Research Article

Complete Convergence for Weighted Sums of Widely Acceptable Random Variables under Sublinear Expectations

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Using different methods than the probability space, under the condition that the Choquet integral exists, we study the complete convergence theorem for weighted sums of widely acceptable random variables under sublinear expectation space. We proved corresponding theorem which was extended to the sublinear expectations' space from the probability space, and similar results were obtained.

1. Introduction

In the study of probability theory and mathematical statistics, limit theory is an important research topic, which is widely used in the financial sector and other fields. However, the establishment of the classical limit theory requires strict conditions for the certainty model, especially in the practice of financial statistics and financial risk measurement, so its limitations are gradually highlighted by many uncertainties. Therefore, Peng [1–3] made an improvement, proposed a sublinear expectations' space that can model the probability and distribution of uncertainty, and gave the corresponding theoretical system, which has aroused the attention of the majority of scholars. At present, the limit theory has obtained many excellent results under sublinear expectation. For example, in the early research on sublinear expectation, Peng [1–3] extended the central limit theory in the traditional probability space to the sublinear expectation space. Zhang [4–6] continued his research on extended negatively dependent random variables and obtained Kolmogorov's strong law of large numbers (SLLN) and a series of inequalities under sublinear expectation. Zhang and Chen [7] obtain the central limit theorem for weighted sums in sublinear expectations' space. Feng et al. [8] obtain a complete convergence of the weighted sum of negatively

dependent (ND) sequences in the sublinear expectations' space. Wang and Wu [9] study on complete convergence and almost sure convergence under the sublinear expectations. Chen [10] obtains a SLLN for an independent identically distributed sequence in the sublinear expectations space. Liang and Wu [11] research on complete convergence and complete integral convergence for extended negatively dependent (END) random variables under sublinear expectations.

Complete convergence is one of the most important problems in limit theory research because of the extensive application of weighted sum in statistics, and its properties attract more scholars to study and discuss. In the study of complete convergence in probability space, statistician Hsu and Robbins [12] first propose the concept of complete convergence in 1947, which aroused the interest of many scholars. So far, complete convergence has been studied very deeply in probability space, for example, Liang and Su [13] obtain the complete convergence theorem for weighted sums of negatively associated (NA) sequences and discuss its necessity. Sung [14], based on the exponential inequality, obtains new complete convergence results for weighted sums of independent random variables. Wu [15, 16] proves the complete convergence theorems for ND sequences and arrays of row-wise ND random variables. Lita [17] explores

the property of complete convergence for END random variables. In this article, we establish the complete convergence theorem for weighted sums of widely acceptable (WA) random variables under sublinear expectations. The results have been obtained by Lang et al. [18] and have been generalized to the sublinear expectation space.

2. Preliminaries

We use the framework and notations of Peng [1–3]. Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, X_2, \dots, X_n \in \mathcal{H}$, then $\varphi(X_1, \dots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l, \text{Lip}}(\mathbb{R}_n)$, where $C_{l, \text{Lip}}(\mathbb{R}_n)$ denotes the linear space of (local Lipschitz) functions φ satisfying

$$|\varphi(\mathbf{x}) - \varphi(\mathbf{y})| \leq c(1 + |\mathbf{x}|^m + |\mathbf{y}|^m)|\mathbf{x} - \mathbf{y}|, \quad \forall \mathbf{x}, \mathbf{y} \in \mathbb{R}_n, \quad (1)$$

for some $c > 0$, $m \in \mathbb{N}$ depending on φ . $\mathcal{H}$ is considered as a space of random variables. In this case, we denote $X \in \mathcal{H}$.

Definition 1 (see [2]). A sublinear expectation $\hat{E}$ on $\mathcal{H}$ is a function $\hat{E}: \mathcal{H} \rightarrow \mathbb{R}$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

- (a) Monotonicity: if $X \geq Y$, then $\hat{E}X \geq \hat{E}Y$
- (b) Constant preserving: $\hat{E}c = c$
- (c) Subadditivity: $\hat{E}(X + Y) \leq \hat{E}X + \hat{E}Y$, whenever $\hat{E}X + \hat{E}Y$ is not of the form $+\infty - \infty$ or $-\infty + \infty$
- (d) Positive homogeneity: $\hat{E}(\lambda X) = \lambda \hat{E}X, \lambda \geq 0$

Here, $\overline{\mathbb{R}} = [-\infty, \infty]$. The triple $(\Omega, \mathcal{H}, \hat{E})$ is called a sublinear expectation space.

Given a sublinear expectation $\hat{E}$, let us denote the conjugate expectation $\hat{e}$ of $\hat{E}$ by

$$\hat{e}X := -\hat{E}(-X), \quad \forall X \in \mathcal{H}. \quad (2)$$

From the definition, it is easily shown that, for all $X, Y \in \mathcal{H}$,

$$\begin{aligned} \hat{e}X &\leq \hat{E}X, \\ \hat{E}(X + c) &= \hat{E}X + c, \\ |\hat{E}(X - Y)| &\leq \hat{E}|X - Y|, \\ \hat{E}(X - Y) &\geq \hat{E}X - \hat{E}Y. \end{aligned} \quad (3)$$

If $\hat{E}Y = \hat{e}Y$, then $\hat{E}(X + aY) = \hat{E}X + a\hat{E}Y$ for any $a \in \mathbb{R}$. Next, we consider the capacities corresponding to the sublinear expectations. Let $\mathcal{G} \subset \mathcal{F}$. A function $V: \mathcal{G} \rightarrow [0, 1]$ is called a capacity if

$$\begin{aligned} V(\emptyset) &= 0, \\ V(\Omega) &= 1, \\ V(A) &\leq V(B), \quad \text{for } \forall A \subseteq B, A, B \in \mathcal{G}. \end{aligned} \quad (4)$$

It is called subadditive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. In the sublinear space $(\Omega, \mathcal{H}, \hat{E})$, we denote a pair $(\mathbb{V}, \mathbb{V}')$ of capacities by

$$\begin{aligned} \mathbb{V}(A) &:= \inf\{\hat{E}\xi; I(A) \leq \xi, \xi \in \mathcal{H}\}, \\ \mathbb{V}'(A) &:= 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F}, \end{aligned} \quad (5)$$

where A^c is the complement set of A . By definition of $\mathbb{V}$ and $\mathbb{V}'$, it is obvious that $\mathbb{V}$ is subadditive, and

$$\mathbb{V}'(A) \leq \mathbb{V}(A), \quad \forall A \in \mathcal{F}. \quad (6)$$

