Research Article

Complete Convergence for END Random Variables under Sublinear Expectations

Qunying Wu

College of Science, Guilin University of Technology, Guilin 541004, China

Correspondence should be addressed to Qunying Wu; wqy666@glut.edu.cn

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In this paper, the complete convergence theorems of partial sums and weighted sums for extended negatively dependent random variables in sublinear expectation spaces have been studied and established. Our results extend the corresponding results of classical probability spaces to the case of sublinear expectation spaces.

1. Introduction

The classical probability and statistics modeling is based on the linear basis of expectation and probability, and their important theoretical basis is the additivity of probability and expectation. However, in many application fields, because many uncertain factors can not be well solved, in order to better simulate this uncertainty phenomenon, we need new tools. The sublinear expectation space established by Peng ([1], [2]) belongs to this kind of tool. Sublinear expectation space is very useful in the simulation of uncertainty. They are widely used in statistics, finance, and insurance and have been studied by many probability statisticians since it came into being. For example, Peng ([1–5]) gave the basic structure, basic properties, and central limit theorem of sublinear expected space. Denis and Martini ([6]) studied a theoretical framework for the pricing of contingent claims in the presence of model uncertainty. Limit theorems for sublinear expectations have raised a large number of issues of interest recently. Marinacci ([7]) and Zhang (2016a [8], 2016b [9], 2016c [10]) established some inequalities and almost sure convergence theorems. Hu (2016 [11]), Chen (2016 [12]), Wu and Jiang (2018 [13]), and Wu and Lu (2020 [14]) also studied strong limit theorem and law of the iterated logarithm.

The concept of complete convergence of sequence of random variables is introduced by Hsu and Robbins (1947 [15]). In view of Borel–Cantelli lemma, complete convergence means almost sure convergence. Therefore, complete

convergence is one of the most important problems in probability limit theory. Many of its related results have been obtained in the probability space, for example, Wu (2010 [16]), Hu et al. (2012 [17]), Wang et al. (2014 [18]), Wang et al. (2015 [19]), Liu et al. (2015 [20]), Chen and Sung (2016 [21]), Tan et al. (2016 [22]), Wu and Jiang (2016 [23]), and Shen et al. (2017 [24]). Recently, Yu and Wu (2018 [25]), Wang and Wu (2019 [26]), and Deng and Wang (2020 [27]) obtained complete convergence theorem under sublinear expectations. In this paper, we study the complete convergence for extended negatively dependent random variables under sublinear expectation. As a result, the corresponding complete convergence theorems of the probability space have been generalized to the sublinear expectation space context.

Because sublinear expectation space relaxes the additivity of probability and expectation of traditional probability space and expands its application scope, and it is of great theoretical and practical significance to extend the limit theorem of traditional probability to space to sublinear expectation space. At the same time, because the sublinear expectation is nonadditive, the study of its limit theory becomes very complex and difficult.

2. Related Concepts and Lemmas

In the sublinear expectation space, many basic concepts need to be defined, such as sublinear expectation, capacity, independence, and identical distribution. This section gives a

brief introduction to these concepts, which can be seen in detail in the works of Peng (1997 [1], 1999 [2]).

Let $(\Omega, \mathcal{F})$ be a given measurable space and let $\mathcal{H}$ be a linear space of real functions defined on $(\Omega, \mathcal{F})$ such that if $X_1, \dots, X_n \in \mathcal{H}$, then $\varphi(X_1, \dots, X_n) \in \mathcal{H}$ for each $\varphi \in C_{l,\text{Lip}}(\mathbb{R}_n)$, where $C_{l,\text{Lip}}(\mathbb{R}_n)$ denotes the linear space of local Lipschitz functions.

Definition 1. A sublinear expectation $\hat{\mathbb{E}}$ on $\mathcal{H}$ is a function $\hat{\mathbb{E}}: \mathcal{H} \rightarrow [-\infty, \infty]$ satisfying the following properties: for all $X, Y \in \mathcal{H}$, we have

- (i) Monotonicity: if $X \geq Y$, then $\hat{\mathbb{E}}X \geq \hat{\mathbb{E}}Y$
- (ii) Constant preserving: $\hat{\mathbb{E}}c = c$
- (iii) Subadditivity: $\hat{\mathbb{E}}(X + Y) \leq \hat{\mathbb{E}}X + \hat{\mathbb{E}}Y$
- (iv) Positive homogeneity: $\hat{\mathbb{E}}(\lambda X) = \lambda \hat{\mathbb{E}}X$, $\lambda \geq 0$

The triple $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$ is called a sublinear expectation space.

The conjugate expectation $\hat{\mathbb{E}}$ of $\hat{\mathbb{E}}$ by $\hat{\mathbb{E}}X := -\hat{\mathbb{E}}(-X)$, $\forall X \in \mathcal{H}$.

Definition 2. Let $\mathcal{G} \subset \mathcal{F}$. A function $V: \mathcal{G} \rightarrow [0, 1]$ is called a capacity if

$$\begin{aligned} V(\emptyset) &= 0, \\ V(\Omega) &= 1, \\ V(A) &\leq V(B), \quad \text{for } \forall A \subseteq B, A, B \in \mathcal{G}. \end{aligned} \quad (1)$$

It is called to be subadditive if $V(A \cup B) \leq V(A) + V(B)$ for all $A, B \in \mathcal{G}$ with $A \cup B \in \mathcal{G}$. In the sublinear space $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, we denote a pair $(\mathbb{V}, \nu)$ of capacities by

$$\begin{aligned} \mathbb{V}(A) &:= \inf \{ \hat{\mathbb{E}}\xi; I(A) \leq \xi, \xi \in \mathcal{H} \}, \\ \nu(A) &:= 1 - \mathbb{V}(A^c), \quad \forall A \in \mathcal{F}, \end{aligned} \quad (2)$$

where $I(\cdot)$ denotes an indicator function and A^c is the complement set of A . By definition of $\mathbb{V}$ and ν , it is obvious that $\mathbb{V}$ is subadditive, and $\nu(A) \leq \mathbb{V}(A)$ for any $A \in \mathcal{F}$.

The corresponding Choquet integrals/expectations $(C_{\mathbb{V}}, C_{\nu})$ are defined by

$$C_V(X) := \int_0^\infty V(X > x)dx + \int_{-\infty}^0 (V(X > x) - 1)dx, \quad (3)$$

where V is replaced by $\mathbb{V}$ and ν , respectively.

Some basic useful properties of the sublinear expected space are given below, which are described in detail in the works of Wu and Jiang (2018 [13]).

