_m(s)$ is approximations of bounded function $f(s)$ by BPFs on $[0, 1)$. Then,

$$\|e(s)\|_{L^2([0,1])}^2 = \int_0^1 e^2(s) ds \leq O(h^2). \tag{27}$$

Lemma 5 (see [3]). Let $e(s, u) = g(s, u) - g_m(s, u)$, where $g_m(s, u)$ is approximations of bounded function $g(s, u)$ by BPFs on $c = [0, 1) \times [0, 1)$. Then,

$$\|e(s, u)\|_{L^2(c)}^2 = \int_0^1 \int_0^1 e^2(s, u) ds du \leq O(h^2). \tag{28}$$

Secondly, consider $e(u) = x(u) - x_m(u)$. We have

$$\begin{aligned}
e(u) &= x(u) - x_m(u) \\
&= x_0(u) - x_{0m}(u) \\
&\quad + \int_0^u [g_1(s, u) \nu(x(s)) - g_{1m}(s, u) \nu(x_m(s))] ds \\
&\quad + \int_0^u [g_2(s, u) \kappa(x(s)) - g_{2m}(s, u) \kappa(x_m(s))] dB_s^H,
\end{aligned} \tag{29}$$

where $x_m(u)$, $x_{0m}(u)$, $g_{1m}(s, u)$, and $g_{2m}(s, u)$ are approximation of $x(u)$, $x_0(u)$, $g_1(s, u)$, and $g_2(s, u)$ by BPFs, respectively.

Then, for each $R > 1$, we define the following stopping time τ_R :

$$\tau_R := \inf\{u \geq 0: \Lambda \geq R\} \wedge T. \tag{30}$$

Finally, we give the main theorem and prove it.

Theorem 1. Assume that bounded analytic functions $\nu(x(u))$ and $\kappa(x(u))$ satisfy Lipschitz conditions:

$$(I) \quad |\nu(q) - \nu(p)| \leq l_1 |q - p| \text{ and } |\kappa(q) - \kappa(p)| \leq l_3 |q - p|$$

$$(II) \quad |\nu(q)| \leq l_2 \text{ and } |\kappa(q)| \leq l_4$$

$$(III) \quad |g_1(s, u)| \leq l_5 \text{ and } |g_2(s, u)| \leq l_6$$

In (I)–(III), $q, p \in \mathbb{R}$, $l_i > 0, i = 1, \dots, 6$ are constants. Then,

$$\begin{aligned}
&\int_0^T \mathbb{E}(|e(u \wedge \tau_R)|^2) du \\
&= \int_0^T \mathbb{E}(|x(u \wedge \tau_R) - x_m(u \wedge \tau_R)|^2) du \\
&\leq C(R) O(h^{2-4\alpha}), \quad T \in [0, 1),
\end{aligned} \tag{31}$$

where $C(R)$ is a constant related to R .

Proof. According to (29), we obtain

$$\begin{aligned}
\mathbb{E}(|e(u \wedge \tau_R)|^2) &\leq 3 \left[\mathbb{E}(|x_0(u) - x_{0m}(u)|^2) \right. \\
&\quad + \mathbb{E} \left(\left| \int_0^{u \wedge \tau_R} [g_1(s, u \wedge \tau_R) \nu(x(s)) - g_{1m}(s, u \wedge \tau_R) \nu(x_m(s))] ds \right|^2 \right) \\
&\quad \left. + \mathbb{E} \left(\left| \int_0^{u \wedge \tau_R} [g_2(s, u \wedge \tau_R) \kappa(x(s)) - g_{2m}(s, u \wedge \tau_R) \kappa(x_m(s))] dB_s^H \right|^2 \right) \right] := 3J_1 + 3J_2 + 3J_3.
\end{aligned} \tag{32}$$

