Research Article

The Effects of Harvesting on the Dynamics of a Leslie–Gower Model

Jingli Xie ¹, Hanyan Liu ¹, and Danfeng Luo ²

¹College of Mathematics and Statistics, Jishou University, Jishou 416000, Hunan, China

²Department of Mathematics, Guizhou University, Guiyang 550025, Guizhou, China

Correspondence should be addressed to Hanyan Liu; liuhanyan1998@163.com

Received 15 January 2021; Accepted 24 April 2021; Published 7 May 2021

Academic Editor: Rigoberto Medina

Copyright © 2021 Jingli Xie et al. This is an open access article distributed under the Creative Commons Attribution License, which permits unrestricted use, distribution, and reproduction in any medium, provided the original work is properly cited.

In this paper, we study a Leslie–Gower predator-prey model with harvesting effects. We carry out local bifurcation analysis and stability analysis. Under certain conditions, the model is shown to undergo a supercritical Hopf bifurcation resulting in a stable limit cycle. Numerical simulations are presented to illustrate our theoretic results.

1. Introduction

In this paper, we consider Leslie–Gower predator-prey model with harvesting effect,

$$\begin{cases} \frac{dx_1}{d\tau} = rx_1 \left(1 - \frac{x_1}{k}\right) - \frac{a_1 x_1 x_2}{n + x_1} - e_1 x_1, \\ \frac{dx_2}{d\tau} = sx_2 \left(1 - \frac{a_2 x_2}{n + x_1}\right) - e_2 x_2, \end{cases} \quad (1)$$

with $x_1(0) \geq 0$ and $x_2(0) \geq 0$, where $x_1(\tau)$ and $x_2(\tau)$ are the prey and predator population densities, respectively, $r, s, a_1, a_2, n, e_1, e_2 > 0$, and τ is the time.

Note that $(a_2 x_2 / (n + x_1))$ is Leslie–Gower term in which the carrying capacity of the predator's environment is a linear function of the prey size $(x_1/a_2) + (n/a_2)$. $(a_1 x_1 x_2 / (n + x_1))$ is the number of prey consumed by the predator in unit time which shows that when the number of the prey x_1 is severe scarcity, and the predators can switch over to other populations as food. Constants r and s are the intrinsic growth rate of the prey and predator, respectively, and e_1 and e_2 denote the harvesting efforts for the prey and predator, respectively.

Since the first prey-predator dynamical models which is the Lotka–Volterra model was built in the 1920s by

Mathematician Lotka and Volterra, more and more researchers are interested in such issues, and they start from different angles to think the problem and many important results have been obtained [1–10]. In particular, in 2003, Aziz-Alaoui and Daher Okiye [11] considered the following Leslie–Gower predator-prey model:

$$\begin{cases} \frac{dx_1}{d\tau} = rx_1 \left(1 - \frac{x_1}{k}\right) - \frac{a_1 x_1 x_2}{n_1 + x_1}, \\ \frac{dx_2}{d\tau} = sx_2 \left(1 - \frac{a_2 x_2}{n_2 + x_1}\right), \end{cases} \quad (2)$$

where x_1 is the numbers of prey and x_2 is the numbers of predators. Existence and stability of the fixed points were studied by using the Lyapunov function. In 2006, Lin and Ho [12] discussed the local and global stability for system (2) by using Poincaré–Bendixson theorem and Dulac's criterion.

Harvesting is an effective way for humans to control the size of predators and prey so that the population has continued to develop healthily and produced good economic benefits [13–16]. Academically, researchers often only consider the harvesting of prey in order to control the size of the population. In 2010, Zhu and Lan [17] investigated the Leslie–Gower predator-prey systems:

$$\begin{cases} \frac{dx_1}{d\tau} = r_1 x_1 \left(1 - \frac{x_1}{k}\right) - ax_1 x_2 - h, \\ \frac{dx_2}{d\tau} = r_2 x_2 \left(1 - \frac{x_2}{bx_1}\right). \end{cases} \quad (3)$$

In 2013, Gupta and Chandra [18] discussed the following Leslie–Gower predator–prey model with harvesting on the prey and the environment providing the same protection to both the predator and prey:

$$\begin{cases} \frac{dx_1}{d\tau} = rx_1 \left(1 - \frac{x_1}{k}\right) - \frac{a_1 x_1 x_2}{n + x_1} - \frac{qEx_1}{m_1 E + m_2 x_1}, \\ \frac{dx_2}{d\tau} = sx_2 \left(1 - \frac{a_2 x_2}{n + x_1}\right). \end{cases} \quad (4)$$

For ecological balance and healthy economic development, for fisheries, wildlife resources, etc., we not only need to consider the harvesting of prey, but also the predator. Therefore, in this paper, we study Leslie–Gower predator–prey model (1) with harvesting on the prey and predator.

In order to make the system dimensionless, we define new dependent variables by $x = (x_1/k)$, $y = ((a_1 x_2)/k)$ and a new time variable by $t = r\tau$. Moreover, let $\alpha = (1/r)$, $\beta = (a_2/a_1)$, $m = (n/k)$, $\rho = (s/r)$, $\gamma = (e_1/r)$, and $\delta = (e_2/(ra_1))$. System (1) becomes

$$\begin{cases} \frac{dx}{dt} = x(1-x) - \frac{\alpha xy}{m+x} - \gamma x, \\ \frac{dy}{dt} = \rho y \left(1 - \frac{\beta y}{m+x}\right) - \delta y, \end{cases} \quad (5)$$

where $\alpha, \beta, \gamma, \rho, \delta$, and m are all positive parameters.