Proposition 1

(i) For all $X, Y \in \mathcal{H}$,

$$\begin{aligned} \hat{\mathbb{E}}X &\leq \hat{\mathbb{E}}X, \\ \hat{\mathbb{E}}(X + c) &= \hat{\mathbb{E}}X + c, \\ |\hat{\mathbb{E}}(X - Y)| &\leq \hat{\mathbb{E}}|X - Y|, \\ \hat{\mathbb{E}}(X - Y) &\geq \hat{\mathbb{E}}X - \hat{\mathbb{E}}Y. \end{aligned} \quad (4)$$

(ii) If $\hat{\mathbb{E}}Y = \hat{\mathbb{E}}Y$, then $\hat{\mathbb{E}}(X + aY) = \hat{\mathbb{E}}X + a\hat{\mathbb{E}}Y$ for any $a \in \mathbb{R}$.

(iii)

$$\begin{aligned} \hat{\mathbb{E}}f &\leq \mathbb{V}(A) \leq \hat{\mathbb{E}}g, \\ \hat{\mathbb{E}}f &\leq \nu(A) \leq \hat{\mathbb{E}}g, \quad \text{if } f \leq I(A) \leq g, f, g \in \mathcal{H}. \end{aligned} \quad (5)$$

(iv) Markov inequality: $\forall X \in \mathcal{H}$:

$$\mathbb{V}(|X| \geq x) \leq \frac{\hat{\mathbb{E}}(|X|^p)}{x^p}, \quad \forall x > 0, p > 0. \quad (6)$$

(v) Hölder inequality: $\forall X, Y \in \mathcal{H}, p, q > 1$, satisfying $p^{-1} + q^{-1} = 1$:

$$\hat{\mathbb{E}}(|XY|) \leq (\hat{\mathbb{E}}(|X|^p))^{1/p} (\hat{\mathbb{E}}(|Y|^q))^{1/q}. \quad (7)$$

(vi) Jensen inequality: $\forall X \in \mathcal{H}$:

$$(\hat{\mathbb{E}}(|X|^r))^{1/r} \leq (\hat{\mathbb{E}}(|X|^s))^{1/s} \quad \text{for } 0 < r \leq s. \quad (8)$$

Definition 3 (see [2, 8])

(i) Identical distribution: let X_1 and X_2 be two random variables defined, respectively, in sublinear expectation spaces $(\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1)$ and $(\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2)$. They are called identically distributed if

$$\hat{\mathbb{E}}_1(\varphi(X_1)) = \hat{\mathbb{E}}_2(\varphi(X_2)), \quad \forall \varphi \in C_{l,\text{Lip}}(\mathbb{R}), \quad (9)$$

whenever the subexpectations are finite. A sequence $\{X_n; n \geq 1\}$ of random variables is said to be identically distributed if, for each $i \geq 1$, X_i and X_1 are identically distributed.

(ii) Extended negatively dependent: a sequence of random variables $\{X_n; n \geq 1\}$ is said to be upper (resp. lower) extended negatively dependent if there is some dominating constant $K \geq 1$ such that

$$\hat{\mathbb{E}}\left(\prod_{i=1}^n \varphi_i(X_i)\right) \leq K \prod_{i=1}^n \hat{\mathbb{E}}(\varphi_i(X_i)), \quad \forall n \geq 2, \quad (10)$$

whenever the nonnegative functions $\varphi_i(x) \in C_{l,\text{Lip}}(\mathbb{R})$, $i = 1, 2, \dots$, are all nondecreasing (resp. all nonincreasing). They are called extended negatively dependent if they are both upper extended negatively dependent and lower extended negatively dependent.

It is obvious that if $\{X_n; n \geq 1\}$ is a sequence of upper (resp. lower) extended negatively dependent random variables and functions $f_1(x), f_2(x), \dots \in C_{l,\text{Lip}}(\mathbb{R})$ are all nondecreasing (resp. all nonincreasing), then $\{f_n(X_n); n \geq 1\}$ is also a sequence of upper (resp. lower) extended negatively dependent random variables.

In the following, let $\{X_n; n \geq 1\}$ be a sequence of random variables in $(\Omega, \mathcal{H}, \hat{\mathbb{E}})$, and $S_n = \sum_{i=1}^n X_i$. The symbol c stands for a generic positive constant which may differ from one

place to another. Let $a_n \ll b_n$ denote that there exists a constant $c > 0$ such that $a_n \leq cb_n$ for sufficiently large n .

To prove our results, we need the following two lemmas.

Lemma 1 (see Theorem 3.1 in [8]). *Let $\{X_k; k \geq 1\}$ be a sequence of upper extended negatively dependent random variables in $(\Omega, \mathcal{H}, \mathbb{E})$ with $\mathbb{E}X_k \leq 0$. Then, for any $x > 0$,*

Lemma 2. *Suppose $X \in \mathcal{H}, \alpha > 0$.*

(i) *Then, for any $c > 0$,*

$$\mathbb{V}(S_n \geq x) \leq (1 + Ke) \frac{\sum_{k=1}^n \mathbb{E}X_k^2}{x^2}. \quad (11)$$

$$C_{\mathbb{V}}(|X|^{2/\alpha}) < \infty \Leftrightarrow \sum_{n=1}^{\infty} \mathbb{V}(|X| > cn^{\alpha/2}) < \infty \Leftrightarrow \sum_{n=1}^{\infty} n\mathbb{V}(|X| > cn^{\alpha}) < \infty. \quad (12)$$

(ii) *If $C_{\mathbb{V}}(|X|^{2/\alpha}) < \infty$, then, for any constant $c > 0$ and integer $\theta > 1$,*

$$\sum_{k=1}^{\infty} \theta^{2k} \mathbb{V}(|X| > c \cdot \theta^{ak}) < \infty, \quad (13)$$

$$\begin{aligned} \lim_{n \rightarrow \infty} n\mathbb{V}(|X| > cn^{\alpha/2}) &= 0, \\ \lim_{n \rightarrow \infty} n^{4/3} \mathbb{V}(|X| > cn^{2\alpha/3}) &= 0. \end{aligned} \quad (14)$$

Proof

(i) For any $c > 0$, it can be obtained by the definition of $C_{\mathbb{V}}(|X|^{2/\alpha})$ and the substitution of integral variables that

$$\begin{aligned} C_{\mathbb{V}}(|X|^{2/\alpha}) &= \int_0^{\infty} \mathbb{V}(|X| > x^{\alpha/2}) dx \\ &= c^{2/\alpha} \int_0^{\infty} \mathbb{V}(|X| > cy^{\alpha/2}) dy \quad (\text{let } y = c^{\alpha/2}x) \\ &= 2c^{2/\alpha} \int_0^{\infty} z\mathbb{V}(|X| > cz^{\alpha}) dz. \quad (\text{let } z = y^{1/2}). \end{aligned} \quad (15)$$