If $I(A) \in \mathcal{H}$, then

$$\begin{aligned} \mathbb{V}(A) &= \hat{E}(I(A)), \\ \mathbb{V}'(A) &= \hat{e}(I(A)). \end{aligned} \quad (7)$$

If $f \leq I(A) \leq g, f, g \in \mathcal{H}$, then

$$\hat{E}f \leq \mathbb{V}(A) \leq \hat{E}g, \quad \hat{e}f \leq \mathbb{V}'(A) \leq \hat{e}g. \quad (8)$$

This implies Markov inequality: $\forall X \in \mathcal{H}$,

$$\mathbb{V}(|X| \geq x) \leq \frac{\hat{E}(|X|^p)}{x^p}, \quad \forall x > 0, p > 0, \quad (9)$$

from $I(|X| \geq x) \leq (|X|^p/x^p) \in \mathcal{H}$.

Definition 2 (see [2]). We define the Choquet integrals/expectations $(C_{\mathbb{V}}, C_{\mathbb{V}'})$ by

$$C_{\mathbb{V}}(X) = \int_0^\infty \mathbb{V}(X \geq t)dt + \int_{-\infty}^0 [\mathbb{V}(X \geq t) - 1]dt, \quad (10)$$

with V being replaced by $\mathbb{V}$ and $\mathbb{V}'$, respectively.

Definition 3 (see [2]).

- (i) $\hat{E}$ countably subadditive: $\hat{E}$ is called to be countably subadditive if it satisfies

$$\hat{E}(X) \leq \sum_{n=1}^\infty \hat{E}(X_n), \quad \text{where } X \leq \sum_{n=1}^\infty X_n, X, X_n \in \mathcal{H}, X \geq 0, X_n \geq 0, n \geq 1. \quad (11)$$

- (ii) V is called to be countably subadditive if

$$V\left\{\bigcup_{n=1}^\infty A_n\right\} \leq \sum_{n=1}^\infty V(A_n), \quad \forall A_n \in \mathcal{F}. \quad (12)$$

Definition 4 (identical distribution, see [2]). Let $\mathbf{X}_1$ and $\mathbf{X}_2$ be two n -dimensional random vectors defined, respectively, in a sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{E}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{E}_2)$. They are called identically distributed if

$$\hat{E}[\varphi(X_1)] = \hat{E}[\varphi(X_2)], \quad \forall \varphi \in C_{l,\text{Lip}}(\mathbb{R}_n), \quad (13)$$

whenever the sublinear expectation is finite. A sequence $\{X_n; n \geq 1\}$ of random variables is said to be identically distributed if, for each $i \geq 1$, X_i and X_1 are identically distributed.

Definition 5 (widely acceptable, see [19]). Let $\{Y_n, n \geq 1\}$ be a sequence of random variables in a sublinear expectation space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$. The sequence $\{Y_n, n \geq 1\}$ is called widely acceptable (WA) if for $t \geq 0$, and for all $n \in \mathbb{N}$,

$$\hat{\mathbb{E}} \exp\left(\sum_{i=1}^n tY_i\right) \leq g(n) \prod_{i=1}^n \hat{\mathbb{E}} \exp(tY_i), \quad (14)$$

where $0 < g(n) < \infty$.

It is obvious that if $\{X_n; n \geq 1\}$ is a sequence of widely acceptable random variables and functions

$f_1(x), f_2(x), \dots \in C_{l,\text{Lip}}(\mathbb{R}_n)$ are all nondecreasing (resp. all nonincreasing), then $\{f_n(X_n); n \geq 1\}$ is also a sequence of widely acceptable random variables.

In the following, let $\{X_n; n \geq 1\}$ be a sequence of random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $S_n = \sum_{i=1}^n X_i$. The symbol c stands for a generic positive constant which may differ from one place to another. Let $a_x \sim b_x$ denote $\lim_{x \rightarrow \infty} a_x/b_x = 1$. $a_x \ll b_x$ denotes that there exists a constant $c > 0$ such that $a_x \leq cb_x$ for sufficiently large x , and $I(\cdot)$ denotes an indicator function.

To prove our results, we need the following two lemmas.

Lemma 1 (see [19]). Let $\{X_1, \dots, X_n\}$ be a sequence of WA random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, with $\hat{\mathbb{E}}(X_k) \leq 0$ for $1 < i \leq n$. Then, for all $x > 0, d > 0$, we have

$$\mathbb{V}(S_n \geq x) \leq \mathbb{V}\left(\max_{1 \leq i \leq n} X_i > d\right) + g(n) \exp\left(\frac{x}{d} - \frac{x}{d} \ln\left(1 + \frac{xd}{\sum_{i=1}^n \hat{\mathbb{E}}|X_i|^2}\right)\right). \quad (15)$$

Proof. We can easily get (15) from Theorem 1 of [19]. $\square$

Lemma 2 Suppose $X \in H$, $p > 0$, for any constant $c > 0$:

(i) Then,

$$C_{\mathbb{V}}(|X|^p) < \infty \Leftrightarrow \sum_{n=1}^{\infty} n^{p-1} \mathbb{V}(|X| > cn) < \infty. \quad (16)$$

Especially, when $p = 1$, we have

$$C_{\mathbb{V}}(|X|) < \infty \Leftrightarrow \sum_{n=1}^{\infty} \mathbb{V}(|X| > cn) < \infty. \quad (17)$$

(ii) If $C_{\mathbb{V}}(|X|^p) < \infty$, then, for any $a > 1$,

$$\sum_{k=1}^{\infty} a^k \mathbb{V}(|X| > ca^{(k/p)}) < \infty. \quad (18)$$

Proof. For (i), similar to the proof of Lemma 3 in [9] and assuming $x^{(1/p)} = y$, then (16) is obtained. For (ii), similar to the proof Theorem 6 in [20], we can get, for any $a > 1$, (18) is satisfied. $\square$

3. The Main Results and Their Proofs

(A.1) Let $g(x)$ be a nondecreasing positive function on $[0, \infty)$, $g(x) = g(n)$, when $x = n$. And, $g(x)/x^\tau \downarrow$ for some $0 < \tau < 1$