Lemma 1 (see 19). *If $a > 0$, $b > 0$, and $\dot{x} \leq x(b - ax^\alpha)$, where α is a positive constant, $t \geq 0$, and $x(0) > 0$, we have*

$$x(t) \leq \left(\frac{b}{a}\right)^{(1/\alpha)} \left(1 + \left(\frac{bx^{-\alpha}(0)}{a} - 1\right)e^{-bat}\right)^{-(1/\alpha)}. \quad (6)$$

Lemma 2 (see [20, 21]). *Consider system $\dot{X} = f(X, \alpha)$ and suppose that $f(X_0, \alpha_0) = 0$, $n \times n$ Jacobian matrix $(J \equiv Df(X_0, \alpha_0))$ has a simple eigenvalue $s = 0$ with eigenvector V , and the transpose of the Jacobian matrix J^T has an eigenvector W to the eigenvalue $s = 0$. Then, the system $\dot{X} = f(X, \alpha)$ experiences a transcritical bifurcation at the equilibrium point X_0 as the control parameter α passes through the bifurcation value $\alpha = \alpha_0$ if the following conditions are satisfied:*

$$\begin{aligned} W^T Df_\alpha(X_0, \alpha_0) &= 0, \\ W^T (Df_\alpha(X_0, \alpha_0)V) &\neq 0, \\ W^T (D^2 f(X_0, \alpha_0)(V, V)) &= 0. \end{aligned} \quad (7)$$

The rest of this paper is organized as follows. In Section 2, we study boundary of solutions. In Section 3, we discuss existence of equilibria points. In Section 4, we discuss stability of the equilibrium points.

2. Boundedness of Solutions

In this section, we prove that every solution of system (5) is positive and uniformly bounded with initial conditions $x(0) = x_0 > 0$ and $y(0) = y_0 > 0$. Denote $\mathbb{R}_+^2 = \{(x(t), y(t)) \in \mathbb{R}^2 | x(t) > 0, y(t) > 0, t > 0\}$.

Theorem 1. *Consider system (5). For any given initial conditions $(x_0, y_0) \in \mathbb{R}_+^2$, the solution $(x(t), y(t))$ of system (5) exists and is unique and positive and ultimate bounded.*

Proof. Let function $f(x(t), y(t)) = (x(t)(1-x(t)) - (\alpha x(t)y(t)/(m+x(t))) - \gamma x(t), \rho y(t)(1 - (\beta y(t)/(m+x(t)))) - \delta y(t))$. Obviously, function $f(x(t), y(t))$ is continuous differentiable on $(x, y) \in \mathbb{R}_+^2$, so for any given the initial conditions $(x_0, y_0) \in \mathbb{R}_+^2$, the solution $(x(t), y(t))$ of the system (5) exists and is unique. Furthermore, x -axis and y -axis are the solutions of the system (5); by the uniqueness of the solution, the solutions $(x(t), y(t))$ of the system (5) with the initial value $x_0 > 0, y_0 > 0$ cannot cross with x -axis and y -axis.

Next, we show the solutions $(x(t), y(t))$ of system (5) with the initial value $x_0 > 0$ and $y_0 > 0$ which is ultimate bounded.

From system (5), we have

$$\frac{dx}{dt} \leq x(1-x). \quad (8)$$

Combining Lemma 1, we have

$$\begin{aligned} x(t) &\leq \left(1 + (x^{-1}(0) - 1)e^{-t}\right)^{-1} \leq M_0, \\ M_0 &= \max\{1, x_0\}. \end{aligned} \quad (9)$$

So, the following inequality is established:

$$\frac{dy}{dt} \leq \rho y \left(1 - \frac{\beta y}{m+x}\right) \leq \rho y \left(1 - \frac{\beta y}{m+1}\right) = y \left(\rho - \frac{\rho\beta y}{m+1}\right). \quad (10)$$

By Lemma 1, we have

$$y(t) \leq M_1, \quad t \geq 0. \quad (11)$$

where $M_1 = \max\{((m+M_0)/\beta), y_0\}$. $\square$

3. Existence of Equilibria

In order to find the equilibrium points of system (5), we let $f(x(t), y(t)) = 0$, i.e.,

$$\begin{cases} x(1-x) - \frac{\alpha xy}{m+x} - \gamma x = 0, \\ \rho y \left(1 - \frac{\beta y}{m+x}\right) - \delta y = 0. \end{cases} \quad (12)$$

It is clear that equation (12) has a trivial solution $E_0 := (0, 0)$. Furthermore, by calculation, we find other solutions of equation (12):

$$(1 - \gamma, 0),$$

$$\left(0, \frac{(\rho - \delta)m}{\rho\beta}\right),$$

$$\left(\frac{(1 - \gamma)\rho\beta - (\rho - \delta)\alpha}{\rho\beta}, \frac{(\rho - \delta)((1 + m - \gamma)\rho\beta - (\rho - \delta)\alpha)}{(\rho\gamma)^2}\right). \quad (13)$$

Remark 1. When $\gamma < 1$, $\delta < \rho$ and $(\rho - \delta)\alpha < (1 - \gamma)\rho\beta$, the position of each point in (13) is shown in Figure 1.

Therefore, we have the following results. where $x_\infty = (((1 - \gamma)\rho\beta - (\rho - \delta)\alpha)/\rho\beta)$, $y_\infty = (((\rho - \delta)((1 + m - \gamma)\rho\beta - (\rho - \delta)\alpha))/(\rho\beta)^2)$.

Theorem 2. Consider system (5) admits x -axial only and y -axial equilibria under following conditions.

(i) The x -axial equilibrium,

$$E_1 := (K, 0), \quad \text{with } K = 1 - \gamma, \quad (14)$$

is a boundary equilibrium of system (5) if and only if

$$1 - \gamma > 0. \quad (15)$$

(ii) The y -axial equilibrium,

$$E_2 := (0, M), \quad \text{with } M = \frac{(\rho - \delta)m}{\rho\beta}, \quad (16)$$

is a boundary equilibrium of system (5) if and only if

$$\rho - \delta > 0. \quad (17)$$

Theorem 3. Consider system (5) admits a unique positive equilibrium,

$$E_3 := (x_\infty, y_\infty), \quad (18)$$

if only if

$$\frac{1 - \gamma}{\alpha} > \frac{\rho - \delta}{\rho\beta} > 0, \quad (19)$$

Remark 2. Existence regions for equilibrium points of system (12) is shown in Figure 2. Equilibrium points E_1 and E_2 exist, but positive equilibrium point E_3 does not in region II and equilibrium points E_1, E_2 , and E_3 coexist in region I. Moreover, if $((1 - \gamma)/\alpha) = ((\rho - \delta)/\rho\beta) > 0$, then positive equilibrium E_3 becomes boundary equilibrium E_2 .