Because $\mathbb{V}(|X| > cy^{\alpha/2})$ is nonnegative and monotonically decreasing, according to the integral convergence method, we can obtain

$$C_{\mathbb{V}}(|X|^{2/\alpha}) < \infty \Leftrightarrow \sum_{n=1}^{\infty} \mathbb{V}(|X| > cn^{\alpha/2}) < \infty. \quad (16)$$

Also, note that

$$\int_0^{\infty} z\mathbb{V}(|X| > cz^{\alpha}) dz = \sum_{n=1}^{\infty} \int_{n-1}^n z\mathbb{V}(|X| > cz^{\alpha}) dz. \quad (17)$$

This implies that

$$\frac{1}{2} \sum_{n=1}^{\infty} n\mathbb{V}(|X| > cn^{\alpha}) \leq \int_0^{\infty} z\mathbb{V}(|X| > cz^{\alpha}) dz \leq 1 + 2 \sum_{n=1}^{\infty} n\mathbb{V}(|X| > cn^{\alpha}). \quad (18)$$

Hence,

$$\int_0^{\infty} z\mathbb{V}(|X| > cz^{\alpha}) dz < \infty \Leftrightarrow \sum_{n=1}^{\infty} n\mathbb{V}(|X| > cn^{\alpha}) < \infty. \quad (19)$$

From combination (16), we get (12) setup.

(ii) Note that, for any integer $\theta > 1$,

$$\begin{aligned} \sum_{n=1}^{\infty} n\mathbb{V}(|X| > cn^{\alpha}) &\geq \sum_{k=1}^{\infty} \sum_{\theta^{k-1} \leq n < \theta^k} \theta^{k-1} \mathbb{V}(|X| > c\theta^{ak}) \\ &= \theta^{-2} (\theta - 1) \sum_{k=1}^{\infty} \theta^{2k} \mathbb{V}(|X| > c\theta^{ak}). \end{aligned} \quad (20)$$

Hence, by (12), $C_{\mathbb{V}}(|X|^{2/\alpha}) < \infty$ implies (13).

By $\int_0^{\infty} \mathbb{V}(|X| > cx^{\alpha/2}) dx < \infty$ and $\mathbb{V}(|X| > cx^{\alpha/2}) \downarrow$, we have

$$\lim_{x \rightarrow \infty} x\mathbb{V}(|X| > cx^{\alpha/2}) = 0. \quad (21)$$

This is equivalent to

$$\lim_{x \rightarrow \infty} x^{4/3} \mathbb{V}(|X| > cx^{2\alpha/3}) = 0. \quad (22)$$

Therefore, (14) holds. $\square$

3. Complete Convergence Theorems

In the sublinear expectation space, we define that $X_n \longrightarrow X$ completely converges as follows.

Definition 4. A sequence of random variables $\{X_n; n \geq 1\}$ is said to completely converge to X , denoted by $X_n \xrightarrow{c} X$ as $n \longrightarrow \infty$ if $\sum_{n=1}^{\infty} \mathbb{V}(|X_n - X| > \epsilon) < \infty$ for any $\epsilon > 0$.

The purpose of this paper is to extend corresponding results from the probability space to sublinear expectation space. Our results are as follows.

Theorem 1. Let $\{X_n; n \geq 1\}$ be a sequence of upper extended negatively dependent and identically distributed random variables. Let, for some $0 < \alpha \leq 1$,

$$\widehat{\mathbb{E}}(|X_1|^{2/\alpha}) \leq C_{\mathbb{V}}(|X_1|^{2/\alpha}) < \infty, \quad (23)$$

$$0 < a_{nk} \leq cn^{-\alpha} \quad \text{for } k \leq n \\ \text{and some } 0 < c < \infty, a_{nk} = 0 \quad \text{for } k > n, \quad (24)$$

$$c_n := \sum_{k=1}^n a_{nk}^2 = o(\log^{-1} n). \quad (25)$$

Then,

$$\sum_{n=1}^{\infty} \mathbb{V} \left(\sum_{k=1}^n a_{nk} (X_k - \widehat{\mathbb{E}} X_k) > \epsilon \right) < \infty \quad \text{for any } \epsilon > 0. \quad (26)$$

Furthermore, if $\{X_n; n \geq 1\}$ is extended negatively dependent, then

$$\sum_{n=1}^{\infty} \mathbb{V} \left(\sum_{k=1}^n a_{nk} (X_k - \widehat{\mathbb{E}} X_k) < -\epsilon \right) < \infty \quad \text{for any } \epsilon > 0. \quad (27)$$

In particular, if $\{X_n; n \geq 1\}$ is extended negatively dependent and $\widehat{\mathbb{E}} X_k = \widehat{\mathbb{E}} X_k$, then

$$\sum_{k=1}^n a_{nk} (X_k - \widehat{\mathbb{E}} X_k) \xrightarrow{c} 0. \quad (28)$$

Remark 1. It is known by Lemma 4.5 (iii) in [9] that if $\lim_{c \rightarrow \infty} \widehat{\mathbb{E}}(|X|^{2/\alpha} - c)^+ = 0$ or $\widehat{\mathbb{E}}$ is countably subadditive, then $\widehat{\mathbb{E}}(|X_1|^{2/\alpha}) \leq C_{\mathbb{V}}(|X_1|^{2/\alpha})$.

Theorem 2. Let $\{X_n; n \geq 1\}$ be a sequence of extended negatively dependent and identically distributed random variables. Let, for some $\alpha > 1$,

$$C_{\mathbb{V}}(|X_1|^{2/\alpha}) < \infty. \quad (29)$$

Then,

$$\frac{S_n}{n^\alpha} \xrightarrow{c} 0. \quad (30)$$

Remark 2. Theorems 1 and 2 extend the corresponding complete convergence results from the probability space to sublinear expectation space.

Proof of Theorem 1. By the Jensen inequality, $\widehat{\mathbb{E}} X_1^2 \leq (\widehat{\mathbb{E}} |X_1|^{2/\alpha})^\alpha < \infty$ for $0 < \alpha \leq 1$. Without loss of generality, we can assume that $\widehat{\mathbb{E}} X_1^2 = 1$.