For the second item, using Lipschitz conditions and Cauchy-Schwarz inequality, we have

$$\begin{aligned}
J_2 &\leq \int_0^u \mathbb{E}(|g_1(s, u \wedge \tau_R) [\nu(x(s)) - \nu(x_m(s))] \\
&\quad + \nu(x_m(s)) [g_1(s, u \wedge \tau_R) - g_{1m}(s, u \wedge \tau_R)]|^2) ds \\
&\leq 2l_1^2 l_5^2 \int_0^u \mathbb{E}(|e(s)|^2) ds \\
&\quad + 2l_2^2 \int_0^u |g_1(s, u \wedge \tau_R) - g_{1m}(s, u \wedge \tau_R)|^2 ds.
\end{aligned} \tag{33}$$

$$\begin{aligned}
J_3 &\leq 2\mathbb{E} \left(\Lambda \int_0^u |g_2(s, u \wedge \tau_R) \kappa(x(s)) \right. \\
&\quad \left. - g_{2m}(s, u \wedge \tau_R) \kappa(x_m(s))| s^{-\alpha} ds \right)^2 \\
&\quad + 2\mathbb{E} \left(\Lambda \int_0^u \int_0^s |g_2(s, u \wedge \tau_R) \kappa(x(s)) \right. \\
&\quad \left. - g_{2m}(x(s)) \kappa(x_m(s)) - g_2(y, u \wedge \tau_R) \kappa(x(y)) \right. \\
&\quad \left. + g_{2m}(y, u \wedge \tau_R) \kappa(x_m(y))| (s-y)^{-\alpha-1} dy ds \right)^2 := J_{31} + J_{32}.
\end{aligned} \tag{34}$$

Then,

For the last item, by (4), we obtain

$$\begin{aligned}
J_{31} &\leq 2R^2 \mathbb{E} \left(\int_0^u |g_2(s, u \wedge \tau_R) [\kappa(x(s)) - \kappa(x_m(s))] \right. \\
&\quad \left. + \kappa(x_m(s)) [g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)]| s^{-\alpha} ds \right)^2 \\
&\leq \left[4R^2 l_3^2 l_6^2 \int_0^u \mathbb{E}(|e(s)|^2) ds + 4R^2 l_4^2 \right. \\
&\quad \left. \times \int_0^u |g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)|^2 ds \right] \int_0^u s^{-2\alpha} ds \\
&\leq 4R^2 l_3^2 l_6^2 C_0 \int_0^u \mathbb{E}(|e(s)|^2) ds \\
&\quad + 4R^2 l_4^2 C_0 \int_0^u |g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)|^2 ds, \\
J_{32} &\leq 2R^2 \mathbb{E} \left(\int_0^u \int_0^s |g_2(s, u \wedge \tau_R) [\kappa(x(s)) - \kappa(x_m(s))] \right. \\
&\quad \left. - g_2(y, u \wedge \tau_R) [\kappa(x(y)) - \kappa(x_m(y))] - \mu(x_m(y)) [g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)] \right. \\
&\quad \left. - g_2(y, u \wedge \tau_R) [\kappa(x(y)) - \kappa(x_m(y))] - \mu(x_m(y)) [g_2(y, u \wedge \tau_R) - g_{2m}(y, u \wedge \tau_R)]| (s-y)^{-\alpha-1} dy ds \right)^2 \\
&\leq 8R^2 l_3^2 l_6^2 \mathbb{E} \left(\int_0^u \int_0^{u_s} |e(s)| (s-y)^{-\alpha-1} dy ds + \int_0^u \int_{u_s}^s |e(s)| (s-y)^{-\alpha-1} dy ds \right)^2 \\
&\quad + 8R^2 l_4^2 \left(\int_0^u \int_0^{u_s} |g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)| (s-y)^{-\alpha-1} dy ds \right. \\
&\quad \left. + \int_0^u \int_{u_s}^s |g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)| (s-y)^{-\alpha-1} dy ds \right)^2 \\
&\quad + 8R^2 l_3^2 l_6^2 \mathbb{E} \left(\int_0^u \int_{u_y}^u |e(y)| (s-y)^{-\alpha-1} dy ds + \int_0^u \int_y^{u_y} |e(y)| (s-y)^{-\alpha-1} dy ds \right)^2 \\
&\quad + 8R^2 l_4^2 \left(\int_0^u \int_{u_y}^u |g_2(y, u \wedge \tau_R) - g_{2m}(y, u \wedge \tau_R)| (s-y)^{-\alpha-1} dy ds \right. \\
&\quad \left. + \int_0^u \int_y^{u_y} |g_2(y, u \wedge \tau_R) - g_{2m}(y, u \wedge \tau_R)| (s-y)^{-\alpha-1} dy ds \right)^2
\end{aligned} \tag{35}$$