FIGURE 1: The position of each point in (13) with $\rho = 0.1$, $\alpha = 1$, $\delta = 0.05$, $\gamma = 1$, $\beta = 5$, and $m = 0.1$.

FIGURE 2: Regions for equilibrium points of system (12).

4. Stability of of Equilibria

Theorem 4. If $1 - \gamma > 0$ and $\rho - \delta > 0$, then the equilibria E_0 and E_1 are unstable.

Proof. Firstly, we show the equilibria E_0 is a unstable equilibrium. The Jacobian matrix about E_0 is given by

$$J_{E_0} = \begin{pmatrix} 1 - \gamma & 0 \\ 0 & \rho - \delta \end{pmatrix}, \quad (20)$$

with trace $1 - \gamma + \rho - \delta$ and determinant $(1 - \gamma)(\rho - \delta)$. From $1 - \gamma > 0$ and $\rho - \delta > 0$, we have $1 - \gamma + \rho - \delta > 0$ and $(1 - \gamma)(\rho - \delta) > 0$. On the contrary, through direct calculation, we obtain $(1 - \gamma + \rho - \delta)^2 - 4(1 - \gamma)(\rho - \delta) = (1 - \gamma - \rho + \delta)^2 > 0$. Therefore, E_0 is an unstable node point.

Secondly, we show the equilibria E_1 is a unstable equilibrium. The Jacobian matrix about E_1 is given by

$$J_{E_1} = \begin{pmatrix} -(1-\gamma) - \frac{\alpha(1-\gamma)}{m+1-\gamma} & \\ 0 & \rho-\delta \end{pmatrix}, \quad (21)$$

with determinant $-(1-\gamma)(\rho-\delta) < 0$. Therefore, E_1 is a saddle point which is unstable. $\square$

Theorem 5. (1) If $((1-\gamma)/\alpha) > ((\rho-\delta)/\rho\beta) > 0$, then the boundary equilibrium E_2 is a unstable equilibrium. (2) If $0 < ((1-\gamma)/\alpha) < ((\rho-\delta)/\rho\beta)$, then the boundary equilibrium E_2 is locally asymptotically stable.

Proof. The Jacobian matrix about E_2 is given by

$$J_{E_2} = \begin{pmatrix} (1-\gamma) - \frac{\alpha(\rho-\delta)}{\rho\beta} & 0 \\ \frac{(\rho-\delta)^2}{\rho\beta} & -(\rho-\delta) \end{pmatrix}, \quad (22)$$

with trace $(1-\gamma) - ((\alpha(\rho-\delta))/\rho\beta) - (\rho-\delta)$ and determinant $-(\rho-\delta)((1-\gamma) - ((\alpha(\rho-\delta))/\rho\beta))$.

- (1) When $((1-\gamma)/\alpha) > ((\rho-\delta)/\rho\beta) > 0$, we have $-(((\rho-\delta)(\rho\beta(1-\gamma) - \alpha(\rho-\delta)))/\rho\beta) < 0$, so E_2 is a saddle point which is an unstable.
- (2) When $0 < ((1-\gamma)/\alpha) < ((\rho-\delta)/\rho\beta)$, we have $(1-\gamma) - ((\alpha(\rho-\delta))/\rho\beta) - (\rho-\delta) = ((\rho\beta(1-\gamma) - \alpha(\rho-\delta))/\rho\beta) - (\rho-\delta) < 0$ and $-(((\rho-\delta)(\rho\beta(1-\gamma) - \alpha(\rho-\delta)))/\rho\beta) > 0$. Furthermore, a direct calculations gives

$$\begin{aligned} & \left((1-\gamma) - \frac{\alpha(\rho-\delta)}{\rho\beta} - (\rho-\delta) \right)^2 + 4(\rho-\delta) \left((1-\gamma) - \frac{\alpha(\rho-\delta)}{\rho\beta} \right) \\ &= \left((1-\gamma) - \frac{\alpha(\rho-\delta)}{\rho\beta} + (\rho-\delta) \right)^2 > 0. \end{aligned} \quad (23)$$

Therefore, E_2 is locally asymptotically stable. $\square$

Theorem 6. (1) If $0 < ((1-\gamma)/\alpha) < ((\rho-\delta)/\rho\beta)$ and $m < ((-2x_\infty^2 + (1-\gamma-\rho+\delta)x_\infty)/(x_\infty + \rho-\delta))$ hold, then the positive equilibrium E_3 is unstable. (2) If $0 < ((1-\gamma)/\alpha) < ((\rho-\delta)/\rho\beta)$ and $m > ((-2x_\infty^2 + (1-\gamma-\rho+\delta)x_\infty)/(x_\infty + \rho-\delta))$ hold, then the positive equilibrium E_3 is locally asymptotically stable. Furthermore, assume that there is $\alpha < \rho\beta$; then, the positive equilibrium E_3 is globally asymptotically stable.

Proof. The Jacobian matrix about E_3 is given by

$$J_{E_3} = \begin{pmatrix} x_\infty \left(-1 + \frac{1-\gamma-x_\infty}{m+x_\infty} \right) - \frac{\alpha x_\infty}{m+x_\infty} & \\ \frac{(\rho-\delta)^2}{\rho\beta} & -(\rho-\delta) \end{pmatrix}, \quad (24)$$

with trace $x_\infty(-1 + ((1-\gamma-x_\infty)/(m+x_\infty))) - (\rho-\delta)$ and determinant $-x_\infty(\rho-\delta) - (1 + ((1-\gamma-x_\infty)/(m+x_\infty))) + (\alpha x_\infty/(m+x_\infty))((\rho-\delta)^2/\rho\beta)$.