For upper extended negatively dependent random variables $\{X_n; n \geq 1\}$, in order to ensure that the truncated random variables are also upper extended negatively dependent, we need that truncated functions belong to $C_{l,\text{Lip}}$

and are nondecreasing. Let $f_c(x) := xI(x \leq c) + cI(x > c)$ for any $c > 0$; for any $1 \leq k \leq n, n \geq 1$, let

$$X_{nk}^{(1)} := f_{n^{-\alpha/3} a_{nk}^{-1}}(X_k) = X_k I(X_k \leq n^{-\alpha/3} a_{nk}^{-1}) \\ + n^{-\alpha/3} a_{nk}^{-1} I(X_k > n^{-\alpha/3} a_{nk}^{-1}), \\ X_{nk}^{(2)} := (X_k - n^{-\alpha/3} a_{nk}^{-1}) I\left(X_k > \frac{\epsilon a_{nk}^{-1}}{4}\right), \quad (31)$$

$$X_{nk}^{(3)} := X_k - X_{nk}^{(1)} - X_{nk}^{(2)} \\ = (X_k - n^{-\alpha/3} a_{nk}^{-1}) I\left(n^{-\alpha/3} a_{nk}^{-1} < X_k < \frac{\epsilon a_{nk}^{-1}}{4}\right).$$

Then, $\{X_{nk}^{(1)}; 1 \leq k \leq n, n \geq 1\}$ is also a sequence of upper extended negatively dependent random variables by $f_c(x) \in C_{l,\text{Lip}}$ and $f_c(x)$ being nondecreasing. Define

$$T_n^{(i)} := \sum_{k=1}^n a_{nk} X_{nk}^{(i)}, \quad i = 1, 2, 3, \\ \bar{T}_n^{(1)} := \sum_{k=1}^n a_{nk} (X_{nk}^{(1)} - \widehat{\mathbb{E}}(X_{nk}^{(1)})). \quad (32)$$

Then,

$$\sum_{k=1}^n a_{nk} (X_k - \widehat{\mathbb{E}} X_k) = \bar{T}_n^{(1)} + T_n^{(2)} + T_n^{(3)} + J_n, \quad (33)$$

where $J_n = \sum_{k=1}^n a_{nk} (\widehat{\mathbb{E}}(X_{nk}^{(1)}) - \widehat{\mathbb{E}} X_k)$.

Therefore, in order to prove (26), it suffices to verify that, for any $\epsilon > 0$,

$$\sum_{n=1}^{\infty} \mathbb{V}(\bar{T}_n^{(1)} > \epsilon) < \infty, \quad (34)$$

$$\sum_{n=1}^{\infty} \mathbb{V}(T_n^{(i)} > \epsilon) < \infty, \quad i = 2, 3, \quad (35)$$

$$J_n \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (36)$$

It shall be noted that, in the probability space, there is an equality: $E I(|X| \leq a) = P(|X| \leq a)$; however, in the sublinear expectation space, $\widehat{\mathbb{E}}$ is defined through continuous functions in $C_{l,\text{Lip}}$ and the indicator function $I(|X| \leq a)$ may be discontinuous. Therefore, the expression $\widehat{\mathbb{E}} I(|X| \leq a)$ may not exist. This needs to modify the indicator function by functions in $C_{l,\text{Lip}}$. To this end, we define the function $g(x) \in C_{l,\text{Lip}}(\mathbb{R})$ as follows.

For $0 < \mu < 1$, let $g(x) \in C_{l,\text{Lip}}(\mathbb{R})$ be a nonincreasing function such that $0 \leq g(x) \leq 1$ for all x and $g(x) = 1$ if $x \leq \mu$ and $g(x) = 0$ if $x > 1$. Then,

$$I(x \leq \mu) \leq g(x) \leq I(x \leq 1), \\ I(x > 1) \leq 1 - g(x) \leq I(x > \mu). \quad (37)$$

Note that

$$\begin{aligned}
|X_k - X_{nk}^{(1)}| &= (X_k - n^{-\alpha/3} a_{nk}^{-1}) I(X_k > n^{-\alpha/3} a_{nk}^{-1}) \\
&\leq |X_k| I(|X_k| > n^{-\alpha/3} a_{nk}^{-1} \geq cn^{2\alpha/3}) \\
&\ll \frac{|X_k|^{2/\alpha}}{n^{(2\alpha(2/(\alpha-1)))/3}}.
\end{aligned} \tag{38}$$

By (23), (24) and (36) follow that

$$\begin{aligned}
|J_n| &\leq \sum_{k=1}^n a_{nk} \widehat{\mathbb{E}} |X_k - X_{nk}^{(1)}| \leq \sum_{k=1}^n a_{nk} \frac{\widehat{\mathbb{E}} |X_1|^{\alpha/2}}{n^{(4/3)-(2\alpha/3)}} \\
&\ll n^{-(\alpha/3)-(1/3)} \longrightarrow 0, \quad n \longrightarrow \infty.
\end{aligned} \tag{39}$$

Now, we prove that (34). Let $\tilde{X}_{nk}^{(1)} = X_{nk}^{(1)} - \widehat{\mathbb{E}} X_{nk}^{(1)}$. Then, $\tilde{T}_n^{(1)} = \sum_{k=1}^n a_{nk} \tilde{X}_{nk}^{(1)}$. Fix $n \geq 1$. Let $u = \min((\epsilon/8c_n), (n^{\alpha/3}/2))$. By $X_{nk}^{(1)} \leq n^{-\alpha/3} a_{nk}^{-1}$, $|X_{nk}^{(1)}| \leq |X_k|$, (24), and Jensen inequality, we obtain

$$\begin{aligned}
ua_{nk} \tilde{X}_{nk}^{(1)} &\leq ua_{nk} X_{nk}^{(1)} + ua_{nk} \widehat{\mathbb{E}} |X_{nk}^{(1)}| \\
&\leq \frac{1}{2} + ua_{nk} (\widehat{\mathbb{E}} X_1^2)^{1/2} \leq \frac{1}{2} + \frac{cn^{-2\alpha/3}}{2} \leq 1.
\end{aligned} \tag{40}$$