(A.2) There exists a nondecreasing positive function $h(x)$ on $[0, \infty)$, such that $h(x)/x \downarrow$ and $\sum_{n=1}^{\infty} g(n)/(nh^\gamma(\mu n)) < \infty$ for some $\gamma > 0, 0 < \mu < 1$

Theorem 1. Let $\{X_n; n \geq 1\}$ be a sequence of WA and identically distributed random variables under sublinear expectations. $g(x)$ satisfies (A.1) or (A.2), and $p \geq 1$. When $p = 1$, assume that (A.1) holds and

$$\hat{\mathbb{E}}(|X|^{1+\delta}) \leq C_{\mathbb{V}}(|X|^{1+\delta}) < \infty, \quad (19)$$

for some $\delta > 0$, or (A.2) holds and

$$\hat{\mathbb{E}}(|X|h(|X|)) \leq C_{\mathbb{V}}(|X|h(|X|)) < \infty. \quad (20)$$

When $p > 1$, assume that

$$\hat{\mathbb{E}}(|X|^p) \leq C_{\mathbb{V}}(|X|^p) < \infty, \quad (21)$$

and (A.1) or (A.2) holds. Let $\{a_{ni}; 1 \leq i \leq n, n \geq 1\}$ be an array of real positive numbers satisfying

$$\max_{1 \leq i \leq n} a_{ni} = O(n^{-1}). \quad (22)$$

Then, for all $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} n^{p-2} \mathbb{V}\left(\sum_{i=1}^n a_{ni}(X_i - \hat{\mathbb{E}}X_i) > \varepsilon\right) < \infty, \quad (23)$$

$$\sum_{n=1}^{\infty} n^{p-2} \mathbb{V}\left(\sum_{i=1}^n a_{ni}(X_i - \hat{\mathbb{E}}X_i) < -\varepsilon\right) < \infty. \quad (24)$$

In particular, if $\hat{\mathbb{E}}X_i = \varepsilon X_i$, then

$$\sum_{n=1}^{\infty} n^{p-2} \mathbb{V}\left(\left|\sum_{i=1}^n a_{ni}(X_i - \hat{\mathbb{E}}X_i)\right| > \varepsilon\right) < \infty. \quad (25)$$

Proof. To simplify our proof, we may assume that $\max_{1 \leq i \leq n} a_{ni} \leq n^{-1}$; thus,

$$\sum_{i=1}^n a_{ni}^2 \leq \left(\max_{1 \leq i \leq n} a_{ni} \right)^2 \sum_{i=1}^n 1 \leq n^{-1}. \quad (26)$$

When $p = 1$, it is obvious that $C_V(|X|) < \infty$ by $C_V(|X|^{1+\delta}) < \infty$ or $C_V(|X|h(|X|)) < \infty$. When $p > 1$, $C_V(|X|) < \infty$ follows from $C_V(|X|^p) < \infty$. Therefore, there exists a positive integer N such that

$$\hat{\mathbb{E}}|X|I(|X| > N) < \frac{\varepsilon}{6}. \quad (27)$$

For widely acceptable random variables $\{X_n; n \geq 1\}$, in order to ensure that the truncated random variables are also widely acceptable, truncated functions should belong to $C_{l,\text{Lip}}$ and should be nondecreasing. Denote for $1 \leq i \leq n$ that

$$\begin{aligned} X'_i &= NI(X_i > N) + X_i I(|X_i| \leq N) - NI(X_i < -N), \\ X''_i &= X_i - X'_i = (X_i - N)I(X_i > N) + (X_i + N)I(X_i < -N). \end{aligned} \quad (28)$$

It is easily checked that

$$\begin{aligned} & \sum_{n=1}^{\infty} n^{p-2} \mathbb{V} \left(\sum_{i=1}^n a_{ni} (X_i - \hat{\mathbb{E}}X_i) > \varepsilon \right) \\ &= \sum_{n=1}^{\infty} n^{p-2} \mathbb{V} \left(\left(\sum_{i=1}^n a_{ni} (X'_i - \hat{\mathbb{E}}X'_i) + \sum_{i=1}^n a_{ni} X''_i + \sum_{i=1}^n a_{ni} (\hat{\mathbb{E}}X'_i - \hat{\mathbb{E}}X_i) \right) > \varepsilon \right) \\ &\leq \sum_{n=1}^{\infty} n^{p-2} \mathbb{V} \left(\sum_{i=1}^n a_{ni} (X'_i - \hat{\mathbb{E}}X'_i) > \frac{\varepsilon}{3} \right) + \sum_{n=1}^{\infty} n^{p-2} \mathbb{V} \left(\sum_{i=1}^n a_{ni} X''_i > \frac{\varepsilon}{3} \right) \\ &\quad + \sum_{n=1}^{\infty} n^{p-2} \mathbb{V} \left(\sum_{i=1}^n a_{ni} (\hat{\mathbb{E}}X'_i - \hat{\mathbb{E}}X_i) > \frac{\varepsilon}{3} \right) \\ &=: I_1 + I_2 + I_3. \end{aligned} \quad (29)$$

Thus, to prove the desired result (23), we only need to show $I_1 < \infty$, $I_2 < \infty$, and $I_3 < \infty$.

Let $\{T_i, 1 \leq i \leq n, n \geq 1\}$ be a sequence of any random variable, and we all have

$$\mathbb{V} \left(\max_{1 \leq i \leq n} T_i > d \right) \leq \sum_{i=1}^n \mathbb{V}(|T_i| > d) \leq \sum_{i=1}^n \frac{1}{d^q} \hat{\mathbb{E}}|T_i|^q. \quad (30)$$

For each $n \geq 1$, $\{X'_i, 1 \leq i \leq n\}$ is a sequence of WA, $\{X'_i - \hat{\mathbb{E}}X'_i\}$ satisfies the conditions of Lemma 1. Let $x = (\varepsilon/3)$ and $d = (\varepsilon/(6(p+\gamma)))$ in Lemma 1; taking $q > p$ and (30), we obtain

$$\begin{aligned} I_1 &\leq \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{i=1}^n \frac{1}{d^q} \hat{\mathbb{E}}|a_{ni} (X'_i - \hat{\mathbb{E}}X'_i)|^q \right) \\ &\quad + g(n) \exp \left(\frac{\varepsilon}{3d} - \frac{\varepsilon}{3d} \ln \left(1 + \frac{\varepsilon d}{3 \sum_{i=1}^n \hat{\mathbb{E}}|a_{ni} (X'_i - \hat{\mathbb{E}}X'_i)|^2} \right) \right) =: I_{11} + I_{12}. \end{aligned} \quad (31)$$