A direct calculation gives

$$\begin{aligned} & x_\infty \left(-1 + \frac{1-\gamma-x_\infty}{m+x_\infty} \right) - (\rho-\delta) \\ &= \frac{x_\infty(-m+1-\gamma-2x_\infty) - m(\rho-\delta) - x_\infty(\rho-\delta)}{m+x_\infty} \\ &= \frac{-2x_\infty^2 - m(x_\infty + \rho-\delta) + (1-\gamma-\rho+\delta)x_\infty}{m+x_\infty}. \end{aligned} \quad (25)$$

- (1) From conditions $0 < ((1-\gamma)/\alpha) < ((\rho-\delta)/\rho\beta)$ and $m < ((-2x_\infty^2 + (1-\gamma-\rho+\delta)x_\infty)/(x_\infty + \rho-\delta))$, we obtain

$$x_\infty \left(-1 + \frac{1-\gamma-x_\infty}{m+x_\infty} \right) - (\rho-\delta) > 0. \quad (26)$$

Therefore, the positive equilibrium E_3 is unstable.

- (2) From conditions $0 < ((1-\gamma)/\alpha) < ((\rho-\delta)/\rho\beta)$ and $m > ((-2x_\infty^2 + (1-\gamma-\rho+\delta)x_\infty)/(x_\infty + \rho-\delta))$, we have

$$x_\infty \left(-1 + \frac{1-\gamma-x_\infty}{m+x_\infty} \right) - (\rho-\delta) < 0,$$

$$\begin{aligned} & -x_\infty(\rho-\delta) \left(-1 + \frac{1-\gamma-x_\infty}{m+x_\infty} \right) + \frac{\alpha x_\infty}{m+x_\infty} \frac{(\rho-\delta)^2}{\rho\beta} \\ &= -x_\infty(\rho-\delta) \left(-1 + \frac{1-\gamma}{m+x_\infty} - \frac{x_\infty}{m+x_\infty} - \frac{\alpha(\rho-\delta)}{\rho\beta(m+x_\infty)} \right) \\ &= x_\infty(\rho-\delta) \left(1 - \frac{1-\gamma}{m+x_\infty} + \frac{x_\infty}{m+x_\infty} + \frac{\alpha(\rho-\delta)}{\rho\beta(m+x_\infty)} \right) \\ &> x_\infty(\rho-\delta) \left(1 + \frac{x_\infty}{m+x_\infty} \right) > 0. \end{aligned} \quad (27)$$

So, the positive equilibrium E_3 is locally asymptotically stable.

Let $\beta(x, y) = (1/xy)$, $F(x, y) = (dx/dt) = x(1-x) - (\alpha xy/(m+x)) - \gamma x$, $G(x, y) = (dy/dt) = \rho y(1 - (\beta y/(m+x))) - \delta y$. By calculations, we have

$$\begin{aligned} & \frac{\partial(\beta(x, y)F(x, y))}{\partial x} + \frac{\partial(\beta(x, y)G(x, y))}{\partial y} \\ &= -\frac{1}{y} + \frac{\alpha}{(m+x)^2} - \frac{\rho\beta}{x(m+x)} \\ &\leq -\frac{1}{y} + \frac{\alpha}{x(m+x)} - \frac{\rho\beta}{x(m+x)} \\ &< -\frac{1}{y} < 0. \end{aligned} \quad (28)$$

Applying Dulac's criterion and Theorem 1, we know that the positive equilibrium E_3 is globally asymptotically stable. $\square$

5. Local Bifurcation

hold, then system (5) undergoes a Hopf bifurcation with respect to bifurcation parameter m around the equilibrium point $E_3 = (x_\infty, y_\infty)$. Furthermore, the direction of the Hopf bifurcation is subcritical and the bifurcation periodic solutions are orbitally asymptotically stable if

$$k_0 + k_1 + k_2 + k_3 < 0. \quad (41)$$

Theorem 7. If $((1-\gamma)/\alpha) = ((\rho-\delta)/\rho\beta) > 0$, then system (5) undergoes a transcritical bifurcation around E_2 .

Proof. From $((1-\gamma)/\alpha) = ((\rho-\delta)/\rho\beta) > 0$, we have

$$\alpha = \frac{(1-\gamma)\rho\beta}{\rho-\delta}. \quad (29)$$

Therefore,

$$f_\alpha(x, y) = \begin{pmatrix} \frac{xy}{m+x} \\ 0 \end{pmatrix}. \quad (30)$$

We have

$$f_\alpha(x_2, y_2) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \quad (31)$$

The eigenvectors V and W associated to zero eigenvalues of matrixes J and J^T , respectively, are

$$V = \begin{pmatrix} \frac{\rho\beta}{\rho-\delta} \\ 1 \end{pmatrix}, \quad (32)$$

$$W = \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \quad (33)$$

So, $W^T f_\alpha(x_2, y_2) = 0$.