It follows that

$$\begin{aligned}
\exp(ua_{nk} \tilde{X}_{nk}^{(1)}) &= \exp(ua_{nk} \tilde{X}_{nk}^{(1)} I(ua_{nk} \tilde{X}_{nk}^{(1)} < -1) + ua_{nk} \tilde{X}_{nk}^{(1)} I(|ua_{nk} \tilde{X}_{nk}^{(1)}| \leq 1)) \\
&= \exp(ua_{nk} \tilde{X}_{nk}^{(1)} I(ua_{nk} \tilde{X}_{nk}^{(1)} < -1)) \exp(ua_{nk} \tilde{X}_{nk}^{(1)} I(|ua_{nk} \tilde{X}_{nk}^{(1)}| \leq 1)) \\
&\leq \begin{cases} 1, & ua_{nk} \tilde{X}_{nk}^{(1)} < -1, \\ \exp(ua_{nk} \tilde{X}_{nk}^{(1)}), & |ua_{nk} \tilde{X}_{nk}^{(1)}| \leq 1, \end{cases} \\
&= \begin{cases} 1 + ua_{nk} \tilde{X}_{nk}^{(1)} + |ua_{nk} \tilde{X}_{nk}^{(1)}|, & ua_{nk} \tilde{X}_{nk}^{(1)} < -1, \\ 1 + \sum_{j=1}^{\infty} \frac{(ua_{nk} \tilde{X}_{nk}^{(1)})^j}{j!}, & |ua_{nk} \tilde{X}_{nk}^{(1)}| \leq 1, \end{cases} \\
&\leq \begin{cases} 1 + ua_{nk} \tilde{X}_{nk}^{(1)} + u^2 a_{nk}^2 (\tilde{X}_{nk}^{(1)})^2, & ua_{nk} \tilde{X}_{nk}^{(1)} < -1, \\ 1 + ua_{nk} \tilde{X}_{nk}^{(1)} + \frac{(ua_{nk} \tilde{X}_{nk}^{(1)})^2}{2} \left(1 + 2 \sum_{j=3}^{\infty} \frac{1}{j!}\right), & |ua_{nk} \tilde{X}_{nk}^{(1)}| \leq 1, \end{cases} \\
&\leq 1 + ua_{nk} \tilde{X}_{nk}^{(1)} + u^2 a_{nk}^2 (\tilde{X}_{nk}^{(1)})^2.
\end{aligned} \tag{41}$$

Hence, by $\widehat{\mathbb{E}} \tilde{X}_{nk}^{(1)} = 0$, and $\widehat{\mathbb{E}} (\tilde{X}_{nk}^{(1)})^2 \leq 2(\widehat{\mathbb{E}} (X_{nk}^{(1)})^2 + (\widehat{\mathbb{E}} X_{nk}^{(1)})^2) \leq 4\widehat{\mathbb{E}} (X_{nk}^{(1)})^2 \leq 4\widehat{\mathbb{E}} X_1^2 = 4$, we obtain

$$\widehat{\mathbb{E}} \left(\exp(ua_{nk} \tilde{X}_{nk}^{(1)}) \right) \leq 1 + 4u^2 a_{nk}^2 \leq \exp(4u^2 a_{nk}^2). \tag{42}$$

Since $\{ua_{nk} \tilde{X}_{nk}^{(1)}, n \geq 1, k \leq n\}$ is also upper extended negatively dependent, it follows from (25) that

$$\begin{aligned}
\widehat{\mathbb{E}} \left(\exp(u \tilde{T}_n^{(1)}) \right) &= \widehat{\mathbb{E}} \left(\prod_{k=1}^n \exp(ua_{nk} \tilde{X}_{nk}^{(1)}) \right) \\
&\leq K \prod_{k=1}^n \widehat{\mathbb{E}} \left(\exp(ua_{nk} \tilde{X}_{nk}^{(1)}) \right) \\
&\ll \prod_{k=1}^n \exp(4u^2 a_{nk}^2) = \exp(4u^2 c_n).
\end{aligned} \tag{43}$$

By the Markov inequality,

$$\mathbb{V}(\tilde{T}_n^{(1)} > \epsilon) \leq \exp(-\epsilon u) \widehat{\mathbb{E}} \exp(u \tilde{T}_n^{(1)}) \ll \exp(-\epsilon u + 4u^2 c_n). \tag{44}$$

Let $\mathbb{N}_1 := \{n; (\epsilon/4c_n) > n^{\alpha/3}\}$ and $\mathbb{N}_2 := \{n; (\epsilon/4c_n) \leq n^{\alpha/3}\}$. If $n \in \mathbb{N}_1$, i.e., $(\epsilon/4c_n) > n^{\alpha/3}$, then, by the definition of u , we have $u = (n^{\alpha/3}/2)$ and $-\epsilon u + 4u^2 c_n \leq (-\epsilon n^{\alpha/3}/4)$. Hence,

$$\sum_{n \in \mathbb{N}_1} \mathbb{V}(\tilde{T}_n^{(1)} > \epsilon) \ll \sum_{n=1}^{\infty} \exp\left(-\frac{\epsilon n^{\alpha/3}}{4}\right) < \infty. \tag{45}$$

If $n \in \mathbb{N}_2$, i.e., $(\epsilon/4c_n) \leq n^{\alpha/3}$, then, by the definition of u , we have $u = \epsilon/8c_n$ and $-\epsilon u + 4u^2 c_n = -(\epsilon^2/16c_n)$. Note that $c_n < (\epsilon^2/32 \log n)$ for sufficiently large n from $c_n = o(\log^{-1} n)$. Hence,

$$\sum_{n \in \mathbb{N}_2} \mathbb{V}(\tilde{T}_n^{(1)} > \epsilon) \leq \sum_{n=1}^{\infty} \exp\left(-\frac{\epsilon^2}{16c_n}\right) \ll \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty. \tag{46}$$

That is that (34) holds from

$$\sum_{n=1}^{\infty} \mathbb{V}(\tilde{T}_n^{(1)} > \epsilon) = \sum_{n \in \mathbb{N}_1} \mathbb{V}(\tilde{T}_n^{(1)} > \epsilon) + \sum_{n \in \mathbb{N}_2} \mathbb{V}(\tilde{T}_n^{(1)} > \epsilon) < \infty. \tag{47}$$