$$\begin{aligned}
&\leq 8R^2 l_3^2 l_6^2 (\alpha^{-1} + 1)^2 \int_0^u \mathbb{E}(|e(s)|^2) ds \int_0^u (s - u_s)^{-2\alpha} ds \\
&\quad + 8R^2 l_4^2 (\alpha^{-1} + 1)^2 \int_0^u |g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)|^2 ds \int_0^u (s - u_s)^{-2\alpha} ds \\
&\quad + 8R^2 l_3^2 l_6^2 (\alpha^{-1} + 1)^2 \mathbb{E} \left(\int_0^u |e(y)| [(u_y - y)^{-\alpha} - (u - y)^{-\alpha}] dy \right)^2 \\
&\quad + 8R^2 l_4^2 (\alpha^{-1} + 1)^2 \left(\int_0^u |g_2(y, u \wedge \tau_R) - g_{2m}(y, u \wedge \tau_R)| [(u_y - y)^{-\alpha} - (u - y)^{-\alpha}] dy \right)^2 \\
&\leq 8R^2 l_3^2 l_6^2 C_1 h^{-2\alpha} \int_0^u \mathbb{E}(|e(s)|^2) ds + 8R^2 l_4^2 C_1 h^{-2\alpha} \int_0^u |g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)|^2 ds \\
&\quad + 8R^2 l_3^2 l_6^2 C_1 h^{-2\alpha} \int_0^u \mathbb{E}(|e(y)|^2) dy + 8R^2 l_4^2 C_1 h^{-2\alpha} \int_0^u |g_2(y, u \wedge \tau_R) - g_{2m}(y, u \wedge \tau_R)|^2 dy,
\end{aligned} \tag{36}$$

where $\varsigma_n = (n/m) = nh, n = 0, \dots, m$, $u_s = \max\{\varsigma_n: \varsigma_n \leq s\}$,
 $u_y = \min\{\varsigma_n: \varsigma_n \geq y\}$, and $C_0 = (1 - 2\alpha)^{-1}$ and $C_1 = C_0(\alpha^{-1} + 1)^2$.

Finally, by combining (33)–(36), we obtain

$$\begin{aligned}
\mathbb{E}(|e(u \wedge \tau_R)|^2) &\leq 3 [|x_0(u) - x_{0m}(u)|^2 \\
&\quad + 2l_1^2 l_5^2 \int_0^u \mathbb{E}(|e(s)|^2) ds + 2l_2^2 \int_0^u |g_1(s, u \wedge \tau_R) - g_{1m}(s, u \wedge \tau_R)|^2 ds \\
&\quad + 4R^2 l_3^2 l_6^2 C_0 \int_0^u \mathbb{E}(|e(s)|^2) ds + 4R^2 l_4^2 C_0 \times \int_0^u |g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)|^2 ds \\
&\quad + 8R^2 l_3^2 l_6^2 C_1 h^{-2\alpha} \int_0^u \mathbb{E}(|e(s)|^2) ds + 8R^2 l_4^2 C_1 h^{-2\alpha} \\
&\quad \times \int_0^u |g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)|^2 ds + 8R^2 l_3^2 l_6^2 C_1 h^{-2\alpha} \int_0^u \mathbb{E}(|e(y)|^2) dy \\
&\quad + 8R^2 l_4^2 C_1 h^{-2\alpha} \times \int_0^u |g_2(y, u \wedge \tau_R) - g_{2m}(y, u \wedge \tau_R)|^2 dy].
\end{aligned} \tag{37}$$

Then,

$$\begin{aligned}
\mathbb{E}(|e(u \wedge \tau_R)|^2) &\leq 3 [|x_0(u) - x_{0m}(u)|^2 \\
&\quad + 2l_2^2 \int_0^u |g_1(s, u \wedge \tau_R) - g_{1m}(s, u \wedge \tau_R)|^2 ds + (4R^2 l_4^2 C_0 + 16R^2 l_4^2 C_1 h^{-2\alpha}) \\
&\quad \times \int_0^u |g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)|^2 ds] + (6l_1^2 l_5^2 + 12R^2 l_3^2 l_6^2 C_0 + 48R^2 l_3^2 l_6^2 C_1 h^{-2\alpha}) \\
&\quad \times \int_0^u \mathbb{E}(|e(s)|^2) ds,
\end{aligned} \tag{38}$$

or

where

$$\mathbb{E}(|e(u \wedge \tau_R)|^2) \leq \sigma(u) + \lambda \int_0^u \mathbb{E}(|e(s)|^2) ds, \tag{39}$$