In addition, $Df_\alpha(x_2, y_2)V$ equals

$$Df_\alpha(x_2, y_2)V = \begin{pmatrix} \frac{\partial(-xy/(m+x))}{\partial x}v_1 + \frac{\partial(-xy/(m+x))}{\partial y}v_2 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 0 \end{pmatrix}. \quad (34)$$

Combining (33) and (34), we have

$$W^T [Df_\alpha(x_2, y_2)V] = (1 \ 0) \begin{pmatrix} -1 \\ 0 \end{pmatrix} = -1 \neq 0. \quad (35)$$

Then, we compute $D^2 f(x_2, y_2, \alpha)(V, V)$ as follows:

$$D^2 f(x_2, y_2, \alpha)(V, V) = \begin{pmatrix} \frac{\partial^2 f_1}{\partial x^2}v_1v_1 + 2\frac{\partial^2 f_1}{\partial x\partial y}v_1v_2 + \frac{\partial^2 f_1}{\partial y^2}v_2v_2 \\ \frac{\partial^2 f_2}{\partial x^2}v_1v_1 + 2\frac{\partial^2 f_2}{\partial x\partial y}v_1v_2 + \frac{\partial^2 f_2}{\partial y^2}v_2v_2 \end{pmatrix} = \begin{pmatrix} -2 \\ \frac{2\rho\beta}{m} \end{pmatrix}. \quad (36)$$

Combining (33) and (36), we have

$$W^T [D^2 f_\alpha(x_2, y_2)(V, V)] = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} -2 \\ -\frac{2\rho\beta}{m} \end{pmatrix} = -2 \neq 0. \quad (37)$$

Since all three conditions of Lemma 2 are satisfied, system (5) undergoes a transcritical bifurcation around E_2 if $((1 - \gamma)/\alpha) = ((\rho - \delta)/\rho\beta) > 0$.
Let

$$\begin{aligned} k_0 &= \frac{\rho - \delta}{8(m + x_\infty)} \left(\frac{2(m - 1)}{m + x_\infty} - \frac{3\alpha m}{\rho\beta(m + x_\infty)^2} + \frac{2\rho\beta}{\alpha} \right), \\ k_1 &= \frac{1}{8} \left(\frac{m}{x_\infty(m + x_\infty)} + \frac{m^2(\rho - \delta)}{x_\infty(m + x_\infty)^2} - \frac{\alpha m^2(\rho - \delta)}{\rho\beta x_\infty(m + x_\infty)^3} \right), \\ k_2 &= \frac{1}{8} \frac{\rho\beta(\rho - \delta)}{m + x_\infty} \left(\frac{2(m + x_\infty)}{\alpha} - \frac{m + 2}{\rho\beta} \right) \left(1 - \frac{\rho\beta}{\alpha x_\infty} + \frac{\alpha m(\rho - \delta)}{(m + x_\infty)^2} \left(\frac{m + x_\infty}{\alpha} - \frac{1}{\rho\beta} \right) \right. \\ &\quad \left. - \frac{\alpha\rho\beta(\rho - \delta)}{(m + x_\infty)^2} \left(\frac{m + x_\infty}{\alpha} - \frac{1}{\rho\beta} \right)^2 \right), \\ k_3 &= \frac{1}{4} \left(1 + \frac{\alpha m(\rho - \delta)}{(m + x_\infty)^2} \left(\frac{m + x_\infty}{\alpha} - \frac{1}{\rho\beta} \right) \right) \left(1 + \frac{\alpha m(\rho - \delta)}{(m + x_\infty)^2} \left(\frac{m + x_\infty}{\alpha} - \frac{1}{\rho\beta} \right) \right. \\ &\quad \left. - \frac{\alpha m(\rho - \delta)}{(m + x_\infty)^2} \left(\frac{m + x_\infty}{\alpha} - \frac{1}{\rho\beta} \right)^2 \right), \end{aligned} \quad (38)$$

and m is m_H of (40). $\square$

Theorem 8. Assume that $0 < ((1 - \gamma)/\alpha) < ((\rho - \delta)/\rho\beta)$ is satisfied. If

$$\rho\beta(1 - \gamma) < (2\alpha - \rho\beta)(\rho - \delta), \quad (39)$$

$$m_H = \frac{-2x_\infty^2 + (1 - \gamma - \rho + \delta)x_\infty}{x_\infty + \rho - \delta}. \quad (40)$$

The direction of the Hopf bifurcation is supercritical and the bifurcation periodic solutions are unstable if

$$k_0 + k_1 + k_2 + k_3 > 0. \quad (42)$$

Proof. From (39), we have

$$2x_\infty < 1 - \gamma - \rho + \delta. \quad (43)$$

So, $m_H > 0$.

The Jacobian matrix of system (5) evaluated at the point E_3 is given by

$$J_{E_3} = \begin{pmatrix} x_\infty \left(-1 + \frac{1 - \gamma - x_\infty}{m + x_\infty} \right) & -\frac{\alpha x_\infty}{m + x_\infty} \\ \frac{(\rho - \delta)^2}{\rho\beta} & -(\rho - \delta) \end{pmatrix}. \quad (44)$$

The trace $z = \text{Tr}(J_{E_3})$ and the determinant $D = \text{Det}(J_{E_3})$ of Jacobian matrix J_{E_3} are given by

$$\begin{aligned} z &= -x_\infty + \frac{x_\infty(1 - \gamma - x_\infty)}{m + x_\infty} - (\rho - \delta) \\ &= \frac{-2x_\infty^2 + (1 - \gamma - m)x_\infty}{m + x_\infty} - (\rho - \delta) \\ &= \frac{-2x_\infty^2 + (1 - \gamma - \rho + \delta)x_\infty - m(x_\infty + \rho - \delta)}{m + x_\infty}, \end{aligned}$$

$$\begin{aligned} D &= -x_\infty(\rho - \delta) \left(-1 + \frac{1 - \gamma - x_\infty}{m + x_\infty} \right) + \frac{(\rho - \delta)^2}{\rho\beta} \frac{\alpha x_\infty}{m + x_\infty} \\ &= x_\infty(\rho - \delta) \frac{\rho\beta m - (\rho - \delta)\alpha + (1 - \gamma)\rho\beta}{\rho\beta(m + x_\infty)} \\ &= x_\infty(\rho - \delta). \end{aligned} \quad (45)$$

By $0 < ((1 - \gamma)/\alpha) < ((\rho - \delta)/\rho\beta)$ and (40), we have $z|_{m_H} = 0$ and $D|_{m_H} > 0$.