Next, we prove that $\sum_{n=1}^{\infty} \mathbb{V}(T_n^{(2)} > \epsilon) < \infty$. By (24),

$$(T_n^{(2)} > \epsilon) \subset \bigcup_{k=1}^n \left(X_k > \frac{\epsilon a_{nk}^{-1}}{4} \right) \subset \bigcup_{k=1}^n (X_k > \epsilon n^\alpha (4c)^{-1} := bn^\alpha), \quad (48)$$

where $b = \epsilon(4c)^{-1} > 0$. It shall be noted that the identical distribution is defined under $\widehat{\mathbb{E}}$, not under $\mathbb{V}$ (see Definition 3). X_i identical distribution implies $\widehat{\mathbb{E}}(f(X_i)) = \widehat{\mathbb{E}}(f(X_1))$ for $f(\cdot) \in C_{l,\text{Lip}}(\mathbb{R})$, but does not imply $\mathbb{V}(f(X_i) \in A) = \mathbb{V}(f(X_1) \in A)$. Therefore, in the calculation of $\mathbb{V}(f(X_i) \in A)$, we need to convert $\mathbb{V}$ to $\widehat{\mathbb{E}}$. Thus, by (5) and (37),

$$\begin{aligned} \mathbb{V}(T_n^{(2)} > \epsilon) &\leq \sum_{k=1}^n \mathbb{V}(X_k > bn^\alpha) \leq \sum_{k=1}^n \widehat{\mathbb{E}}\left(1 - g\left(\frac{X_k}{bn^\alpha}\right)\right) \\ &= \sum_{k=1}^n \widehat{\mathbb{E}}\left(1 - g\left(\frac{X_1}{bn^\alpha}\right)\right) \leq \sum_{k=1}^n \mathbb{V}(X_1 > \mu bn^\alpha) \\ &\leq n\mathbb{V}(|X_1| > \mu bn^\alpha). \end{aligned} \quad (49)$$

Therefore, from (12) and (23),

$$\sum_{n=1}^{\infty} \mathbb{V}(T_n^{(2)} > \epsilon) \leq \sum_{n=1}^{\infty} n\mathbb{V}(|X_1| > \mu bn^\alpha) < \infty. \quad (50)$$

Lastly, we prove that $\sum_{n=1}^{\infty} \mathbb{V}(T_n^{(3)} > \epsilon) < \infty$. Since

$$\begin{aligned} (T_n^{(3)} > \epsilon) &:= \sum_{k=1}^n a_{nk} (X_k - n^{-\alpha/3} a_{nk}^{-1}) I\left(n^{-\alpha/3} a_{nk}^{-1} < X_k < \frac{\epsilon a_{nk}^{-1}}{4}\right) > \epsilon \\ &\subset \left(\sum_{k=1}^n a_{nk} X_k I\left(n^{-\alpha/3} a_{nk}^{-1} < X_k < \frac{\epsilon a_{nk}^{-1}}{4}\right) > \epsilon \right) \\ &\subset \left(\text{there exist at least 4 indices } k \text{ such that } X_k > a_{nk}^{-1} n^{-\alpha/3} \right). \end{aligned} \quad (51)$$

The last inclusion relation above holds that if there are only less than 4 indices k , such that $X_k > a_{nk}^{-1} n^{-\alpha/3}$, then, in the sum $\sum_{k=1}^n a_{nk} X_k I(n^{-\alpha/3} a_{nk}^{-1} < X_k < (\epsilon a_{nk}^{-1}/4))$, there are only less than 4 indices k , such that $a_{nk} X_k I(n^{-\alpha/3} a_{nk}^{-1} < X_k < (\epsilon a_{nk}^{-1}/4)) \neq 0$, and each term $a_{nk} X_k I(n^{-\alpha/3} a_{nk}^{-1} < X_k < (\epsilon a_{nk}^{-1}/4)) < (\epsilon/4)$, so $\sum_{k=1}^n a_{nk} X_k I(n^{-\alpha/3} a_{nk}^{-1} < X_k < (\epsilon a_{nk}^{-1}/4)) < \epsilon$. This is in contradiction with $\sum_{k=1}^n a_{nk} X_k I(n^{-\alpha/3} a_{nk}^{-1} < X_k < (\epsilon a_{nk}^{-1}/4)) > \epsilon$.

Therefore, by the definition of extended negatively dependent (5), (24), and (37), we have

$$\begin{aligned} \mathbb{V}(T_n^{(3)} > \epsilon) &\leq \mathbb{V}(\text{there exist at least 4 indices } k \text{ such that } a_{nk} X_k > n^{-\alpha/3}) \\ &\leq \sum_{1 \leq i_1 < \dots < i_4 \leq n} \mathbb{V}(a_{ni_1} X_{i_1} > n^{-\alpha/3}, a_{ni_2} X_{i_2} > n^{-\alpha/3}, a_{ni_3} X_{i_3} > n^{-\alpha/3}, a_{ni_4} X_{i_4} > n^{-\alpha/3}) \\ &\leq \sum_{1 \leq i_1 < \dots < i_4 \leq n} \widehat{\mathbb{E}}\left(\prod_{j=1}^4 \left(1 - g\left(a_{ni_j} n^{\alpha/3} X_{i_j}\right)\right)\right) \\ &\leq \sum_{1 \leq i_1 < \dots < i_4 \leq n} K \prod_{j=1}^4 \widehat{\mathbb{E}}\left(1 - g\left(a_{ni_j} n^{\alpha/3} X_{i_j}\right)\right) \\ &\leq \sum_{1 \leq i_1 < \dots < i_4 \leq n} K \prod_{j=1}^4 \widehat{\mathbb{E}}\left(1 - g\left(a_{ni_j} n^{\alpha/3} |X_{i_j}|\right)\right) \\ &\ll \sum_{1 \leq i_1 < \dots < i_4 \leq n} [\widehat{\mathbb{E}}(1 - g(cn^{-2\alpha/3} |X_1|))]^4 \\ &\leq C_n^4 \mathbb{V}^4(|X_1| > \mu c^{-1} n^{2\alpha/3}) \leq n^4 \mathbb{V}^4(|X_1| > \mu c^{-1} n^{2\alpha/3}). \end{aligned} \quad (52)$$

Thus,

$$\sum_{n=1}^{\infty} \mathbb{V}(T_n^{(3)} > \epsilon) \ll \sum_{n=1}^{\infty} n^{-4/3} (n^{4/3} \mathbb{V}(|X_1| > \mu c^{-1} n^{2\alpha/3}))^4 \ll \sum_{n=1}^{\infty} n^{-4/3} < \infty, \quad (53)$$

from (14). Together with (50), we obtain (35). This completes the proof of (26).