TABLE 1: When $m = 16$ and $H = 0.6$, the table exhibits numerical results of Example 1.

u	X_E	X_S	95% confidence interval for error mean	
			Lower	Upper
1/16	$1.23683212E-05$	$6.18416058E-06$	$2.47366423E-07$	$2.44892759E-05$
2/16	$2.15162737E-05$	$1.07581368E-05$	$4.30325473E-07$	$4.26022219E-05$
3/16	$2.89145059E-05$	$1.44572530E-05$	$5.78290119E-07$	$5.72507217E-05$
4/16	$3.48716777E-05$	$1.74358389E-05$	$6.97433554E-07$	$6.90459219E-05$
5/16	$4.00955361E-05$	$2.00477681E-05$	$8.01910722E-07$	$7.93891615E-05$

TABLE 2: When $m = 16$ and $H = 0.8$, the table exhibits numerical results of Example 1.

u	X_E	X_S	95% confidence interval for error mean	
			Lower	Upper
1/16	$3.87689012E-06$	$1.93844506E-06$	$7.75378025E-08$	$7.67624244E-06$
2/16	$8.79318940E-06$	$4.39659470E-06$	$1.75863788E-07$	$1.74105150E-05$
3/16	$1.29658037E-05$	$6.48290186E-06$	$2.59316074E-07$	$2.56722914E-05$
4/16	$1.86048116E-05$	$9.30240580E-06$	$3.72096232E-07$	$3.68375270E-05$
5/16	$2.14168523E-05$	$1.07084262E-05$	$4.28337047E-07$	$4.24053677E-05$

FIGURE 1: Simulation result of Example 1 for $H = 0.6$ and $m = 16$. FIGURE 2: Simulation result of Example 1 for $H = 0.6$ and $m = 32$.

TABLE 3: When $m = 16$ and $H = 0.6$, the table exhibits numerical results of Example 2.

u	X_E	X_S	95% confidence interval for error mean	
			Lower	Upper
1/16	$4.82207134E-06$	$2.41103567E-06$	$9.64414269E-08$	$9.54770126E-05$
2/16	$9.11303123E-06$	$4.55651561E-06$	$1.82260625E-07$	$1.80438018E-05$
3/16	$1.31576729E-05$	$6.57883646E-06$	$2.63153458E-07$	$2.60521924E-05$
4/16	$1.60910855E-05$	$8.04554277E-06$	$3.21821711E-07$	$3.18603494E-05$
5/16	$2.03276041E-05$	$1.01638020E-05$	$4.06552081E-07$	$4.02486560E-05$

TABLE 4: When $m = 16$ and $H = 0.8$, the table exhibits numerical results of Example 2.

u	X_E	X_S	95% confidence interval for error mean	
			Lower	Upper
1/16	$3.08317271E-06$	$1.54158636E-06$	$6.16634543E-08$	$6.10468197E-06$
2/16	$6.35447091E-06$	$3.17723546E-06$	$1.27089418E-07$	$1.25818524E-05$
3/16	$7.04978090E-06$	$3.52489045E-06$	$1.40995618E-07$	$1.39585662E-05$
4/16	$8.48079420E-06$	$4.24039710E-06$	$1.69615884E-07$	$1.67919725E-05$
5/16	$1.14691035E-05$	$5.73455173E-06$	$2.29382069E-07$	$2.27088249E-05$

FIGURE 3: Simulation result of Example 2 for $H = 0.8$ and $m = 16$. FIGURE 4: Simulation result of Example 2 for $H = 0.8$ and $m = 32$.

$$\begin{aligned}
\sigma(u) &= 3 \left[|x_0(u) - x_{0m}(u)|^2 + 2l_2^2 \int_0^u |g_1(s, u \wedge \tau_R) - g_{1m}(s, u \wedge \tau_R)|^2 ds \right. \\
&\quad + (4R^2 l_4^2 C_0 + 16R^2 l_4^2 C_1 h^{-2\alpha}) \\
&\quad \left. \times \int_0^u |g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)|^2 ds \right], \\
\lambda &= 6l_1^2 l_5^2 + 12R^2 l_3^2 l_6^2 C_0 + 48R^2 l_3^2 l_6^2 C_1 h^{-2\alpha}.
\end{aligned} \tag{40}$$