In addition, we have

$$\begin{aligned} \frac{\partial z}{\partial m} &= \frac{-(x_{\infty} + \rho - \delta)(m + x_{\infty}) - (-2x_{\infty}^2 + (1 - \gamma - \rho + \delta)x_{\infty} - m(x_{\infty} + \rho - \delta))}{(m + x_{\infty})^2} \\ &= \frac{x_{\infty}^2 - (1 - \gamma)x_{\infty}}{(m + x_{\infty})^2} \\ &= -\frac{x_{\infty}(1 - \gamma - x_{\infty})}{(m + x_{\infty})^2} \\ &= -\frac{\alpha x_{\infty}(\rho - \delta)}{\rho\beta(m + x_{\infty})^2} \\ &< 0. \end{aligned} \tag{46}$$

So, $(\partial z / \partial m)|_{m_H} < 0$. Therefore, this guarantees the existence of Hopf bifurcation around E_3 .

We translate the equilibrium E_3 to the origin by the translation $\tilde{x} = x - x_{\infty}$ and $\tilde{y} = y - y_{\infty}$. For the sake of

convenience, we still denote $\tilde{x}$ and $\tilde{y}$ by x and y , respectively. So, the system (5) becomes

$$\begin{cases} \dot{x} = (x + x_{\infty})(1 - x - x_{\infty}) - \frac{\alpha(x + x_{\infty})(y + y_{\infty})}{m + x + x_{\infty}} - \gamma(x + x_{\infty}), \\ \dot{y} = \rho(y + y_{\infty})\left(1 - \frac{\beta(y + y_{\infty})}{m + x + x_{\infty}}\right) - \delta(y + y_{\infty}). \end{cases} \tag{47}$$

Rewrite system (47) to

where

$$\begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} = J_{E_3} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_1(x, y, m) \\ f_2(x, y, m) \end{pmatrix}, \tag{48}$$

$$\begin{aligned} f_1(x, y, m) &= (x + x_{\infty})(1 - x - x_{\infty}) - \frac{\alpha(x + x_{\infty})(y + y_{\infty})}{m + x + x_{\infty}} - \gamma(x + x_{\infty}) \\ &\quad - x_{\infty}\left(-1 + \frac{1 - \gamma - x_{\infty}}{m + x_{\infty}}\right)x + \frac{\alpha x_{\infty}}{m + x_{\infty}}y, \\ f_2(x, y, m) &= \rho(y + y_{\infty})\left(1 - \frac{\beta(y + y_{\infty})}{m + x + x_{\infty}}\right) - \delta(y + y_{\infty}) \\ &\quad - \frac{(\rho - \delta)^2}{\rho\beta}x + (\rho - \delta)y. \end{aligned} \tag{49}$$

Denote the eigenvalues of J_{E_3} by $\varphi + i\omega$ with $\varphi = (z/2)$ and $\omega = (\sqrt{4D - z^2}/2)$.

Define matrix

$$P = \begin{pmatrix} 1 & 0 \\ N & G \end{pmatrix}, \tag{50}$$

where $G = ((\omega(m + x_\infty))/\alpha x_\infty)$ and $N = ((m + x_\infty)(x_\infty(-1 + ((1 - \gamma - x_\infty)/(m + x_\infty))) - \varphi))/\alpha x_\infty$. Obviously,

$$P^{-1} = \begin{pmatrix} 1 & 0 \\ -\frac{N}{G} & \frac{1}{G} \end{pmatrix}, \quad (51)$$

and when $m = m_H$, i.e., $-2x_\infty^2 + (1 - \gamma - m_H)x_\infty = (\rho - \delta)(x_\infty + m_H)$, we have

$$\begin{aligned} \omega_0 &:= \omega_{m_H} = \sqrt{(\rho - \delta)x_\infty}, \\ N_0 &:= N|_{m_H} = \frac{(m_H + x_\infty)(-1 + ((1 - \gamma - x_\infty)/(m_H + x_\infty)))}{\alpha} \\ &= \frac{(m_H + x_\infty)(\rho - \delta)}{\alpha}, \\ G_0 &:= G|_{m_H} = \frac{\omega_0(m_H + x_\infty)}{\alpha x_\infty}, \\ \frac{N_0}{G_0} &= \frac{(\rho - \delta)x_\infty}{\omega_0} = \omega_0, \\ \frac{1}{G_0} &= \frac{\alpha x_\infty}{\omega_0(m_H + x_\infty)}. \end{aligned} \quad (52)$$

By the transformation,

$$\begin{pmatrix} x \\ y \end{pmatrix} = P \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ Nu + Gv \end{pmatrix}, \quad (53)$$

system (48) becomes

$$\begin{pmatrix} \dot{u} \\ \dot{v} \end{pmatrix} = \begin{pmatrix} \varphi & -\omega \\ \omega & \varphi \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} F_1(u, v, m) \\ F_2(u, v, m) \end{pmatrix}, \quad (54)$$

where

$$\begin{aligned} F_1(u, v, m) &= A_{20}u^2 + A_{11}uv + A_{21}u^2v + A_{30}u^3 \\ &\quad + o(|u|^4, |u|^4|v|), \\ F_2(u, v, m) &= B_{20}u^2 + B_{11}uv + B_{02}v^2 + B_{21}u^2v \\ &\quad + o(|u|^4, |u|^4|v|), \end{aligned} \quad (55)$$

with

$$\begin{aligned} A_{20} &= -1 - \frac{\alpha m(Nx_\infty + Nm - y_\infty)}{(m + x_\infty)^3}, \\ A_{11} &= -\frac{\alpha Gm}{(m + x_\infty)^2}, \\ A_{21} &= \frac{\alpha Gm}{(m + x)^3}, \\ A_{30} &= \frac{\alpha m(Nx_\infty + Nm - y_\infty)}{(m + x_\infty)^4}, \\ B_{20} &= \frac{N}{G} \left(1 + \frac{\alpha m(Nx_\infty + Nm - y_\infty)}{(m + x_\infty)^3} \right) \\ &\quad - \frac{1}{G} \left(\frac{\rho\beta(N^2x_\infty^2 + 2N^2x_\infty m - 2Nx_\infty y_\infty) + (Nm - y_\infty)^2}{(m + x_\infty)^3} \right), \\ B_{11} &= \frac{\alpha Nm}{(m + x_\infty)^2} - \frac{2\rho\beta(Nx_\infty + Nm - y_\infty)}{(m + x_\infty)^2}, \\ B_{02} &= -\frac{\rho\beta G}{m + x_\infty}, \\ B_{21} &= \frac{2\rho\beta(Nx_\infty + Nm - y_\infty) - \alpha Nm}{(m + x_\infty)^3}. \end{aligned} \quad (56)$$