Furthermore, if $\{X_n; n \geq 1\}$ is extended negatively dependent, then $\{-X_n; n \geq 1\}$; also, satisfying the conditions of Theorem 1 and considering $\{-X_n; n \geq 1\}$ instead of $\{X_n; n \geq 1\}$ in (26), for any $\epsilon > 0$, we can obtain by $\hat{\mathbb{E}}(-X_k) = -\hat{\mathbb{E}}X_k$,

$$\sum_{n=1}^{\infty} \mathbb{V} \left(\sum_{k=1}^n a_{nk} (-X_k - \hat{\mathbb{E}}(-X_k)) > \epsilon \right) = \sum_{n=1}^{\infty} \mathbb{V} \left(- \sum_{k=1}^n a_{nk} (X_k - \hat{\mathbb{E}}X_k) > \epsilon \right) < \infty. \quad (54)$$

That is, (27) is established. $\square$

Proof of Theorem 2. We first prove that

$$\sum_{n=1}^{\infty} \mathbb{V}(S_n > \epsilon n^\alpha) < \infty, \quad \forall \epsilon > 0. \quad (55)$$

For any $c > 0$, let $h_c(x) := -cI(x < -c) + xI(|x| \leq c) + cI(x > c)$, for all $n \geq 1, 1 \leq k \leq n$, and let

$$\begin{aligned} Y_{nk} &:= h_{n^\alpha}(X_k) = -n^\alpha I(X_k < -n^\alpha) \\ &\quad + X_k I(|X_k| \leq n^\alpha) + n^\alpha I(X_k > n^\alpha), \\ Z_{nk} &:= X_k - Y_{nk} = (X_k + n^\alpha)I(X_k < -n^\alpha) \\ &\quad + (X_k - n^\alpha)I(X_k > n^\alpha). \end{aligned} \quad (56)$$

Then, $\{Y_{nk}; 1 \leq k \leq n, n \geq 1\}$ is also a sequence of upper extended negatively dependent random variables by $h_c(x) \in C_{l,\text{Lip}}$ and $h_c(x)$ being nondecreasing. Note that

$$S_n = \sum_{k=1}^n (Y_{nk} - \hat{\mathbb{E}}Y_{nk}) + \sum_{k=1}^n Z_{nk} + \sum_{k=1}^n \hat{\mathbb{E}}Y_{nk} := U_{n1} + U_{n2} + U_{n3}. \quad (57)$$

In order to prove (55), it suffices to verify that

$$\sum_{n=1}^{\infty} \mathbb{V}(U_{ni} > \epsilon n^\alpha) < \infty \quad \text{for any } \epsilon > 0, i = 1, 2. \quad (58)$$

$$\frac{U_{n3}}{n^\alpha} \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \quad (59)$$

We first prove that (59). For any $0 < a < b$, by (37), we have

$$I(a < x \leq b) = I(x \leq b) - I(x \leq a) \leq g\left(\frac{\mu x}{b}\right) - g\left(\frac{x}{a}\right), \quad (60)$$

$$I(-b \leq x < -a) = I\left(\frac{-x}{b} \leq 1\right) - I\left(\frac{-x}{a} \leq 1\right) \leq g\left(\frac{-\mu x}{b}\right) - g\left(\frac{-x}{a}\right). \quad (61)$$

For every n , there exists a k such that $2^{k-1} \leq n < 2^k$; thus, for any $r > 0$, by $g(x) \downarrow$, c_r inequality, (37) and (60), and (61), we obtain

$$\begin{aligned} |Y_{n1}|^r &\ll |X_1|^r I(|X_1| \leq n^\alpha) + n^{\alpha r} I(|X_1| > n^\alpha) \\ &\leq 1 + |X_1|^r \left(I(1 < X_1 \leq 2^{\alpha k}) + I(-2^{\alpha k} \leq X_1 < 1) \right) \\ &\quad + 2^{\alpha r k} \left(I(X_1 > 2^{\alpha(k-1)}) + I(-X_1 > 2^{\alpha(k-1)}) \right) \\ &= 1 + \sum_{j=1}^k |X_1|^r \left(I(2^{\alpha(j-1)} < X_1 \leq 2^{\alpha j}) + I(-2^{\alpha j} \leq X_1 < -2^{\alpha(j-1)}) \right) \\ &\quad + 2^{\alpha r k} \left(I(X_1 > 2^{\alpha(k-1)}) + I(-X_1 > 2^{\alpha(k-1)}) \right) \\ &\leq 1 + \sum_{j=1}^k |X_1|^r \left\{ \left(g\left(\frac{\mu X_1}{2^{\alpha j}}\right) - g\left(\frac{X_1}{2^{\alpha(j-1)}}\right) \right) + \left(g\left(\frac{-\mu X_1}{2^{\alpha j}}\right) - g\left(\frac{-X_1}{2^{\alpha(j-1)}}\right) \right) \right\} \\ &\quad + 2^{\alpha r k} \left\{ \left(1 - g\left(\frac{X_1}{2^{\alpha(k-1)}}\right) \right) + \left(1 - g\left(\frac{-X_1}{2^{\alpha(k-1)}}\right) \right) \right\}. \end{aligned} \quad (62)$$

Hence, by (5) and (37) and $g(\mu X_1/2^{\alpha j}) - g(X_1/2^{\alpha(j-1)}) \leq I(\mu 2^{\alpha(j-1)} \leq X_1 \leq \mu^{-1} 2^{\alpha j})$, $g(-\mu X_1/2^{\alpha j}) - g(-X_1/2^{\alpha(j-1)}) \leq I(-\mu^{-1} 2^{\alpha j} \leq X_1 \leq -\mu 2^{\alpha(j-1)})$,

$$\begin{aligned}
\widehat{\mathbb{E}}(|Y_{n1}|^r) &\ll 1 + \sum_{j=1}^k \left\{ \widehat{\mathbb{E}}|X_1|^r \left(g\left(\frac{\mu X_1}{2^{\alpha j}}\right) - g\left(\frac{X_1}{2^{\alpha(j-1)}}\right) \right) + \widehat{\mathbb{E}}|X_1|^r \left(g\left(\frac{-\mu X_1}{2^{\alpha j}}\right) - g\left(\frac{-X_1}{2^{\alpha(j-1)}}\right) \right) \right\} \\
&\quad + 2^{\alpha r k} \left\{ \widehat{\mathbb{E}}\left(1 - g\left(\frac{X_1}{2^{\alpha(k-1)}}\right)\right) + \widehat{\mathbb{E}}\left(1 - g\left(\frac{-X_1}{2^{\alpha(k-1)}}\right)\right) \right\} \\
&\leq 1 + \sum_{j=1}^k \left(\mu^{-r} 2^{\alpha r j} \mathbb{V}(|X_1| \geq \mu 2^{\alpha(j-1)}) + \mu^{-r} 2^{\alpha r j} \mathbb{V}(|X_1| \geq \mu 2^{\alpha(j-1)}) \right) \\
&\quad + 2^{\alpha r k} (\mathbb{V}(X_1 > \mu 2^{\alpha(k-1)}) + \mathbb{V}(X_1 < -\mu 2^{\alpha(k-1)})) \\
&\ll 1 + \sum_{j=1}^k 2^{\alpha r j} \mathbb{V}(|X_1| \geq \mu 2^{-\alpha} 2^{\alpha j}) + 2^{\alpha r k} \mathbb{V}(|X_1| > \mu 2^{-\alpha} 2^{\alpha k}) \\
&\ll \sum_{j=0}^k 2^{\alpha r j} \mathbb{V}(|X_1| \geq \mu 2^{-\alpha} 2^{\alpha j}).
\end{aligned} \tag{63}$$