Let $V(u \wedge \tau_R) = \mathbb{E}(|e(u \wedge \tau_R)|^2)$; we obtain

$$V(u \wedge \tau_R) \leq \sigma(u) + \lambda \int_0^u V(\theta) d\theta, \quad \theta \in [0, u]. \tag{41}$$

According to Gronwall's inequality, we have

$$V(u \wedge \tau_R) \leq \sigma(u) + \lambda \int_0^u e^{\lambda(u \wedge \tau_R - \theta)} \sigma(\theta) d\theta, \quad u \in [0, T]. \tag{42}$$

Then,

$$\begin{aligned}
\int_0^T V(u \wedge \tau_R) du &= \int_0^T \mathbb{E}(|e(u \wedge \tau_R)|^2) du \\
&\leq \int_0^T \left(\sigma(u) + \lambda \int_0^u e^{\lambda(u \wedge \tau_R - \theta)} \sigma(\theta) d\theta \right) du \\
&= \int_0^T \sigma(u) du + \lambda \int_0^T \int_0^u e^{\lambda(u \wedge \tau_R - \theta)} \sigma(\theta) d\theta du \\
&\leq \int_0^T \sigma(u) du + \lambda e^{\lambda T} \int_0^T \int_0^u \sigma(\theta) d\theta du = 3 \int_0^T |x_0(u) - x_{0m}(u)|^2 du \\
&\quad + 6l_2^2 \int_0^T \int_0^u |g_1(s, u \wedge \tau_R) - g_{1m}(s, u \wedge \tau_R)|^2 ds du \\
&\quad + (12R^2 l_4^2 C_0 + 48R^2 l_4^2 C_1 h^{-2\alpha}) \int_0^T \int_0^u |g_2(s, u \wedge \tau_R) - g_{2m}(s, u \wedge \tau_R)|^2 ds du \\
&\quad + \lambda e^{\lambda T} \left[3 \int_0^T \int_0^u |x_0(\theta) - x_{0m}(\theta)|^2 d\theta du \right. \\
&\quad + 6l_2^2 \int_0^T \int_0^u \int_0^\theta |g_1(s, \theta \wedge \tau_R) - g_{1m}(s, \theta \wedge \tau_R)|^2 ds d\theta du \\
&\quad \left. + (12R^2 l_4^2 C_0 + 48R^2 l_4^2 C_1 h^{-2\alpha}) \int_0^T \int_0^u \int_0^\theta |g_2(s, \theta \wedge \tau_R) - g_{2m}(s, \theta \wedge \tau_R)|^2 ds d\theta du \right] \\
&:= 3N_1 + 6l_2^2 N_2 + (12R^2 l_4^2 C_0 + 48R^2 l_4^2 C_1 h^{-2\alpha}) N_3 \\
&\quad + \lambda e^{\lambda T} [3N_4 + 6l_2^2 N_5 + (12R^2 l_4^2 C_0 + 48R^2 l_4^2 C_1 h^{-2\alpha}) N_6].
\end{aligned} \tag{43}$$

By (27) and (28), we have

$$N_i \leq w_i h^2, \quad i = 1, \dots, 6. \tag{44}$$

So,

$$\begin{aligned}
\int_0^T \mathbb{E}(|e(u \wedge \tau_R)|^2) du &\leq [3w_1 + 6l_2^2 w_2 \\
&\quad + (12R^2 l_4^2 C_0 + 48R^2 l_4^2 C_1 h^{-2\alpha}) w_3 \\
&\quad + \lambda e^{\lambda T} (3w_4 + 6l_2^2 w_5 + 12R^2 l_4^2 C_0 w_6 \\
&\quad + 48R^2 l_4^2 C_1 h^{-2\alpha} w_6)] h^2 \leq C(R) O(h^{2-4\alpha}),
\end{aligned} \tag{45}$$

where $w_i > 0, i = 1, \dots, 6$, are constants.

The proof is accomplished. $\square$

5. Numerical Example

We verify the effectiveness and precision of the above method by two numerical examples in this section.