In order to determine the stability of the periodic solution, we need to calculate the sign of the coefficient $b(m_H)$, which is given by

$$\begin{aligned} b(m_H) &:= \frac{1}{16} \{ F_{uuu}^1 + F_{uvv}^1 + F_{uuv}^2 + F_{vvv}^2 \} \\ &\quad + \frac{1}{16\omega(m_H)} \{ F_{uv}^1 (F_{uu}^1 + F_{vv}^1) - F_{uv}^2 (F_{uu}^2 + F_{vv}^2) \\ &\quad - F_{uu}^1 F_{uu}^2 + F_{vv}^1 F_{vv}^2 \}. \end{aligned} \quad (57)$$

where all partial derivatives are evaluated at the bifurcation point $(0, 0, m_H)$.

Combining $F^1(u, v)$, $F^2(u, v)$, ω_0 , G_0 , and N_0 , we have

$$\begin{aligned}
F_{uuu}^1(0,0) &= \frac{6m(\rho-\delta)}{(m+x_\infty)^2} - \frac{6\alpha m(\rho-\delta)}{\rho\beta(m+x_\infty)^3}, \\
F_{uvv}^1(0,0) &= 0, \\
F_{uv}^1(0,0) &= -\frac{m\omega_0}{x_\infty(m+x_\infty)}, \\
F_{uu}^1(0,0) &= -2 - 2\alpha m \left(\frac{\rho-\delta}{\alpha(m+x_\infty)} - \frac{\rho-\delta}{\rho\beta(m+x_\infty)^2} \right), \\
F_{vv}^1(0,0) &= 0, \\
F_{vvv}^2(0,0) &= 0, \\
F_{uuu}^2(0,0) &= \frac{4\rho\beta(\rho-\delta)}{\alpha(m+x_\infty)} - \frac{(2m+4)(\rho-\delta)}{(m+x_\infty)^2}, \\
F_{uv}^2(0,0) &= \frac{(m+2)(\rho-\delta)}{(m+x_\infty)} - \frac{2\rho\beta(\rho-\delta)}{\alpha}, \\
F_{uu}^2(0,0) &= 2\omega_0 \left(1 + \frac{\alpha m(\rho-\delta)}{(m+x_\infty)^2} \left(\frac{m+x_\infty}{\alpha} - \frac{1}{\rho\beta} \right) - \frac{\alpha\rho\beta(\rho-\delta)}{(m+x_\infty)^2} \left(\frac{m+x_\infty}{\alpha} - \frac{1}{\rho\beta} \right)^2 \right), \\
F_{vv}^2(0,0) &= -\frac{2\rho\beta\omega_0}{\alpha x_\infty},
\end{aligned} \tag{58}$$

where $m = m_H$ and

$$\begin{aligned}
b(m_H) &= \frac{\rho-\delta}{8(m+x_\infty)} \left(\frac{2(m-1)}{m+x_\infty} - \frac{3\alpha m}{\rho\beta(m+x_\infty)^2} + \frac{2\rho\beta}{\alpha} \right) \\
&\quad + \frac{1}{8} \left(\frac{m}{x_\infty(m+x_\infty)} + \frac{m^2(\rho-\delta)}{x_\infty(m+x_\infty)^2} - \frac{\alpha m^2(\rho-\delta)}{\rho\beta x_\infty(m+x_\infty)^3} \right. \\
&\quad \left. + \frac{\rho\beta(\rho-\delta)}{m+x_\infty} \left(\frac{2(m+x_\infty)}{\alpha} - \frac{m+2}{\rho\beta} \right) \right) \\
&\quad \times \left(1 - \frac{\rho\beta}{\alpha x_\infty} + \frac{\alpha m(\rho-\delta)}{(m+x_\infty)^2} \left(\frac{m+x_\infty}{\alpha} - \frac{1}{\rho\beta} \right) \right. \\
&\quad \left. - \frac{\alpha\rho\beta(\rho-\delta)}{(m+x_\infty)^2} \left(\frac{m+x_\infty}{\alpha} - \frac{1}{\rho\beta} \right)^2 \right) \\
&\quad + 2 \left(1 + \frac{\alpha m(\rho-\delta)}{(m+x_\infty)^2} \left(\frac{m+x_\infty}{\alpha} - \frac{1}{\rho\beta} \right) \right) \left(1 + \frac{\alpha m(\rho-\delta)}{(m+x_\infty)^2} \left(\frac{m+x_\infty}{\alpha} - \frac{1}{\rho\beta} \right) \right. \\
&\quad \left. - \frac{\alpha m(\rho-\delta)}{(m+x_\infty)^2} \left(\frac{m+x_\infty}{\alpha} - \frac{1}{\rho\beta} \right)^2 \right) \\
&= k_0 + k_1 + k_2 + k_3.
\end{aligned} \tag{59}$$

FIGURE 3: Phase portrait for system (12) with $\alpha = \beta = \rho = m = 1$, $\gamma = 0.5$, and $\delta = 0.1$.