Therefore, by $n^{-\alpha+1} \downarrow 0$ from $\alpha > 1$, for $2^{k-1} \leq n < 2^k$, we obtain

$$\begin{aligned}
n^{-\alpha} |U_{n3}| &\leq n^{1-\alpha} \widehat{\mathbb{E}}(|Y_{n1}|) \leq 2^{(k-1)(1-\alpha)} \widehat{\mathbb{E}}(|Y_{n1}|) \ll 2^{(1-\alpha)k} \sum_{j=0}^k 2^{\alpha j} \mathbb{V}(|X_1| \geq \mu 2^{-\alpha} 2^{\alpha j}) \\
&:= U_{n31}.
\end{aligned} \tag{64}$$

Next, we estimate U_{n31} . Noting that, by (13),

$$\sum_{j=0}^{\infty} \frac{2^{\alpha j}}{2^{(\alpha-1)j}} \mathbb{V}(|X_1| > 2^{\alpha(j-1)}) = \sum_{j=0}^{\infty} 2^j \mathbb{V}(|X_1| > 2^{-\alpha} 2^{\alpha j}) < \infty. \tag{65}$$

It follows that

$$U_{n31} \longrightarrow 0, \quad \text{as } n \longrightarrow \infty, \tag{66}$$

from $2^{(\alpha-1)j} \uparrow \infty$ and the Kronecker lemma. This combination, (59) and (64), was established.

Since $\{Y_{nk} - \widehat{\mathbb{E}}Y_{nk}, n \geq 1, k \leq n\}$ is also upper extended negatively dependent with $\widehat{\mathbb{E}}(Y_{nk} - \widehat{\mathbb{E}}Y_{nk}) = 0$. It follows from Lemma 1, (63), and (13) that

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{V}(U_{n1} > \epsilon n^{\alpha}) &\ll \sum_{n=1}^{\infty} n^{-2\alpha} \sum_{k=1}^n \widehat{\mathbb{E}}Y_{nk}^2 \\
&= \sum_{n=1}^{\infty} n^{1-2\alpha} \widehat{\mathbb{E}}Y_{n1}^2 = \sum_{k=1}^{\infty} \sum_{2^{k-1} \leq n < 2^k} n^{1-2\alpha} \widehat{\mathbb{E}}Y_{n1}^2
\end{aligned}$$

$$\begin{aligned}
&\ll \sum_{k=1}^{\infty} \sum_{2^{k-1} \leq n < 2^k} 2^{(k-1)(1-2\alpha)} \sum_{j=0}^k 2^{2\alpha j} \mathbb{V}(|X_1| \geq \mu 2^{-\alpha} 2^{\alpha j}) \\
&\ll \sum_{k=1}^{\infty} 2^{2(1-\alpha)} \sum_{j=0}^k 2^{2\alpha j} \mathbb{V}(|X_1| \geq \mu 2^{-\alpha} 2^{\alpha j}) \\
&\ll \sum_{j=0}^{\infty} 2^{2\alpha j} \sum_{k=j}^{\infty} 2^{2(1-\alpha)k} \mathbb{V}(|X_1| > \mu 2^{-\alpha} 2^{\alpha j}) \\
&\ll \sum_{j=0}^{\infty} 2^{2j} \mathbb{V}(|X_1| > \mu 2^{-\alpha} 2^{\alpha j}) \\
&< \infty.
\end{aligned} \tag{67}$$

By (5), (12), and (37),

$$\begin{aligned}
\sum_{n=1}^{\infty} \mathbb{V}(U_{n2} > \epsilon n^{\alpha}) &\ll \sum_{n=1}^{\infty} \mathbb{V}\left(\sum_{k=1}^n |Z_k| > \epsilon n^{\alpha}\right) \\
&\leq \sum_{n=1}^{\infty} \mathbb{V}(\text{there exists a } k \in [1, n] \text{ such that } X_k > n^{\alpha} \text{ or } X_k < -n^{\alpha}) \\
&\leq \sum_{n=1}^{\infty} \sum_{k=1}^n (\mathbb{V}(X_k > n^{\alpha}) + \mathbb{V}(X_k < -n^{\alpha})) \\
&\leq \sum_{n=1}^{\infty} \sum_{k=1}^n \left(\hat{\mathbb{E}}\left[1 - g\left(\frac{X_k}{n^{\alpha}}\right)\right] + \hat{\mathbb{E}}\left[1 - g\left(\frac{-X_k}{n^{\alpha}}\right)\right] \right) \\
&= \sum_{n=1}^{\infty} n \left(\hat{\mathbb{E}}\left[1 - g\left(\frac{X_1}{n^{\alpha}}\right)\right] + \hat{\mathbb{E}}\left[1 - g\left(\frac{-X_1}{n^{\alpha}}\right)\right] \right) \\
&\leq \sum_{n=1}^{\infty} n \mathbb{V}(|X_1| > \mu n^{\alpha}) \\
&< \infty.
\end{aligned} \tag{68}$$

From this combination, (67), we get (58). Therefore, (55) holds.

Because of $\{-X_n; n \geq 1\}$, also satisfying the conditions of Theorem 2 and considering $\{-X_n; n \geq 1\}$ instead of $\{X_n; n \geq 1\}$ in (55), we can obtain

$$\sum_{n=1}^{\infty} \mathbb{V}(-S_n > \epsilon n^{\alpha}) = \sum_{n=1}^{\infty} \mathbb{V}(S_n < -\epsilon n^{\alpha}) < \infty. \tag{69}$$

Hence,

$$\sum_{n=1}^{\infty} \mathbb{V}(|S_n| > \epsilon n^{\alpha}) \leq \sum_{n=1}^{\infty} \mathbb{V}(S_n > \epsilon n^{\alpha}) + \sum_{n=1}^{\infty} \mathbb{V}(S_n < -\epsilon n^{\alpha}) < \infty, \tag{70}$$

namely, (30) is established. This completes the Proof of Theorem 2. $\square$

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.

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