For I_{11} , it follows from the definition of $\{X'_i, i \geq 1\}$ that $\hat{\mathbb{E}}|X'_i| \leq N$. By Cr inequality and (22), we obtain

$$\begin{aligned}
I_{11} &\ll \sum_{n=1}^{\infty} n^{p-2} \sum_{i=1}^n \frac{1}{a_{ni}^q} \hat{\mathbb{E}} |X'_i|^q \\
&\leq c \sum_{n=1}^{\infty} n^{p-2} \sum_{i=1}^n a_{ni}^q \leq c \sum_{n=1}^{\infty} n^{p-2} \left(\max_{1 \leq i \leq n} a_{ni} \right)^q \sum_{i=1}^n 1 \\
&\leq c \sum_{n=1}^{\infty} n^{p-q-1} \\
&< \infty.
\end{aligned} \tag{32}$$

For I_{12} , we get, by (26), Cr inequality, $\tau < 1 < 1 + \gamma$, and (A.1) or (A.2) holds, such that

$$\begin{aligned}
I_{12} &\leq c \sum_{n=1}^{\infty} n^{p-2} g(n) \left(\sum_{i=1}^n \hat{\mathbb{E}} |a_{ni} (X'_i - \hat{\mathbb{E}} X'_i)|^2 \right)^{(\varepsilon/3d)} \\
&\leq c \sum_{n=1}^{\infty} n^{p-2} g(n) \left(\sum_{i=1}^n a_{ni}^2 \hat{\mathbb{E}} (X'_i)^2 \right)^{(\varepsilon/3d)} \\
&\leq c \sum_{n=1}^{\infty} n^{p-2} g(n) \left(\sum_{i=1}^n a_{ni}^2 \right)^{(\varepsilon/3d)} \\
&\leq \begin{cases} c \sum_{n=1}^{\infty} n^{p-2+\tau} \left(\sum_{i=1}^n a_{ni}^2 \right)^{(\varepsilon/3d)}, & \text{if (A.1) holds,} \\ c \sum_{n=1}^{\infty} n^{p-2-(\varepsilon/3d)} \frac{g(n) h^\gamma(n) n^{\gamma+1}}{n h^\gamma(n) n^\gamma}, & \text{if (A.2) holds,} \end{cases} \\
&\leq \begin{cases} c \sum_{n=1}^{\infty} n^{p-2+\tau-(\varepsilon/3d)}, & \text{if (A.1) holds,} \\ c \sum_{n=1}^{\infty} n^{p-1+\gamma-(\varepsilon/3d)}, & \text{if (A.2) holds,} \end{cases} \\
&= \begin{cases} c \sum_{i=1}^{\infty} n^{p-2+\tau-(\varepsilon/(3(\varepsilon/(6(p+\gamma)))))}, & \text{if (A.1) holds,} \\ c \sum_{i=1}^{\infty} n^{p-1+\gamma-(\varepsilon/(3(\varepsilon/(6(p+\gamma)))))}, & \text{if (A.2) holds,} \end{cases} \\
&\leq \sum_{n=1}^{\infty} n^{-1-p-\gamma} < \infty.
\end{aligned} \tag{33}$$

Thus, we have proved $I_1 < \infty$.

Now, we deal with I_2 . For $n > N$, denote for $1 \leq i \leq n$ that

$$\begin{aligned}
Y_{ni} &= (X_i - N)I(N < X_i \leq n) + (n - N)I(X_i > n) \\
&\quad + (X_i + N)I(-n < X_i \leq -N) + (-n + N)I(X_i < -n).
\end{aligned} \tag{34}$$

Then,

$$\begin{aligned}
I_2 &= \sum_{n=1}^{\infty} n^{p-2} \mathbb{V} \left(\sum_{i=1}^n a_{ni} X''_i > \frac{\varepsilon}{3} \right) \\
&\leq \sum_{n=1}^{\infty} n^{p-2} \sum_{i=1}^n \mathbb{V}(|X_i| > n) + \sum_{n=1}^{\infty} n^{p-2} \sum_{i=1}^n \mathbb{V} \left(\sum_{i=1}^n a_{ni} Y_{ni} > \frac{\varepsilon}{3} \right) \\
&=: I_{21} + I_{22}.
\end{aligned} \tag{35}$$

For the μ of condition (A.2), let $z(x) \in C_{l,\text{Lip}}(\mathbb{R})$, $0 \leq z(x) \leq 1$, for all x , and $z(x) = 1$ if $|x| \leq \mu$, $z(x) = 0$ if $|x| > 1$, and $z(x)$ are nonincreasing for any $x > 0$. Then,

$$\begin{aligned}
I(|x| \leq \mu) &\leq z(|x|) \leq I(|x| \leq 1), I(|x| > 1) \\
&\leq 1 - z(|x|) \leq I(|x| > \mu).
\end{aligned} \tag{36}$$

For I_{21} , we can get by (36); then,

$$\begin{aligned}
I_{21} &\leq \sum_{n=1}^{\infty} n^{p-2} \sum_{i=1}^n \hat{\mathbb{E}} \left(1 - z \left(\frac{|X_i|}{n} \right) \right) \\
&= \sum_{n=1}^{\infty} n^{p-2} \sum_{i=1}^n \hat{\mathbb{E}} \left(1 - z \left(\frac{|X|}{n} \right) \right) \leq \sum_{n=1}^{\infty} n^{p-1} \mathbb{V}(|X| > \mu n).
\end{aligned} \tag{37}$$

When $p = 1$, we can get $I_{21} \leq \sum_{n=1}^{\infty} \mathbb{V}(|X| > \mu n) < \infty$ by (17). When $p > 1$, we can get $I_{21} < \infty$ by (16). Next, we prove $I_{22} < \infty$. Obviously,

$$\begin{aligned}
I_{22} &= \sum_{n=1}^{\infty} n^{p-2} \mathbb{V} \left(\sum_{i=1}^n a_{ni} Y_{ni} > \frac{\varepsilon}{3} \right) \\
&\leq \sum_{n=1}^{\infty} n^{p-2} \mathbb{V} \left(\sum_{i=1}^n a_{ni} (Y_{ni} - \hat{\mathbb{E}} Y_{ni}) > \frac{\varepsilon}{6} \right) \\
&\quad + \sum_{n=1}^{\infty} n^{p-2} \mathbb{V} \left(\sum_{i=1}^n a_{ni} \hat{\mathbb{E}} Y_{ni} > \frac{\varepsilon}{6} \right).
\end{aligned} \tag{38}$$