FIGURE 4: Phase portrait for system (12) with $\alpha = \rho = 1$, $\beta = 0.5$, $m = 0.01$, $\gamma = 0.5$, and $\delta = 0.8$.

Combining (41), we have $b(m_H) < 0$. Therefore, according to Poincaré–Andronow’s Hopf bifurcation theory, we have the direction of the Hopf bifurcation is subcritical and the bifurcation periodic solutions are orbitally asymptotically stable.

In addition, combining (42), we have $b(m_H) > 0$. Therefore, according to Poincaré–Andronow’s Hopf bifurcation theory, we have the direction of the Hopf bifurcation is supercritical and the bifurcation periodic solutions are unstable. $\square$

6. Numerical Illustrations

In this section, we perform numerical simulations about system (5). Figure 3 shows that E_0 is an unstable node point, E_1 is a saddle point, E_3 does not exist, and E_2 is asymptotically stable and every orbit tends to it. Figure 4 shows that E_0 is an unstable node point, E_1 is a saddle point, E_2 is unstable, E_3 is unstable, and there is a limit cycle around E_3 to which every orbit tends. Figure 5 shows that E_0 is an unstable node point, E_1 is unstable and E_2 is also unstable, but E_3 is asymptotically stable and every orbit approaches this equilibrium.

FIGURE 5: Phase portrait for system (12) with $\alpha = \rho = 1$, $\beta = 0.5$, $m = 0.3$, $\gamma = 0.5$, and $\delta = 0.8$.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was partially supported by National Natural Science Foundation of P.R.China (no. 11661037).

References

- [1] N. F. Britton, *Essential Mathematical Biology*, Springer, Berlin, Germany, 2013.
- [2] J. D. Murray, *Mathematical Biology I: An Introduction*, Springer, Berlin, Germany, 2002.
- [3] J. D. Murray, *Mathematical Biology: Spatial Models and Biomedical Applications*, Springer, Berlin, Germany, 2002.
- [4] F. Brauer and C. Castillo-Chavez, *Mathematical Models in Population Biology and Epidemiology*, Springer, Berlin, Germany, 2011.
- [5] W. Yang and Y. Li, “Dynamics of a diffusive predator-prey model with modified Leslie-Gower and Holling-type III schemes,” *Computers & Mathematics with Applications*, vol. 65, no. 11, pp. 1727–1737, 2013.
- [6] F. Wu and Y. Jiao, “Stability and Hopf bifurcation of a predator-prey model,” *Boundary Value Problems*, vol. 2019, 2019.
- [7] J. Huang, S. Ruan, and J. Song, “Bifurcations in a predator-prey system of Leslie type with generalized Holling type III functional response,” *Journal of Differential Equations*, vol. 257, no. 6, pp. 1721–1752, 2014.
- [8] L. Chen and F. Chen, “Global stability of a Leslie-Gower predator-prey model with feedback controls,” *Applied Mathematics Letters*, vol. 22, no. 9, pp. 1330–1334, 2009.
- [9] G. Lin, J. Ji, L. Wang, and J. Yu, “Multitype bistability and long transients in a delayed spruce budworm population model,” *Journal of Differential Equations*, vol. 283, pp. 263–289, 2021.

- [10] N. Mart-Jeraldo and P. Aguirre, "Allee effect acting on the prey species in a Leslie-Gower predation model," *Nonlinear Analysis: Real World Applications*, vol. 45, pp. 895–917, 2019.
- [11] M. Aziz-Aloui and M. Daher Okiye, "Boundedness and global stability for a predator-prey model with modified Leslie-Gower and holling-type II schemes," *Applied Mathematics Letters*, vol. 16, pp. 1069–1075, 2003.
- [12] C. Lin and C. Ho, "Local and global stability for a predator-prey model of modified Leslie-Gower and holling-type II with time-delay," *Tunghai Science*, vol. 8, pp. 33–61, 2006.
- [13] J. Huang, Y. Gong, Y. Gong, and S. Ruan, "Bifurcation analysis in a predator-prey model with constant-yield predator harvesting," *Discrete & Continuous Dynamical Systems-B*, vol. 18, no. 8, pp. 2101–2121, 2013.
- [14] S. Chakraborty, S. Pal, and N. Bairagi, "Predator-prey interaction with harvesting: mathematical study with biological ramifications," *Applied Mathematical Modelling*, vol. 36, no. 9, pp. 4044–4059, 2012.
- [15] P. Lenzini and J. Rebaza, "Nonconstant predator harvesting on ratio-dependent predator-prey models," *Applied Mathematical Sciences*, vol. 4, pp. 791–803, 2010.
- [16] J. Ali, R. Shivaji, and K. Wampler, "Population models with diffusion, strong Allee effect and constant yield harvesting," *Journal of Mathematical Analysis and Applications*, vol. 352, no. 2, pp. 907–913, 2009.
- [17] C. Zhu and K. Lan, "Phase portraits, "Hopf bifurcations and limit cycles of Leslie-Gower predator-prey systems with harvesting rates"" *Discrete and Continuous Dynamical Systems Series B*, vol. 14, pp. 289–306, 2010.
- [18] R. P. Gupta and P. Chandra, "Bifurcation analysis of modified Leslie-Gower predator-prey model with michaelis-menten type prey harvesting," *Journal of Mathematical Analysis and Applications*, vol. 398, no. 1, pp. 278–295, 2013.
- [19] F. Chen, "On a nonlinear nonautonomous predator-prey model with diffusion and distributed delay," *Journal of Computational and Applied Mathematics*, vol. 180, no. 1, pp. 33–49, 2005.
- [20] L. Peko, *Differential Equations and Dynamical Systems*, Springer, Berlin, Germany, 2001.
- [21] B. Pirayesh, A. Pazirandeh, and M. Akbari, "Local bifurcation analysis in nuclear reactor dynamics by Sotomayor's theorem," *Annals of Nuclear Energy*, vol. 94, pp. 716–731, 2016.