Example 1. The following nonlinear SIVIE is discussed [8]:

$$\begin{aligned}
x(u) &= x_0 - 2H\beta^2 \int_0^u s^{2H-1} x(s) (1 - x^2(s)) ds \\
&\quad + \beta \int_0^u (1 - x^2(s)) dB_s^H, \quad u \in [0, 1],
\end{aligned} \tag{46}$$

where $x(u) = \tanh(\beta B_u^H + \operatorname{arctanh}(x_0))$ is the exact solution.

Let $\beta = (1/30)$ and $x_0 = (1/10)$. For different values of H , the error means X_E , error standard deviations X_S , and confidence intervals for error means of Example 1 with $m = 16$ are given in Tables 1 and 2, where X_E and X_S are obtained by 10^2 trajectories. Also, when $m = 16$ and $H = 0.6$, the simulation result of the approximate and precise solutions of this example are shown in Figure 1, and for $m = 32$ and $H = 0.6$, the results are exhibited in Figure 2.

Example 2. The following nonlinear SIVIE is discussed [8]:

$$\begin{aligned} x(u) = x_0 - H\beta^2 \int_0^u \tanh(x(s))s^{2H-1} \operatorname{sech}^2(x(s))ds \\ + \beta \int_0^t \operatorname{sech}(x(s))dB_s^H, \quad u \in [0, 1), \end{aligned} \quad (47)$$

where $x(u) = \operatorname{arcsinh}(\beta B_u^H + \sinh(x_0))$ is the exact solution.

Let $\beta = (1/30)$ and $x_0 = (1/10)$. Similarly, for different values of H , X_E , and X_S and confidence intervals for error means of Example 2 with $m = 16$ are given in Tables 3 and 4, where X_E and X_S are obtained by 10^2 trajectories. Also, when $m = 16$ and $H = 0.8$, simulation result of the approximate and precise solutions of this example is shown in Figure 3, and for $m = 32$ and $H = 0.8$, the results are shown in Figure 4.

6. Conclusion

For nonlinear SIVIEs driven by FBM, this paper presents a valid method to solve them by BPFs. The simulation results show that the approximate solution of this method is very close to the exact solution. In Section 4, error analysis is given, and the convergence rate of the approximate solution is faster than the method of [21] when $H \rightarrow 1$.

Appendix

In this section, we give detailed proofs of $\mathbf{M}_1$ and $\mathbf{M}_2$.

Let G_1^i and G_2^i be the i th row of the matrix $\mathbf{G}_1$ and $\mathbf{G}_2$, respectively, R^i and R_H^i be the i th row of the integral operation matrix $\mathbf{P}$ and the stochastic integral matrix $\mathbf{P}_H$, respectively, and $\mathbf{D}_{G_1^i}$ and $\mathbf{D}_{G_2^i}$ be a diagonal matrix with G_1^i and G_2^i as the diagonal elements, respectively. By (16) and (18), we obtain

$$\begin{aligned} \left(\int_0^u \Phi(s)\Phi(s)^T ds \right) \mathbf{G}_1 \Phi(u) &= \begin{pmatrix} R^1 \Phi(u) G_1^1 \Phi(u) \\ R^2 \Phi(u) G_1^2 \Phi(u) \\ \vdots \\ R^m \Phi(u) G_1^m \Phi(u) \end{pmatrix} \\ &= \begin{pmatrix} R^1 \mathbf{D}_{G_1^1} \\ R^2 \mathbf{D}_{G_1^2} \\ \vdots \\ R^m \mathbf{D}_{G_1^m} \end{pmatrix} \Phi(u) = \mathbf{M}_1 \Phi(u), \end{aligned}$$

$$\begin{aligned} \left(\int_0^u \Phi(s)\Phi(s)^T dB_s^H \right) \mathbf{G}_2 \Phi(u) &= \begin{pmatrix} R_H^1 \Phi(u) G_2^1 \Phi(u) \\ R_H^2 \Phi(u) G_2^2 \Phi(u) \\ \vdots \\ R_H^m \Phi(u) G_2^m \Phi(u) \end{pmatrix} \\ &= \begin{pmatrix} R_H^1 \mathbf{D}_{G_2^1} \\ R_H^2 \mathbf{D}_{G_2^2} \\ \vdots \\ R_H^m \mathbf{D}_{G_2^m} \end{pmatrix} \Phi(u) = \mathbf{M}_2 \Phi(u). \end{aligned} \quad (\text{A.1})$$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors state that there are no conflicts of interest in the publication of this paper.

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