And, it follows from (22) and (27); when $n > N$, we have

$$\begin{aligned}
\sum_{i=1}^n a_{ni} \hat{\mathbb{E}} Y_{ni} &\leq \sum_{i=1}^n a_{ni} \hat{\mathbb{E}} |X_i| I(|X_i| > N) \\
&\leq \max_{1 \leq i \leq n} a_{ni} \sum_{i=1}^n \hat{\mathbb{E}} |X| I(|X| > N) \\
&\leq \hat{\mathbb{E}} |X| I(|X| > N) < \frac{\varepsilon}{6}.
\end{aligned} \tag{39}$$

So, we obtain $\mathbb{V}(\sum_{i=1}^n a_{ni} \hat{\mathbb{E}} Y_{ni} > (\varepsilon/6)) = 0$, when $n > N$. It follows that $\sum_{n=1}^{\infty} n^{p-2} \mathbb{V}(\sum_{i=1}^n a_{ni} \hat{\mathbb{E}} Y_{ni} > (\varepsilon/6)) < \infty$. Thus, to estimate I_{22} , it suffices to show $\sum_{n=1}^{\infty} n^{p-2} \mathbb{V}(\sum_{i=1}^n a_{ni} (Y_{ni} - \hat{\mathbb{E}} Y_{ni}) > (\varepsilon/6)) < \infty$. Next, we consider the following two cases. $\square$

Case 1. $p > 1$.

$\{Y_{ni}, i \geq 1, n > N\}$ also is a sequence of WA random variables, and $\{Y_{ni} - \hat{\mathbb{E}}Y_{ni}\}$ satisfies the conditions of Lemma

1. Applying Lemma 1 with $x = (\varepsilon/6)$, $d > 0$, and (30), taking $q > p$, we have

$$\begin{aligned} I_{22} &\ll \sum_{n=1}^{\infty} n^{p-2} \mathbb{V} \left(\sum_{i=1}^n a_{ni} (Y_{ni} - \hat{\mathbb{E}}Y_{ni}) > \frac{\varepsilon}{6} \right) \\ &\leq \sum_{n=1}^{\infty} n^{p-2} \left(\sum_{i=1}^n \frac{1}{d^q} \hat{\mathbb{E}} |a_{ni} (Y_{ni} - \hat{\mathbb{E}}Y_{ni})|^q \right. \\ &\quad \left. + g(n) \exp \left(\frac{\varepsilon}{6d} - \frac{\varepsilon}{6d} \ln \left(1 + \frac{\varepsilon d}{6 \sum_{i=1}^n \hat{\mathbb{E}} |a_{ni} (Y_{ni} - \hat{\mathbb{E}}Y_{ni})|^2} \right) \right) \right) = I_{221} + I_{222}. \end{aligned} \quad (40)$$

Firstly, we prove $I_{221} < \infty$. For any $r > 0$, by the Cr inequality and (36), we have

$$\begin{aligned} |Y_{ni}|^r &\leq |X_i|^r I(|X_i| \leq n) + n^r I(|X_i| > n) \\ &\leq |X_i|^r z \left(\frac{\mu |X_i|}{n} \right) + n^r \left(1 - z \left(\frac{|X_i|}{n} \right) \right). \end{aligned} \quad (41)$$

Thus,

$$\begin{aligned} \hat{\mathbb{E}} |Y_{ni}|^r &\leq \hat{\mathbb{E}} \left(|X_i|^r z \left(\frac{\mu |X_i|}{n} \right) \right) + n^r \hat{\mathbb{E}} \left(1 - z \left(\frac{|X_i|}{n} \right) \right) \\ &= \hat{\mathbb{E}} \left(|X|^r z \left(\frac{\mu |X|}{n} \right) \right) + n^r \hat{\mathbb{E}} \left(1 - z \left(\frac{|X|}{n} \right) \right) \\ &\leq \hat{\mathbb{E}} \left(|X|^r z \left(\frac{\mu |X|}{n} \right) \right) + n^r \mathbb{V}(|X| > \mu n). \end{aligned} \quad (42)$$

For I_{221} , it follows from (42) and (22) that

$$\begin{aligned} I_{221} &\leq c \sum_{n=1}^{\infty} n^{p-2} \sum_{i=1}^n a_{ni}^q \hat{\mathbb{E}} |Y_{ni}|^q \\ &\leq c \sum_{n=1}^{\infty} n^{p-2} \sum_{i=1}^n a_{ni}^q \left(\hat{\mathbb{E}} \left(|X|^q z \left(\frac{\mu |X|}{n} \right) \right) + n^q \mathbb{V}(|X| > \mu n) \right) \\ &\leq c \sum_{n=1}^{\infty} n^{p-2} \left(\max_{1 \leq i \leq n} a_{ni} \right)^q \sum_{i=1}^n 1 \left(\hat{\mathbb{E}} \left(|X|^q z \left(\frac{\mu |X|}{n} \right) \right) + n^q \mathbb{V}(|X| > \mu n) \right) \\ &\leq c \sum_{n=1}^{\infty} n^{p-1-q} \left(\hat{\mathbb{E}} \left(|X|^q z \left(\frac{\mu |X|}{n} \right) \right) + n^q \mathbb{V}(|X| > \mu n) \right) \\ &\ll \sum_{n=1}^{\infty} n^{p-1-q} \hat{\mathbb{E}} \left(|X|^q z \left(\frac{\mu |X|}{n} \right) \right) + \sum_{n=1}^{\infty} n^{p-1} \mathbb{V}(|X| > \mu n) \\ &=: I'_{221} + I''_{221}. \end{aligned} \quad (43)$$

When $p > 1$, we can get $I'_{221} < \infty$ by (16). Next, we prove $I''_{221} < \infty$.

Let $z_j(x)$ be an even function and $z_j(x) \in C_{l,\text{Lip}}(\mathbb{R})$, $j \geq 1$ such that $0 \leq z_j(x) \leq 1$ or all x and $z_j(x/2^j) = 1$ if $2^{j-1} < |x| \leq 2^j$ and $z_j(x/2^j) = 0$ if $|x| \leq \mu 2^{j-1}$ or $|x| > (1 + \mu)2^j$. Then,

$$z_j \left(\frac{X}{2^j} \right) \leq I(\mu 2^{(j-1)} < |X| < (1 + \mu)2^j), \quad (44)$$

$$|X|^r z \left(\frac{X}{2^k} \right) \leq 1 + \sum_{j=1}^k |X|^r z_j \left(\frac{X}{2^j} \right), \quad \text{for } \forall r > 0.$$

For every n , there exists k such that $2^{k-1} \leq n < 2^k$; thus, by (8) and (44) and from $z(x)$ which is nonincreasing for any $x > 0$, we obtain

$$\begin{aligned} I'_{221} &= \sum_{k=1}^{\infty} \sum_{2^{k-1} \leq n < 2^k} 2^{k(p-1-q)} \hat{\mathbb{E}} \left(|X|^q z \left(\frac{\mu |X|}{2^k} \right) \right) \\ &\ll \sum_{k=1}^{\infty} 2^{k(p-q)} \hat{\mathbb{E}} \left(|X|^q z \left(\frac{\mu |X|}{2^k} \right) \right) \\ &\ll \sum_{k=1}^{\infty} 2^{k(p-q)} \sum_{j=1}^k \hat{\mathbb{E}} \left(|X|^q z_j \left(\frac{\mu |X|}{2^j} \right) \right) \\ &\ll \sum_{k=1}^{\infty} 2^{k(p-q)} \sum_{j=1}^k 2^{qj} \hat{\mathbb{E}} \left(z_j \left(\frac{\mu |X|}{2^j} \right) \right) \\ &= \sum_{j=1}^{\infty} 2^{qj} \hat{\mathbb{E}} \left(z_j \left(\frac{\mu |X|}{2^j} \right) \right) \sum_{k=j}^{\infty} 2^{k(p-q)} \\ &\ll \sum_{j=1}^{\infty} 2^{pj} \hat{\mathbb{E}} \left(z_j \left(\frac{\mu |X|}{2^j} \right) \right) \\ &\leq \sum_{j=1}^{\infty} 2^{pj} \mathbb{V}(|X| > \mu 2^{(j-1)}) \\ &= \sum_{j=1}^{\infty} (2^p)^j \mathbb{V}(|X| > \mu (2^p)^{(j/p)}) < \infty. \end{aligned} \quad (45)$$

It follows that $I_{221} < \infty$. For I_{222} , we consider the following two cases.

Case 2. $p \geq 2$.

By (8), we have $\hat{\mathbb{E}}|Y_{ni}|^2 \leq C_V(|Y_{ni}|^2) \leq C_V(|X|^2) < \infty$, $\tau < 1 < 1 + \gamma$, and (26), and taking $d = (\varepsilon/(12(p + \gamma)))$ and (A.1) or (A.2) hold that

$$\begin{aligned}
 I_{222} &\leq c \sum_{n=1}^{\infty} n^{p-2} g(n) \exp\left(-\frac{\varepsilon}{6d} \ln\left(\frac{\varepsilon d}{c \sum_{i=1}^n a_{ni}^2}\right)\right) \\
 &\leq c \sum_{n=1}^{\infty} n^{p-2} g(n) \exp\left(-\frac{\varepsilon}{6d} \ln\left(\frac{\varepsilon d}{cn^{-1}}\right)\right) \\
 &\leq c \sum_{n=1}^{\infty} n^{p-2} g(n) n^{-(\varepsilon/6d)} \\
 &\leq \begin{cases} c \sum_{n=1}^{\infty} n^{p-2+\tau-(\varepsilon/6d)}, & \text{if (A.1) holds,} \\ c \sum_{n=1}^{\infty} n^{p-2-(\varepsilon/6d)} \frac{g(n)h^\gamma(n)n^{\gamma+1}}{nh^\gamma(n)n^\gamma}, & \text{if (A.2) holds,} \end{cases} \\
 &\leq \begin{cases} c \sum_{n=1}^{\infty} n^{p-2+\tau-(\varepsilon/6d)}, & \text{if (A.1) holds,} \\ c \sum_{n=1}^{\infty} n^{p-1-(\varepsilon/6d)+\gamma}, & \text{if (A.2) holds,} \end{cases} \\
 &= \begin{cases} c \sum_{n=1}^{\infty} n^{p-2+\tau-(\varepsilon/(6(\varepsilon/(12(p+\gamma)))))}, & \text{if (A.1) holds,} \\ c \sum_{n=1}^{\infty} n^{p-1+\gamma-(\varepsilon/(6(\varepsilon/(12(p+\gamma)))))}, & \text{if (A.2) holds,} \end{cases} \\
 &\ll \sum_{n=1}^{\infty} n^{-1-p-\gamma} < \infty.
 \end{aligned} \tag{46}$$

Case 3. $1 < p < 2$.

Similar to Case 2, taking $d = (\varepsilon/(12(1 + ((\gamma + 1)/(p - 1)))))$ and (A.1) or (A.2) hold, we have

$$\begin{aligned}
 I_{222} &\leq c \sum_{n=1}^{\infty} n^{p-2} g(n) \exp\left(-\frac{\varepsilon}{6d} \ln\left(\frac{\varepsilon d}{6 \sum_{i=1}^n a_{ni}^2 \hat{\mathbb{E}}|Y_{ni}|^p |Y_{ni}|^{2-p}}\right)\right) \\
 &\leq c \sum_{n=1}^{\infty} n^{p-2} g(n) \exp\left(-\frac{\varepsilon}{6d} \ln\left(\frac{\varepsilon d}{cn^{2-p} \sum_{i=1}^n a_{ni}^2}\right)\right) \\
 &\leq c \sum_{n=1}^{\infty} n^{p-2} g(n) \exp\left(-\frac{\varepsilon}{6d} \ln\left(\frac{\varepsilon d}{cn^{1-p}}\right)\right) \\
 &\leq c \sum_{n=1}^{\infty} n^{p-2} g(n) n^{-(\varepsilon/6d)(p-1)} \\
 &\leq \begin{cases} c \sum_{n=1}^{\infty} n^{p-2+\tau-(\varepsilon/6d)(p-1)}, & \text{if (A.1) holds,} \\ c \sum_{n=1}^{\infty} n^{p-2-(\varepsilon/6d)(p-1)} \frac{g(n)h^\gamma(n)n^{\gamma+1}}{nh^\gamma(n)n^\gamma}, & \text{if (A.2) holds,} \end{cases} \\
 &\leq \begin{cases} c \sum_{n=1}^{\infty} n^{p-2+\tau-(\varepsilon/6d)(p-1)}, & \text{if (A.1) holds,} \\ c \sum_{n=1}^{\infty} n^{p-1-(\varepsilon/6d)(p-1)+\gamma}, & \text{if (A.2) holds,} \end{cases} \\
 &= \begin{cases} c \sum_{n=1}^{\infty} n^{p-2+\tau-(\varepsilon/(6(\varepsilon/(12(1+((\gamma+1)/(p-1)))))))(p-1)}, & \text{if (A.1) holds,} \\ c \sum_{n=1}^{\infty} n^{p-1+\gamma-(\varepsilon/(6(\varepsilon/(12(1+((\gamma+1)/(p-1)))))))(p-1)}, & \text{if (A.2) holds,} \end{cases} \\
 &\ll \sum_{n=1}^{\infty} n^{-1-p-\gamma} < \infty.
 \end{aligned} \tag{47}$$

So, when $p > 1$, we have $I_{22} < \infty$.

Case 4. $p = 1$.

Let $x = (\varepsilon/6)$ and $d = (\varepsilon\delta/(12(1 + \gamma)))$ in Lemma 1; taking $q > p$ and (30), we still can obtain (40). And, similar to Case 1, we also get $I_{221} < \infty$. Next, we discuss the situation of (A.1) or (A.2) holds.

If (A.1) holds, according to (19), we can get $\hat{\mathbb{E}}|Y_{ni}|^{1+\delta} \leq C_V(|X|^{1+\delta}) < \infty$. Then,

$$\begin{aligned}
I_{22} &\ll \sum_{n=1}^{\infty} n^{-1} g(n) \exp \left(\frac{\varepsilon}{6d} - \frac{\varepsilon}{6d} \ln \left(1 + \frac{\varepsilon d}{6 \sum_{i=1}^n \hat{\mathbb{E}} |a_{ni}(Y_{ni} - \hat{\mathbb{E}} Y_{ni})|^2} \right) \right) \\
&\leq c \sum_{n=1}^{\infty} n^{-1} g(n) \exp \left(-\frac{\varepsilon}{6d} \ln \left(\frac{\varepsilon d}{c \sum_{i=1}^n a_{ni}^2 \hat{\mathbb{E}} |Y_{ni}|^2} \right) \right) \\
&\leq c \sum_{n=1}^{\infty} n^{-1} g(n) \exp \left(-\frac{\varepsilon}{6d} \ln \left(\frac{\varepsilon d}{c \sum_{i=1}^n a_{ni}^2 \hat{\mathbb{E}} |Y_{ni}|^{1+\delta} |Y_{ni}|^{1-\delta}} \right) \right) \\
&\leq c \sum_{n=1}^{\infty} n^{-1} g(n) \exp \left(-\frac{\varepsilon}{6d} \ln \left(\frac{\varepsilon d}{c n^{1-\delta} \sum_{i=1}^n a_{ni}^2} \right) \right) \\
&\leq c \sum_{n=1}^{\infty} n^{-1} g(n) \exp \left(-\frac{\varepsilon}{6d} \ln \left(\frac{\varepsilon d}{c n^{-\delta}} \right) \right) \\
&\leq c \sum_{n=1}^{\infty} n^{-1} g(n) n^{-(\varepsilon/6d)\delta} \leq c \sum_{n=1}^{\infty} n^{-1+\tau} n^{-(\varepsilon/6d)\delta} \\
&\leq c \sum_{n=1}^{\infty} n^{-1+\tau-(\varepsilon/6d)\delta} = c \sum_{i=1}^{\infty} n^{-1+\tau-(\varepsilon/(6\delta/(12(1+\gamma))))\delta} \\
&\leq c \sum_{n=1}^{\infty} n^{-2-\gamma} < \infty.
\end{aligned} \tag{48}$$

If (A.2) holds, by (42) and Cr inequality, we have

$$\begin{aligned}
I_{22} &\leq c \sum_{n=1}^{\infty} n^{-1} g(n) \exp \left(-\frac{\varepsilon}{6d} \ln \left(\frac{\varepsilon d}{c \sum_{i=1}^n a_{ni}^2 \hat{\mathbb{E}} |Y_{ni}|^2} \right) \right) \\
&\ll \sum_{n=1}^{\infty} n^{-1} g(n) \left(\sum_{i=1}^n a_{ni}^2 \hat{\mathbb{E}} |Y_{ni}|^2 \right)^{(\varepsilon/6d)} \\
&\leq \sum_{n=1}^{\infty} n^{-1-(\varepsilon/6d)} g(n) \left(\hat{\mathbb{E}} |X|^2 z \left(\frac{\mu |X|}{n} \right) + n^2 \hat{\mathbb{E}} \left(1 - z \left(\frac{|X|}{n} \right) \right) \right)^{(\varepsilon/6d)} \\
&\leq \sum_{n=1}^{\infty} n^{-1-(\varepsilon/6d)} g(n) \left(\hat{\mathbb{E}} |X|^2 z \left(\frac{\mu |X|}{n} \right) \right)^{(\varepsilon/6d)} \\
&\quad + \sum_{n=1}^{\infty} n^{-1-(\varepsilon/6d)} g(n) \left(n^2 \hat{\mathbb{E}} \left(1 - z \left(\frac{|X|}{n} \right) \right) \right)^{(\varepsilon/6d)} \\
&=: I_4 + I_5.
\end{aligned} \tag{49}$$

Then, in order to prove $I_{22} < \infty$, it suffices to show $I_4 < \infty$ and $I_5 < \infty$.

By (36), we can obtain $z(\mu |X|/n) \leq I(|X| \leq (n/\mu))$. Since $h(x) \uparrow$ and $h(x)/x \downarrow$, we can get $xh(x) \uparrow$ and $x/h(x) \uparrow$. Combine $\sum_{n=1}^{\infty} g(n)/(nh^\gamma(\mu n)) < \infty$ and (20), and taking $d = (\varepsilon/6\gamma)$, we can obtain

$$\begin{aligned}
I_4 &= \sum_{n=1}^{\infty} n^{-1-\gamma} g(n) \left(\frac{\hat{\mathbb{E}} |X| h(|X|) |X|}{h(|X|)} z \left(\frac{\mu |X|}{n} \right) \right)^\gamma \\
&\leq \sum_{n=1}^{\infty} n^{-1-\gamma} g(n) \left(\frac{(n/\mu)}{h(n/\mu)} \hat{\mathbb{E}} |X| h(|X|) \right)^\gamma \\
&\ll \sum_{n=1}^{\infty} \frac{g(n)}{nh^\gamma(\mu n)} (\hat{\mathbb{E}} |X| h(|X|))^\gamma \\
&< \infty.
\end{aligned} \tag{50}$$

And, by (36), we also can get $1 - z(|X|/n) \leq I(|X| \geq \mu n)$. And, combine $x/h(x) \uparrow$; then,

$$\begin{aligned}
I_5 &= \sum_{n=1}^{\infty} n^{-1-\gamma} g(n) \left(n^2 \hat{\mathbb{E}} \frac{|X| h(|X|)}{|X| h(|X|)} \left(1 - z \left(\frac{|X|}{n} \right) \right) \right)^\gamma \\
&\leq \sum_{n=1}^{\infty} n^{-1-\gamma} g(n) \left(n^2 \hat{\mathbb{E}} \frac{|X| h(|X|)}{(\mu n) h(\mu n)} \right)^\gamma \\
&\ll \sum_{n=1}^{\infty} n^{-1-\gamma} g(n) \left(n \frac{\hat{\mathbb{E}} |X| h(|X|)}{h(\mu n)} \right)^\gamma \\
&= \sum_{n=1}^{\infty} n^{-1} g(n) \left(\frac{\hat{\mathbb{E}} |X| h(|X|)}{h(\mu n)} \right)^\gamma \\
&= \sum_{n=1}^{\infty} \frac{g(n)}{nh^\gamma(\mu n)} (\hat{\mathbb{E}} |X| h(|X|))^\gamma \\
&< \infty.
\end{aligned} \tag{51}$$

So, we have proved $I_{22} < \infty$.

Next, we prove $I_3 < \infty$. From (3), (22), and (27), we have

$$\begin{aligned}
\sum_{i=1}^n a_{ni} |\hat{\mathbb{E}} X'_i - \hat{\mathbb{E}} X_i| &\leq \sum_{n=1}^n a_{ni} \hat{\mathbb{E}} |X'_i - X_i| \\
&\leq \max_{1 \leq i \leq n} a_{ni} \sum_{n=1}^n \hat{\mathbb{E}} |X| I(|X| > N) \\
&\leq \hat{\mathbb{E}} |X| I(|X| > N) < \frac{\varepsilon}{6}.
\end{aligned} \tag{52}$$

So, we can obtain $\mathbb{V}(\sum_{i=1}^n a_{ni} (\hat{\mathbb{E}} X'_i - \hat{\mathbb{E}} X_i)) > (\varepsilon/3) = 0$, when $n > N$. It follows that $I_3 = \sum_{n=1}^{\infty} n^{p-2} \mathbb{V}(\sum_{i=1}^n a_{ni} (\hat{\mathbb{E}} X'_i - \hat{\mathbb{E}} X_i)) > (\varepsilon/3) < \infty$. Then, (23) holds.

Obviously, $\{-X_n, n \geq 1\}$ also satisfies the conditions of Theorem 1. Considering $\{-X_n, n \geq 1\}$ instead of $\{X_n, n \geq 1\}$ in (23), we can obtain (24).

In particular, if $\hat{\mathbb{E}} X_i = \hat{\mathbb{E}} X_i$, then (25) follows from (23), and

$$\mathbb{V}\left(\left|\sum_{i=1}^n a_{ni}(X_i - \hat{E}X_i)\right| > \varepsilon\right) \leq \mathbb{V}\left(\sum_{i=1}^n a_{ni}(X_i - \hat{E}X_i) > \varepsilon\right) + \mathbb{V}\left(\sum_{i=1}^n a_{ni}(X_i - \hat{E}X_i) < -\varepsilon\right). \quad (53)$$

Together with (23)–(25) holds. The proof of Theorem 1 is completed.

We give an example satisfying the conditions of Theorem 1 and obtain corresponding results.

Example 1. Let $\{X_n; n \geq 1\}$ be a sequence of WA and identically distributed random variables with dominating coefficient $g(n) = n^{(1/2)}$ under sublinear expectations and $\mathbb{E}$ is countably subadditive. Assume $a_{ni} = (in)^{-1}$ and $\mathbb{V}(|X| > x) \sim (1/x^{1+\delta_1})$ for some $\delta_1 > 0$. Then, for all $\varepsilon > 0$, it will show that

$$\begin{aligned} \sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\sum_{i=1}^n (in)^{-1}(X_i - \hat{E}X_i) > \varepsilon\right) &< \infty, \\ \sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\sum_{i=1}^n (in)^{-1}(X_i - \hat{E}X_i) < -\varepsilon\right) &< \infty. \end{aligned} \quad (54)$$

If $\hat{E}X_i = \hat{E}X_j$, then

$$\sum_{n=1}^{\infty} n^{-1} \mathbb{V}\left(\left|\sum_{i=1}^n (in)^{-1}(X_i - \hat{E}X_i)\right| > \varepsilon\right) < \infty. \quad (55)$$

Obviously, $g(x) = x^{(1/2)}$ satisfies (A.1). $\exists 0 < \delta < \delta_1$, we have $C_{\mathbb{V}}(|X|^{1+\delta}) \sim \int_1^{\infty} \mathbb{V}(|X|^{1+\delta} > x) dx \sim \int_1^{\infty} (1/x^{(1+\delta)/2}) (1+\delta) dx < \infty$. Furthermore, $\hat{E}$ is countably subadditive, and by [6], then we have $\hat{E}(|X|^{1+\delta}) \leq C_{\mathbb{V}}(|X|^{1+\delta}) < \infty$. Therefore, all the conditions of Theorem 1 are satisfied; then, Theorem 1 holds.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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